



**HAESE MATHEMATICS**

# **Mathematics**

## **Applications and Interpretation HL**



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for use with

# **IB Diploma Programme**

# **REVISION GUIDE**



# MATHEMATICS: APPLICATIONS AND INTERPRETATION HL REVISION GUIDE

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National Library of Australia Card Number & ISBN 978-1-925489-91-0

© Haese & Harris Publications 2020

Published by Haese Mathematics.  
152 Richmond Road, Marleston, SA 5033, AUSTRALIA

First Edition                      2020

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Typeset in Australia by Deanne Gallasch and Charlotte Frost. Typeset in Times Roman 10.

This book is available on Snowflake only.

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## FOREWORD

The aim of this Guide is to help you prepare for tests and the final examination for the Mathematics: Applications and Interpretation HL course.

This Guide covers all five Topics in the Mathematics: Applications and Interpretation HL syllabus. All of the relevant material from the Mathematics: Core Topics HL and the Mathematics: Applications and Interpretation HL textbooks is covered. However, it is a resource that can be used by any student, regardless of their main textbook.

For each Topic, there is a theory summary and a set of skill builder questions.

- The theory summaries highlight the important facts and concepts. They are intended to complement your textbook and International Baccalaureate formula booklet. When a formula can be found in the formula booklet, it may not be repeated in this Guide.
- The set of skill builder questions are designed to help consolidate your understanding of each Topic. They should be used to reinforce key ideas, and to identify any areas of weakness. Within each Topic, the questions are logically ordered according to the chapters of the textbook, so they can be used for test preparation.

Following the coverage of all five Topics, the Guide has 20 mixed questions sets, each containing 10 questions. Each set contains questions from every Topic, as well as cross-topic questions. It is recommended that you attempt all of the questions in a mixed questions set in one sitting, as this will give you practice in answering questions from a range of topics in a short time frame.

The Guide contains four trial examinations, written by IB teachers from around the world. Each trial examination contains three papers: Paper 1, which contains shorter questions, Paper 2, which contains longer questions, and Paper 3, which contains extended response problem-solving questions. This format is consistent with the Mathematics: Applications and Interpretation HL final examination. Full solutions are provided, but it is recommended that you work through a complete paper before checking the solutions.

The Guide concludes with some extra Paper 3 questions. These have been included to give you more practice at answering the extended, investigation-style questions you will encounter in your Paper 3 exam.

We recommend completing each trial examination under exam conditions. You are encouraged to print the formulae summary (see page 5), and have it alongside you as you complete the trial examinations.

- If you are having trouble with a question, it is often a good strategy to move on to other questions, and return to it later. Time management is very important during the examination, and too much time spent on a difficult question may mean that you do not leave yourself sufficient time to complete other questions.
- Set out your work clearly with full explanations. A correct answer with no working will not necessarily receive full marks.

- If you make a mistake, draw a single line through the work you want to replace. Do not cross out work until you have replaced it with something you consider better.
- Diagrams and graphs should be sufficiently large, well labelled, and clearly drawn.
- Remember to leave answers correct to three significant figures unless an exact answer is more appropriate or a different level of accuracy is requested in the question.
- Check for key words. If the word “hence” appears, then you must use the result you have just obtained. “Hence, or otherwise” means that you can use any method you like, although it is likely that the best method uses the previous result.
- It is important to read the question carefully. Rushing into a question may mean that you miss subtle points. Underlining key words may help.
- Remember that questions in the examination are often set so that, even if you cannot complete one part, the question can still be picked up in a later part.

After completing a trial examination, you should identify areas of weakness.

- Return to your notes or textbook and review any material you found challenging.
- Ask your teacher or a friend for help if further explanation is needed.
- Summarise each Topic. Summaries that you make yourself are the most valuable.
- If you have had difficulty with a question, try it again later. Do not just assume that you know how to do it once you have read the solution.

In addition to the formula booklet, your graphics display calculator is an essential aid.

- Make sure you are familiar with the model you will be using.
- In trigonometry questions, remember to check whether the graphics calculator should be in degrees or radians.
- Important features of graphs may be revealed by zooming in or out.
- When using your graphics calculator, it is always important to reflect on the reasonableness of the results.

We hope this Guide will help you structure your revision program effectively. Remember that good examination techniques will come from good examination preparation.

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# FORMULAE SUMMARY



## PRIOR LEARNING

Area of a parallelogram	$A = bh$ , where $b$ is the base, $h$ is the height
Area of a triangle	$A = \frac{1}{2}(bh)$ , where $b$ is the base, $h$ is the height
Area of a trapezoid	$A = \frac{1}{2}(a + b)h$ , where $a$ and $b$ are the parallel sides, $h$ is the height
Area of a circle	$A = \pi r^2$ , where $r$ is the radius
Circumference of a circle	$C = 2\pi r$ , where $r$ is the radius
Volume of a cuboid	$V = lwh$ , where $l$ is the length, $w$ is the width, $h$ is the height
Volume of a cylinder	$V = \pi r^2 h$ , where $r$ is the radius, $h$ is the height
Volume of a prism	$V = Ah$ , where $A$ is the area of the cross-section, $h$ is the height
Area of the curved surface of a cylinder	$A = 2\pi rh$ , where $r$ is the radius, $h$ is the height
Distance between two points $(x_1, y_1)$ and $(x_2, y_2)$	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints $(x_1, y_1)$ and $(x_2, y_2)$	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

## TOPIC 1: NUMBER AND ALGEBRA

### ARITHMETIC SEQUENCES

$$u_n = u_1 + (n - 1)d$$

$$S_n = \frac{n}{2}(2u_1 + (n - 1)d) \quad \text{or} \quad S_n = \frac{n}{2}(u_1 + u_n)$$

### GEOMETRIC SEQUENCES

$$u_n = u_1 r^{n-1}$$

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$$

$$S_\infty = \frac{u_1}{1 - r}, \quad |r| < 1$$

### COMPOUND INTEREST

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}, \quad \text{where}$$

$FV$  is the future value

$PV$  is the present value

$n$  is the number of years

$k$  is the number of compounding periods per year

$r\%$  is the nominal annual rate of interest

### EXPONENTS AND LOGARITHMS

$$a^x = b \Leftrightarrow x = \log_a b, \quad \text{where } a > 0, b > 0, a \neq 1$$

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^m = m \log_a x$$



PERCENTAGE ERROR

$\varepsilon = \left| \frac{V_A - V_E}{V_E} \right| \times 100\%$ ,    where  $V_E$  is the exact value and  $V_A$  is the approximate value.

COMPLEX NUMBERS

$$\begin{aligned} z &= a + bi \\ &= r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta \quad \{\text{Modulus-argument (polar) form}\} \\ &= re^{i\theta} \quad \{\text{Exponential (Euler) form}\} \end{aligned}$$

MATRICES

For  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ :

- $\det \mathbf{A} = |\mathbf{A}| = ad - bc$
- $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad \neq bc$

$\mathbf{M}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$ ,    where  $\mathbf{P}$  is the matrix of eigenvectors of  $\mathbf{M}$ , and  $\mathbf{D}$  is the diagonal matrix of its eigenvalues.

TOPIC 2: FUNCTIONS

STRAIGHT LINES

$$y = mx + c \quad \text{or} \quad ax + by + d = 0 \quad \text{or} \quad y - y_1 = m(x - x_1)$$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

QUADRATIC FUNCTIONS AND EQUATIONS

$$f(x) = ax^2 + bx + c \Rightarrow \text{axis of symmetry is } x = -\frac{b}{2a}$$
$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a \neq 0$$

Discriminant     $\Delta = b^2 - 4ac$

LOGISTIC FUNCTION

$$f(x) = \frac{L}{1 + Ce^{-kx}}, \quad L, k, C > 0$$



# TOPIC 3: GEOMETRY AND TRIGONOMETRY

## MEASUREMENT

Distance between two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
Coordinates of the midpoint of a line segment with endpoints $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
Volume of a right-pyramid	$V = \frac{1}{3}Ah$ , where $A$ is the area of the base, $h$ is the height
Volume of a right cone	$V = \frac{1}{3}\pi r^2h$ , where $r$ is the radius, $h$ is the height
Area of the curved surface of a cone	$A = \pi rl$ , where $r$ is the radius, $l$ is the slant height
Volume of a sphere	$V = \frac{4}{3}\pi r^3$ , where $r$ is the radius
Surface area of a sphere	$A = 4\pi r^2$ , where $r$ is the radius
Length of an arc	$l = \frac{\theta}{360} \times 2\pi r$ , where $\theta$ is the angle measured in degrees, $r$ is the radius $l = r\theta$ , where $\theta$ is the angle measured in radians, $r$ is the radius
Area of a sector	$A = \frac{\theta}{360} \times \pi r^2$ , where $\theta$ is the angle measured in degrees, $r$ is the radius $A = \frac{1}{2}r^2\theta$ , where $\theta$ is the angle measured in radians, $r$ is the radius

## TRIGONOMETRY

Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine rule	$c^2 = a^2 + b^2 - 2ab \cos C$ and $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
Area of a triangle	$A = \frac{1}{2}ab \sin C$
Identity for $\tan \theta$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$
Pythagorean identity	$\cos^2 \theta + \sin^2 \theta = 1$

## TRANSFORMATION MATRICES

Reflection in the line $y = (\tan \theta)x$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
Horizontal stretch with scale factor $k$	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Vertical stretch with scale factor $k$	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Enlargement with centre $(0, 0)$ , scale factor $k$	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Anticlockwise rotation of $\theta$ about $(0, 0)$	$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$



VECTORS

Magnitude of a vector	$ \mathbf{v}  = \sqrt{v_1^2 + v_2^2 + v_3^2}$ , where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
Scalar product	$\mathbf{v} \bullet \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$ , where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ , $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\mathbf{v} \bullet \mathbf{w} =  \mathbf{v}   \mathbf{w}  \cos \theta$ , where $\theta$ is the angle between $\mathbf{v}$ and $\mathbf{w}$
Angle between two vectors	$\cos \theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{ \mathbf{v}   \mathbf{w} }$
Vector equation of a line	$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$
Parametric form of the equation of a line	$x = x_0 + \lambda l$ , $y = y_0 + \lambda m$ , $z = z_0 + \lambda n$
Vector product	$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$ , where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ , $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $ \mathbf{v} \times \mathbf{w}  =  \mathbf{v}   \mathbf{w}  \sin \theta$ , where $\theta$ is the angle between $\mathbf{v}$ and $\mathbf{w}$
Area of a parallelogram	$A =  \mathbf{v} \times \mathbf{w} $ where $\mathbf{v}$ and $\mathbf{w}$ form two adjacent sides of a parallelogram

TOPIC 4: STATISTICS AND PROBABILITY

Interquartile range =  $Q_3 - Q_1$

Mean of a set of data     $\bar{x} = \frac{\sum_{i=1}^k f_i x_i}{n}$ ,    where     $n = \sum_{i=1}^k f_i$

PROBABILITY

Probability of an event  $A$      $P(A) = \frac{n(A)}{n(U)}$

$P(A) + P(A') = 1$   
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$   
 $P(A \cup B) = P(A) + P(B)$     for mutually exclusive events  
 $P(A | B) = \frac{P(A \cap B)}{P(B)}$   
 $P(A \cap B) = P(A) P(B)$     for independent events

Expected value of a discrete random variable  $X$ ,  $E(X) = \sum x P(X = x)$

BINOMIAL DISTRIBUTION

For  $X \sim B(n, p)$ :

- Mean  $E(X) = np$
- Variance  $\text{Var}(X) = np(1 - p)$

RANDOM VARIABLES

Linear transformation of a single random variable	$E(aX + b) = a E(X) + b$ $\text{Var}(aX + b) = a^2 \text{Var}(X)$
Linear combinations of $n$ independent random variables, $X_1, X_2, \dots, X_n$	$E(a_1 X_1 \pm a_2 X_2 \pm \dots \pm a_n X_n)$ $= a_1 E(X_1) \pm a_2 E(X_2) \pm \dots \pm a_n E(X_n)$ $\text{Var}(a_1 X_1 \pm a_2 X_2 \pm \dots \pm a_n X_n)$ $= a_1^2 \text{Var}(X_1) + a_2^2 \text{Var}(X_2) + \dots + a_n^2 \text{Var}(X_n)$



SAMPLE STATISTICS

Unbiased estimate of population variance  $s_{n-1}^2 = \frac{n}{n-1} s_n^2$

POISSON DISTRIBUTION

For  $X \sim \text{Po}(m)$ :

- $E(X) = m$
- $\text{Var}(X) = m$

TRANSITION MATRICES

$\mathbf{T}^n \mathbf{s}_0 = \mathbf{s}_n$ , where  $\mathbf{s}_0$  is the initial state.

TOPIC 5: CALCULUS

DIFFERENTIATION

$f(x)$	$f'(x)$
$x^n$	$nx^{n-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$e^x$	$e^x$
$\ln x$	$\frac{1}{x}$

Chain rule	$y = g(u)$ , where $u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
Product rule	$y = uv \Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$
Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

INTEGRATION

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x} dx = \tan x + C$$

$$\int e^x dx = e^x + C$$

Area between a curve $y = f(x)$ , and the $x$ -axis, where $f(x) > 0$	$A = \int_a^b y dx$
Area of region enclosed by a curve and $x$ -axis	$A = \int_a^b  y  dx$
Area of region enclosed by a curve and $y$ -axis	$A = \int_a^b  x  dy$
Volume of revolution about the $x$ or $y$ -axes	$V = \int_a^b \pi y^2 dx \quad \text{or} \quad V = \int_a^b \pi x^2 dy$



TRAPEZOIDAL RULE

$\int_a^b y \, dx \approx \frac{1}{2}h((y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})), \quad \text{where} \quad h = \frac{b-a}{n}$

KINEMATICS

$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v \frac{dv}{ds}$

Distance travelled from  $t_1$  to  $t_2 = \int_{t_1}^{t_2} |v(t)| \, dt$

Displacement from  $t_1$  to  $t_2 = \int_{t_1}^{t_2} v(t) \, dt$

DIFFERENTIAL EQUATIONS

Euler’s method	$y_{n+1} = y_n + h \times f(x_n, y_n); \quad x_{n+1} = x_n + h, \text{ where } h \text{ is a constant (step length)}$
Euler’s method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n); \quad y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n); \quad t_{n+1} = t_n + h,$ where $h$ is a constant (step length)
Exact solution for coupled linear differential equations	$\mathbf{x} = Ae^{\lambda_1 t} \mathbf{p}_1 + Be^{\lambda_2 t} \mathbf{p}_2$



## TOPIC 1: NUMBER AND ALGEBRA

### APPROXIMATION AND ESTIMATION

A **measurement** is accurate to  $\pm \frac{1}{2}$  of the smallest division on the scale.

An **approximation** is a value given to a number which is close to, but not equal to, its true value.

An **estimation** is a value which is found by judgement or prediction instead of carrying out a more accurate measurement.

If the exact value is  $V_E$  and the approximate value is  $V_A$  then:

- **absolute error** =  $|V_A - V_E|$
- **percentage error** =  $\frac{|V_A - V_E|}{V_E} \times 100\%$

### SCIENTIFIC NOTATION (STANDARD FORM)

A number is in **scientific notation** if it is written in the form  $a \times 10^k$  where  $1 \leq a < 10$  and  $k \in \mathbb{Z}$ .

### SEQUENCES AND SERIES

A **number sequence** is a set of numbers defined by a rule. Often, the rule is a formula for the **general term** or  **$n$ th term** of the sequence.

A sequence which continues forever is called an **infinite sequence**. A sequence which terminates is called a **finite sequence**.

#### Arithmetic sequences

In an **arithmetic sequence**, each term differs from the previous one by the same fixed number.

$u_{n+1} - u_n = d$  for all  $n \in \mathbb{Z}^+$ , where  $d$  is a constant called the **common difference**.

For an arithmetic sequence with first term  $u_1$  and common difference  $d$ , the  $n$ th term is  $u_n = u_1 + (n - 1)d$ .

#### Geometric sequences

In a **geometric sequence**, each term is obtained from the previous one by multiplying by the same non-zero constant, called the **common ratio**  $r$ .

$u_{n+1} = ru_n$ , so we can find  $r = \frac{u_{n+1}}{u_n}$  for all  $n \in \mathbb{Z}^+$ .

For a geometric sequence with first term  $u_1$  and common ratio  $r$ , the  $n$ th term is  $u_n = u_1 r^{n-1}$ .

#### Series

A **series** is the sum of the terms of a sequence.

For a finite sequence with  $n$  terms, the corresponding series is  $S_n = u_1 + u_2 + \dots + u_n$ .

For an infinite sequence, the corresponding series  $u_1 + u_2 + \dots + u_n + \dots$  can only be calculated in some cases.

Using **sigma notation** or **summation notation** we write  $u_1 + u_2 + u_3 + \dots + u_n$  as  $\sum_{k=1}^n u_k$ .

For a **finite arithmetic series**,  $S_n = \frac{n}{2}(u_1 + u_n)$  or  $S_n = \frac{n}{2}(2u_1 + (n - 1)d)$ .

For a **finite geometric series** with  $r \neq 1$ ,  $S_n = \frac{u_1(r^n - 1)}{r - 1}$ .

The sum of an **infinite geometric series** is  $S = \frac{u_1}{1 - r}$  provided  $|r| < 1$ .

If  $|r| > 1$  the series is **divergent**.

#### Compound interest

The value of a compound interest investment after  $n$  time periods is

$$u_n = u_0(1 + i)^n$$

where  $u_0$  is the initial value of the investment

and  $i$  is the interest rate per compounding period.

To find the **real value** of the investment, we divide by the inflation multiplier each year.

You should be able to use the TVM solver on your calculator to solve problems involving compound interest investments and loans.



Depreciation

**Depreciation** is the loss in value of an item over time.

The value of an item after  $n$  years is  $u_n = u_0(1 - d)^n$   
where  $u_0$  is the initial value of the item  
and  $d$  is the rate of depreciation per year.

POLYNOMIAL EQUATIONS

The highest power of  $x$  in a polynomial equation is called its **degree**.

If a polynomial equation has degree  $n$  then it may have up to  $n$  real solutions.

You should be able to use your graphics calculator to solve polynomial equations.

EXPONENTIALS AND LOGARITHMS

Laws of exponents	
$a^m \times a^n = a^{m+n}$	$a^0 = 1, a \neq 0$
$\frac{a^m}{a^n} = a^{m-n}$	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
$(a^m)^n = a^{mn}$	$a^{\frac{1}{n}} = \sqrt[n]{a}$
$(ab)^n = a^n b^n$	$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	

The **logarithm in base 10** of a positive number is the power that 10 must be raised to in order to obtain that number.

If  $10^x = b$  for  $b > 0$ , we say that  $x$  is the logarithm of  $b$  in base 10, and write  $x = \log b$ .

$\log 10^x = x$  and  $10^{\log x} = x$  for any  $x > 0$ .

The **natural logarithm** is the logarithm in base  $e$ . The natural logarithm of  $x$  is written as  $\ln x$  or  $\log_e x$ .

$\ln e^x = x$  and  $e^{\ln x} = x$  for all  $x > 0$ .

Laws of logarithms	
Base 10	Base $e$
$\log xy = \log x + \log y$	$\ln xy = \ln x + \ln y$
$\log\left(\frac{x}{y}\right) = \log x - \log y$	$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
$\log(x^m) = m \log x$	$\ln(x^m) = m \ln x$
$\log 1 = 0$	$\ln 1 = 0$

COMPLEX NUMBERS

Any number of the form  $a + bi$  where  $a$  and  $b$  are real and  $i = \sqrt{-1}$  is called a **complex number**.

Complex numbers allow us to obtain solutions to quadratic equations of the form  $ax^2 + bx + c = 0$  with  $b^2 - 4ac < 0$ .

If  $z = a + bi$  where  $a$  and  $b$  are real then:

- $a$  is the **real part** of  $z$ , written  $\text{Re}(z)$
- $b$  is the **imaginary part** of  $z$ , written  $\text{Im}(z)$ .

You should be able to calculate sums, differences, products, quotients, and integer powers of complex numbers.

Two complex numbers are equal if their real parts are equal *and* their imaginary parts are equal

$$a + bi = c + di \Leftrightarrow a = c \text{ and } b = d.$$

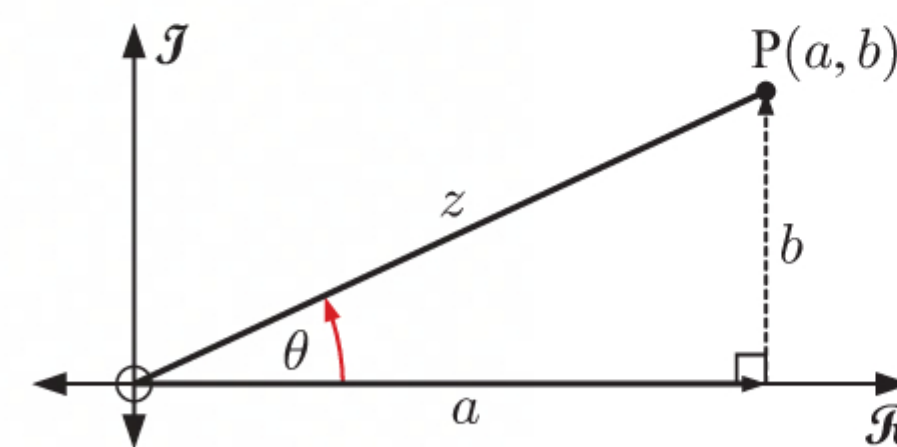
The **complex conjugate** of  $z = a + bi$  is  $z^* = a - bi$ .



## The complex plane (Argand plane)

On the complex plane, the  $x$ -axis is called the **real axis** and the  $y$ -axis is called the **imaginary axis**.

The complex number  $z = a + bi$  is represented by the vector  $\overrightarrow{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$ .



## Modulus and argument

- The **modulus** of the complex number  $z = a + bi$  is the length of the vector  $\begin{pmatrix} a \\ b \end{pmatrix}$ , which is  $|z| = \sqrt{a^2 + b^2}$ .
- If  $z$  is represented by  $P(a, b)$  on the Cartesian plane, the **argument** of  $z$  is the angle  $\theta$ , where  $-\pi < \theta \leq \pi$  is measured anticlockwise between the positive real axis and  $\overrightarrow{OP}$ .

Properties of modulus and argument:

- $|wz| = |w||z|$  and  $\arg(wz) = \arg w + \arg z$
- $\left| \frac{w}{z} \right| = \frac{|w|}{|z|}$  and  $\arg\left(\frac{w}{z}\right) = \arg w - \arg z$  provided  $z \neq 0$
- $|z^*| = |z|$  and  $\arg(z^*) = -\arg z$
- $zz^* = |z|^2$

For points  $P_1$  and  $P_2$  on the complex plane defined by  $z_1 \equiv \overrightarrow{OP_1}$  and  $z_2 \equiv \overrightarrow{OP_2}$ , the distance between  $P_1$  and  $P_2$  is  $|z_1 - z_2|$ .

## Polar and exponential form

The complex number  $z$  can be written in **polar form**  $z = |z| \operatorname{cis} \theta = |z|(\cos \theta + i \sin \theta)$  or **exponential form**  $z = |z|e^{i\theta}$ , where  $\theta$  is the argument of  $z$ .

$$\operatorname{cis} \theta \times \operatorname{cis} \phi = \operatorname{cis}(\theta + \phi)$$

$$\frac{\operatorname{cis} \theta}{\operatorname{cis} \phi} = \operatorname{cis}(\theta - \phi)$$

$$\operatorname{cis}(\theta + k2\pi) = \operatorname{cis} \theta \text{ for all } k \in \mathbb{Z}.$$

If a complex number is multiplied by  $r \operatorname{cis} \theta$  then its modulus is *multiplied* by  $r$ , and its argument is *increased* by  $\theta$ .

You should be able to use the exponential form to add trigonometric functions with the same frequency but different phase, such as  $\sin(3t - \frac{\pi}{3}) + \sin(3t + \frac{\pi}{6})$ .

## MATRICES

A matrix with  $m$  rows and  $n$  columns has order  $m \times n$ .

Two matrices are **equal** if they have the **same order** and the elements in corresponding positions are equal.

## Operations with matrices

To **add** two matrices, they must be of the **same order**, and we **add corresponding elements**.

To **subtract** matrices, they must be of the **same order**, and we **subtract corresponding elements**.

To **multiply** a matrix by a scalar  $k$ , we multiply each element of the matrix by  $k$ .

The **product** of an  $m \times n$  matrix **A** with an  $n \times p$  matrix **B**, is the  $m \times p$  matrix **AB** in which the element in the  $r$ th row and  $c$ th column is the product of the  $r$ th row of **A** (as a row matrix) and the  $c$ th column of **B** (as a column matrix).

In general, **AB**  $\neq$  **BA**.

## Zero and identity matrices

A **zero matrix** is a matrix in which all the elements are zero.

If **A** is a matrix of any order and **O** is the corresponding zero matrix, then **A** + **O** = **O** + **A** = **A**.

An **identity matrix** is a square matrix with 1s in the leading diagonal and 0s everywhere else.

If **A** is a square matrix and **I** is the corresponding identity matrix then **AI** = **IA** = **A**.



## Inverse of a matrix

The **multiplicative inverse** of  $\mathbf{A}$ , denoted  $\mathbf{A}^{-1}$ , satisfies  $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ .

For the  $2 \times 2$  matrix  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ :

- The value  $ad - bc$  is called the **determinant** of matrix  $\mathbf{A}$ , denoted  $\det \mathbf{A}$  or  $|\mathbf{A}|$ .
- If  $\det \mathbf{A} \neq 0$ , then  $\mathbf{A}$  is **invertible** or **non-singular**, and  $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ .
- If  $\det \mathbf{A} = 0$ , then  $\mathbf{A}$  is **singular**, and  $\mathbf{A}^{-1}$  does not exist.

You should be able to use technology to find determinants and inverses of larger matrices.

## Simultaneous linear equations

A system of linear equations can be written in the form  $\mathbf{Ax} = \mathbf{b}$  where  $\mathbf{A}$  is the matrix of coefficients,  $\mathbf{x}$  is the vector of unknowns, and  $\mathbf{b}$  is a vector of constants.

Provided the inverse matrix  $\mathbf{A}^{-1}$  exists, we can solve the matrix equation  $\mathbf{Ax} = \mathbf{b}$  for  $\mathbf{x}$  by finding  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ .

## EIGENVALUES AND EIGENVECTORS

Suppose  $\mathbf{A}$  is a square matrix.

If  $\mathbf{x}$  is a non-zero vector and  $\lambda$  is a constant such that  $\mathbf{Ax} = \lambda\mathbf{x}$ , then  $\lambda$  is an **eigenvalue** of  $\mathbf{A}$  and  $\mathbf{x}$  is its corresponding **eigenvector**.

The **characteristic polynomial** of  $\mathbf{A}$  is  $p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$ .

The eigenvalues of  $\mathbf{A}$  are the solutions to  $p(\lambda) = 0$ .

For a given eigenvalue  $\lambda$ , the corresponding eigenvectors are the solutions  $\mathbf{x}$  to  $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ .

## Matrix diagonalisation

A non-zero square matrix is said to be **diagonal** if the elements *not* on its leading diagonal are zero.

A square matrix  $\mathbf{A}$  is **diagonalisable** if there exists a matrix  $\mathbf{P}$  such that  $\mathbf{P}^{-1}\mathbf{AP}$  is a diagonal matrix. We say that  $\mathbf{P}$  diagonalises  $\mathbf{A}$ .

If  $\mathbf{A}$  is a  $2 \times 2$  matrix with *distinct* eigenvalues  $\lambda_1, \lambda_2$  and corresponding eigenvectors  $\mathbf{x}_1, \mathbf{x}_2$ , then  $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2)$  diagonalises  $\mathbf{A}$ , and  $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ .

In this case, we can calculate powers of  $\mathbf{A}$  using  $\mathbf{A}^n = \mathbf{P} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \mathbf{P}^{-1}$ .



## SKILL BUILDER QUESTIONS

- 1 The speed of a cricket player's bowl is recorded as  $141.6 \text{ km h}^{-1}$ , rounded to 1 decimal place. In what range of values does the actual speed  $s$  lie?
- 2 The dimensions of a block of land are measured to be  $17 \text{ m} \times 22 \text{ m}$ , rounded to the nearest metre. What are the boundary values for the actual area  $A$  of the block of land?
- 3 Terry measures the dimensions of a box as  $15 \text{ cm}$  by  $12 \text{ cm}$  by  $8 \text{ cm}$ , rounded to the nearest centimetre.
  - a Use Terry's measurements to estimate the volume of the box.
  - b The actual dimensions of the box are  $15.3 \text{ cm}$  by  $11.8 \text{ cm}$  by  $8.4 \text{ cm}$ .
    - i Find the actual volume of the box.
    - ii Find the absolute and percentage error in Terry's estimate.
- 4 The radius of a circle is measured as  $7 \text{ cm}$ , rounded to the nearest centimetre.
  - a Use this measurement to estimate the area of the circle.
  - b Find the boundary values for the area of the circle.
  - c Hence find the maximum percentage error in the estimate.
- 5 Express in exponent form with a prime number base:
  - a  $64$
  - b  $125 \times 5^k$
  - c  $\frac{9^m}{81^n}$
- 6 Write without brackets:
  - a  $(-3m^3)^4$
  - b  $\left(\frac{xy^2}{2}\right)^5$
  - c  $7s^2t \times (4st^3)^3$
- 7 Simplify:
  - a  $4^0 + 4^{-1}$
  - b  $(2\frac{3}{4})^{-2}$
  - c  $2^2 + 2^1 + 2^{-1}$
- 8 Expand the brackets and write in simplest form:
  - a  $(x^2 + x^{-2})^2$
  - b  $(x^4 - x^2)(x^3 + 3)$
- 9 Write without negative exponents:
  - a  $a^2b^{-3}$
  - b  $\frac{2m^{-2}n^3}{m^5n^{-5}}$
  - c  $\frac{12a^{-3}}{b^{-5}}$
- 10 Without using a calculator, write in simplest rational form:
  - a  $4^{\frac{5}{2}}$
  - b  $49^{-\frac{3}{2}}$
  - c  $27^{\frac{5}{3}}$
- 11 Expand and simplify:
  - a  $x^{\frac{1}{2}}(x^{-\frac{1}{2}} + 2x - x^{\frac{1}{2}})$
  - b  $5^x(5^{-x} + 5^{3x})$
  - c  $2^{-2x}(2^{2x+3} - 2^{-4x} + 3)$
- 12 Write in scientific notation:
  - a  $42\,000$
  - b  $0.000\,067\,8$
  - c  $526\,000\,000$
- 13 Use your calculator to evaluate the following, giving your answer in scientific notation:
  - a  $(3.57 \times 10^6) \times (2.38 \times 10^3)$
  - b  $\frac{4.61 \times 10^{-7}}{3.45 \times 10^8}$
  - c  $(0.000\,08)^4$
- 14 Use technology to find the real solutions of:
  - a  $3x^3 + 7x^2 - 3x = 2$
  - b  $x^4 + 3x^3 + 2 = 4x^2 - 8x$
- 15 Solve using technology:
  - a  $\begin{cases} 2x - 3y = 2 \\ 5x + 3y = 5 \end{cases}$
  - b  $\begin{cases} 3x - 7y = -8 \\ 6x + 11y = 12 \end{cases}$
  - c  $\begin{cases} 2x + y + 3z = -3 \\ x - y + 2z = 1 \\ 3x - 2y + 5z = 4 \end{cases}$
- 16 A triangle is defined by the lines with equations  $y = x + 2$ ,  $x + y = 9$ , and  $x = 4$ .
  - a Find the coordinates of the triangle's vertices.
  - b Find the area of the triangle.



- 17** Consider the sequence 8, 13, 18, 23, 28, ....
- Show that the sequence is arithmetic.
  - Find the formula for its general term.
  - Find the 42nd term.
  - Determine whether each number is a member of the sequence:      **i** 153      **ii** 4067
- 18** Find  $k$  given the consecutive arithmetic terms:
- 3,  $k$ , 11
  - $-2$ ,  $k + 4$ ,  $k^2 + 11$
  - $k - 5$ ,  $2k$ ,  $2k^2$
- 19** An empty hamster cage has mass 800 g. When 5 hamsters are placed in the cage, the total mass is 1400 g.
- Find the average mass of the hamsters in the cage.
  - Hence write an arithmetic sequence for  $u_n$ , the approximate total mass when  $n$  hamsters are placed in the cage.
- 20** Find the general term  $u_n$  of the geometric sequence which has:
- $u_5 = 324$  and  $u_{10} = 78\,732$
  - $u_8 = -10$  and  $u_{12} = -160$
- 21** Consider the sequence  $2, 2\sqrt{3}, 6, 6\sqrt{3}, \dots$
- Show that the sequence is geometric.
  - Find the formula for its general term.
  - Find the 10th term.
  - Find the first term which exceeds 1000.
- 22** An endangered species of bird has population 217. However, with a successful breeding program it is expected to increase by 42% each year.
- Find the expected population size after:      **i** 5 years      **ii** 10 years.
  - How long will it take for the population to reach 30 000?
- 23** Paige invests €500 in an account that pays 7.2% p.a. compounded monthly.
- The amount of money in Paige's account at the end of each month follows a geometric sequence with common ratio  $r$ .
- Find the value of  $r$ .
  - Find the value of the account after 3 years.
  - Given that inflation averages 2% p.a. over the 3 years, find the real value of the investment after 3 years.
- 24** Stan invests £3500 for 33 months at 8% p.a. interest compounded quarterly. Find its maturing value.
- 25** How much should I invest now to produce \$30 000 in 5 years' time, if the money can be invested at a fixed rate of 4.8% p.a. interest compounded monthly?
- 26** A television was purchased for £2000, and depreciates at 30% p.a. for 3 years.
- Find the value of the television at the end of this period.
  - By how much has the television depreciated?
- 27** Lauren deposits \$10 000 in an account that compounds interest monthly. 4.5 years later, the account has balance \$12 000.
- What annual rate of interest did the account pay?
  - How long will it take Lauren to double her deposit?
- 28** Find the sum of:
- $11 + 15 + 19 + 23 + \dots$  to 20 terms
  - $7 + 12.5 + 18 + 23.5 + \dots + 106$
  - $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$  to 100 terms
  - the integers from 1 to 200 not divisible by 3.
- 29** An arithmetic sequence has terms  $u_7 = 1$  and  $u_{15} = -23$ .
- Find the first term  $u_1$  and common difference  $d$ .
  - Find the 27th term  $u_{27}$ .
  - Find the sum of the first 27 terms of the series.
- 30** The first term of a finite arithmetic series is 18 and the sum of the series is  $-210$ . The common difference is  $-3$ . Suppose there are  $n$  terms in the series.
- Show that  $\frac{n}{2}(39 - 3n) = -210$ .
  - Hence find  $n$ .
- 31** Consider the arithmetic sequence 7, 10, 13, 16, 19, ....
- Write down an expression for the sum of the first  $n$  terms  $S_n$ .
  - Find  $n$  such that  $S_n = 140$ .



- 32** Find the sum of:
- a**  $10 + 5 + 2\frac{1}{2} + 1\frac{1}{4} + \dots$  to 8 terms **b**  $2 + 10 + 50 + 250 + \dots$  to 10 terms
- c**  $\sum_{k=1}^{20} 3 \times (-2)^{k+2}$ .
- 33** Find the sum of the series:
- a**  $10 + 14 + 18 + 22 + \dots + 138$  **b**  $6 - 12 + 24 - 48 + 96 - \dots + 1536$
- 34** **a** An infinite geometric series is defined by  $\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^k$ .
- i** Find the first term  $u_1$  and common ratio  $r$ . **ii** Find the sum of the series.
- b** A finite arithmetic series is defined by  $\sum_{k=1}^n (k - 4)$ .
- i** Find the first term  $u_1$  and common difference  $d$ . **ii** Find the sum of the series, in terms of  $n$ .
- c** Find  $n$  such that the sums of the series in **a** and **b** are equal.
- 35** Consider the series  $\sum_{k=1}^{\infty} 12(x - 2)^{k-1}$ .
- a** For what values of  $x$  will the series converge? **b** Evaluate the sum of the series when  $x = \sqrt{5}$ .
- 36** Emma takes out a home loan of \$120 000 at 7.2% p.a. interest compounded monthly. The loan is to be repaid over 20 years.
- a** Calculate the monthly repayment.
- b** Calculate the amount of money still owing on the loan after one year.
- c** **i** Calculate the amount paid in the first year of the loan.
- ii** By how much has the principal been reduced at the end of the first year?
- iii** Explain why the loan does not reduce by the full amount of your first year's repayment.
- d** Suppose that after 1 year, the interest rate falls to 6.95% p.a.
- i** Calculate the new monthly repayment.
- ii** If Emma is able to keep paying the original repayments, how much earlier will the loan be paid off?
- 37** Oscar decides to start a new business venture which involves taking out a bank loan. The bank charges an interest rate of 6.55% p.a. compounded quarterly.
- His quarterly repayments are \$933.62, and must be repaid over 8 years.
- a** How much did Oscar borrow?
- b** How much interest will he pay over the 8 year period?
- c** **i** Calculate the outstanding balance at the end of the sixth year.
- ii** At the end of the 6th year, Oscar pays a lump-sum of \$3000 off the loan. Assuming his repayments remain the same, how much sooner will Oscar repay the loan?
- 38** Cassie made an initial investment of €2000 into a savings account, and followed it with regular deposits of €500 per quarter. The account pays 1.2% interest per quarter, and inflation is 0.3% per quarter.
- a** Explain why the real interest rate is approximately 0.9% per quarter.
- b** Find the real value of Cassie's investment after 5 years.
- 39** Bill collects \$81 000 as his share of a lottery win. He decides to retire from work and buy an annuity to provide \$2000 per month, until he gets a pension in four years' time.
- a** What annual interest rate, compounded monthly, is needed for Bill's plan to work?
- b** How much will Bill actually receive each month over the period of the annuity if he receives 7% p.a. interest compounded monthly?
- 40** Celia and Mike want to provide each of their two children with \$200 per month for the next ten years. Interest on the investment is 5.2% p.a. calculated monthly.
- a** How much should be invested now to provide such an annuity?
- b** Calculate the total interest earned over the term of the annuity.



**41** Simplify:

**a**  $\log(10^9 \times 1000^b)$                       **b**  $\log\left(\frac{10^n}{100}\right)$                       **c**  $\log(2^t \times 5^t)$

**42** Use your calculator to write the following in the form  $10^x$  where  $x$  is correct to 4 decimal places:

**a** 2    **b** 200    **c** 0.02

**43** Simplify:

**a**  $\ln(e^k \times e^4)$                       **b**  $\ln\left(\frac{e}{e^m}\right)$                       **c**  $e^{2 \ln 6}$                       **d**  $e^{-\ln 3}$

**44** Use your calculator to write the following in the form  $e^k$  where  $k$  is correct to 4 decimal places:

**a** 47    **b** 500    **c** 0.023

**45** Simplify by writing as a single logarithm:

**a**  $\frac{1}{4} \ln 81 + \ln 12 - \ln 4$                       **b**  $3 \log 2 - \log 24$                       **c**  $5 + \log 3 - \frac{1}{2} \log 49$

**46** If  $x = \log 5$ , write in terms of  $x$ :

**a**  $\log 50$     **b**  $\log\left(\frac{125}{100}\right)$     **c**  $\log \sqrt[3]{5}$

**47** Suppose  $A = \log P$ ,  $B = \log Q$ , and  $C = \log R$ . Write in terms of  $A$ ,  $B$ , and  $C$ :

**a**  $\log(PQ)$     **b**  $\log(P^2 Q \sqrt{R})$     **c**  $\log\left(\frac{PQ^3}{R^2}\right)$

**48** Simplify without using a calculator:

**a**  $\frac{\log 9}{\log 3}$     **b**  $\frac{\log 8}{\log 4}$     **c**  $\frac{\log 0.25}{\log 64}$

**49** Write as a single logarithm:

**a**  $\ln 20 - \ln 10$     **b**  $-\ln 13 - 3$     **c**  $\frac{1}{3} \ln 64 + 2 \ln 2$

**50** Find  $x$  such that:

**a**  $\log x = -2$     **b**  $\log x = \frac{1}{3}$     **c**  $\log x \approx 5.2831$

**51** Solve for  $x$ :

**a**  $3 \log x = \log 24 + \log\left(\frac{1}{3}\right)$     **b**  $\ln x = \ln 12 - \ln(7 - x)$   
**c**  $\ln(x^2 - 3) - \ln(2x) = 0$     **d**  $\log x + \log(x - 3) = 1$

**52** The apparent magnitude  $M$  of an object's brightness in the sky is given by  $M = -2.5 \log\left(\frac{F}{F_0}\right)$ , where  $F$  is the observed flux density of the object and  $F_0$  is a reference flux density.

Jupiter has apparent magnitude  $-2.6$ , and the observed flux density of Venus is 5.2 times greater than that of Jupiter. Find the apparent magnitude of Venus.

**53** Solve for  $x$ :

**a**  $3x^2 = -27$     **b**  $x^2 + x + 1 = 0$     **c**  $2x^2 + x + 5 = 0$

**54** If  $z = 2 + i$  and  $w = 3 + 5i$ , find:

**a**  $\operatorname{Re}(z - 3w)$     **b**  $\operatorname{Im}(iw^2)$     **c**  $\operatorname{Re}\left(\frac{z}{w}\right)$

**55** Suppose  $\frac{z+2}{z-2} = i$ . Find  $z$  in the form  $a + bi$  where  $a, b \in \mathbb{R}$ .

**56** Find the exact values of  $x, y \in \mathbb{R}$  such that:

**a**  $(3 - 2i)(x - yi) = -i$     **b**  $(x + yi)^2 - (x - yi)^2 = x - y + 16i$

**57** Use technology to find:

**a**  $(6 - 5i)^3$     **b**  $\frac{1}{(1 - 3i)^2}$     **c**  $\frac{(4 + 3i)^4}{2 - i}$

**58** Suppose  $z = 2 + i$  and  $w = 3 - 2i$ . Find:

**a**  $2z + w$     **b**  $w^* - z$     **c**  $z^* + 2w + 2i$ .

Illustrate your answers on separate Argand diagrams.

**59** Given  $|z| = 3$ , find:

**a**  $|3z|$     **b**  $|(2 + i)z|$     **c**  $\left|\frac{2i}{z^2}\right|$



**60** Use the properties of  $\text{cis}$  to simplify the following. Convert your answer to exact Cartesian form if possible.

**a**  $2 \text{cis} \frac{\pi}{7} \text{cis} \frac{6\pi}{7}$

**b**  $(\text{cis} \frac{5\pi}{12})^2$

**c**  $\frac{\sqrt{8} \text{cis} \frac{3\pi}{16}}{\sqrt{2} \text{cis}(-\frac{5\pi}{16})}$

**d**  $\text{cis}(\theta + 15\pi)$

**61** Suppose  $z = \sqrt{3} + i$  and  $w = 2 - 2i$ .

**a** Write  $z$  and  $w$  in polar form.

**b** Hence find  $zw$  in polar form.

**c** Describe the transformation to  $z$  when it is multiplied by  $w$ .

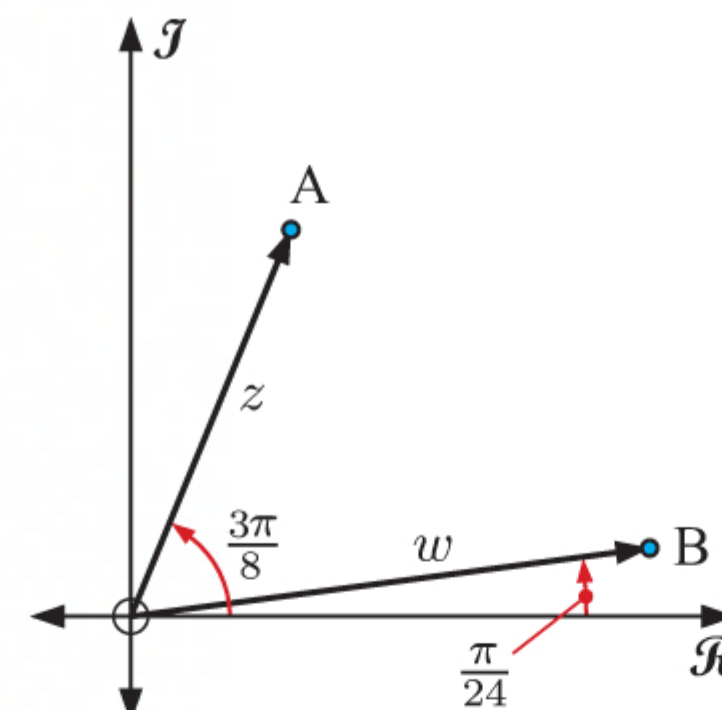
**62**  $z$  and  $w$  correspond to the points A and B respectively.  $|z| = 4$  and  $|w| = 5$ .

**a** Find the exact value of:

**i**  $\arg\left(\frac{z}{w}\right)$

**ii**  $|z - w|$ .

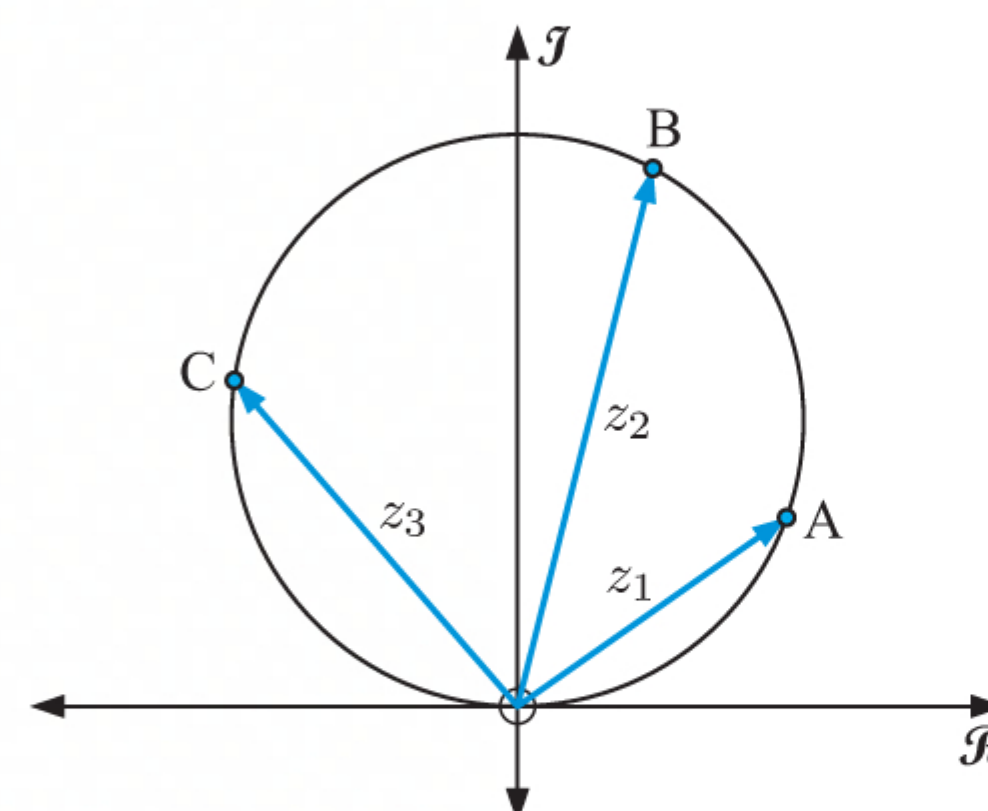
**b** Hence find the exact perimeter and area of triangle OAB.



**63** Points O, A, B, and C lie on a circle. Suppose  $z_1$  represents  $\overrightarrow{OA}$ ,  $z_2$  represents  $\overrightarrow{OB}$ , and  $z_3$  represents  $\overrightarrow{OC}$ .

**a** What vectors are represented by  $z_1 - z_2$  and  $z_3 - z_2$ ?

**b** Hence find the value of  $\arg\left(\frac{z_3}{z_1}\right) + \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$ .



**64** Write:

**a**  $\sqrt{3} + i$  in polar form and exponential form

**b**  $2 \text{cis} \frac{5\pi}{6}$  in Cartesian form and exponential form

**c**  $5e^{-i\frac{\pi}{4}}$  in Cartesian form and polar form.

**65** Use your calculator to:

**a** convert  $7 + 2i$  to polar form

**b** convert  $\sqrt{7} \text{cis} \frac{\pi}{5}$  to Cartesian form.

**66** **a** Express  $1 + i$  and  $\sqrt{3} - i$  in the form  $re^{i\theta}$ .

**b** Hence write  $z = \frac{-1 - i}{\sqrt{3} - i}$  in the form  $re^{i\theta}$ .

**c** Find the smallest positive integer  $n$  such that  $z^n$  is a real number.

**67** **a** Write  $-1 + i$  in exponential form.

**b** Sketch  $z = e^{\frac{5\pi}{6}i}$  and  $w = 2e^{\frac{\pi}{12}i}$  on an Argand diagram.

**c** Find the smallest positive integer  $n$  such that  $(-1 + i)w^n$  is perpendicular to  $z$ .

**68** Write:

**a**  $2 \sin(4t + 2) + 3 \sin(4t + 5)$  in the form  $A \sin(4t + B)$

**b**  $1.5 \cos(10t - 3) + 4.5 \cos(10t + 7)$  in the form  $A \cos(10t + B)$ .

**69** Suppose  $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 5 & 0 & 1 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ -1 & 3 \end{pmatrix}$ , and  $\mathbf{C} = \begin{pmatrix} 4 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$ .

**a** State the order of:

**i**  $\mathbf{A}$

**ii**  $\mathbf{B}$

**iii**  $\mathbf{C}$ .

**b** Find, if possible:

**i**  $\mathbf{A} + \mathbf{C}$

**ii**  $\mathbf{B} + \mathbf{C}$

**iii**  $3\mathbf{C}$

**iv**  $\frac{1}{2}\mathbf{A} - \mathbf{C}$

**v**  $\mathbf{BA}$

**vi**  $\mathbf{AB}$

**70** Find  $k$  such that  $\begin{pmatrix} k+2 & -1 & 0 \\ 3 & k^2-7 & 1 \end{pmatrix} \begin{pmatrix} k+1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}$ .



**71** Given  $\mathbf{P} = \begin{pmatrix} 5 & 2 \\ 6 & 4 \end{pmatrix}$  and  $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ -1 & 4 \\ 2 & 1 \end{pmatrix}$ , find:

**a**  $-5\mathbf{Q}$

**b**  $\mathbf{QP}$

**c**  $\det \mathbf{P}$

**d**  $\mathbf{P}^{-1}$

**72** Let  $\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -3 & 4 \\ 2 & 2 & -3 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} -1 & 0 & 1 \\ -17 & -3 & 13 \\ -12 & -2 & 9 \end{pmatrix}$ .

**a** Calculate  $\mathbf{BA}$ .

**b** Hence solve the system 
$$\begin{cases} x + 2y - 3z = 5 \\ 3x - 3y + 4z = -2 \\ 2x + 2y - 3z = 6 \end{cases}$$
.

**73** Suppose  $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ .

**a** Find  $\mathbf{A}^{-1}$ .

**b** Hence solve the system of equations 
$$\begin{cases} x + 3y = 6 \\ 2x + 7y = 13 \end{cases}$$
.

**74** **a** Write the system of equations 
$$\begin{cases} x + 3y - 4z = -5 \\ 2x + y + z = 7 \\ x - 4y + 2z = -1 \end{cases}$$
 in matrix form.

**b** Hence solve the system.

**75** Let  $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -6 & -2 \end{pmatrix}$ .

**a** Find the characteristic polynomial of  $\mathbf{A}$ .

**b** Find the eigenvalues and corresponding eigenvectors of  $\mathbf{A}$ .

**c** Hence find the eigenvalues and corresponding eigenvectors of: **i**  $2\mathbf{A}$  **ii**  $\mathbf{A}^{-1}$ .

**76** The matrix  $\mathbf{A} = \begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix}$  has eigenvectors  $\mathbf{x}_1 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$  and  $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

**a** Find a matrix  $\mathbf{P}$  which diagonalises  $\mathbf{A}$ .

**b** Hence find the eigenvalues of  $\mathbf{A}$ .

**77** Suppose  $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ -2 & -1 \end{pmatrix}$ .

**a** Find the characteristic polynomial.

**b** Show that the eigenvalues are  $1 \pm 2\sqrt{3}$ , and find the corresponding eigenvectors.

**c** Write down a matrix  $\mathbf{P}$  such that  $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 1 - 2\sqrt{3} & 0 \\ 0 & 1 + 2\sqrt{3} \end{pmatrix}$ .

**78** The matrix  $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$  has eigenvalues  $-2$  and  $5$ .

**a** Find the corresponding eigenvectors of  $\mathbf{A}$ .

**b** Find a matrix  $\mathbf{P}$  which diagonalises  $\mathbf{A}$ .

**c** Hence find the matrix  $\mathbf{A}^{51}$  exactly.

**79** The matrix  $\mathbf{A}$  has eigenvalues  $\lambda_1 = 3$  and  $\lambda_2 = 4$  with corresponding eigenvectors  $\mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$  and  $\mathbf{x}_2 = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$  respectively.

**a** Find a matrix  $\mathbf{P}$  which diagonalises  $\mathbf{A}$ .

**b** Hence find the matrix  $\mathbf{A}$ .

**c** Find the matrix  $\mathbf{A}^{30}$  exactly.

**80** Let  $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ .

**a** Find the eigenvalues and corresponding eigenvectors of  $\mathbf{A}$ .

**b i** Find  $\mathbf{A}^{-1}$ .

**ii** Show that the eigenvectors of  $\mathbf{A}$  are also eigenvectors of  $\mathbf{A}^{-1}$ , and state the corresponding eigenvalues.



## TOPIC 2: FUNCTIONS

### PROPERTIES OF LINES

The **gradient** of the line passing through  $A(x_1, y_1)$  and  $B(x_2, y_2)$  is  $m = \frac{y\text{-step}}{x\text{-step}} = \frac{y_2 - y_1}{x_2 - x_1}$ .

The gradient of any horizontal line is zero. The gradient of any vertical line is undefined.

The **y-intercept** of a line is the value of  $y$  where the line cuts the  $y$ -axis.

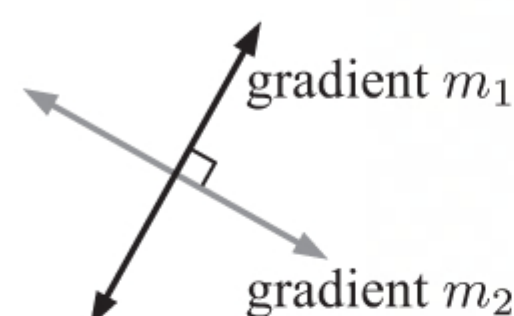
The **x-intercept** of a line is the value of  $x$  where the line cuts the  $x$ -axis.

### PARALLEL AND PERPENDICULAR LINES

The gradients of parallel lines are equal.

The gradients of perpendicular lines are negative reciprocals.

$$m_1 = -\frac{1}{m_2}$$



### EQUATION OF A LINE

The equation of a line can be presented in:

- **gradient-intercept form**  $y = mx + c$  where  $m$  is the gradient and  $c$  is the  $y$ -intercept.
- **general form**  $ax + by = d$
- **point-gradient form**  $y - y_1 = m(x - x_1)$

You should be able to find the equation of a line given:

- its gradient and the coordinates of any point on the line
- the coordinates of two distinct points on the line.

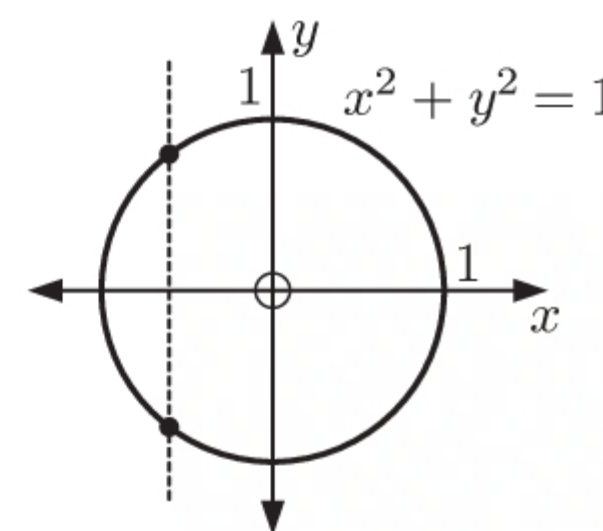
### FUNCTIONS $f : x \mapsto f(x)$ OR $y = f(x)$

A **relation** between variables  $x$  and  $y$  is any set of points in the  $(x, y)$  plane.

A **function** is a relation in which no two different ordered pairs have the same  $x$ -coordinate or first component. For each value of  $x$  there is at most one value of  $y$  or  $f(x)$ . We sometimes refer to  $y$  or  $f(x)$  as the **image** of  $x$ .

We test for functions using the **vertical line test**. A graph is a function if no vertical line intersects the graph more than once.

For example, the graph of the circle  $x^2 + y^2 = 1$  shows that this relation is not a function.



The **domain** of a relation is the set of values that  $x$  can take.

To find the domain of a function, remember that we cannot:

- divide by zero
- take the square root of a negative number
- take the logarithm of a non-positive number.

The **range** of a relation is the set of values that  $y$  or  $f(x)$  can take.

Given  $f : x \mapsto f(x)$  and  $g : x \mapsto g(x)$ , the **composite function** of  $f$  and  $g$  is  $f \circ g : x \mapsto f(g(x))$ .

In general,  $f(g(x)) \neq g(f(x))$ , so  $f \circ g \neq g \circ f$ .

The **identity function** is  $f(x) = x$ .



## INVERSE FUNCTIONS

A function is:

- **one-to-one** if there is only one value of  $x$  for each value of  $y$
- **many-to-one** if there is more than one value of  $x$  with the same value of  $y$ .

The function  $y = f(x)$  has an **inverse function**  $y = f^{-1}(x)$  if and only if it is one-to-one.

Many-to-one functions do not have an inverse function. However, we can often restrict the domain of a many-to-one function to make it a one-to-one function. This restricted function will have an inverse function.

If  $y = f(x)$  has an inverse function  $y = f^{-1}(x)$ , then the inverse function:

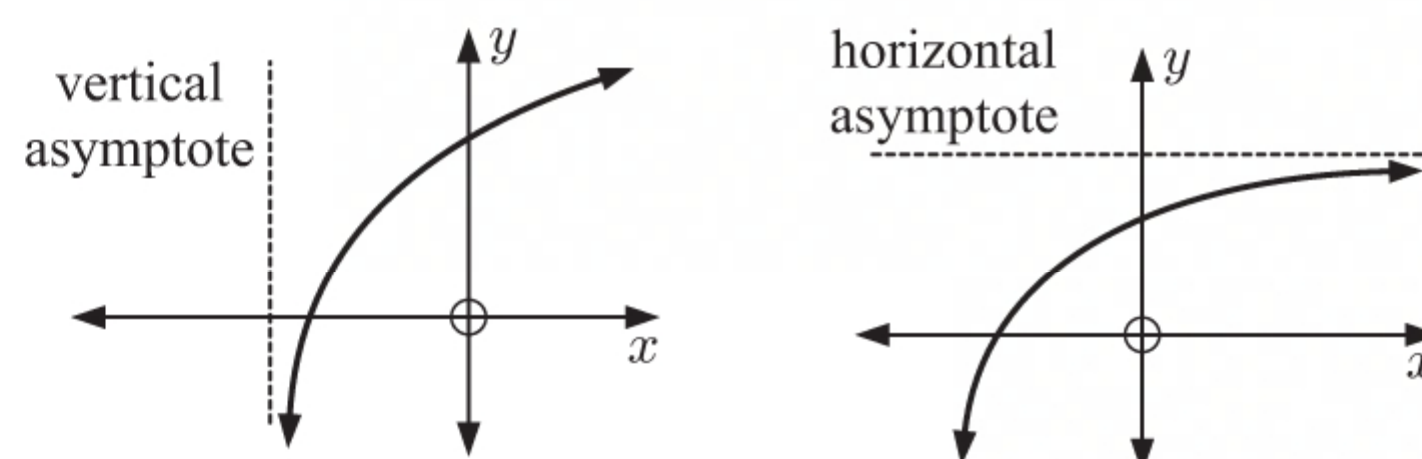
- must satisfy the vertical line test
- is a reflection of  $y = f(x)$  in the line  $y = x$
- satisfies  $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- has range equal to the domain of  $f(x)$
- has domain equal to the range of  $f(x)$ .

## GRAPHS OF FUNCTIONS

The  **$x$ -intercepts** of a function are the values of  $x$  for which  $y = 0$ . They are the **zeros** of the function.

The  **$y$ -intercept** of a function is the value of  $y$  when  $x = 0$ .

An **asymptote** is a line that the graph *approaches* or begins to look like as it tends to infinity in a particular direction.



To find vertical asymptotes, look for values of  $x$  for which the function is undefined:

- If  $y = \frac{f(x)}{g(x)}$ , find where  $g(x) = 0$ .
- If  $y = \log_a(f(x))$ , find where  $f(x) = 0$ .

To find horizontal asymptotes, consider the behaviour as  $x \rightarrow \pm\infty$ .

You should be able to use technology to:

- graph a function
- find the domain and range
- find axes intercepts
- find turning points
- find asymptotes
- find where functions meet.

## MODELLING

Mathematical models are developed using a **modelling cycle**:

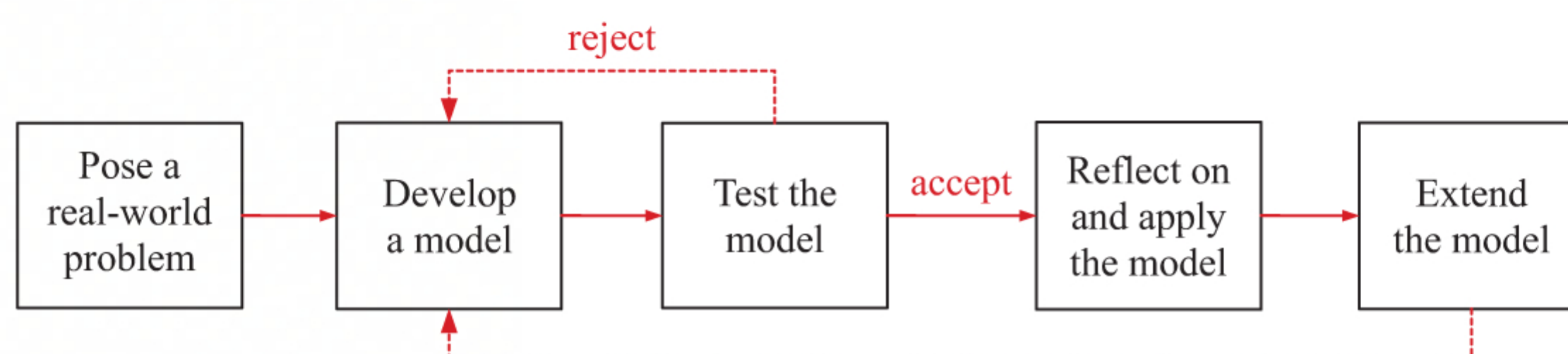
**Step 1:** **Pose** a real-world problem. Make **assumptions** which simplify the problem without missing key features.

**Step 2:** **Develop** a model which represents the problem with mathematics. This may involve a formula or an equation.

**Step 3:** **Test** the model by comparing its predictions with known data. If the model is unsatisfactory, return to **Step 2**.

**Step 4:** **Reflect** on your model and **apply** it to your original problem, interpreting the solution in its real-world context.

**Step 5:** If appropriate, **extend** your model to make it more general or accurate as needed.



You should be able to solve systems of equations using technology to find unknown parameters in models.





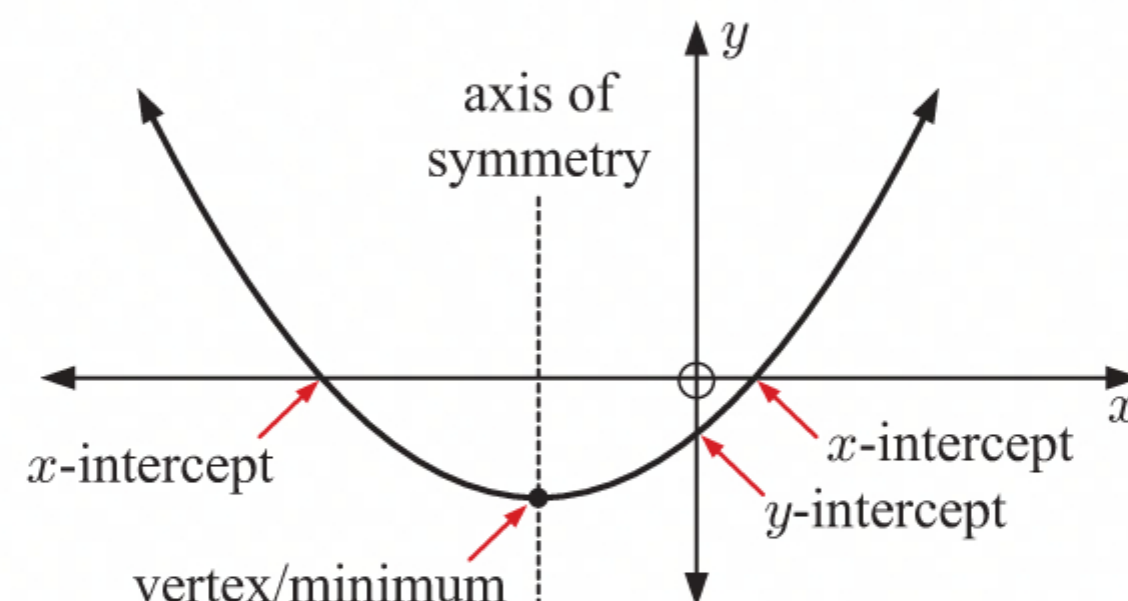
## QUADRATICS

### Quadratic functions

A **quadratic function** has the form  $y = ax^2 + bx + c$ ,  $a \neq 0$ .

The graph is a parabola with the following properties:

- It is *concave up* if  $a > 0$   and *concave down* if  $a < 0$ . 
- Its axis of symmetry is  $x = \frac{-b}{2a}$ .
- Its vertex has  $x$ -coordinate  $\frac{-b}{2a}$ . The  $y$ -coordinate of its vertex is found by substituting  $x = \frac{-b}{2a}$  into the function.
  - ▶ If  $a > 0$  the vertex is a minimum turning point.
  - ▶ If  $a < 0$  the vertex is a maximum turning point.



$y = a(x - p)(x - q)$ $x$ -intercepts $p, q$ axis of symmetry $x = \frac{p + q}{2}$	
$y = a(x - h)^2 + k$ vertex $(h, k)$ axis of symmetry $x = h$	
$y = ax^2 + bx + c$ axis of symmetry $x = \frac{-b}{2a}$ $x$ -intercepts $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $b^2 - 4ac \geq 0$	

### Quadratic equations

A quadratic equation of the form  $ax^2 + bx + c = 0$ ,  $a \neq 0$ , can be solved by:

- factorisation
- completing the square
- the quadratic formula  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .

### Quadratic inequalities

A **quadratic inequality** can be written in either the form  $ax^2 + bx + c \geq 0$  or  $ax^2 + bx + c > 0$  where  $a \neq 0$ .

You should be able to use sign diagrams to solve quadratic inequalities.

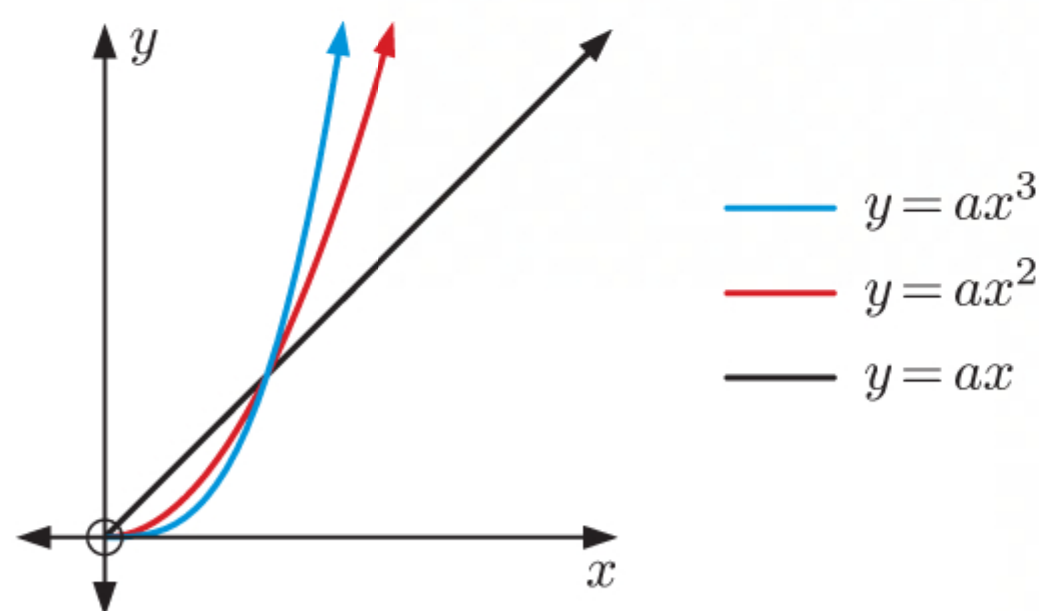


## VARIATION MODELS

Variation models have the form  $y = ax^n$ ,  $n \in \mathbb{Z}$ ,  $n \neq 0$ .

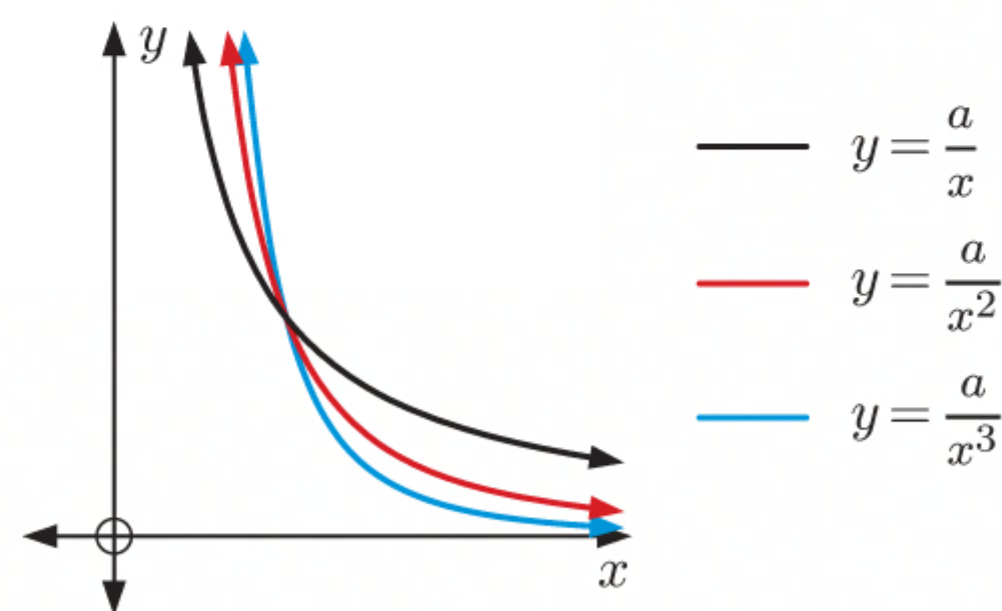
- If  $n > 0$  we have **direct variation**.

The graph passes through the origin  $(0, 0)$ .



- If  $n < 0$  we have **inverse variation**.

The graph is asymptotic to both the  $x$  and  $y$  axes.



You should be able to:

- use a point which lies on the graph of a variation model to find the exact equation of the variation model
- use technology to find the variation model which best fits a set of data.

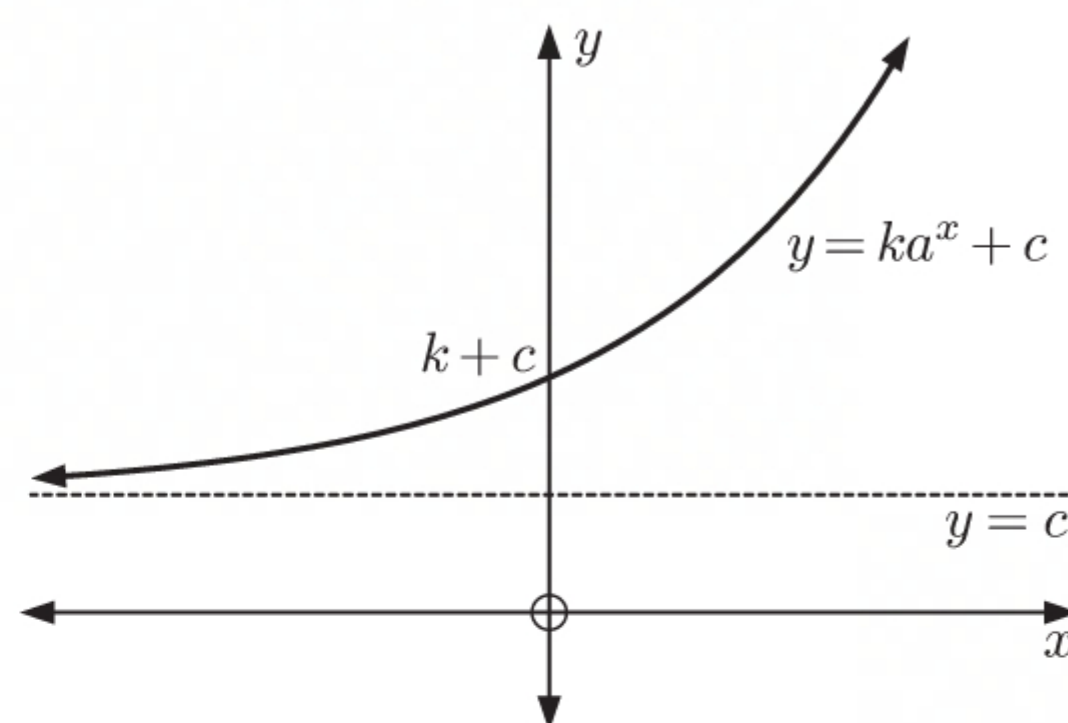
## EXPONENTIAL FUNCTIONS

In this course you need to deal with exponential functions of the form:

- $y = ka^x + c$
- $y = ka^{-x} + c$

In each case:

- $a$  and  $k$  control the steepness of the curve
- $y = c$  is the equation of the **horizontal asymptote**.

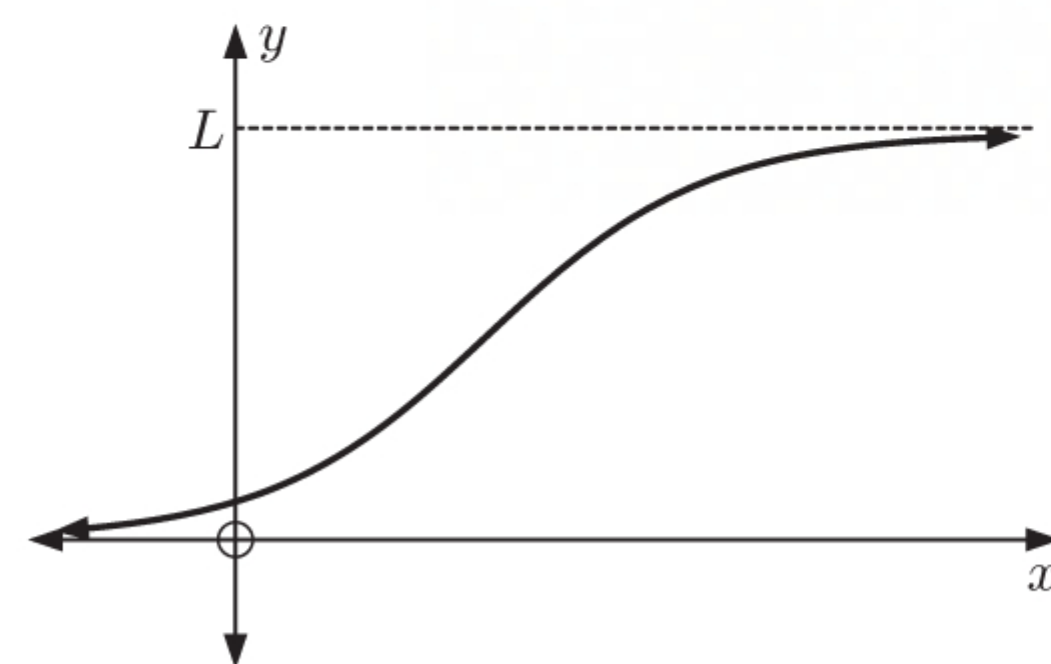


You will also need to deal with natural exponential functions of the form  $y = ke^{rx} + c$ .

Exponential functions are commonly used to model **growth** and **decay** problems.

**Exponential equations** are equations where the variable appears in an index or exponent. You should be able to solve exponential equations using technology.

The **logistic model** has the form  $y = \frac{L}{1 + Ce^{-kx}}$  where  $L > 0$  is the limiting value, and  $C$  and  $k$  are positive constants.

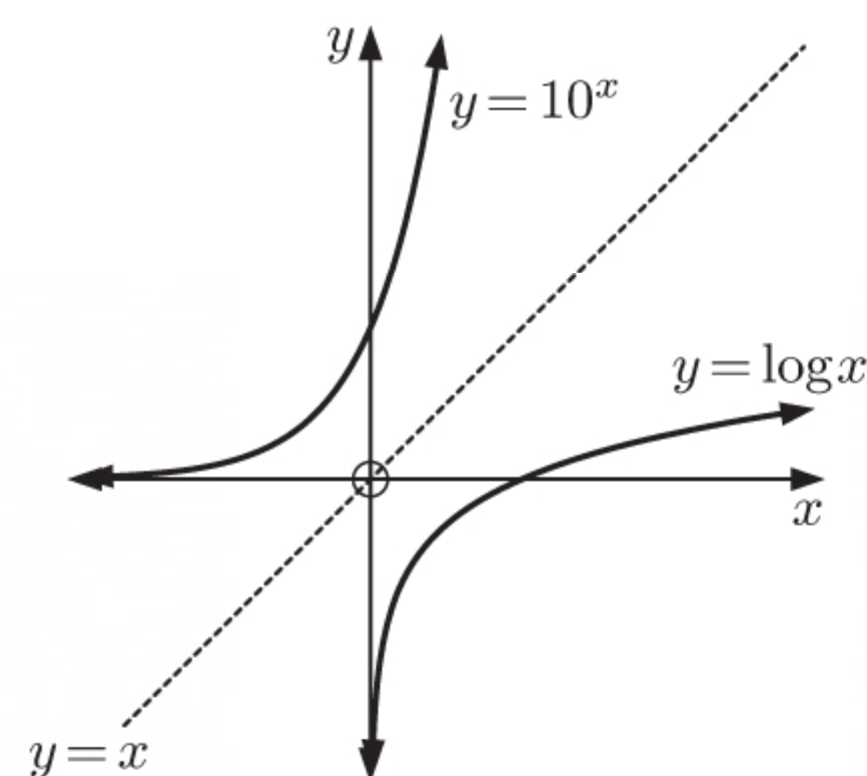


## LOGARITHMIC FUNCTIONS

The **logarithmic function**  $y = \log x$ ,  $x > 0$  is the inverse function of  $y = 10^x$ .

The graph of  $y = \log x$  has the vertical asymptote  $x = 0$ .

The natural logarithmic function  $y = \ln x$ ,  $x > 0$  is the inverse function of  $y = e^x$ .





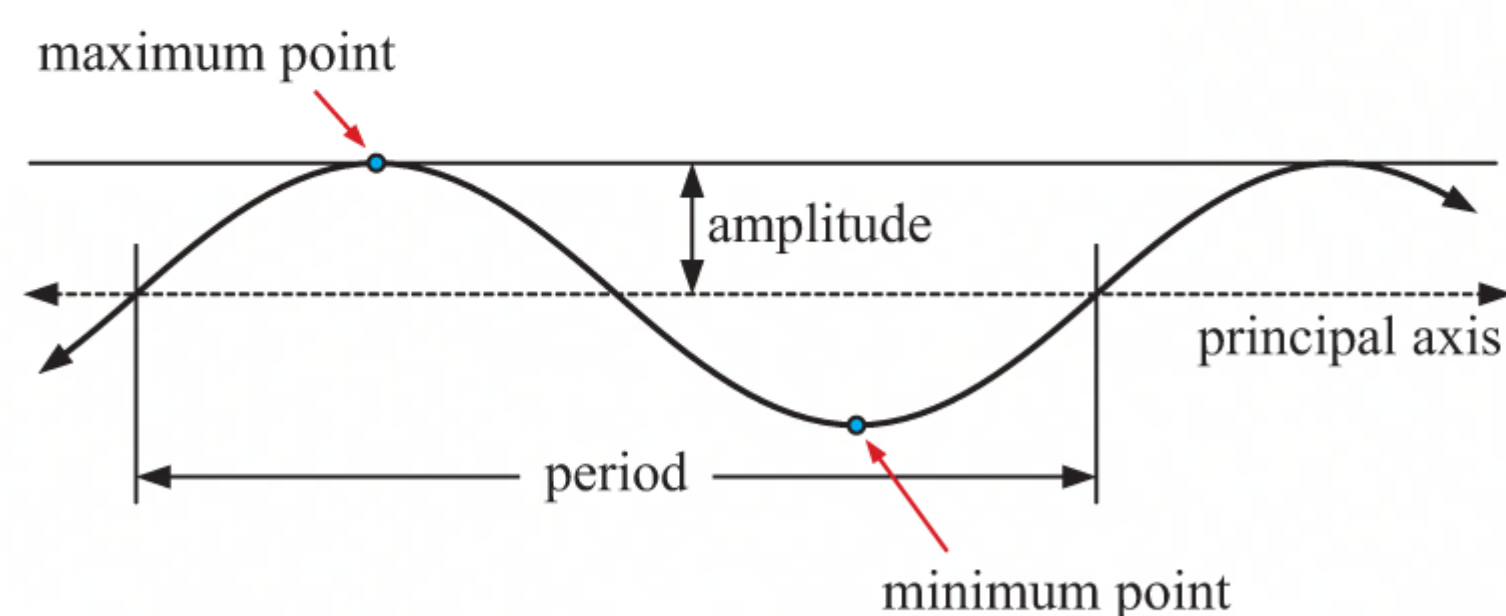
## TRANSFORMATIONS OF FUNCTIONS

- $y = f(x) + b$  **translates**  $y = f(x)$  vertically  $b$  units.
- $y = f(x - a)$  **translates**  $y = f(x)$  horizontally  $a$  units.
- $y = f(x - a) + b$  **translates**  $y = f(x)$  by the vector  $\begin{pmatrix} a \\ b \end{pmatrix}$ .
- $y = pf(x)$ ,  $p > 0$  is a **vertical stretch** of  $y = f(x)$  with scale factor  $p$ .
- $y = f(qx)$ ,  $q > 0$  is a **horizontal stretch** of  $y = f(x)$  with scale factor  $\frac{1}{q}$ .
- $y = -f(x)$  is a **reflection** of  $y = f(x)$  in the  $x$ -axis.
- $y = f(-x)$  is a **reflection** of  $y = f(x)$  in the  $y$ -axis.
- If  $f^{-1}(x)$  exists,  $y = f^{-1}(x)$  is a **reflection** of  $y = f(x)$  in the line  $y = x$ .

## PERIODIC FUNCTIONS

A **periodic function** is one which repeats itself over and over in a horizontal direction.

For example, a **wave** is a periodic function which oscillates about a horizontal line called the **principal axis**.



The **period** of a periodic function is the length of one cycle.

The **amplitude** is the distance between a maximum or minimum point and the principal axis.

## THE SINE FUNCTION

If we begin with  $y = \sin x$ , we can perform transformations to produce the **general sine function**  $f(x) = a \sin(b(x - c)) + d$ , where  $b > 0$ .

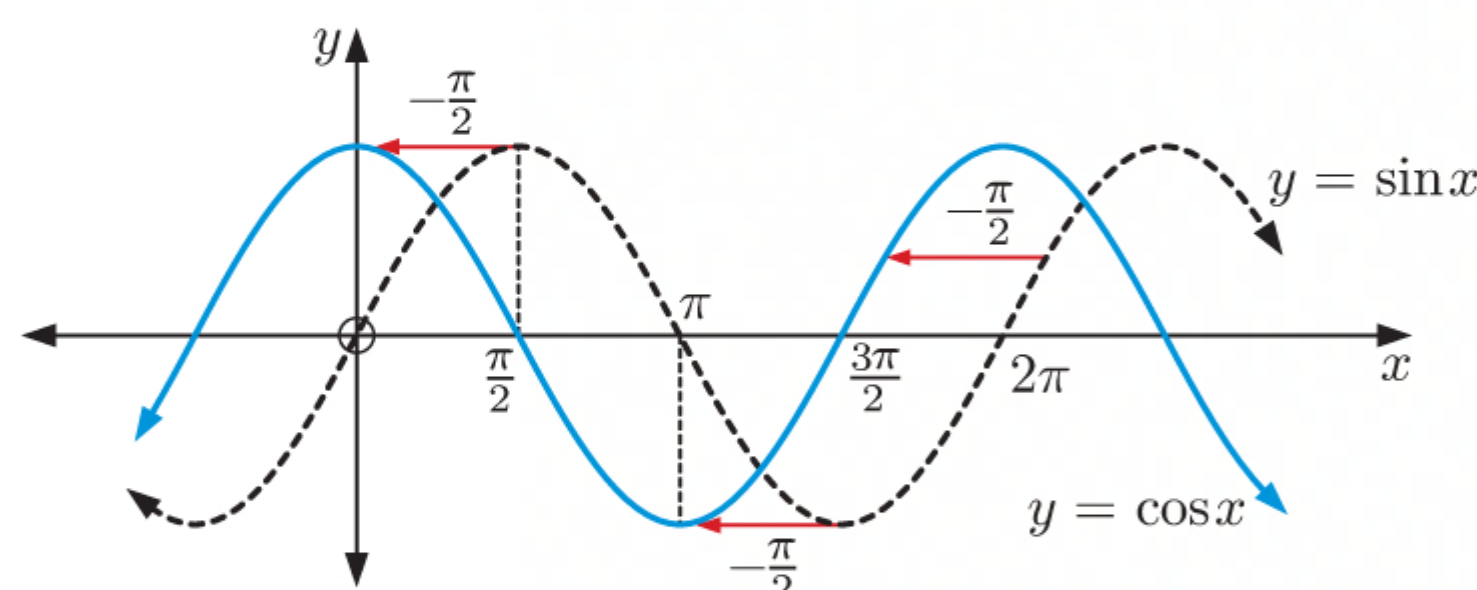
We have a vertical stretch with scale factor  $|a|$  and a horizontal stretch with scale factor  $\frac{1}{b}$ , a reflection in the  $x$ -axis if  $a < 0$ , and a translation through  $\begin{pmatrix} c \\ d \end{pmatrix}$ .

The general sine function has the following properties:

- the **amplitude** is  $|a|$
- the **principal axis** is  $y = d$
- the **period** is  $\frac{2\pi}{b}$ .

## THE COSINE FUNCTION

Since  $\cos x = \sin(x + \frac{\pi}{2})$ , the graph of  $y = \cos x$  is a horizontal translation of  $y = \sin x$ ,  $\frac{\pi}{2}$  units to the left.



The properties of the **general cosine function**  $y = a \cos(b(x - c)) + d$  are the same as those of the general sine function.

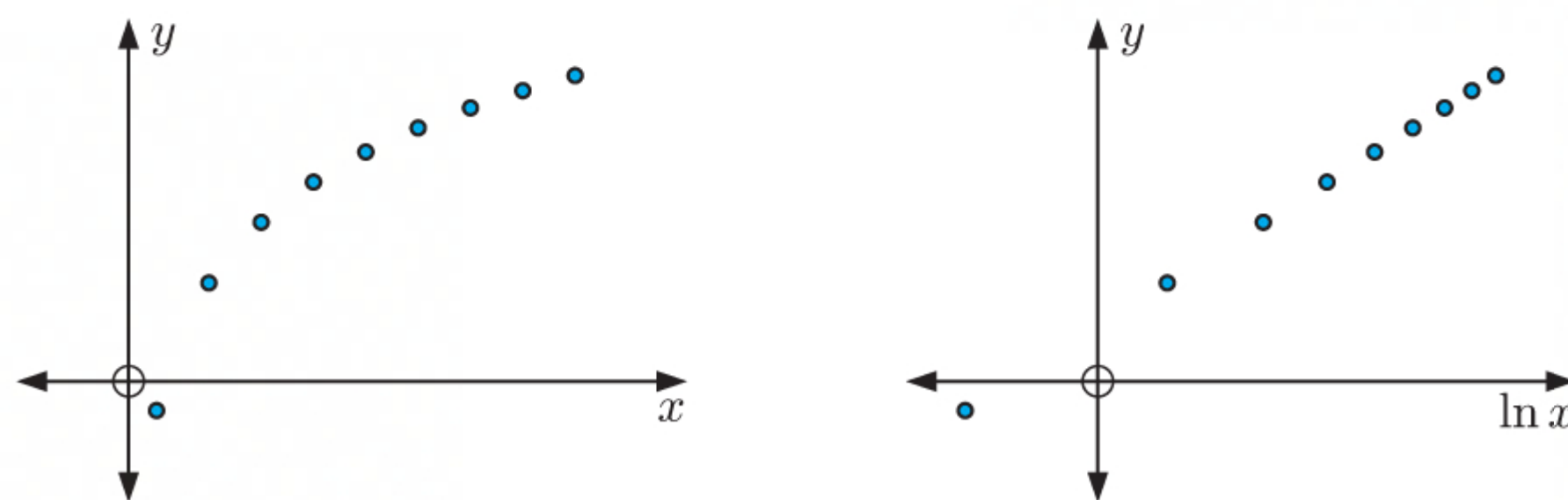


## NON-LINEAR MODELLING

### Logarithmic models

A **logarithmic model** has the form  $y = a + b \ln x$ .

If the variables  $x$  and  $y$  are connected by a logarithmic model, the graph of  $y$  against  $\ln x$  is linear.

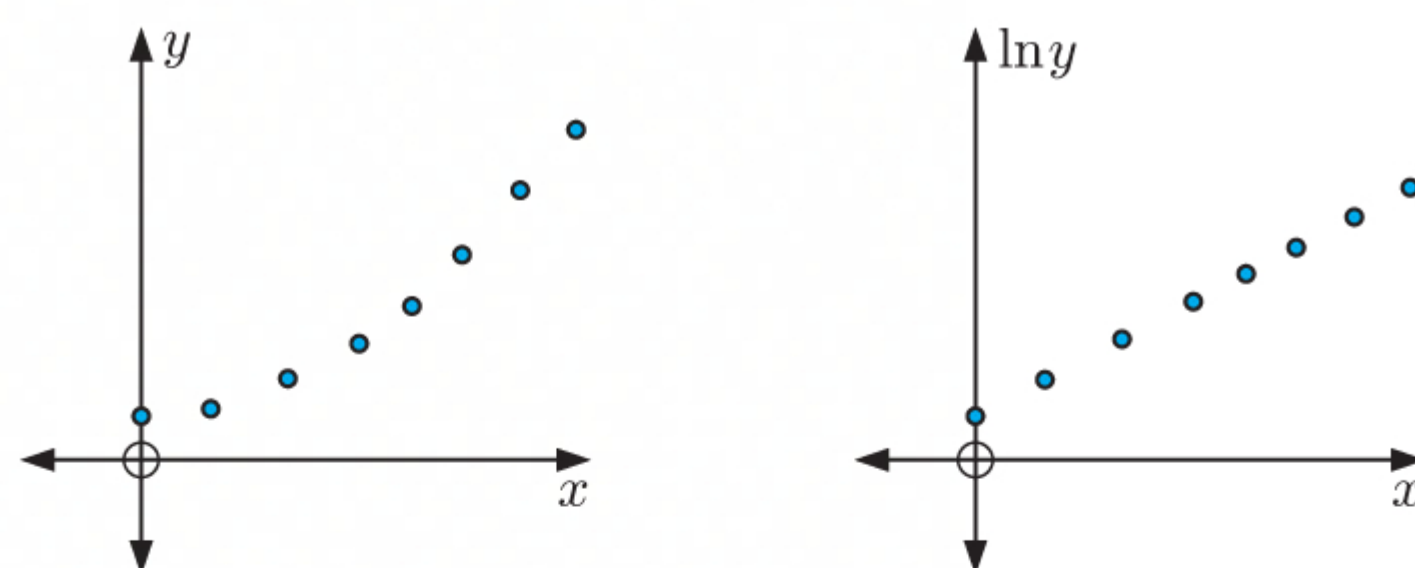


We can use linear regression to find an equation connecting  $y$  and  $\ln x$ .

### Exponential models

An **exponential model** has the form  $y = kb^x$  where  $k > 0$  and  $b > 0$ ,  $b \neq 1$  are constants.

If the variables  $x$  and  $y$  are connected by an exponential model, the graph of  $\ln y$  against  $x$  is linear.

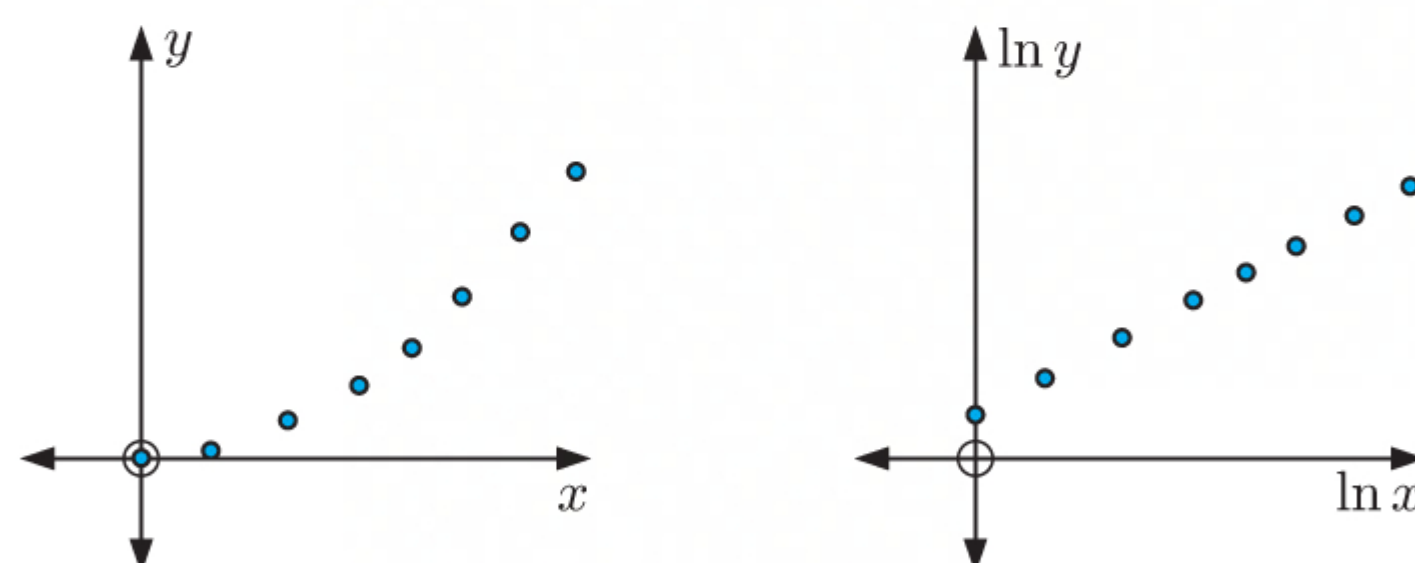


We can use linear regression to find a *linear* equation connecting  $\ln y$  and  $x$ , which we can rearrange into an *exponential* equation connecting  $y$  and  $x$ .

### Power models

A **power model** has the form  $y = ax^n$  where  $a > 0$  and  $n \neq 0$  are constants.

If the variables  $x$  and  $y$  are connected by a power model, the graph of  $\ln y$  against  $\ln x$  is linear.



We can use linear regression to find a *linear* equation connecting  $\ln y$  and  $\ln x$ , which we can rearrange into a *power* equation connecting  $y$  and  $x$ .

Since  $y = ax^n$ , all power models pass through the origin. However, since we cannot take the logarithm of 0, we will need to remove data such as  $(0, 0)$  before starting our analysis.

### Problem solving

To determine whether a linear, logarithmic, exponential, or power model is most appropriate to connect variables  $x$  and  $y$ :

**Step 1:** Draw scatter diagrams of  $y$  against  $x$ ,  $y$  against  $\ln x$ ,  $\ln y$  against  $x$ , and  $\ln y$  against  $\ln x$ .

**Step 2:** Find the regression line for each scatter diagram in which the variables appear linearly related. If more than one model appears appropriate, choose the one with the highest  $r^2$  value.

**Step 3:** Write the equation connecting the variables  $x$  and  $y$  without logarithms.



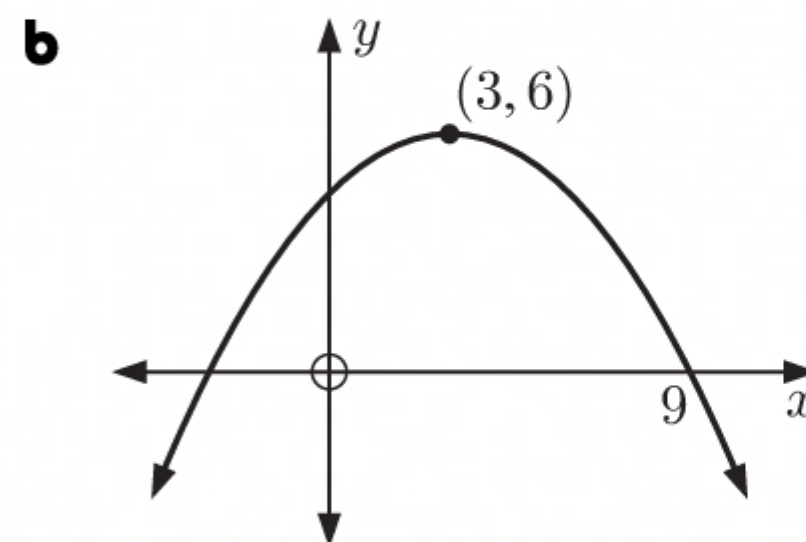
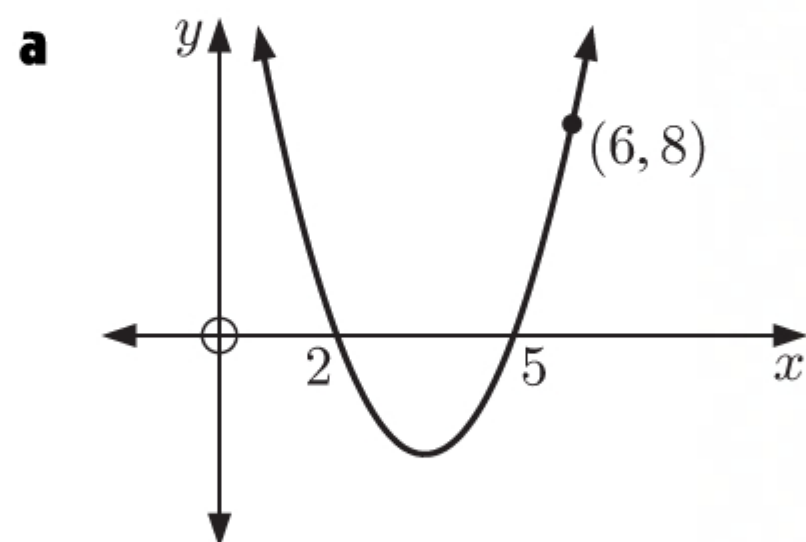
## SKILL BUILDER QUESTIONS

- 1 Find the equation of the line which is:
  - a parallel to  $2x - y = -3$  and passes through  $(5, 3)$
  - b perpendicular to  $y = -4x + 3$  and passes through  $(-1, 5)$ .
- 2 A line passes through the points  $(-3, 4)$  and  $(-1, 10)$ . For this line, find:
  - a the gradient
  - b the equation
  - c the axes intercepts.
- 3 Draw the graph of:
  - a  $y = -4x + 8$
  - b  $7x + 4y = 14$
- 4 Tammy buys tickets to a stage show. Tickets cost \$30 for adults, and \$15 for children. She spends a total of \$120 buying tickets for  $x$  adults and  $y$  children.
  - a Explain why  $30x + 15y = 120$ .
  - b If Tammy bought tickets for 4 children, how many adult tickets did she buy?
  - c Find the  $x$ -intercept of the line  $30x + 15y = 120$ , and interpret your answer.
  - d Draw the graph of  $30x + 15y = 120$ . Mark two points on your graph to indicate your answers to **b** and **c**.
- 5 The line  $ax + by = 20$  is perpendicular to  $y = \frac{3}{2}x + 1$ , and passes through the point  $(2, 2)$ . Find  $a$  and  $b$ .
- 6 Solve for  $x$ :
  - a  $2x^2 - 9x = 0$
  - b  $x^2 + 8x - 20 = 0$
  - c  $4x^2 + 11x = 3$
  - d  $(x + 3)(1 - 2x) = -9$
- 7  $x = -2$  is a solution to  $x^2 + bx + (b - 2) = 0$ .
  - a Find the value of  $b$ .
  - b Find the other solution to the equation.
- 8 Use technology to solve:
  - a  $2^x - x^3 = 0$
  - b  $x^2 - \sqrt{x} = 2$
- 9 Consider the quadratic function  $f(x) = -x^2 + 2x + 5$ .
  - a Copy and complete the table of values.
 

$x$	-3	-2	-1	0	1	2	3
$f(x)$							
  - b Hence sketch the graph of  $y = f(x)$ .
- 10 Find the zeros of the following functions:
  - a  $y = x^2 - x - 12$
  - b  $f(x) = 5x - x^2$
  - c  $y = 8x^2 - 2x - 3$
- 11 Find the axes intercepts of:
  - a  $y = (2x - 1)(x + 3)$
  - b  $f(x) = (x + 1)^2$
  - c  $y = 3x^2 + 4x - 4$
- 12 Sketch the graph of each function by considering the coefficient of  $x^2$  and the axes intercepts.
  - a  $y = x^2 - 2x - 8$
  - b  $f(x) = -(2x + 1)(x - 3)$
  - c  $y = -\frac{1}{2}(x - 4)^2$
- 13 A quadratic function has axis of symmetry  $x = -1$ , and one of its  $x$ -intercepts is 2. Find the other  $x$ -intercept.
- 14 The quadratic function  $f(x) = 2x^2 + bx - 3$  has axis of symmetry  $x = 6$ .
  - a Find the value of  $b$ .
  - b Find the coordinates of the vertex.
- 15 For each of the following quadratics:
  - i Find the axes intercepts.
  - ii Find the axis of symmetry.
  - iii Find the coordinates of the vertex, and state whether it is a maximum turning point or a minimum turning point.
  - iv Sketch the quadratic.
  - v State the domain and range.
  - a  $y = -(x - 1)(x + 3)$
  - b  $y = 2(x + 7)(x - 2)$



- 16** Find the equation of the quadratic with graph:



- 17** Find the coordinates of the point(s) of intersection of:

**a**  $y = x^2 - 4x - 5$  and  $y = 3x - 11$

**b**  $y = -2x^2 + 5x$  and  $y = 5 - 2x$

- 18** The daily profit made by a local baker selling  $x$  homemade pies is given by  $P = -0.05x^2 + 9x - 60$  dollars.

- a** Copy and complete this table.

$x$	0	20	40	60	80	100
$P$		100		300		340

- b** Use the points in **a** to sketch the graph of  $P$  against  $x$ .

- c** Find:

- i** the number of pies that need to be sold to maximise the profit
- ii** the maximum possible daily profit
- iii** the number of pies that need to be sold to make a profit of \$200
- iv** the amount of money the baker loses if no pies are sold.

- 19** Andreas is making an aquarium in the shape of an equilateral triangular prism. The sum of all side lengths of the prism must be 1.8 m.

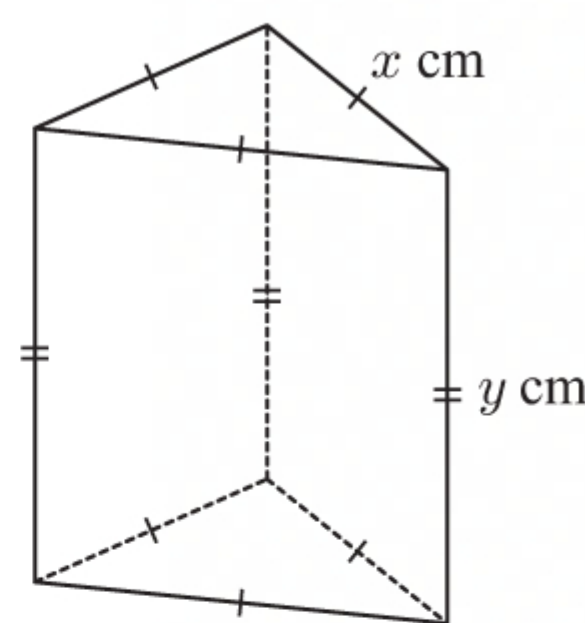
Let the equilateral triangle ends have sides of length  $x$  cm, and the aquarium have height  $y$  cm.

- a** Show that the area of the end is  $\frac{\sqrt{3}}{4}x^2$  cm<sup>2</sup>.

- b** Hence show that the total surface area of the aquarium is

$$A = \left(\frac{\sqrt{3}}{2} - 6\right)x^2 + 180x \text{ cm}^2.$$

- c** What dimensions should Andreas choose for the aquarium to maximise its surface area?



- 20** Solve for  $x$ :

**a**  $(x - 1)(5 - x) \leq 0$

**b**  $x^2 + 8x - 20 < 0$

**c**  $-9x^2 + 4x + 5 \geq 0$

- 21** Solve for  $x$ :

**a**  $x^2 > 9$

**b**  $x^2 - 15 \leq 2x$

**c**  $3x^2 < 2(5x + 4)$

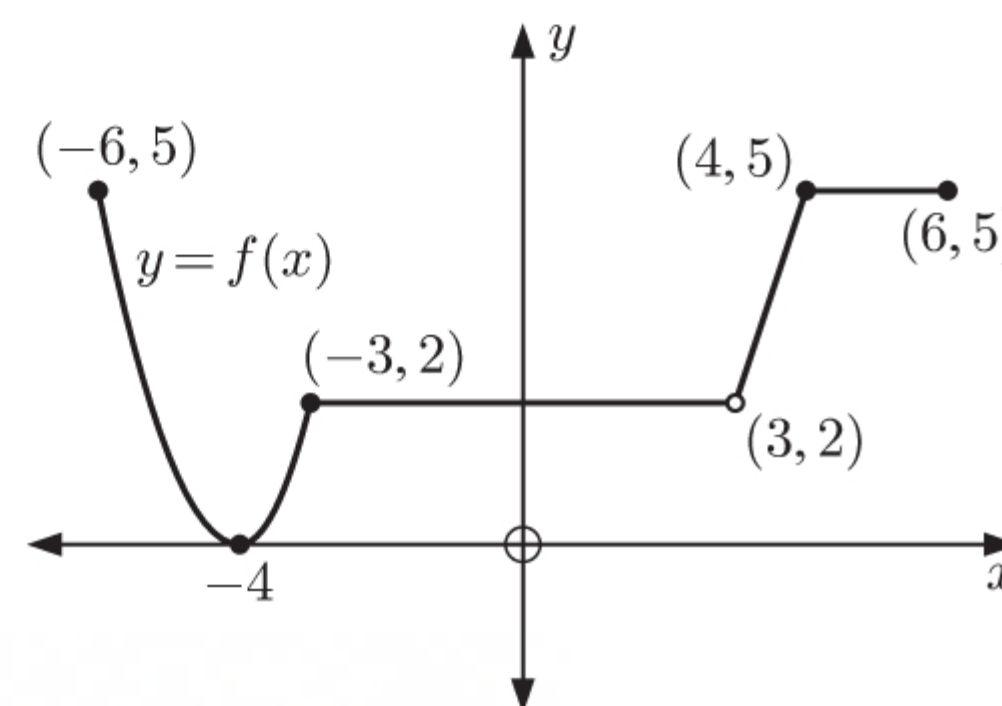
- 22** Jacob's rainwater tank started leaking. The amount of water in the tank after  $t$  hours is given by  $W(t) = 1000 - 0.5t$  litres.

- a** Find  $W(0)$ , and interpret your answer.
- b** Find  $t$  when  $W(t) = 700$ , and explain what this represents.
- c** How long will it take for the tank to empty?

- 23** Consider the graph of  $y = f(x)$  alongside.

Decide whether each statement is true or false:

- a** 0 is in the domain of  $f$ .
- b** 0 is in the range of  $f$ .
- c** 6 is in the range of  $f$ .
- d** 3 is in the domain of  $f$ .
- e** 2 is in the range of  $f$ .



- 24** A function  $f$  is defined as  $f(x) = \sqrt{x + 4}$  for  $-4 \leq x \leq 12$ ,  $x \in \mathbb{R}$ .

- a** Find: **i**  $f(-4)$  **ii**  $f(0)$  **iii**  $f(12)$ .
- b** Sketch  $y = f(x)$ .
- c** Hence write down the range of  $f(x)$ .



**25** State the domain and range of each function:

**a**  $f(x) = \sqrt{5-x}$

**b**  $f(x) = \frac{2}{x+4}$

**c**  $f(x) = \frac{1}{(x-1)^2}$

**26** Consider  $h(x) = x^2 - 2^{-x} + \frac{1}{x}$ .

**a** Determine  $h(-2)$ .

**b** Solve  $h(x) = 2$ .

**c** Write down the equation of the vertical asymptote.

**d** Sketch  $y = h(x)$ , illustrating your results from **a**, **b**, and **c**.

**e** Determine the range of  $y = h(x)$ .

**27** Consider the functions  $f(x) = \frac{1}{x-1} + \sqrt{x+1}$  and  $g(x) = x^2$ .

**a** State the domain of  $f$ .

**b** Find  $(f \circ g)(x)$ .

**c** Is the domain of  $(f \circ g)$  the same as the domain of either  $f$  or  $g$ ? Explain your answer.

**28** Let  $f(x) = \sqrt{x+4}$  and  $g(x) = x^2 - 3$ .

**a** Find  $(f \circ g)(x)$  and state its domain and range.

**b** Find  $(g \circ f)(x)$  and state its domain and range.

**29** A cylindrical water tank has radius 1 m, and initially contains 400 L of water. Rain begins to fall and fills the tank at 600 mL per minute.

**a** Find the function for the volume  $V$  L of water in the tank  $t$  minutes after the rain began to fall.

**b** Find an expression for the height  $h$  m of water in the tank in terms of  $V$ .

**c** Find  $h \circ V$  and explain what it means.

**d** Find the height of water in the tank 1 hour after the rain began to fall.

**30** Functions  $f$  and  $g$  are given by  $f: x \mapsto e^{x+1}$  and  $g: x \mapsto \ln x - 1$ .

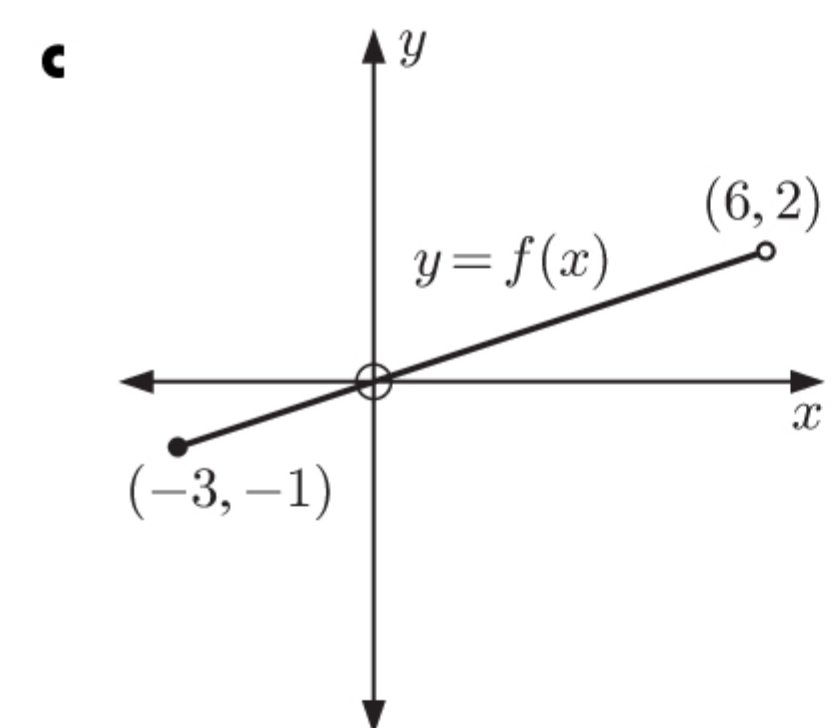
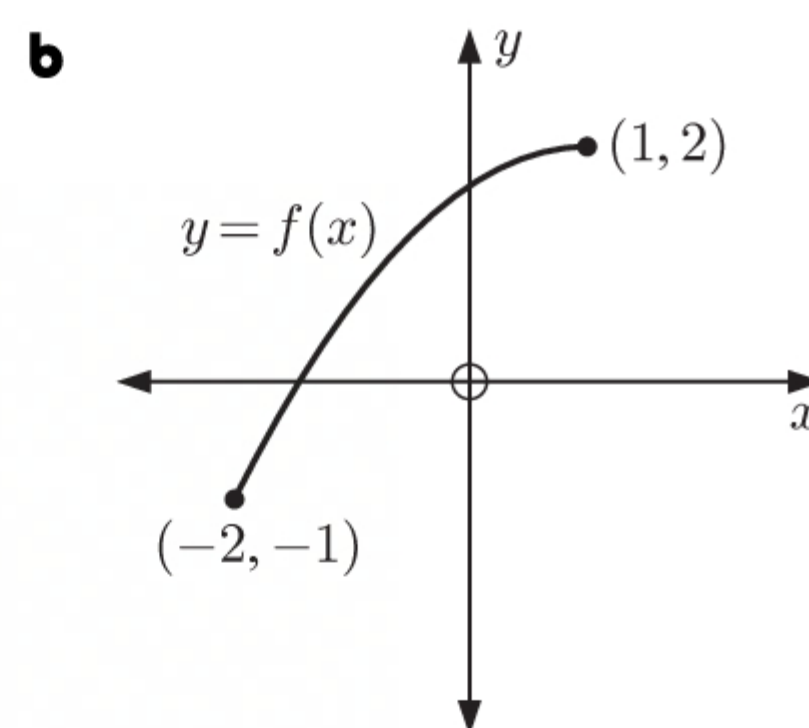
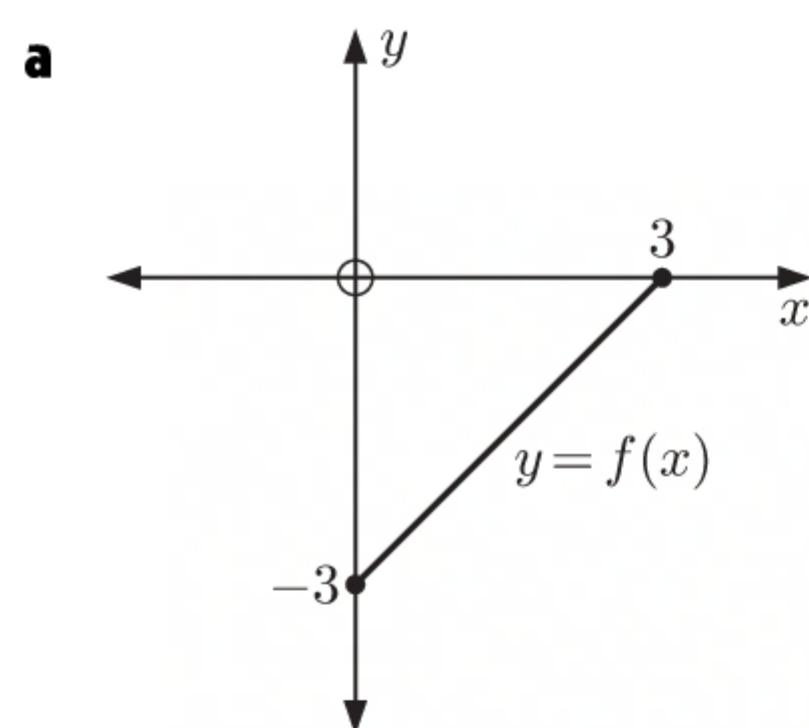
**a** Find  $(f \circ g)(x)$  and state its domain and range.

**b** Find  $(g \circ f)(x)$  and state its domain and range.

**c** Graph  $y = f(x)$  and  $y = g(x)$  on the same set of axes.

**d** State the relationship between  $f$  and  $g$ .

**31** Copy the graphs of the following functions and draw the graphs of  $y = x$  and  $y = f^{-1}(x)$  on the same set of axes. In each case, state the domain and range of both  $f$  and  $f^{-1}$ .



**32** Functions  $f$  and  $g$  are defined by  $f: x \mapsto 3x + 1$  and  $g: x \mapsto 4 - x$ . Find:

**a**  $f(g(x))$

**b**  $(g \circ f)(-4)$

**c**  $f^{-1}(\frac{1}{2})$

**33** Suppose  $f: x \mapsto \ln x$  and  $g: x \mapsto 3 + x$ . Find:

**a**  $f^{-1}(2) \times g^{-1}(2)$

**b**  $(f \circ g)^{-1}(2)$

**34** Suppose  $f: x \mapsto x + 5$  and  $g: x \mapsto 7 - 3x$ .

**a** Find: **i**  $f^{-1}(x)$  **ii**  $g^{-1}(x)$  **iii**  $(f \circ g)(x)$

**b** Show that  $(g^{-1} \circ f^{-1})(x) = (f \circ g)^{-1}(x)$ .

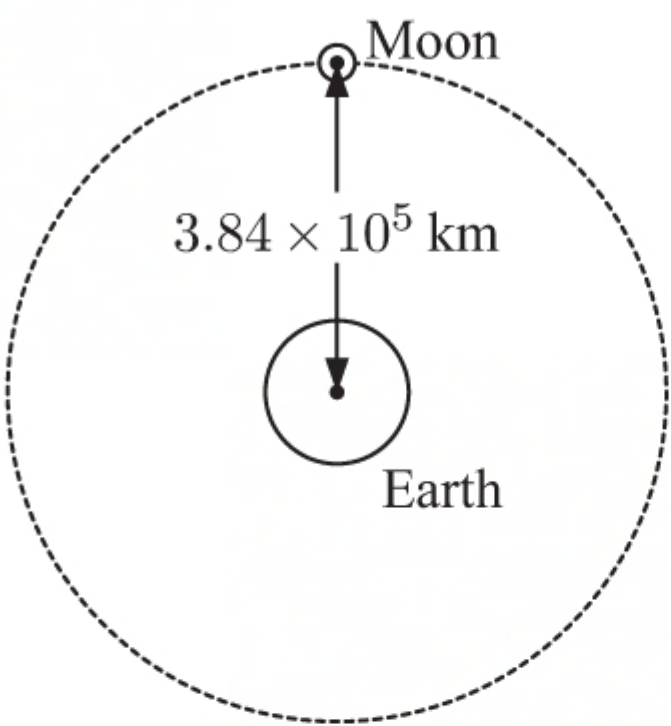
**35** Suppose  $f(x) = x^2 + 2x$ ,  $x \leq -1$ .

Find  $f^{-1}(x)$ , and state its domain and range.



- 36** Let  $f(x) = -x^2 - 4x + 7$ ,  $x \leq k$ .
- a** Find the largest value of  $k$  such that  $f^{-1}(x)$  exists.
  - b** For this value of  $k$ :
    - i** Find  $f^{-1}(x)$ .
    - ii** State the domain and range of  $f^{-1}(x)$ .

- 37** The moon is approximately  $3.84 \times 10^5$  km from the Earth, and completes one full orbit in about 28 days.
- a** Find a model for the distance  $D$  km travelled by the moon in  $t$  days.
  - b** List any assumptions you have made in **a**.
  - c** Use your model in **a** to estimate the distance travelled by the moon:
    - i** in one full orbit
    - ii** in 7 days.
  - d** The moon actually completes an orbit of the Earth in 27 days, 7 hours, and 43 minutes. Discuss whether your answers in **c** are overestimates or underestimates.



- 38** In one hour, Isabelita can prepare 120 spring rolls, and Arturo can prepare 100 spring rolls.
- a** How long will it take to prepare 600 spring rolls if:
    - i** Isabelita works by herself
    - ii** Arturo works by himself
    - iii** Isabelita and Arturo work together?
  - b** Discuss any assumptions you have made in **a**, and whether or not they are reasonable.
- 39** An oven technician charges a call-out fee of \$50 and an additional \$20 per hour for the duration of the appointment.
- Let  $\$C$  be the amount charged for an appointment lasting  $t$  hours.
- a** Sketch the graph of  $C$  against  $t$  for  $0 \leq t \leq 3$ .
  - b** Find the linear model connecting  $C$  and  $t$ .
  - c** Use your model to calculate the amount charged for a  $1\frac{1}{2}$  hour appointment.
  - d** Mikhal schedules an appointment to repair his oven. The technician arrives at 9:00 am, and the total amount charged is \$118. At what time did the technician complete the repairs?

- 40** The cost of hiring a car is summarised in the table alongside.
- Let  $\$C$  be the total cost of hiring a car for  $t$  days.

Hire period ( $t$ days)	Cost
1 - 2	\$40 per day
3 - 6	\$33 per day
7+	\$29 per day

- a** Draw a graph of  $C$  against  $t$  for  $0 \leq t \leq 10$ .
  - b** Find the cost of hiring a car for:
    - i** 2 days
    - ii** 5 days
    - iii** 9 days.
  - c** Georgia and Tim are planning an 8 day holiday. They require a car for the first 2 days, and the last 3 days.  
Is it cheaper for them to hire one car for the first 2 days and a separate car for the last 3 days, or to hire one car for the whole holiday?
- 41** The temperature  $T^\circ\text{C}$  of a casserole  $t$  minutes after being placed in an oven is given by

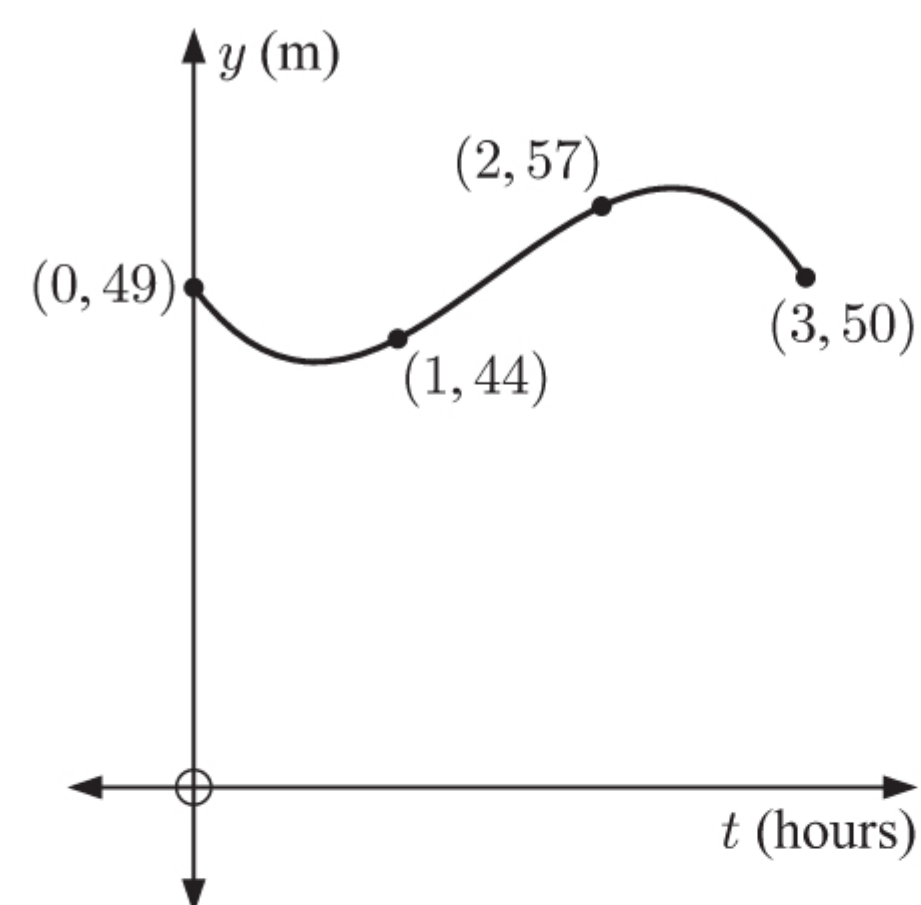
$$T(t) = \begin{cases} 260 - 240 \times 2^{-\frac{t}{a}}, & 0 \leq t < 10 \\ 200, & 10 \leq t < 30 \\ 30 + b \times 5^{\frac{30-t}{10}}, & t \geq 30 \end{cases}$$

- where  $a$  and  $b$  are constants.
- a** Find the temperature of the casserole when it is placed in the oven.
  - b** Find  $a$  and  $b$ .
  - c** Sketch the graph of  $T(t)$  for  $0 \leq t \leq 50$ .
  - d** How long was the casserole in the oven?
  - e** From the time the casserole was removed from the oven, how long did it take for the casserole's temperature to fall to  $45^\circ\text{C}$ ?



- 42** Stephen and Hugh take a walk in a national park. They measure their elevation in metres above sea level before they start, and every hour throughout their walk. Their elevation after  $t$  hours can be modelled by the function  $y = at^3 + bt^2 + ct + d$  for  $0 \leq t \leq 3$ .

- State the value of  $d$ .
- Use technology to find  $a$ ,  $b$ , and  $c$ .
- Estimate their elevation after  $2\frac{1}{2}$  hours.
- Their actual elevation after  $2\frac{1}{2}$  hours is 60 m.  
Calculate the percentage error in your estimate in **c**.



- 43** Suppose  $w \propto z$  and that  $w = 27$  when  $z = 9$ . Find:

- $w$  when  $z = 2$
- $z$  when  $w = 45$ .

- 44** The potential difference  $V$  across a resistor is proportional to the current  $I$  running through it. When the potential difference is 9 volts, the current is 0.01 amps.

- Find the proportionality constant.
- Find the current running through the resistor if the potential difference is 12 volts.
- Find the potential difference if there is 0.018 amps of current running through the resistor.

- 45** The table below shows the cost of purchasing  $x$  kg of tomatoes.

Weight ( $x$ kg)	0	1	2	3	4	5
Cost ( $\$C$ )	0	3.5	7	10.5	14	17.5

- Draw the graph of  $C$  against  $t$ .
- Explain why  $C$  and  $x$  are directly proportional.
- Find a formula connecting  $C$  and  $x$ .
  - Hence find the cost of purchasing 10 kg of tomatoes.

- 46** The mass of an orange is directly proportional to the cube of its diameter.

The diameter of orange A is 14.6% larger than that of orange B. What percentage heavier is orange A than orange B?

- 47** Suppose  $y$  is inversely proportional to  $x$ . Explain what happens to  $y$  if:

- $x$  is tripled
- $x$  is divided by 4
- $x$  is multiplied by  $\frac{5}{3}$
- $x$  is decreased by 60%.

- 48** When a constant force is applied by an object, the *pressure* applied by the object is inversely proportional to the *area* over which the force is applied.

When applied to an area of  $2 \text{ m}^2$ , the object applies 300 Pa of pressure.

- Find the pressure when the force is applied over  $0.8 \text{ m}^2$ .
- Find the percentage increase in area required to reduce the pressure by 15%.

- 49** Suppose  $A$  is inversely proportional to the cube of  $r$ , and that  $A = 7$  when  $r = 2$ .

- Find  $A$  when  $r = 5$ .
- Find  $r$  when  $A = 16$ .

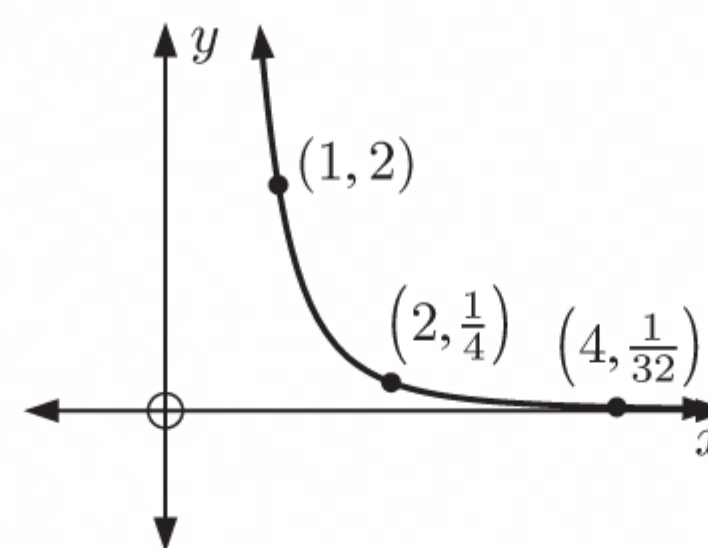
- 50** An ice cream cone company wants to adjust the existing dimensions of their cones, but maintain the same volume. Their cones currently have radius 2.8 cm and height 14.3 cm.

- Explain why the height of the cone is inversely proportional to the square of its radius.
- Find the height of the cone if the radius chosen is 3.2 cm.
- Find the radius of the cone if the height chosen is 10.8 cm.
- It is decided that the possible values for the radius should be between 2.5 cm and 3.5 cm. Can you suggest why this was done?



- 51** It is suspected that  $y$  is inversely proportional to a power of  $x$ .

- Calculate  $x^2y$ ,  $x^3y$ , and  $x^4y$  for the marked points.
- Hence determine the correct model connecting  $y$  and  $x$ .
- Find the value of  $y$  when  $x = 5$ .



- 52** The diameters of circular rugs and their corresponding masses are recorded in the table below.

Diameter ( $d$ m)	0.77	1.22	1.69	2.25
Mass ( $m$ kg)	0.97	2.44	4.68	8.30

- Do you think there is direct variation or inverse variation between the variables? Explain your answer.
- Use technology to obtain a power model which best fits the data.
- Estimate the mass of a rug with diameter 1.5 m.

- 53** For each data set, obtain the power model which best fits the data:

**a**

$x$	2	3	5	6
$y$	1.13	8.60	111	275

**b**

$x$	1	4	5	7	8
$y$	72.1	4.6	2.9	1.5	1.1

- 54** If  $f(x) = 2 \times 3^{-x}$ , find:

- $f(0)$
- $f(1)$
- $f(-2)$

- 55** Consider  $y = -1 + 2^{-x}$ .

- Find the  $y$ -intercept.
- Find any asymptotes of the function.
- State the domain and range of the function.
- Hence sketch the function.

- 56** The exponential function  $y = a \times 2^x + b$  passes through the points alongside:

- Write down two linear equations which could be used to determine the values of  $a$  and  $b$ .
- Solve the linear equations simultaneously to find  $a$  and  $b$ .
- Hence find the values of  $p$  and  $q$ .

$x$	0	1	2	3
$y$	20	$p$	35	$q$

- 57** Consider the exponential function  $f(x) = 2 \times \left(\frac{1}{3}\right)^x + 1$ .

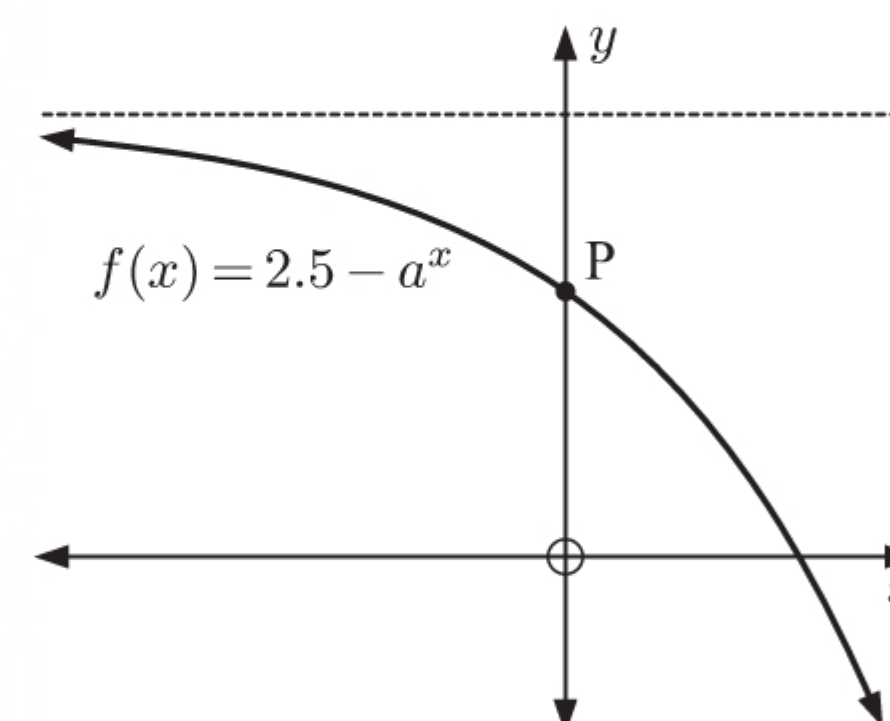
- Find:
  - $f(0)$
  - $f(2)$
  - $f(-1)$
- State the equation of the horizontal asymptote.
- Sketch the graph of the function.
- State the domain and range of the function.

- 58** Use technology to solve:

- $5^x = 14$
- $3 \times 2^{x-1} = 60$
- $40 \times (0.9)^x = 25$

- 59** The graph shows the function  $f(x) = 2.5 - a^x$  where  $a$  is a positive constant. The point  $(3, -5.5)$  lies on the graph.

- Write down the coordinates of P.
- Find the value of  $a$ .
- Find the equation of the horizontal asymptote.



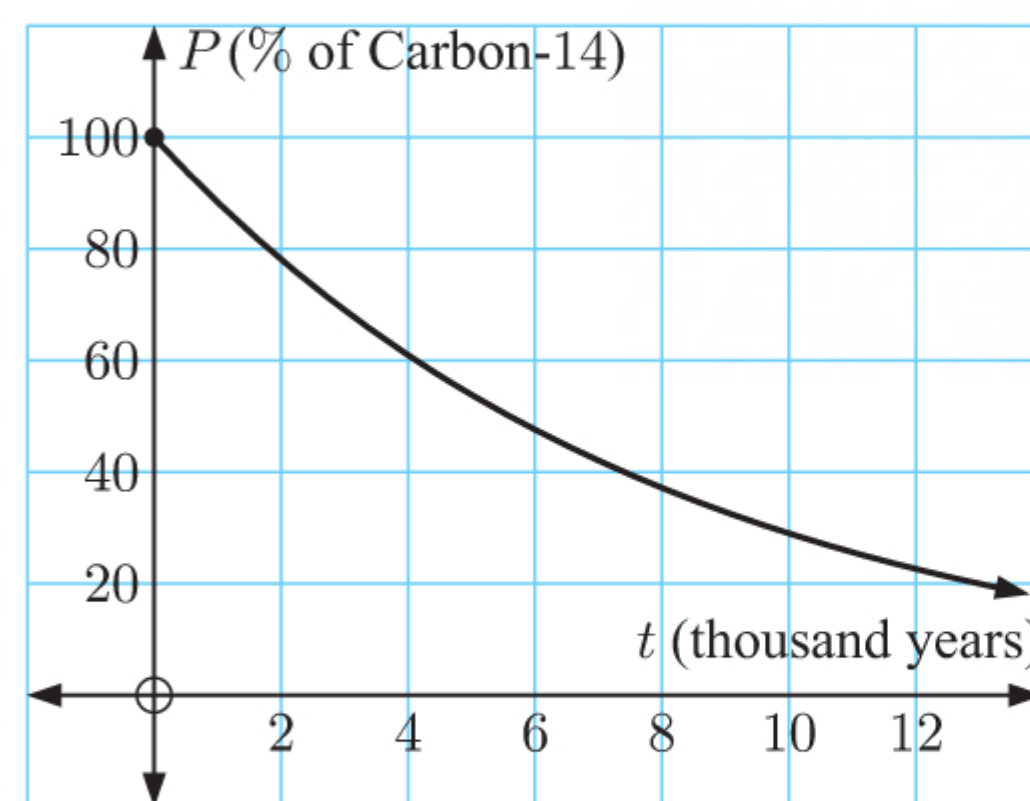
- 60** The population of bees in a hive after  $t$  weeks is given by  $P(t) = a(0.95)^t + b$ .

Initially, there are 2500 bees, and after 2 weeks there are 2383 bees.

- Find the value of:
  - $a$
  - $b$ .
- Find the population after:
  - 3 weeks
  - 5 weeks.
- State the horizontal asymptote, and interpret this value.



- 61** The graph alongside shows the percentage  $P$  of radioactive Carbon-14 remaining in an organism  $t$  thousands of years after it dies.



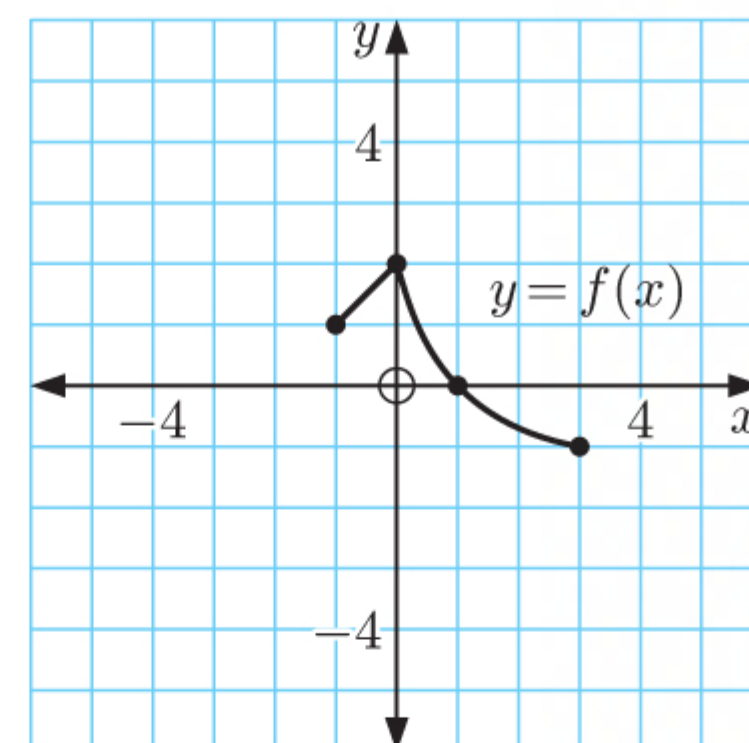
- a** Use the graph to estimate:
- the percentage of Carbon-14 remaining after 4000 years
  - the number of years for the percentage of Carbon-14 to fall to 50%.
- b** The equation of the graph is  $P = 100 \times (1.1318)^{-t}$ ,  $t \geq 0$ .
- Calculate the percentage of Carbon-14 remaining after 8000 years.
  - How long will it take for the percentage to fall to 15%?
- 62** The number of people  $N$  on a small island  $t$  years after settlement, increases according to the formula  $N = 120 \times (1.04)^t$ .
- Find the number of people who started the settlement.
  - Find the number of people on the island after 4 years.
  - How many years will it take for the number of people to double?
- 63** A radioactive substance has a half-life of 4 days. The weight of a sample of this substance after  $t$  days is  $W(t) = 100 \times a^t$  mg, where  $a > 0$ .
- Find the initial weight of the sample.
  - Calculate the value of  $a$ , correct to 4 decimal places, and interpret this value.
  - Find the weight of the sample after 6 days.
  - How long will it take for the weight of the sample to fall to:   
    - 60 mg
    - 30 mg?
- 64** Sketch on the same set of axes as the graph of  $y = e^x$ :
- $y = e^x - 1$
  - $y = 2e^x$
  - $y = e^{\frac{x}{3}}$
- For each graph, state the  $y$ -intercept and equation of the horizontal asymptote.
- 65**
- Use technology to help sketch the graph of  $f(x) = 6 - e^{-0.5x}$ .
  - State the domain and range of  $f(x)$ .
  - Describe the behaviour of  $y = f(x)$  as  $x \rightarrow \pm\infty$ .
  - Find  $k$  such that  $f(x) = k$  has:   
    - one solution
    - no solutions.
- 66** The population of an island  $t$  years after it is first inhabited, is given by  $P(t) = \frac{2000}{1 + Ce^{-0.2t}}$ .
- Given the initial population was 500, find  $C$ .
  - State the limiting population of the island.
  - Find the population after:   
    - 10 years
    - 30 years.
  - Use technology to help sketch the graph of  $P$  against  $t$ .
  - How long did it take for the population to reach 1800?
- 67** For each of the functions  $f$ :
- State the domain and range.
  - Find the  $x$ -intercept, using technology if necessary.
  - Sketch the graph of  $y = f(x)$ , showing all important features.
  - Find the inverse function.
- $f(x) = 2 \ln x$
  - $f(x) = \ln x + 3$
  - $f(x) = 2 - 5 \ln x$
- 68**
- Write  $\ln\left(\frac{e^2}{x^3}\right)$  in the form  $a + b \ln x$ .
  - Hence sketch the graph of  $y = \ln\left(\frac{e^2}{x^3}\right)$ .
- 69** Find the equation of the resulting graph  $g(x)$  when:
- $f(x) = x^2 - 5x + 6$  is translated 8 units upwards
  - $f(x) = -2x^2 + x + 3$  is translated 1 unit to the right.



- 70** **a** State the domain and range of  $f(x) = \frac{1}{\sqrt{x-4}} + 3$ .  
**b** What transformation maps  $y = \frac{1}{\sqrt{x}}$  onto the function  $f$ ?  
**c** Write down the equations of the asymptotes of  $y = f(x)$ .

**71** Copy this graph of  $y = f(x)$ , and draw the graph of:

- a**  $y = f(x+2)$       **b**  $y = 2f(x) - 3$       **c**  $y = 4 - f(x)$   
**d**  $y = f(-x)$       **e**  $y = f(2x)$



**72** Find the equation of the resulting image when  $y = \frac{2}{x}$  is:

- a** reflected in the  $y$ -axis      **b** translated through  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$   
**c** stretched horizontally with scale factor 3.

**73** Find the amplitude, principal axis, and period of:

- a**  $f(x) = \sin 4x$       **b**  $f(x) = -2 \sin \frac{x}{2} - 1$ .

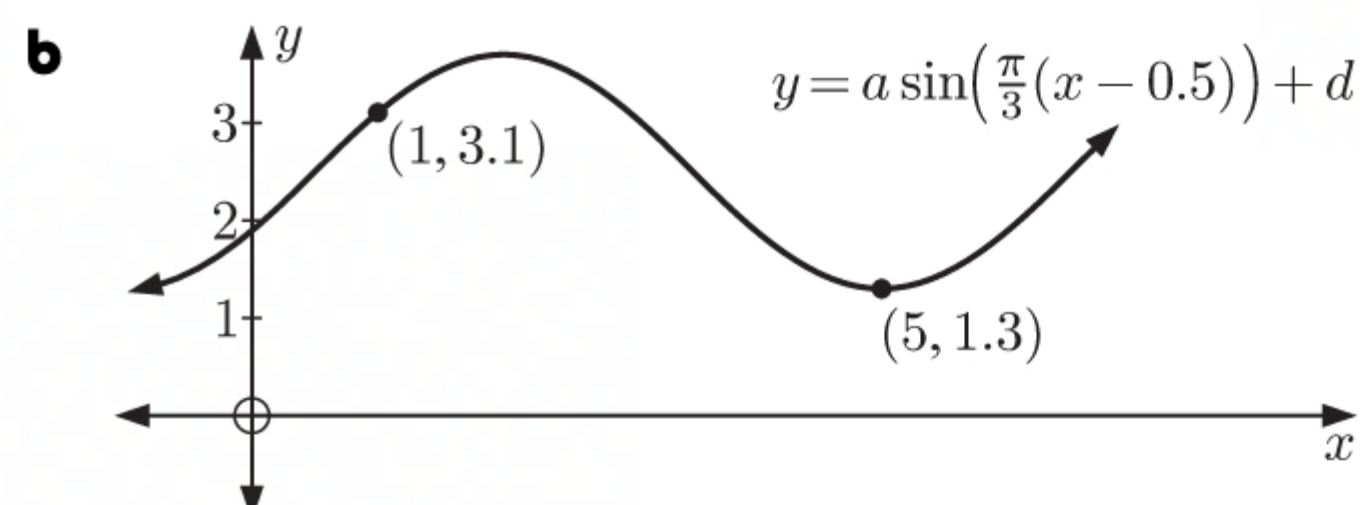
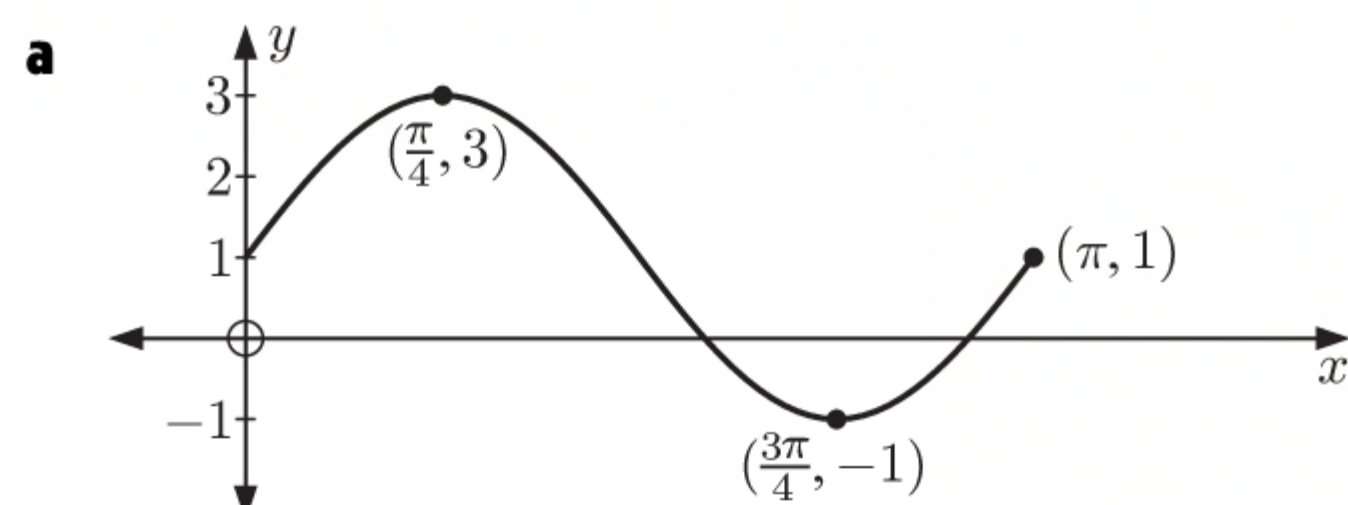
**74** For each of the following functions:

- i** State the amplitude.      **ii** State the principal axis.  
**iii** State the period.      **iv** Sketch the function.  
**a**  $y = 2 \sin(x - \frac{\pi}{3})$  for  $0 \leq x \leq 2\pi$       **b**  $y = \sin x + 2$  for  $-\pi \leq x \leq \pi$   
**c**  $y = 3 \cos 2x$  for  $0 \leq x \leq 2\pi$       **d**  $y = \cos \frac{x}{2} - 1$  for  $0 \leq x \leq 2\pi$   
**e**  $y = \sin(2(x + \frac{\pi}{4}))$  for  $0 \leq x \leq 2\pi$       **f**  $y = 10 - 6 \sin 3x$  for  $0 \leq x \leq 2\pi$

**75** State the transformations which map  $y = \sin x$  onto:

- a**  $y = 2 \sin \frac{x}{3}$       **b**  $y = \sin(x + \frac{\pi}{3}) - 4$

**76** Find the equation of each sine function.



**77** Consider the data:

$x$	1	3	9	27	81
$y$	0.5	7.3	14.5	20.1	24.8

- a** Draw a scatter diagram of  $y$  against  $x$  and  $y$  against  $\ln x$ .  
**b** Explain why  $y$  and  $x$  are related by a logarithmic model.  
**c** Find the equation connecting  $y$  and  $x$ .

**78** The intensity of sunlight  $I$  has been measured at various depths below the surface of a lake.

- a** Draw a scatter diagram of  $\ln I$  against  $x$ .  
**b** Explain why  $I$  and  $x$  are related by an exponential model.  
**c** Find the equation connecting  $x$  and  $I$ .  
**d** At what depth is the intensity of sunlight 1000 units?

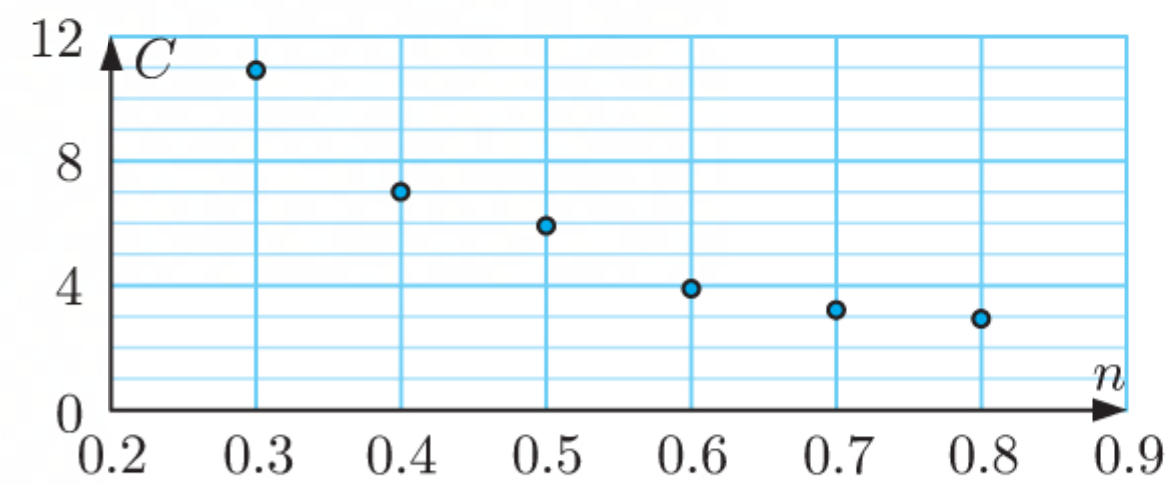
Depth ( $x$ metres)	2	5	7	10
Intensity ( $I$ units)	2980	365	90	11



- 79** During a grazing experiment, data was collected on the mean percentage of ground  $C$  covered by pasture, and the stock-rate  $n$  grazing on the pasture, in sheep per hectare.

$n$ (sheep per hectare)	0.3	0.4	0.5	0.6	0.7	0.8
$C$ (%)	10.9	7.0	5.9	3.9	3.2	2.9

A scatter diagram of  $C$  versus  $n$  is given below.



- a** Explain why a linear model is not appropriate for the data.
- b** Draw scatter diagrams of:
  - i**  $C$  against  $\ln n$
  - ii**  $\ln C$  against  $n$
  - iii**  $\ln C$  against  $\ln n$ .
- c** Which model is most appropriate for the data? Explain your answer.
- d** Find the model connecting  $n$  and  $C$ .
- e** Estimate the mean percentage of ground covered by pasture if there are 0.1 sheep per hectare grazing on the pasture.
- f** Estimate the stock-rate the owner should allow so that 9% of the ground is covered by pasture.



## TOPIC 3: GEOMETRY AND TRIGONOMETRY

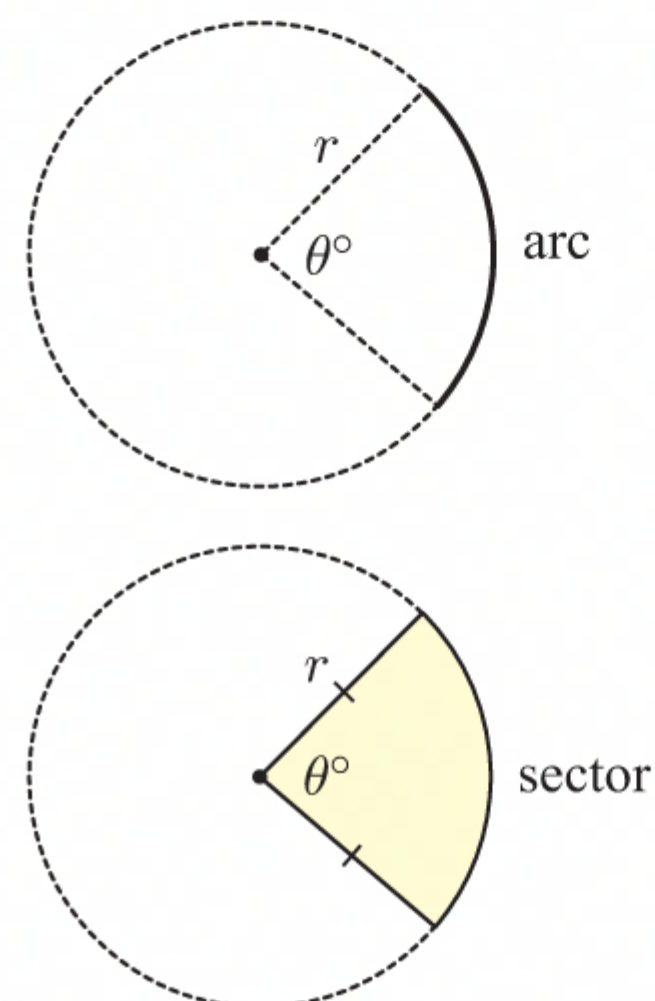
### ARCS AND SECTORS

An **arc** is a part of a circle which joins any two different points.

$$\text{Arc length} = \frac{\theta}{360} \times 2\pi r$$

A **sector** is the region between two radii of a circle and the arc between them.

$$\text{Area} = \frac{\theta}{360} \times \pi r^2$$



### GEOMETRY OF 3-DIMENSIONAL FIGURES

The **surface area** of a three-dimensional figure with plane faces is the sum of the areas of the faces.

The **volume** of a solid is the amount of space it occupies.

The **capacity** of a container is the quantity of fluid it is capable of holding. You should understand how the units of volume and capacity are related.

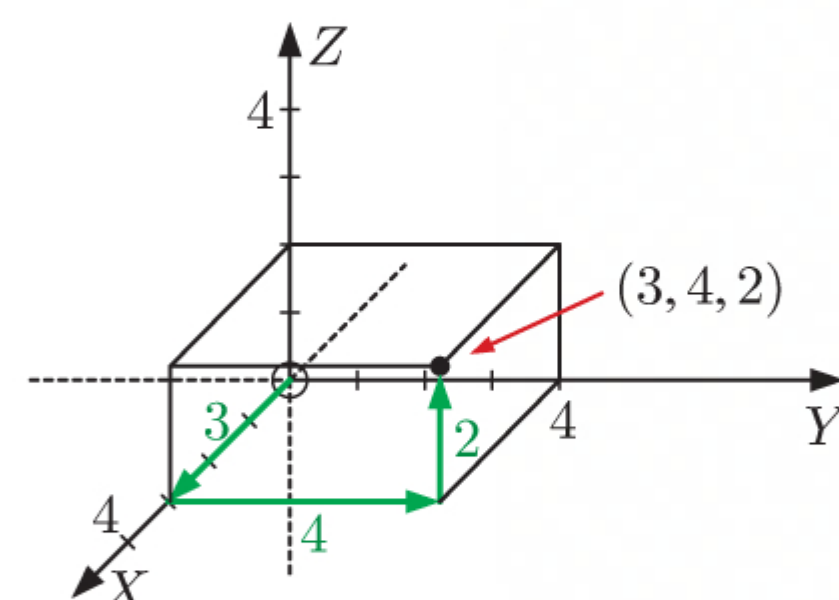
You should be able to calculate the surface area and volume of 3-dimensional figures, including solids of uniform cross-section, pyramids, spheres, and cones.

### 3-DIMENSIONAL COORDINATE GEOMETRY

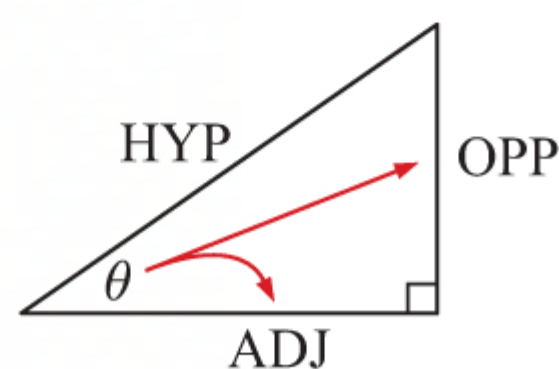
In 3-dimensional coordinate geometry, we specify an origin O, and three mutually perpendicular axes called the *X*-axis, the *Y*-axis, and the *Z*-axis.

For points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$ :

- the **distance**  $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- the **midpoint** of  $[AB]$  is  $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$ .



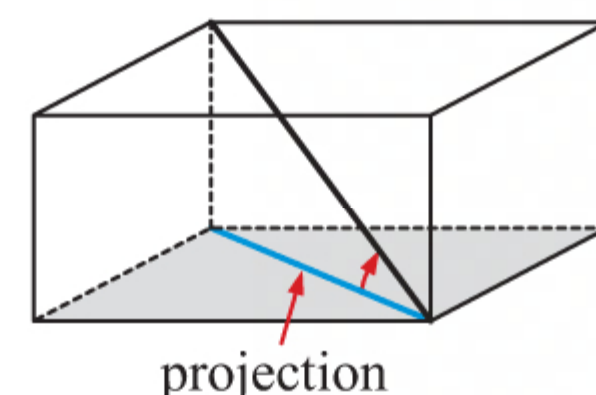
### RIGHT ANGLED TRIANGLE TRIGONOMETRY



$$\begin{aligned}\cos \theta &= \frac{\text{ADJ}}{\text{HYP}} \\ \sin \theta &= \frac{\text{OPP}}{\text{HYP}} \\ \tan \theta &= \frac{\text{OPP}}{\text{ADJ}}\end{aligned}$$

### Angle between a line and a plane

The **angle between a line and a plane** is the angle between the line and its **projection** on the plane.



### True bearings

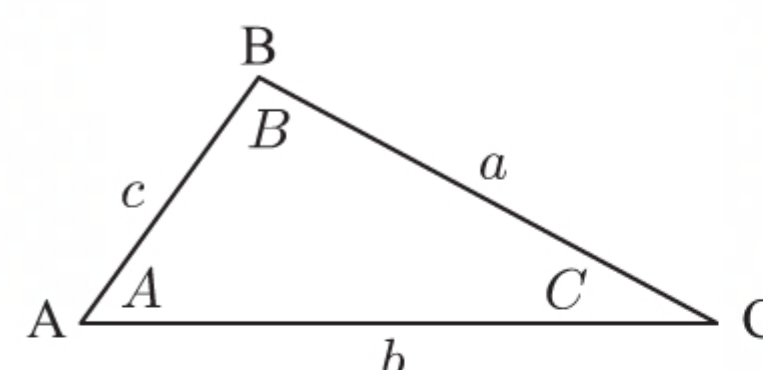
**True bearings** are used to describe the direction of one object from another. The direction is measured clockwise from true north.

### NON-RIGHT ANGLED TRIANGLE TRIGONOMETRY

**Area formula:**  $\text{Area} = \frac{1}{2}ab \sin C$

**Cosine rule:**  $a^2 = b^2 + c^2 - 2bc \cos A$

**Sine rule:**  $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$  or  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$





## RADIAN MEASURE

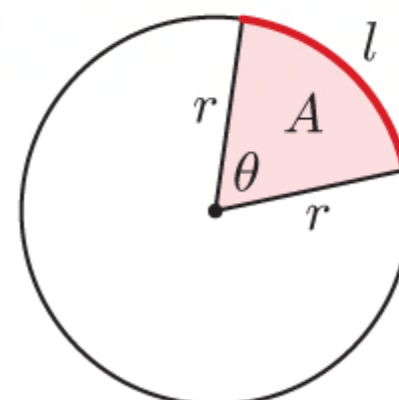
There are  $360^\circ \equiv 2\pi$  radians in a circle.

To convert from degrees to radians, multiply by  $\frac{\pi}{180}$ .

To convert from radians to degrees, multiply by  $\frac{180}{\pi}$ .

For  $\theta$  in radians:

- the length of an arc of radius  $r$  and angle  $\theta$  is  $l = \theta r$
- the area of a sector of radius  $r$  and angle  $\theta$  is  $A = \frac{1}{2}\theta r^2$ .



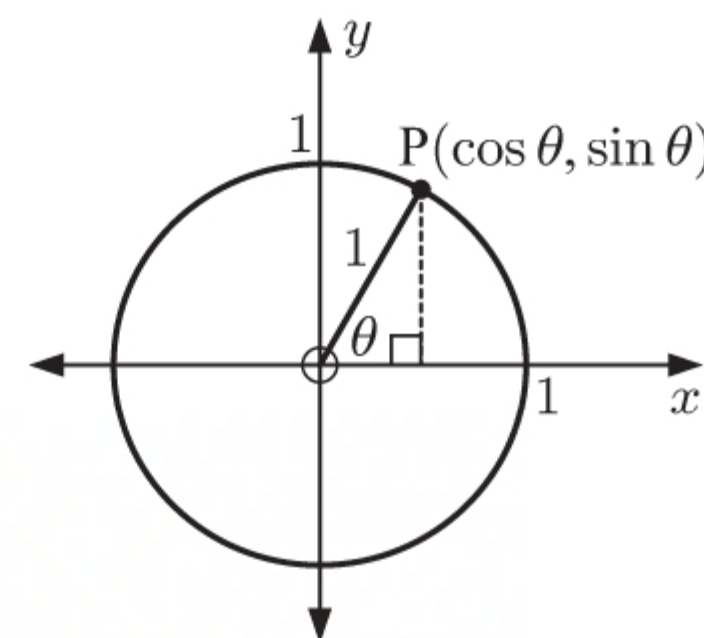
## THE UNIT CIRCLE

The **unit circle** is the circle centred at the origin O and with radius 1 unit.

Consider point P on the unit circle where [OP] makes angle  $\theta$  with the positive  $x$ -axis. The coordinates of P are  $(\cos \theta, \sin \theta)$ .

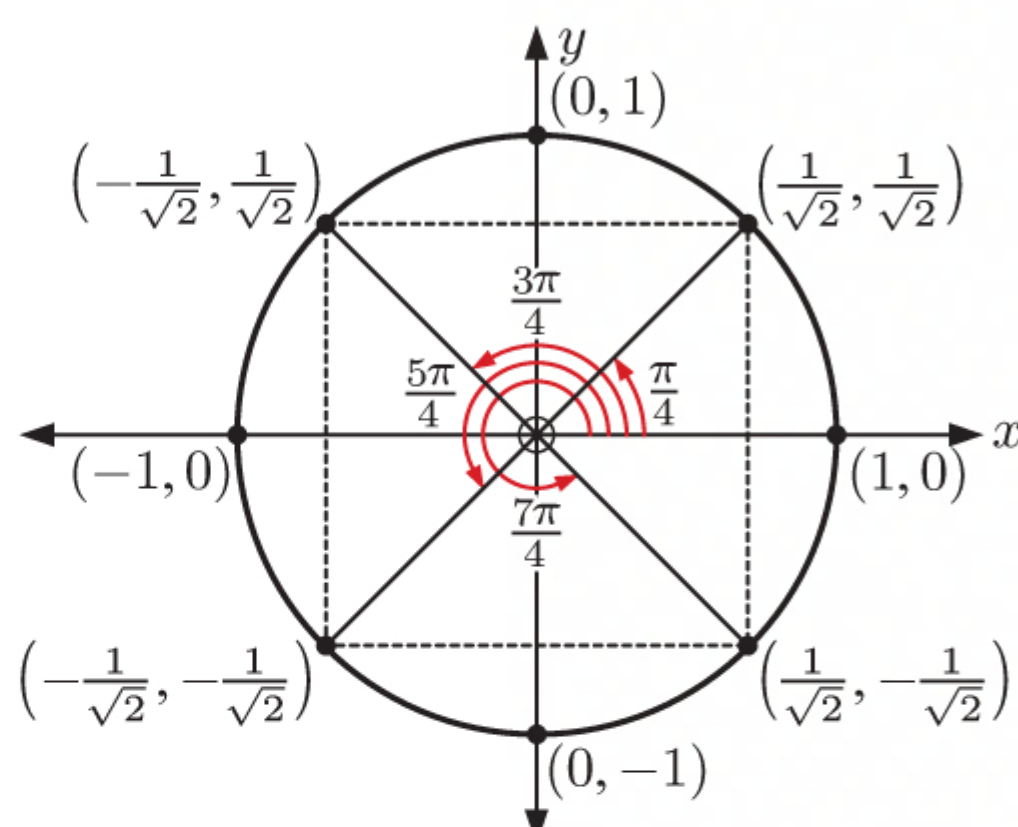
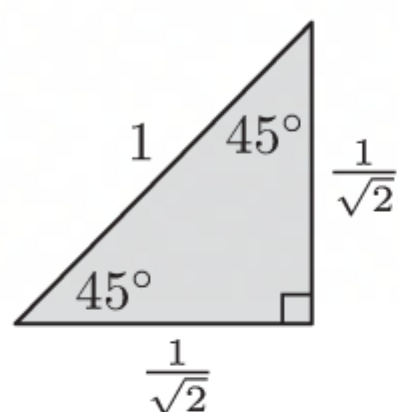
$\theta$  is **positive** when measured in an **anticlockwise** direction from the positive  $x$ -axis.

$\tan \theta$  is defined as  $\frac{\sin \theta}{\cos \theta}$ .  $\tan \theta$  is the **gradient** of [OP].

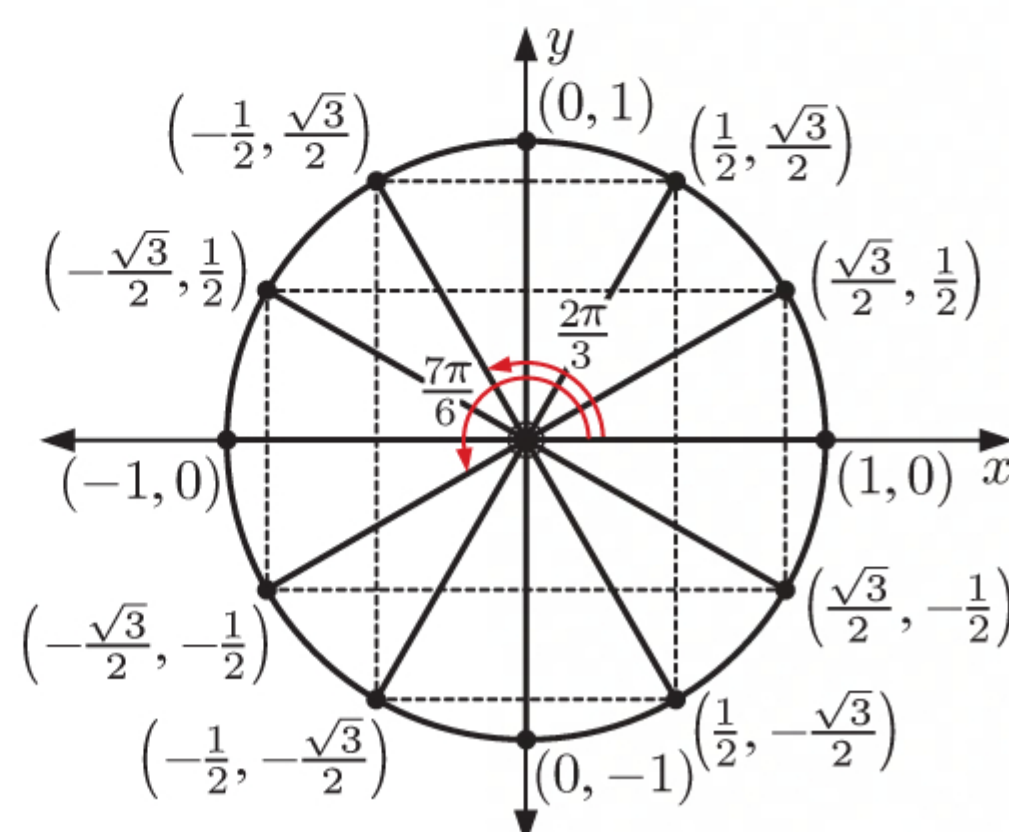
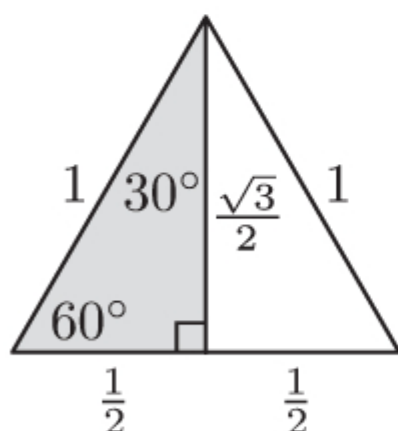


You should memorise or be able to quickly find the values of  $\cos \theta$ ,  $\sin \theta$ , and  $\tan \theta$  that are multiples of  $\frac{\pi}{4}$  or  $\frac{\pi}{6}$ .

### Multiples of $\frac{\pi}{4}$ or $45^\circ$



### Multiples of $\frac{\pi}{6}$ or $30^\circ$



The **Pythagorean identity**  $\cos^2 \theta + \sin^2 \theta = 1$  can be used to find one trigonometric ratio from another.

## TRIGONOMETRIC EQUATIONS

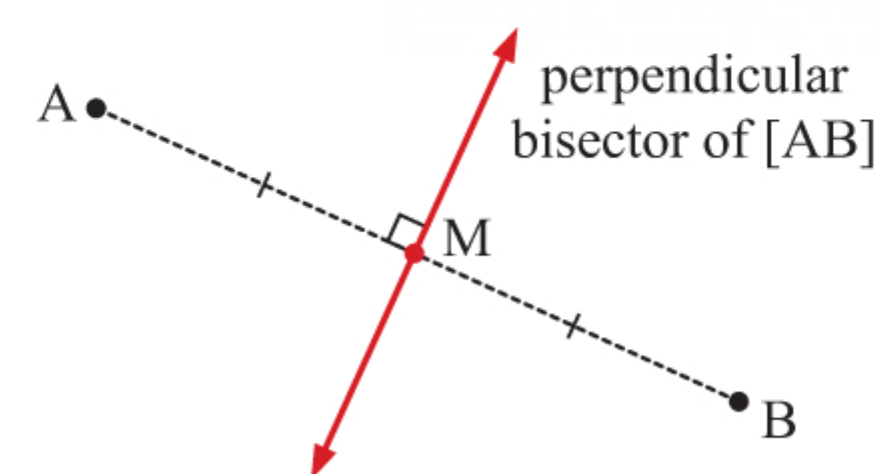
Trigonometric equations may be solved graphically, using pre-prepared graphs or technology.



## PERPENDICULAR BISECTORS

The **perpendicular bisector** of a line segment  $[AB]$  is the line perpendicular to  $[AB]$  which passes through its midpoint.

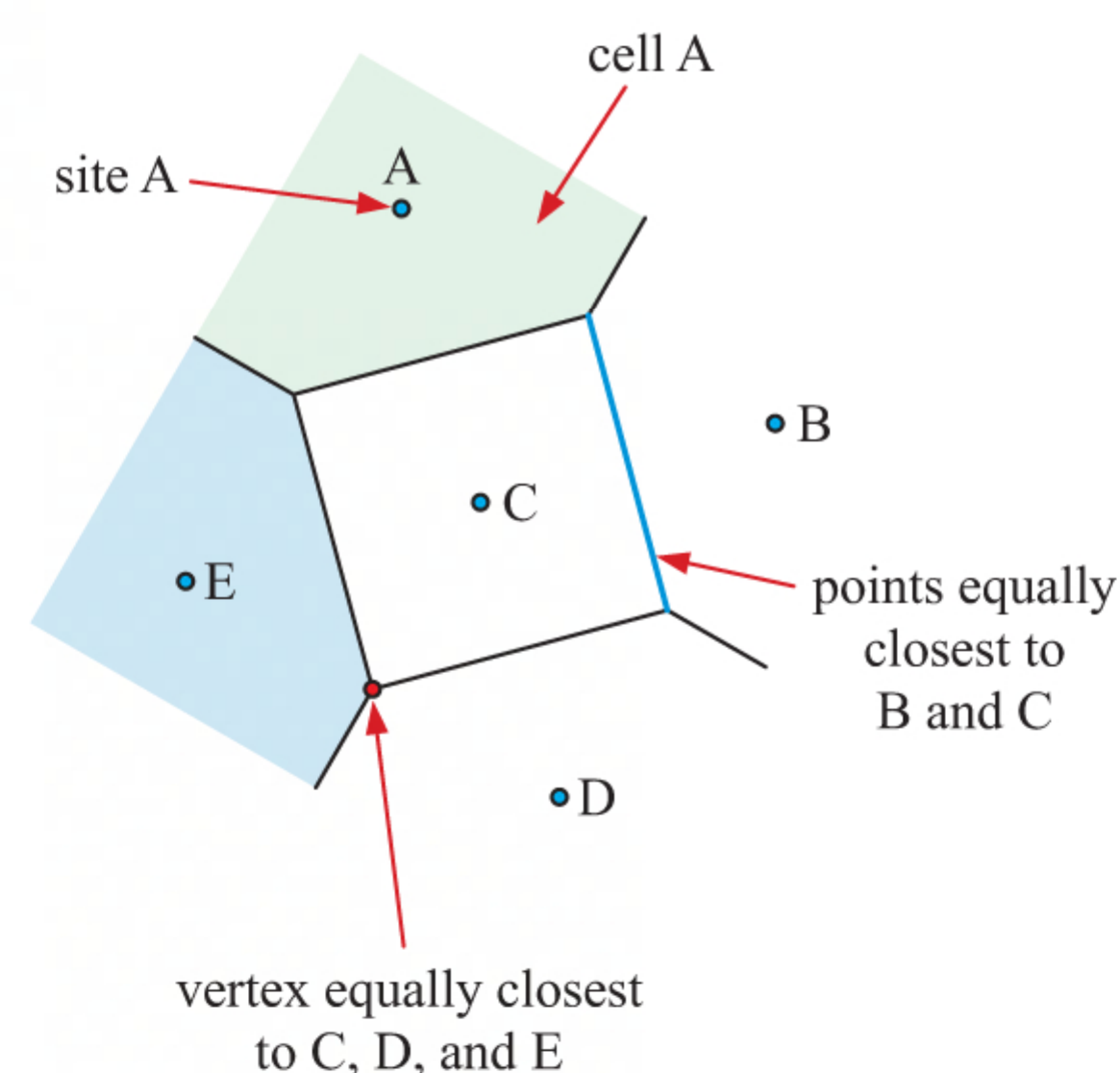
Points on the perpendicular bisector are equidistant from A and B.



## VORONOI DIAGRAMS

In a Voronoi diagram:

- Important locations are called **sites**.
- Each site is surrounded by a region or **cell** which contains the points which are closer to that site than to any other site.
- The lines which separate the cells are called **edges**. Each point on an edge is equally closest to the two sites whose cells are adjacent to that edge.
- The points at which the edges meet are called **vertices**. Each vertex is equally closest to the sites whose cells meet at that vertex.



### Adding a new site to a Voronoi diagram

To add the cell for a new site X to an existing Voronoi diagram with sites  $P_1, P_2, P_3, \dots, P_n$ , we follow these steps:

- Step 1:* Identify the site  $P_i$  whose cell contains the new site X. Construct the perpendicular bisector of  $[P_i X]$ , within this cell. At any point where this line meets an existing edge, create a new vertex.
- Step 2:* For each site  $P_j$  whose cell is adjacent to a new vertex, construct the perpendicular bisector of  $[P_j X]$  within that cell through the vertex. Continue to create new vertices as in *Step 1*. Repeat this process until no more new vertices are created. At this time cell X is complete.
- Step 3:* Remove any segments of edges from the original Voronoi diagram which now lie within cell X.

### Nearest neighbour interpolation

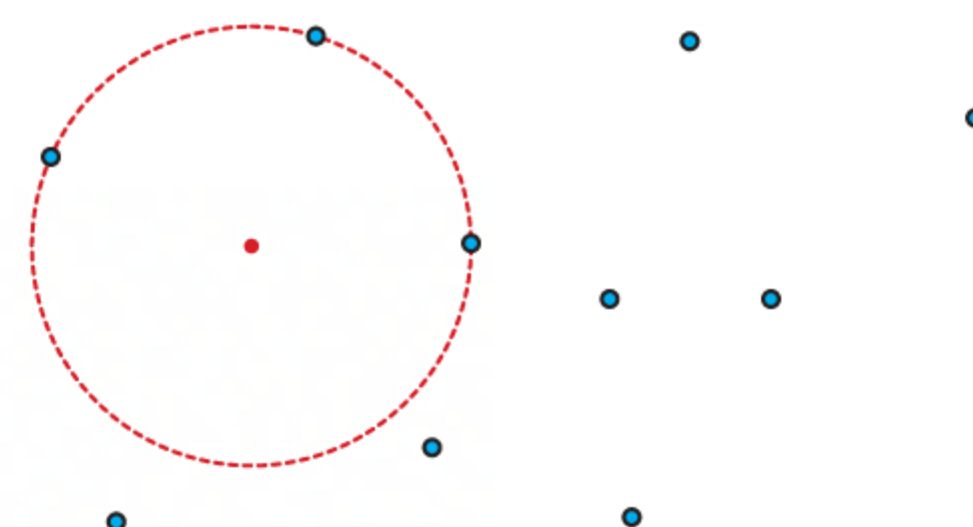
If we are given the values of a variable at a set of known data points, we can *estimate* the value of the variable at some other point. We use the variable's value at the *nearest* known data point.

From a Voronoi diagram with the known data points as sites, we can quickly identify the nearest known data point to any given point.

If the given point lies on an edge or at a vertex, we take the average of the closest known data points.

### The Largest Empty Circle problem

The **Largest Empty Circle problem** is the problem of finding the largest circle whose interior does not contain any sites.



In the problems considered in this course, the optimal position for the circle's centre will occur at one of the vertices of the Voronoi diagram. The vertex with the greatest distance from its nearest site is the optimal position for the circle's centre.



## VECTORS

A **vector** is a quantity with both **magnitude** and **direction**.

Two vectors are **equal** if and only if they have the same magnitude *and* direction.

In examinations:

- scalars are written in italics  $a$
- vectors are written in bold  $\mathbf{a}$ .

On paper, you should write vector  $\mathbf{a}$  as  $\underline{a}$ .

The 3-dimensional base unit vectors are  $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ , and  $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .

The 3-dimensional **zero vector**  $\mathbf{0}$  is  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ .

The general 3-dimensional vector  $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ .

You should understand the following for vectors in both algebraic and geometric forms:

- vector addition
- vector subtraction  $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$
- multiplication by a scalar  $k$  to produce vector  $k\mathbf{v}$  which is parallel to  $\mathbf{v}$
- the magnitude of vector  $\mathbf{v}$ ,  $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- the distance between two points in space is the magnitude of the vector which joins them.

The **position vector** of  $A(x, y, z)$  is  $\overrightarrow{OA}$  or  $\mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ .

The **displacement vector** of  $B(b_1, b_2, b_3)$  relative to  $A(a_1, a_2, a_3)$  is  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$ .

A, B, and C are **collinear** if  $\overrightarrow{AB} = k\overrightarrow{BC}$  for some scalar  $k$ .

The unit vector in the direction of  $\mathbf{a}$  is  $\frac{1}{|\mathbf{a}|}\mathbf{a}$ .

## THE SCALAR OR DOT PRODUCT OF TWO VECTORS

$$\mathbf{v} \bullet \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$$

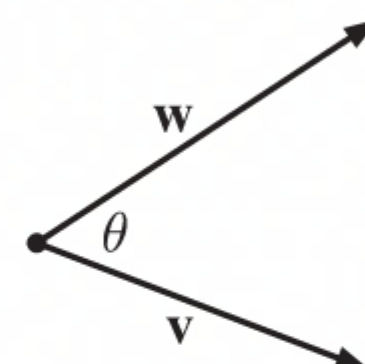
For non-zero vectors  $\mathbf{v}$  and  $\mathbf{w}$ :

- $\mathbf{v} \bullet \mathbf{w} = 0 \Leftrightarrow \mathbf{v}$  and  $\mathbf{w}$  are perpendicular
- $|\mathbf{v} \bullet \mathbf{w}| = |\mathbf{v}||\mathbf{w}| \Leftrightarrow \mathbf{v}$  and  $\mathbf{w}$  are parallel

The angle  $\theta$  between vectors  $\mathbf{v}$  and  $\mathbf{w}$  can be found using  $\cos \theta = \frac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$ .

If  $\mathbf{v} \bullet \mathbf{w} > 0$  then  $\theta$  is acute.

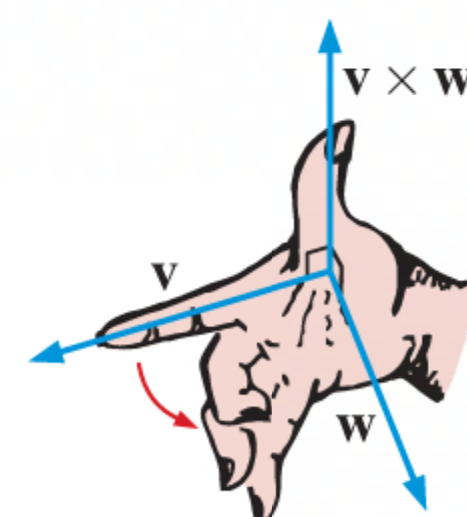
If  $\mathbf{v} \bullet \mathbf{w} < 0$  then  $\theta$  is obtuse.



## THE VECTOR CROSS PRODUCT OF TWO VECTORS

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \\ &= (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k} \end{aligned}$$

$\mathbf{v} \times \mathbf{w}$  is perpendicular to both  $\mathbf{v}$  and  $\mathbf{w}$ . Its direction is found using the right hand rule.





Geometric properties of the vector product:

$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}| |\mathbf{w}| \sin \theta$  where  $\theta$  is the angle between the vectors.

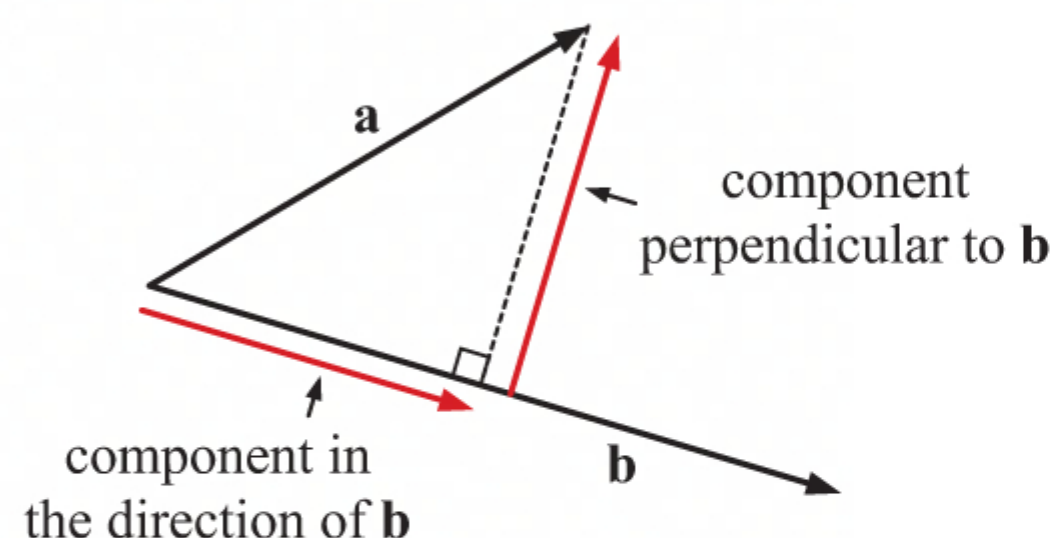
$|\mathbf{v} \times \mathbf{w}| = \text{area of parallelogram formed by vectors } \mathbf{v} \text{ and } \mathbf{w}.$

$\frac{1}{2} |\mathbf{v} \times \mathbf{w}| = \text{area of triangle formed by vectors } \mathbf{v} \text{ and } \mathbf{w}.$

## VECTOR COMPONENTS

For 3-dimensional vectors  $\mathbf{a}$  and  $\mathbf{b}$ :

- the component of  $\mathbf{a}$  in the direction of  $\mathbf{b}$  is  $\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$
- the component of  $\mathbf{a}$  perpendicular to  $\mathbf{b}$  is  $\frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$ .

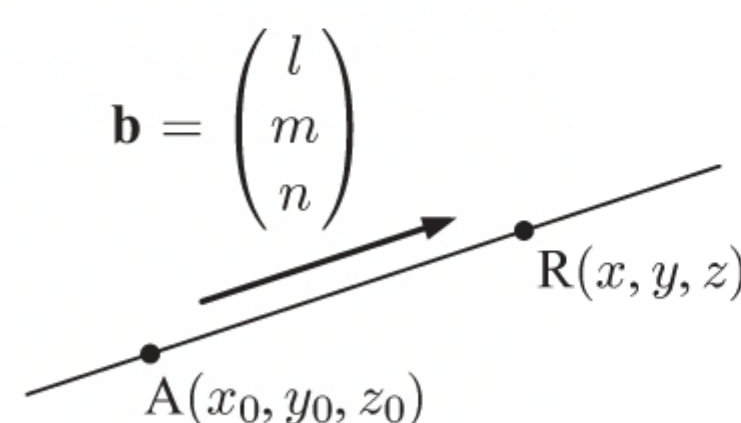


## LINES

Suppose  $R(x, y, z)$  is any point on the line,

$A(x_0, y_0, z_0)$  is a known point on the line,

and  $\mathbf{b} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$  is the **direction vector** of the line.



Then:

- The **vector equation** of the line is  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}, \lambda \in \mathbb{R}$
- The **parametric equations** of the line are: 
$$\begin{cases} x = x_0 + \lambda l \\ y = y_0 + \lambda m \\ z = z_0 + \lambda n \end{cases}, \lambda \in \mathbb{R}$$

The **acute angle**  $\theta$  **between two lines** is given by  $\cos \theta = \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|}$  where  $\mathbf{b}_1$  and  $\mathbf{b}_2$  are the direction vectors of the lines.

The shortest distance from point P to a line with direction vector  $\mathbf{b}$  occurs at the point R on the line such that  $\overrightarrow{PR}$  is perpendicular to  $\mathbf{b}$ .

You should be able to find the shortest distance between two objects moving with constant velocity.

You should also be able to determine whether a pair of lines are **parallel**, **coincident**, **intersecting**, or **skew**.

## MOTION WITH VARIABLE VELOCITY

For an object moving with parametric equations  $P(x(t), y(t))$ :

- the velocity vector  $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$
- the speed  $= |\mathbf{v}| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ .

## GEOMETRIC TRANSFORMATIONS

A **linear transformation** is a geometric transformation which can be described by the matrix equation  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  where  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

Linear transformations include **stretches**, **rotations**, **reflections**, and any combination or **composition** of these.

An **affine transformation** is a geometric transformation which can be described by the matrix equation  $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$  where  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}$ .

In addition to the linear transformations, affine transformations include **translations**.



## Translations

Suppose the point  $(x, y)$  is **translated** through  $\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}$  to the image point  $(x', y')$ .

The image is given by  $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$ .

## Rotations

For a **rotation** anticlockwise about  $O(0, 0)$  through  $\theta$ , the transformation matrix is

$$\mathbf{A} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ with } |\mathbf{A}| = 1.$$

## Reflections

For a **reflection** in the mirror line  $y = (\tan \alpha)x$ , the transformation matrix is

$$\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \text{ with } |\mathbf{A}| = -1.$$

## Stretches and enlargements

For a **horizontal stretch** with scale factor  $k$ , the transformation matrix is  $\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ .

For a **vertical stretch** with scale factor  $k$ , the transformation matrix is  $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ .

For an **enlargement** with scale factor  $k$ , the transformation matrix is  $\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ .

## Composite transformations

If we perform a linear transformation with transformation matrix  $\mathbf{A}$ , followed by another linear transformation with transformation matrix  $\mathbf{B}$ , then the resulting composite transformation is itself a linear transformation, with transformation matrix  $\mathbf{BA}$ .

The affine transformation  $\mathbf{x}' = \mathbf{Ax} + \mathbf{b}$  is the composition of the linear transformation with transformation matrix  $\mathbf{A}$ , followed by a translation through  $\mathbf{b}$ .

## Area

For the transformation  $\mathbf{x}' = \mathbf{Ax} + \mathbf{b}$ , area of image =  $|\det \mathbf{A}| \times \text{area of object}$ .

## GRAPH THEORY

A **graph** or **network** is a structure consisting of **vertices** and **edges**, which shows the physical connections or relationships between things of interest.

If we are allowed to move in either direction along the edges, the graph is **undirected**.

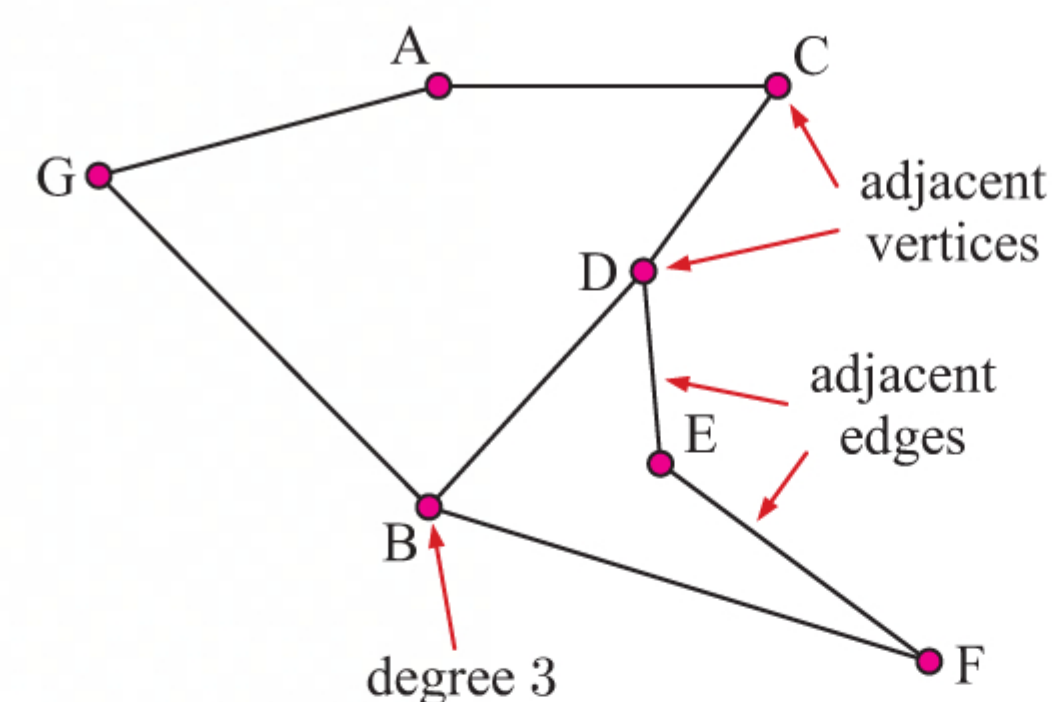
In an undirected graph:

- **Adjacent vertices** are vertices which are connected by an edge.
- **Adjacent edges** are edges which share a common vertex.
- The **degree** of a vertex is the number of edges connected to it.

A **directed graph** contains arrows indicating the direction we can move along the edges.

In a directed graph:

- The **in degree** of a vertex is the number of edges coming in to the vertex.
- the **out degree** of a vertex is the number of edges going out from the vertex.





## Properties of graphs

- A graph is called **simple** if it contains no loops, and if there is a maximum of one edge joining any pair of distinct vertices.
- An undirected graph is **connected** if it is possible to travel from every vertex to every other vertex by following edges.
- A directed graph is **strongly connected** if it is possible to travel from every vertex to every other vertex by following edges in the correct direction.
- A **complete graph** is a simple undirected graph in which every vertex is connected by an edge to every other vertex.

## Routes on graphs

- A **walk** is a sequence of vertices such that each successive pair of vertices is adjacent.
- A **trail** is a walk in which no *edge* is repeated.
- A **path** is a walk in which no *edge* or *vertex* is repeated.
- A **circuit** is a trail which starts and finishes at the same vertex.
- A **cycle** is a circuit in which no vertex is repeated (except the finish coinciding with the start).

The **length** of a route is the number of edges traversed.

## Adjacency matrices

For a given graph:

- The **adjacency matrix**  $A$  shows the number of direct routes between each pair of vertices.
- $A^n$  shows the number of routes of length  $n$  between each pair of vertices.

## Minimum spanning trees

A **tree** is a connected, simple graph with no cycles.

A **spanning tree** of a graph is a tree which contains all the vertices of the graph.

On a weighted graph, the spanning tree with minimum weight is called the **minimum spanning tree**.

We can use **Kruskal's algorithm** or **Prim's algorithm** to find the minimum spanning tree.

## Kruskal's algorithm

For a graph with  $n$  vertices:

*Step 1:* Start with the shortest (or **least weight**) edge. If there are several, choose one at random.

*Step 2:* Choose the shortest edge remaining that does not complete a cycle. If there is more than one possible choice, pick one at random.

*Step 3:* Repeat *Step 2* until  $n - 1$  edges have been chosen.

## Prim's algorithm

*Step 1:* Start with any vertex.

*Step 2:* Join this vertex to the nearest vertex. If two or more vertices are an equal distance away, choose one at random.

*Step 3:* Join the vertex which is nearest to *either* of those already connected.

*Step 4:* Continue joining connected vertices to the nearest unconnected vertex until all vertices are connected.

You should also be able to apply Prim's algorithm to a weighted adjacency table.

## Eulerian graphs

An **Eulerian circuit** is a circuit which traverses every edge exactly once.

An **Eulerian trail** is a trail which traverses every edge exactly once, but does not start and end at the same vertex.

A graph is:

- **Eulerian** if it contains an Eulerian circuit
- **semi-Eulerian** if it contains an Eulerian trail but not an Eulerian circuit.



We can use the degrees of a graph's vertices to identify Eulerian and semi-Eulerian graphs.

- A connected graph is **Eulerian** if there are *no* vertices of odd degree.
- A connected graph is **semi-Eulerian** if there are *exactly two* vertices of odd degree. The Eulerian trail starts at one of these vertices and ends at the other.

### Chinese Postman Problem

The **Chinese Postman Problem** is to find the route of minimum weight which starts and finishes at the same vertex, and traverses every edge of the graph.

If the graph is Eulerian, any Eulerian circuit is a solution to the Chinese Postman Problem.

If a graph is not Eulerian, some of the edges must be traversed twice to solve the Chinese Postman Problem. We need to traverse twice the edges forming the shortest paths between pairs of odd vertices.

### Hamiltonian graphs

A **Hamiltonian cycle** is a cycle which visits each vertex (except the starting and ending vertex) exactly once.

A **Hamiltonian path** is a path which visits each vertex exactly once, but does not start and end at the same vertex.

A graph is:

- **Hamiltonian** if it contains a Hamiltonian cycle
- **semi-Hamiltonian** if it contains a Hamiltonian path, but not a Hamiltonian cycle.

### Travelling Salesman Problem

The **Travelling Salesman Problem** (TSP) is to find the route of minimum weight which starts and finishes at the same vertex, and visits every vertex of the graph.

The **nearest neighbour algorithm** can be used to find an *upper bound* for the TSP.

*Step 1:* Choose a starting vertex.

*Step 2:* Follow the edge of least weight to an unvisited vertex. If there is more than one such edge, choose one at random.

*Step 3:* Repeat *Step 2* until all vertices have been visited.

*Step 4:* Return to the starting vertex by adding the corresponding edge.

The **deleted vertex algorithm** can be used to find a *lower bound* for the TSP.

*Step 1:* Delete a vertex, together with all edges connected to it, from the original graph.

*Step 2:* Find the minimum spanning tree for the remaining graph.

*Step 3:* Add to the length of the minimum spanning tree, the lengths of the two shortest deleted edges.

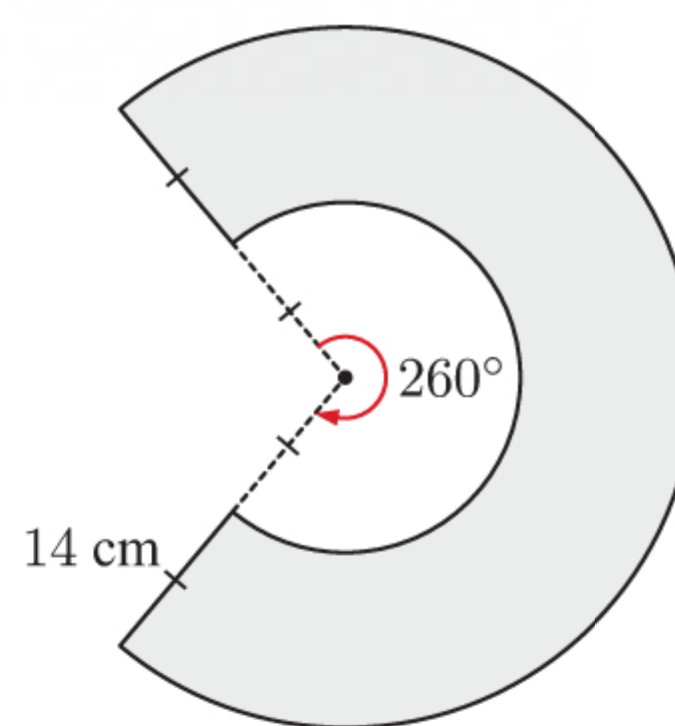
You should understand the difference between the **classical TSP** and the **practical TSP**. A practical TSP can be converted into a classical TSP by adding or replacing edges to show the least weight between vertices.



## SKILL BUILDER QUESTIONS

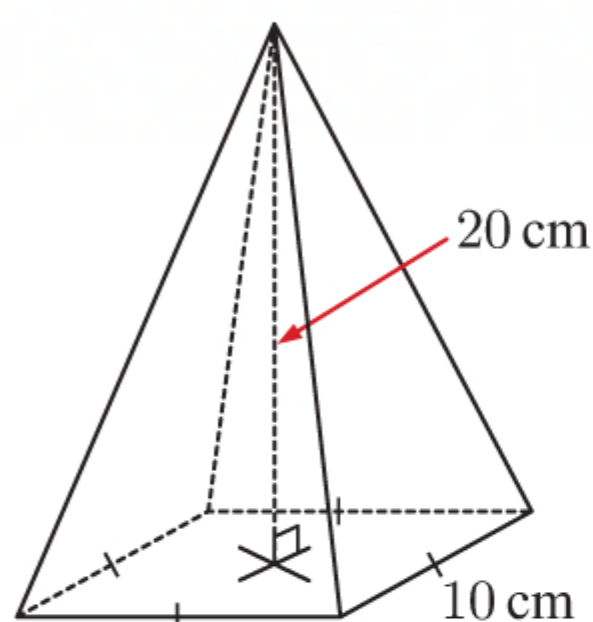
1 For the given figure, find to 3 significant figures:

- a the perimeter  
b the area.

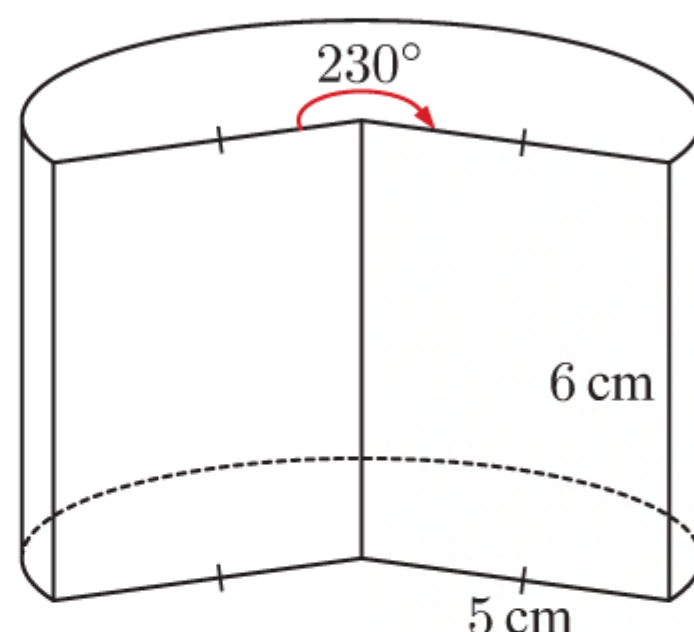


2 Find the surface area of each solid:

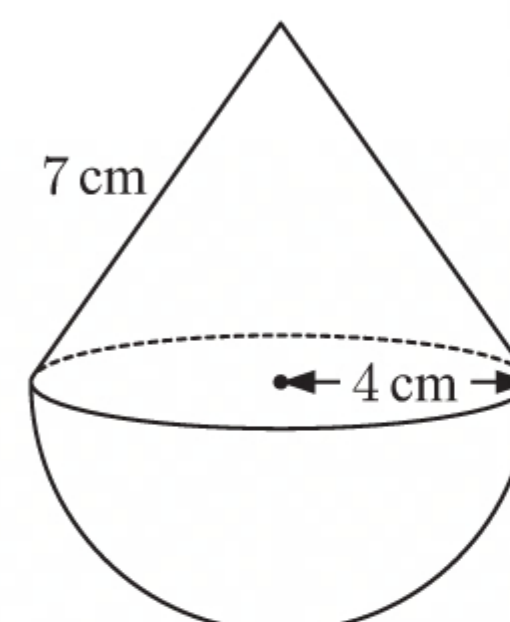
a



b

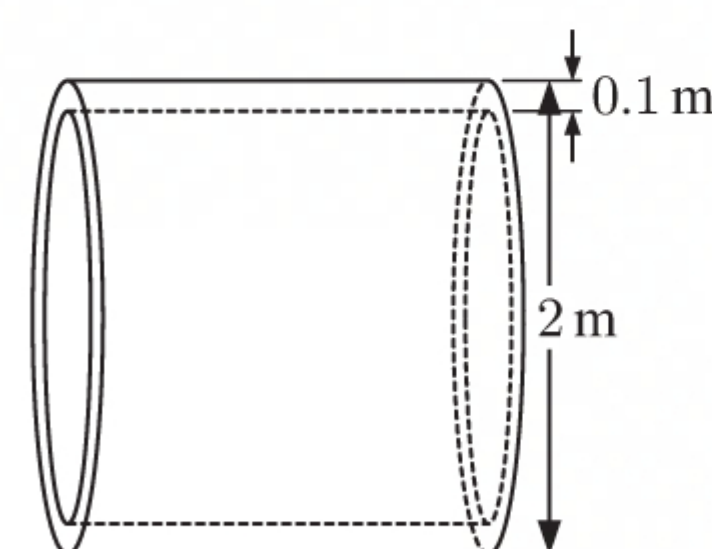


c



3 The surface area of a beach ball is  $2800 \text{ cm}^2$ . Find the radius of the beach ball.

4 A pipe used to drain stormwater is made from  $3 \text{ m}^3$  of concrete. Find the length of the pipe.



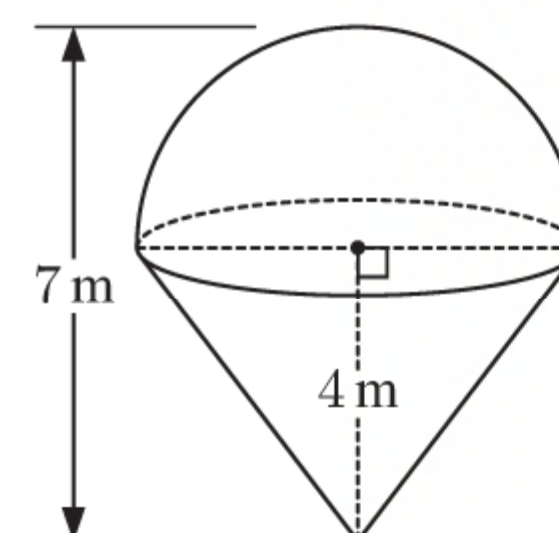
5 A sector of a circle of radius 10 cm has perimeter 40 cm. Find:

- a the arc length of the sector  
b the area of the sector.

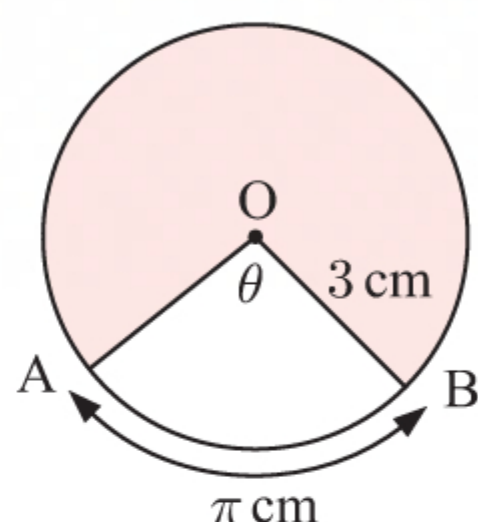
6 A large artificial ice cream for a shop front display is to be made with a hemisphere on top of an inverted cone.

The total height of the structure is 7 m, and the cone is 4 m high.

- a Show that the radius of the cone is 3 m.  
b Calculate the total volume of the ice cream.  
c Find the slant height of the cone.  
d Find the total surface area of the ice cream.  
e The ice cream is to be made from a lightweight polymer, weighing  $1.23 \text{ kg per m}^2$ . Calculate the total weight of the ice cream.



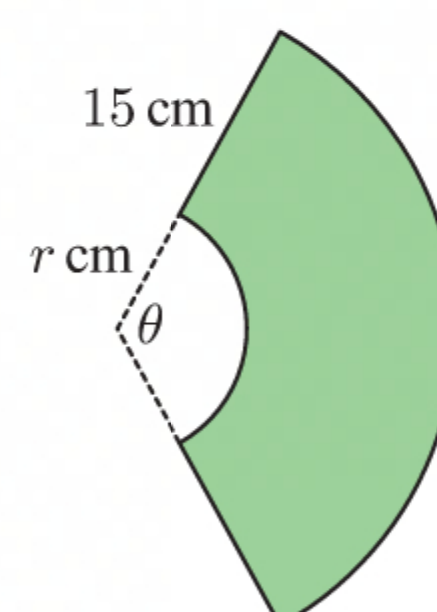
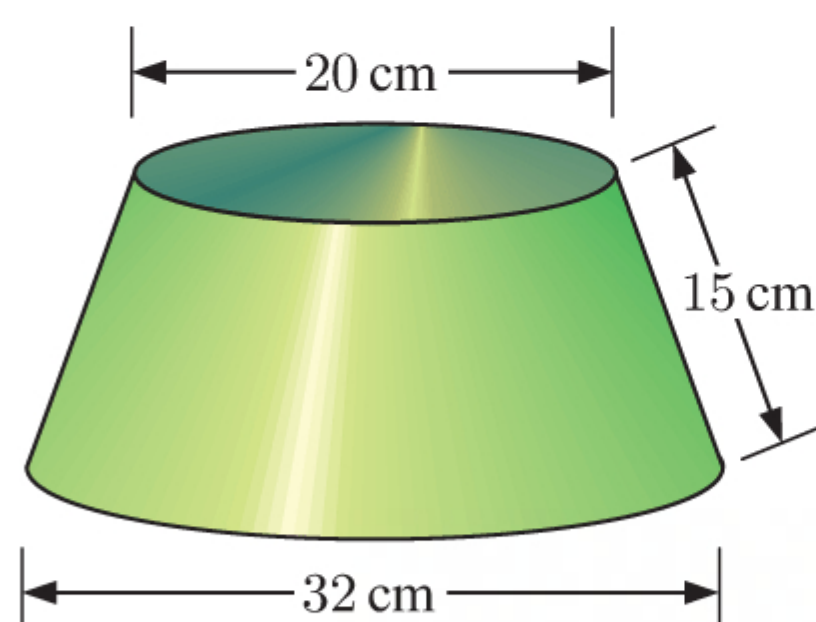
7



Find:

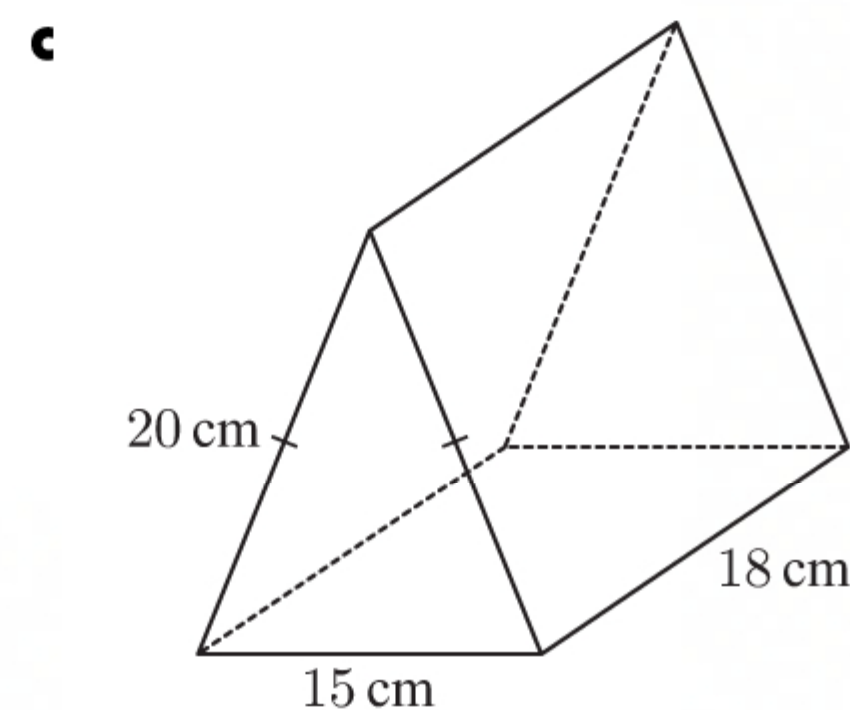
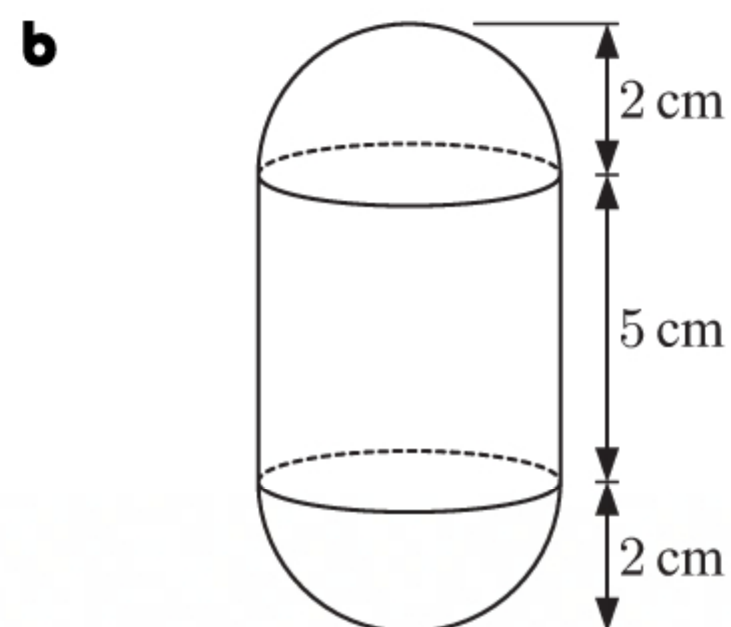
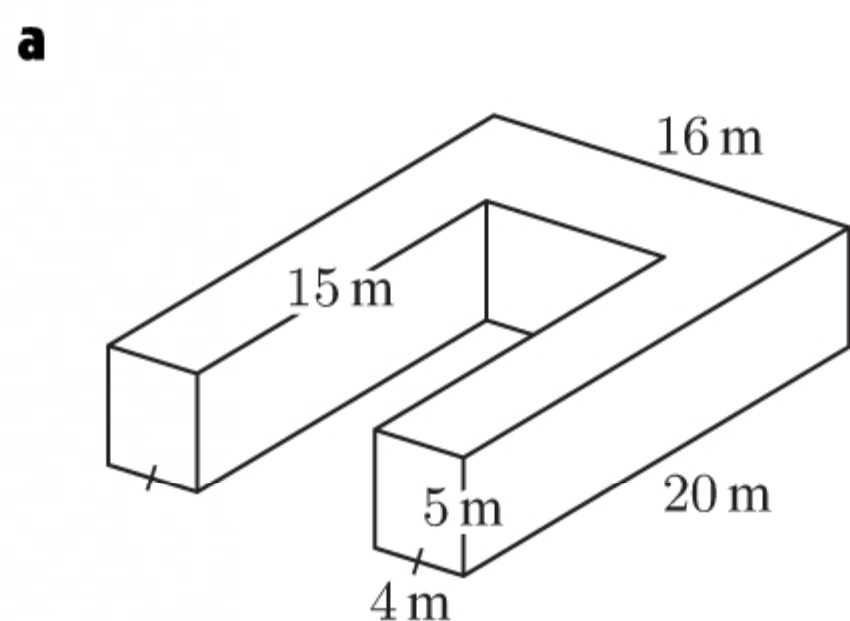
- a  $\theta$  in radians  
b the shaded area.

8 The lampshade on the left is made from the sheet of metal on the right. Find  $r$  and  $\theta$ .

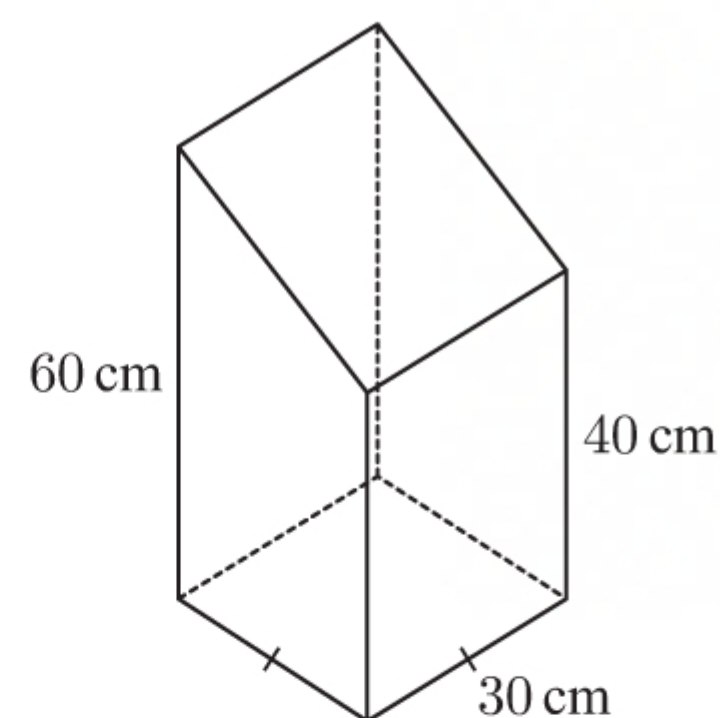




9 Find the volume of:



10 How many of these petrol containers can be completely filled with 300 L of petrol?



11 For each pair of points, find:

i the distance AB

ii the midpoint of [AB].

**a**  $A(2, 4, 1)$  and  $B(4, 0, 7)$

**b**  $A(3, -5, 2)$  and  $B(-1, 2, -3)$

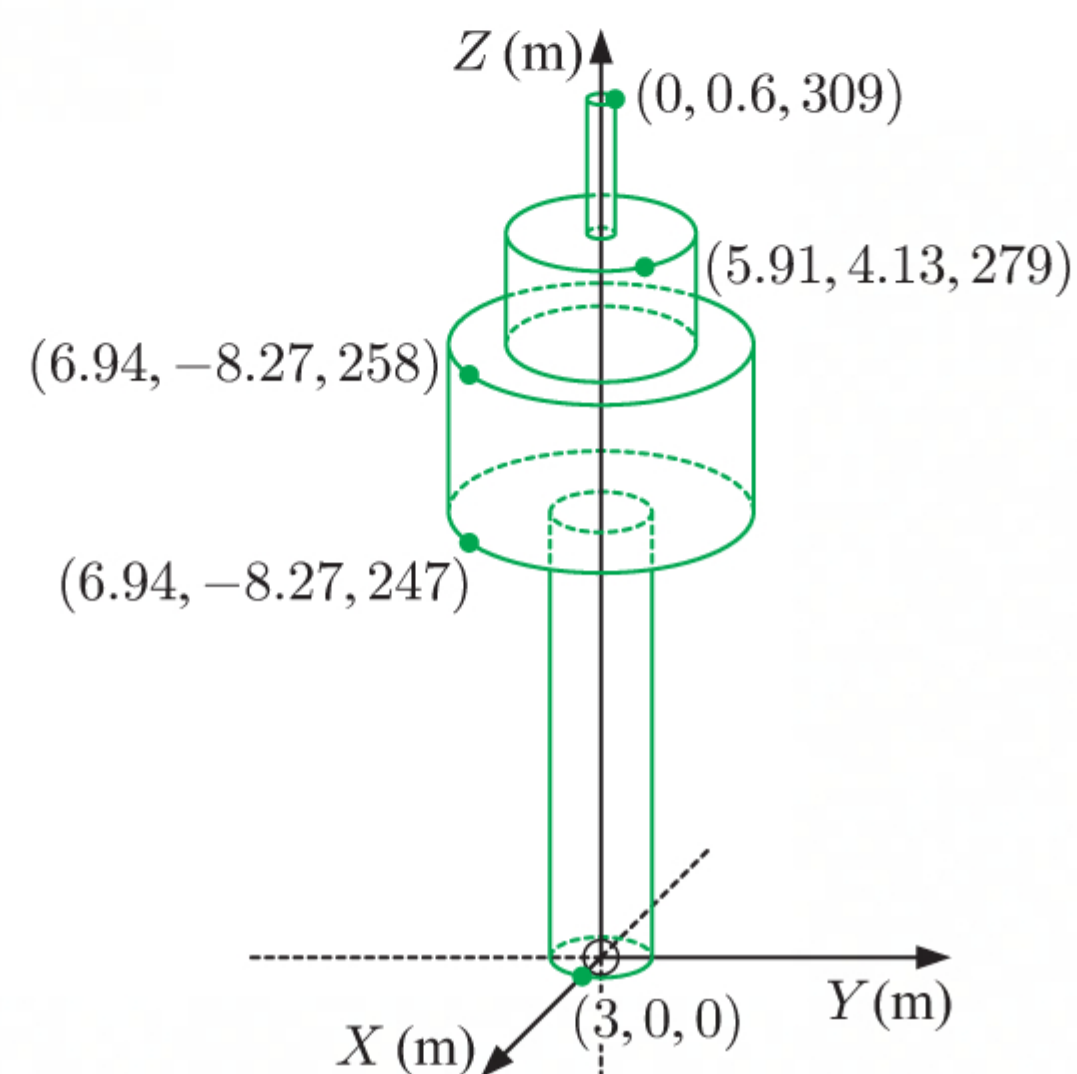
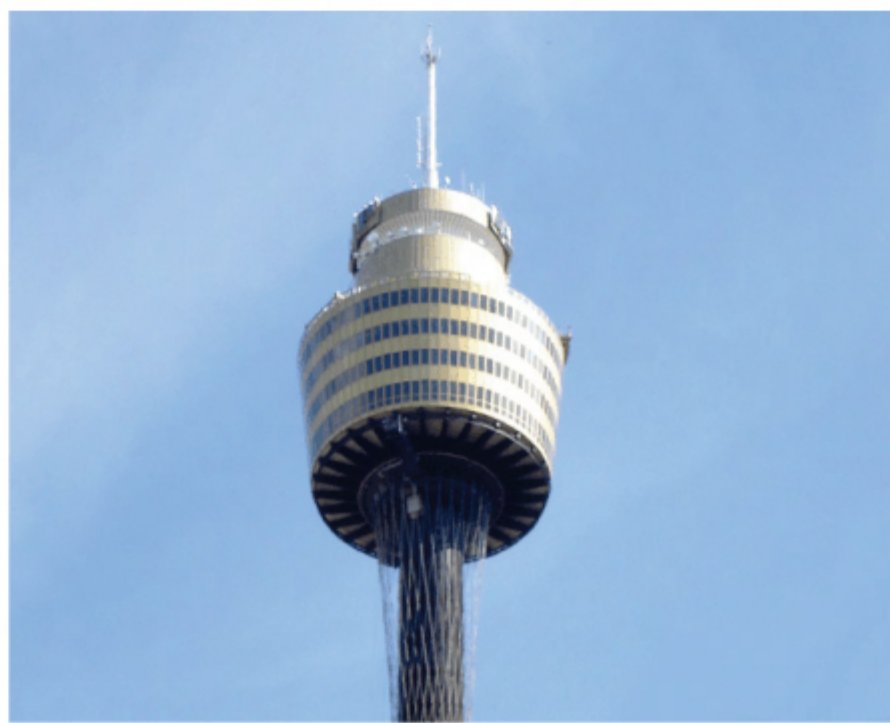
**c**  $A(-6, 0, 5)$  and  $B(-3, -3, 1)$

12 The distance from  $P(k, 6, -5)$  to  $Q(2, -1, -8)$  is 9 units. Find the possible values of  $k$ .

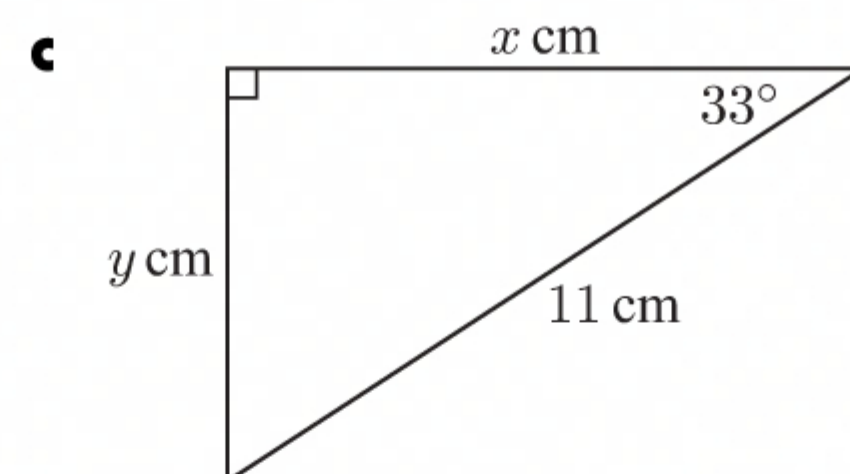
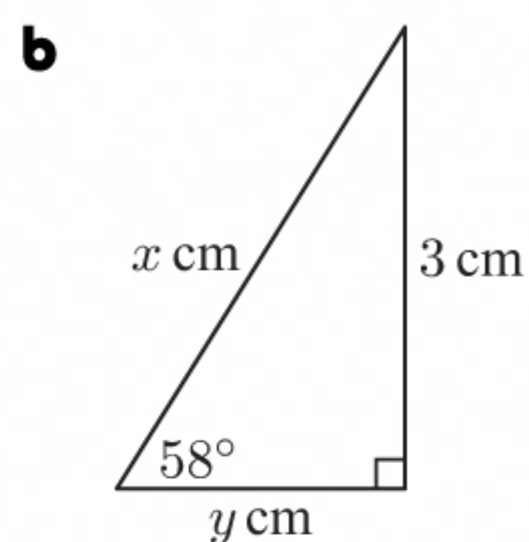
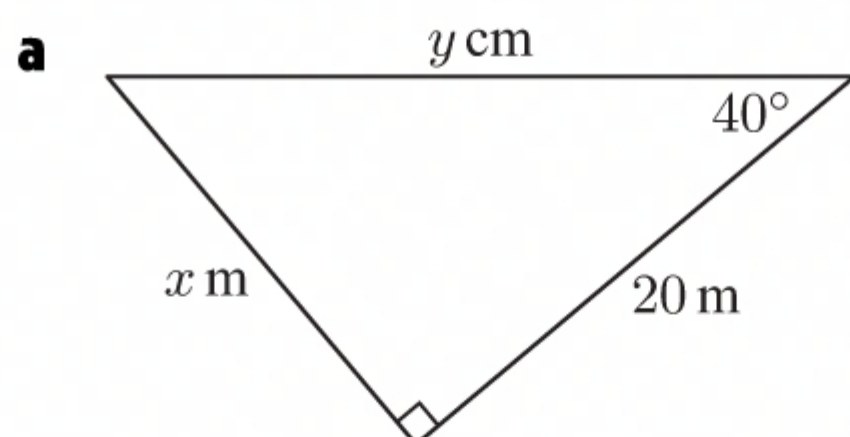
13 Suppose A is  $(3, 4, -6)$ , and  $M(-\frac{1}{2}, 9, -7)$  is the midpoint of [AB]. Find the coordinates of B.

14 Sydney tower in Australia is the second tallest observation tower in the Southern Hemisphere.

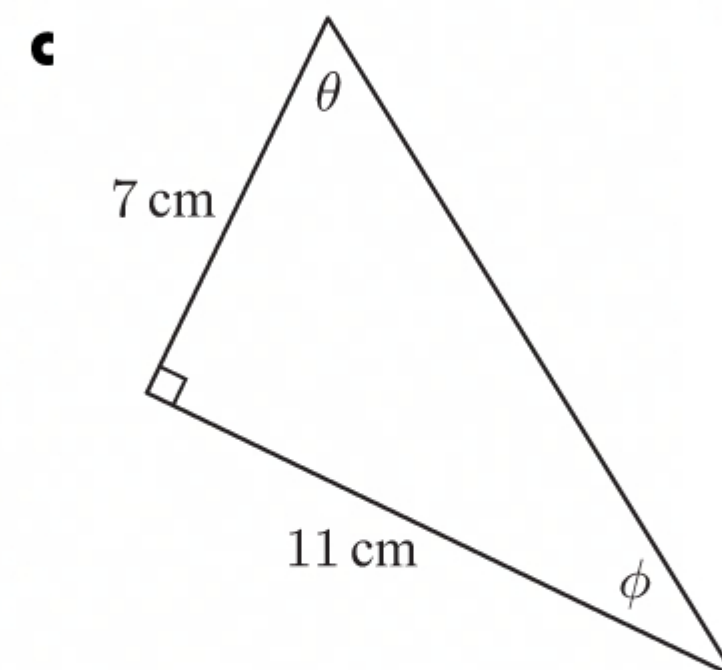
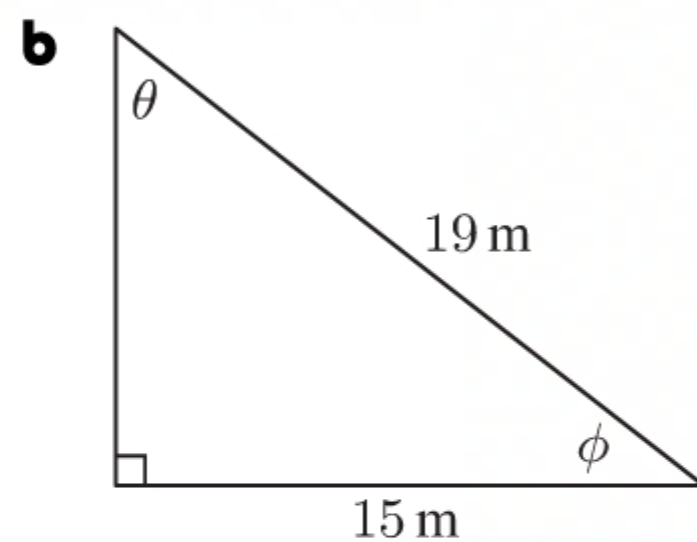
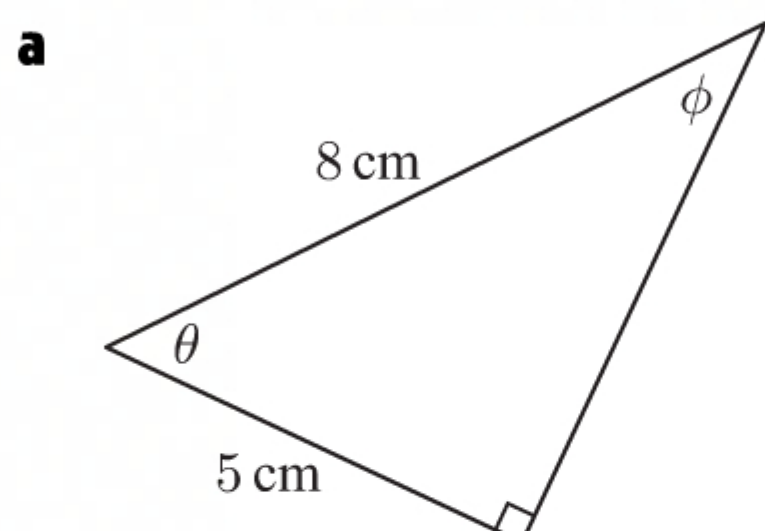
Find the volume of the tower.



15 Find, correct to 3 significant figures, all unknown sides:

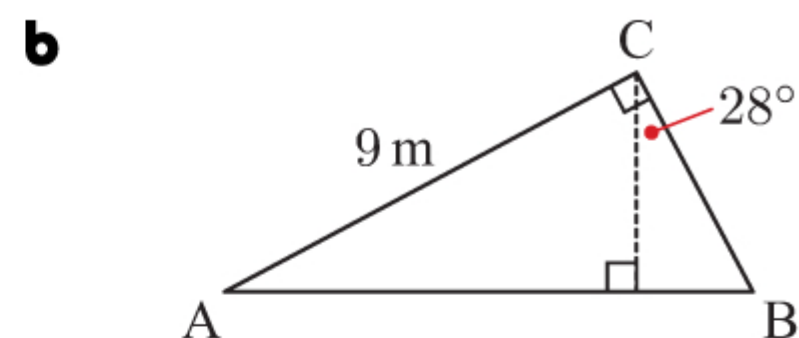
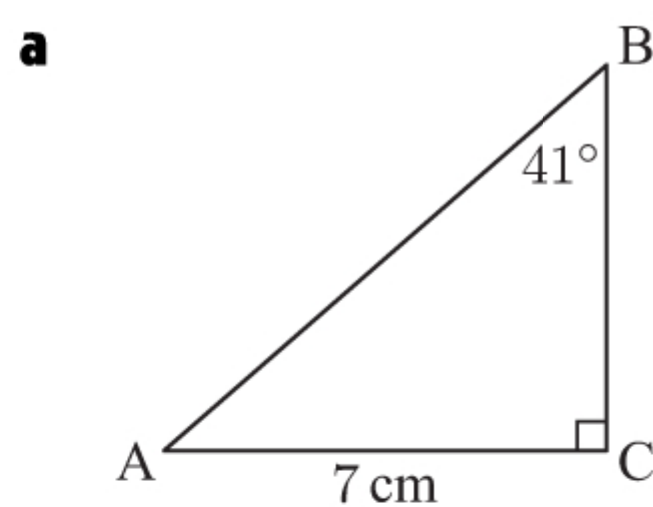


16 Find all unknown angles:

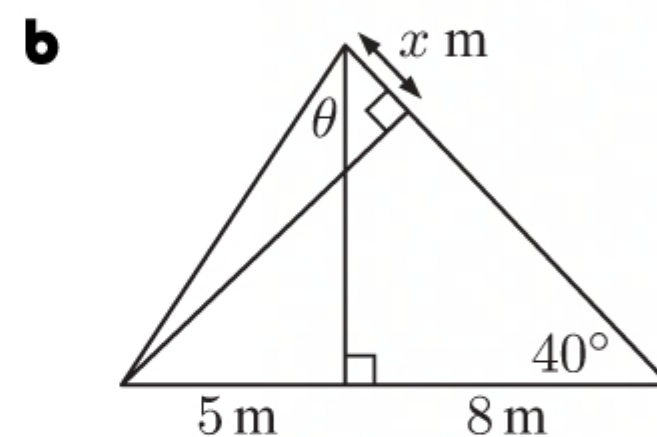
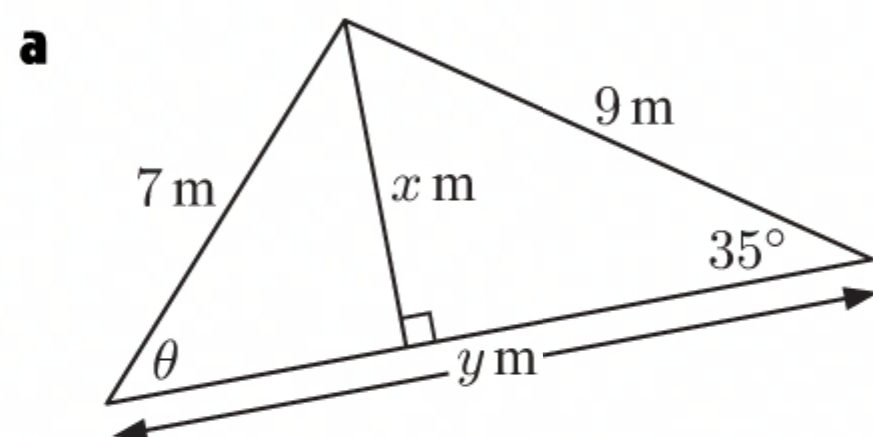




**17** Find the perimeter and area of triangle ABC:



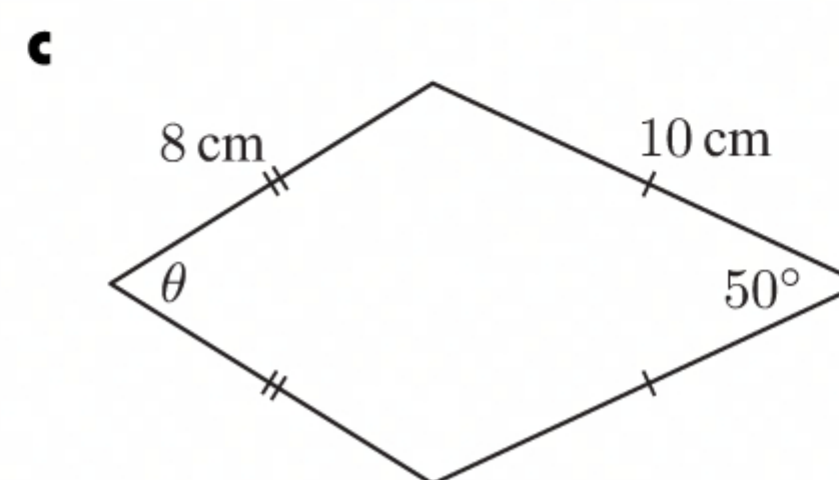
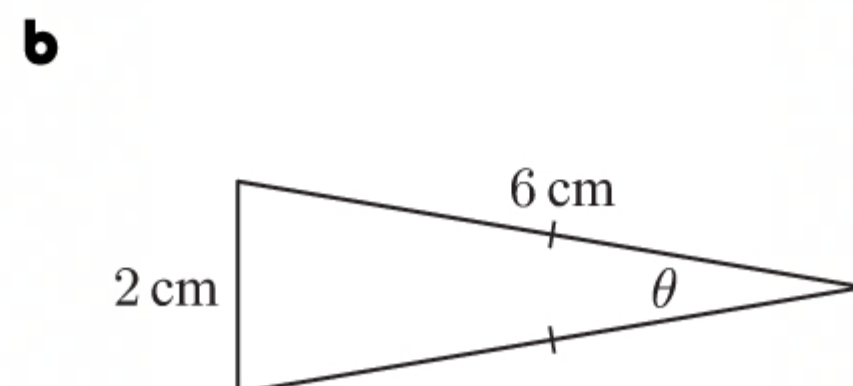
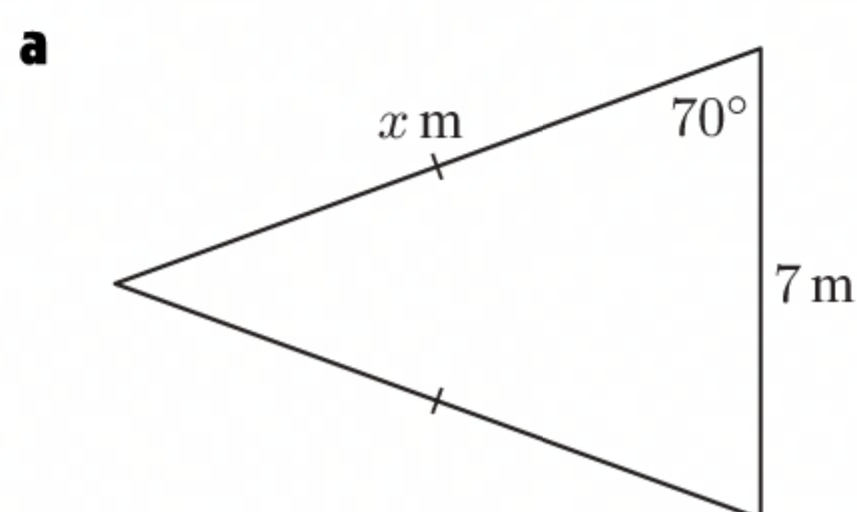
**18** Find all unknowns in these figures:



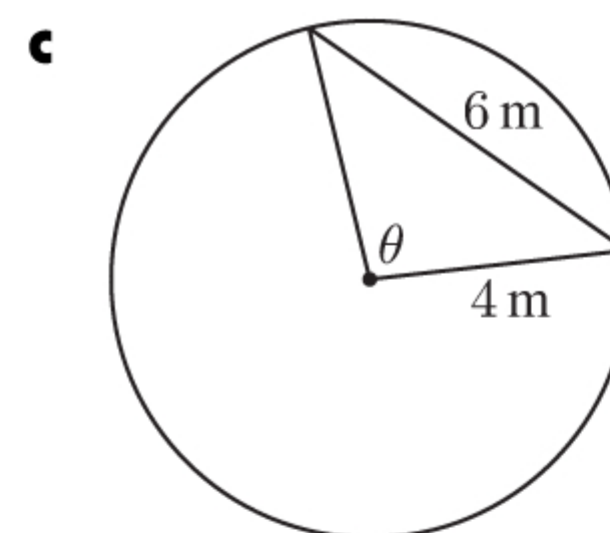
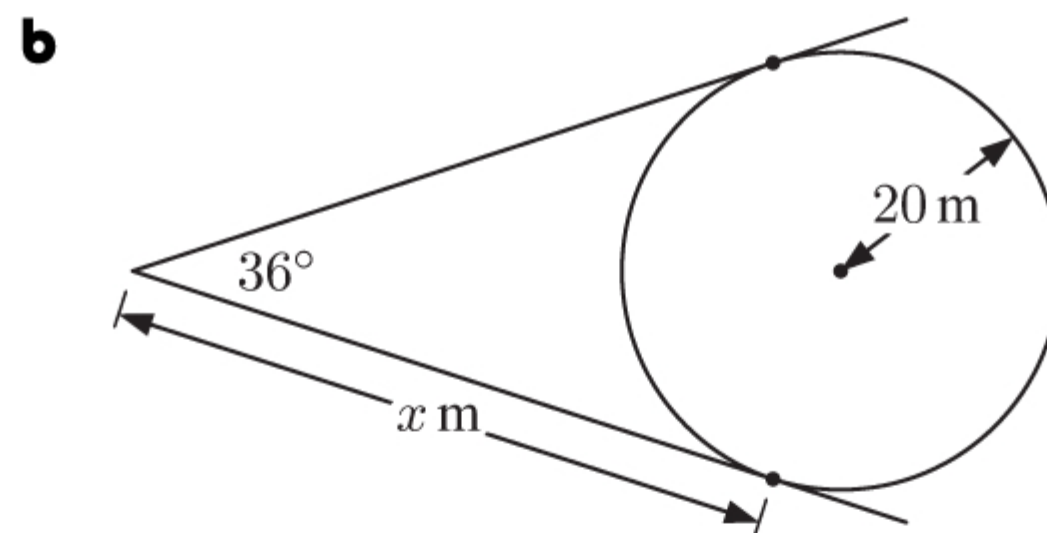
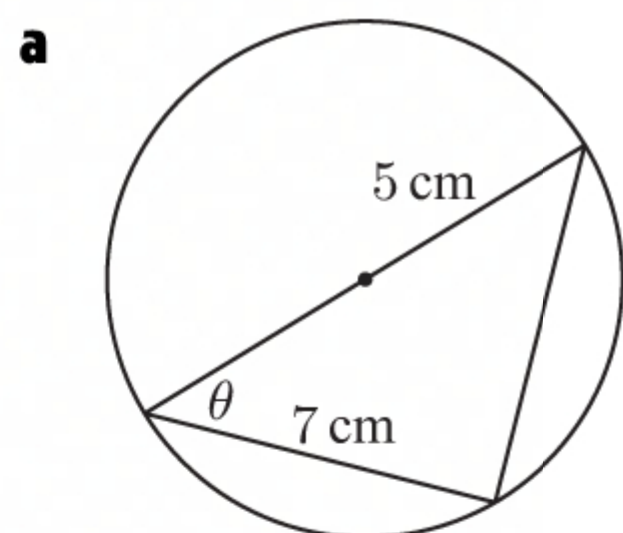
**19** A rhombus has diagonals of length 5 cm and 8 cm.

- Draw a diagram and label it with the given information.
- Find the length of the sides of the rhombus.
- Find the measure of the larger angle in the rhombus.

**20** Find the unknowns, correct to 3 significant figures:



**21** Find the value of the unknown:



**22** At 2:35 pm Fari sees an airplane directly overhead. At 2:38 pm he estimates that the angle of elevation to the plane is  $15^\circ$ . The plane is travelling in a straight line at  $110 \text{ m s}^{-1}$ . Calculate:

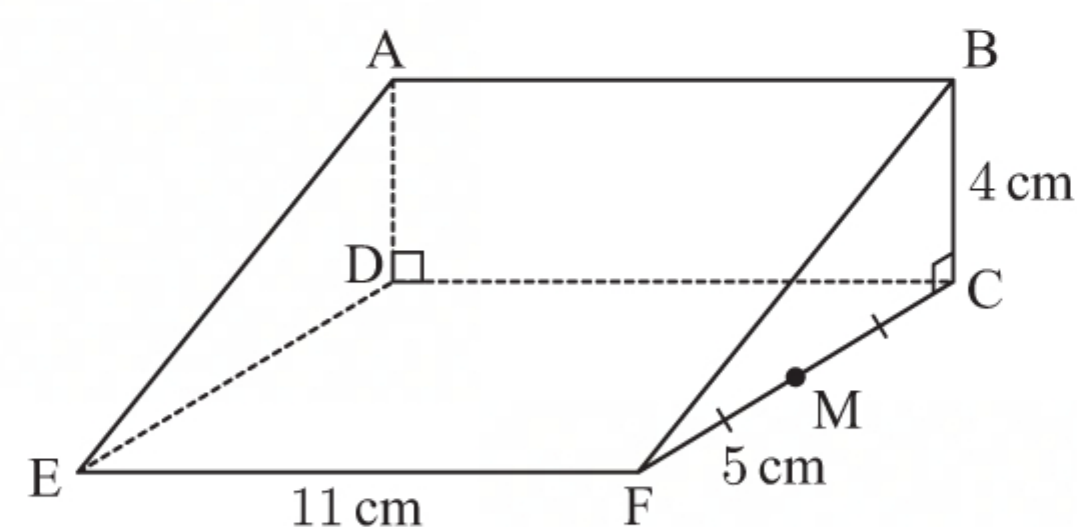
- the height of the plane above the ground
- the angle of elevation to the plane at 2:42 pm.

**23** A helicopter lands 5 km east and 7 km south of its starting point.

- Find the helicopter's distance from its starting point.
- Find the helicopter's bearing from its starting point.

**24** Find the angle between the following line segments and the base plane of the triangular prism:

- [AE]
- [BD]
- [BE]
- [AM]

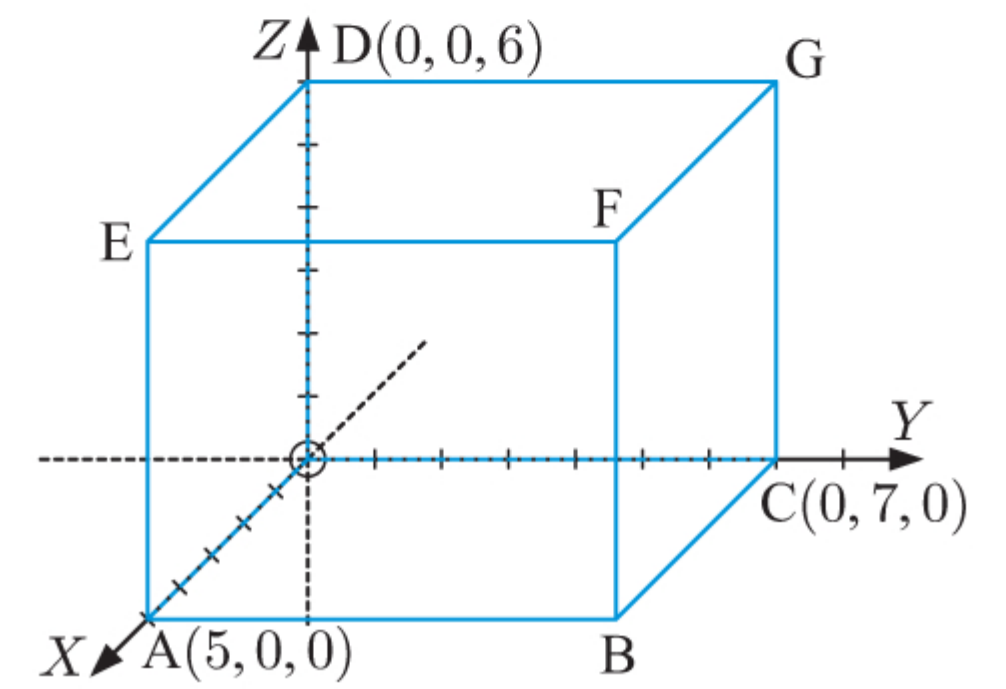




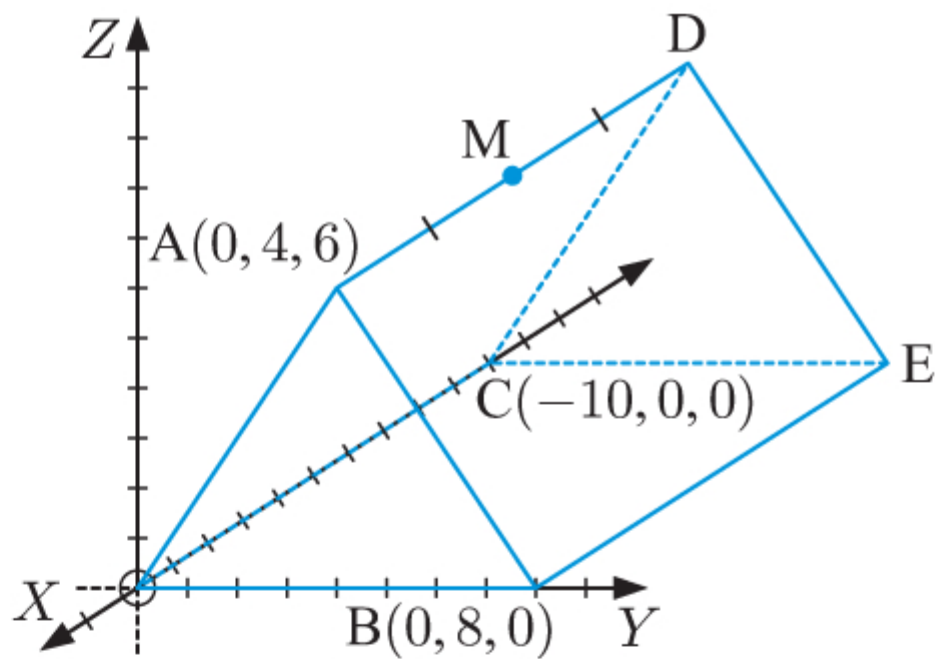
- 25** Consider the rectangular prism shown.

Find the angle between the following line segments and the base plane ABCO:

- a** [CD]                      **b** [OF]                      **c** [AG]



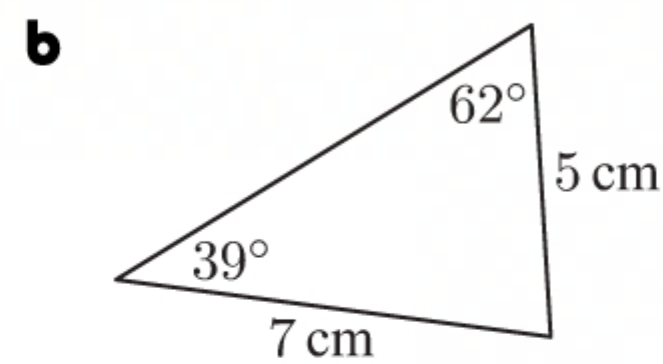
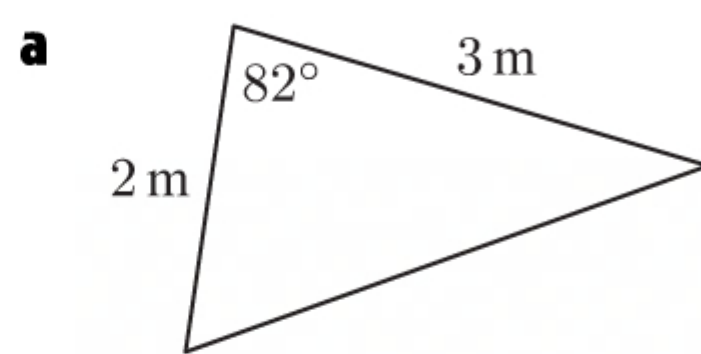
- 26**



Consider the triangular prism shown.

- a** State the coordinates of M.  
**b** Find the measure of  $\widehat{CMD}$ .  
**c** Find the angle between the following line segments and the base plane BECO:  
     **i** [OD]                      **ii** [EM]

- 27** Find the area of:



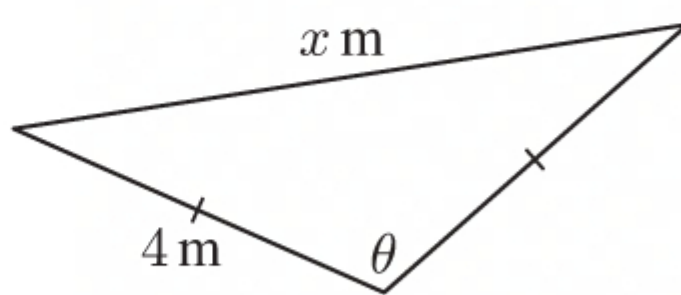
- 28** Triangle ABC has  $AB = 8$  cm,  $BC = 10$  cm, and  $AC = 12$  cm.

- a** Draw a diagram clearly showing this information.                      **b** Find the smallest angle in triangle ABC.  
**c** Find the area of triangle ABC.

- 29** In triangle ABC,  $AB = 72$  cm,  $BC = 61$  cm, and  $\widehat{ABC} = 43^\circ$ .

- a** Calculate the length of AC.                      **b** Find the measure of  $\widehat{ACB}$ .

- 30**

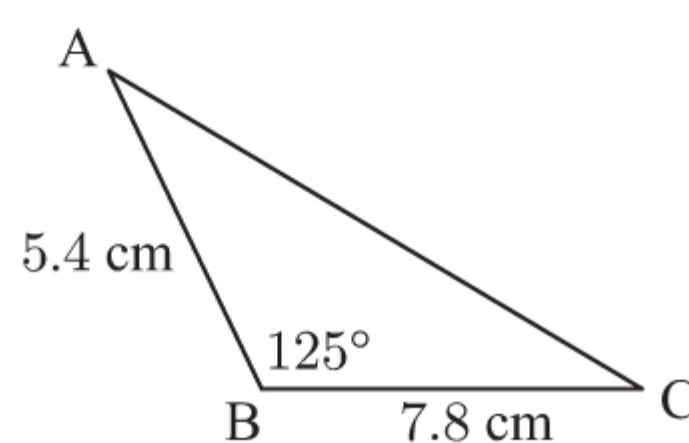


The area of the triangle shown is  $4 \text{ m}^2$ .

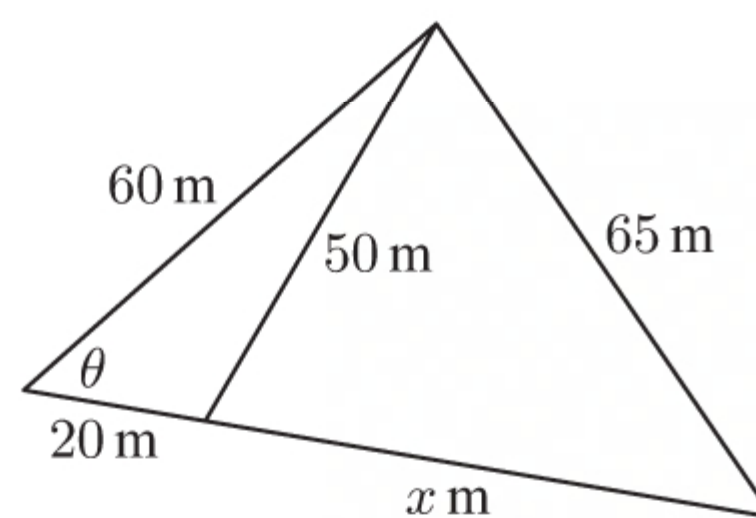
- a** Given that  $\theta$  is obtuse, find the value of  $\theta$ .  
**b** Find  $x$ .

- 31** Find:

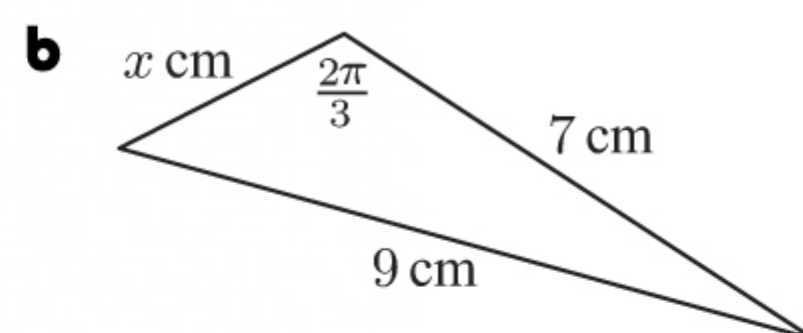
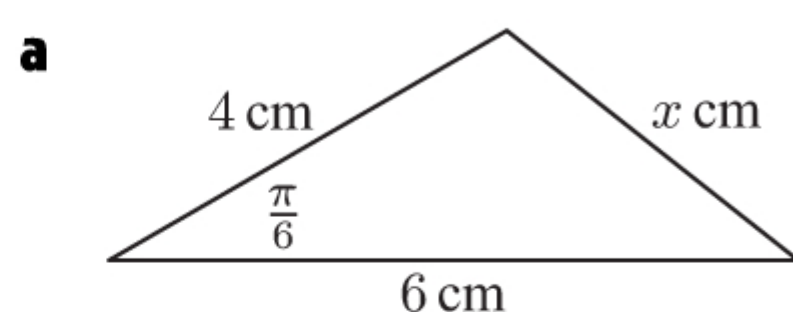
- a** the area of the triangle  
**b** the length of [AC].



- 32** **a** Find  $\cos \theta$  but not  $\theta$ .  
**b** Hence find the value of  $x$ .

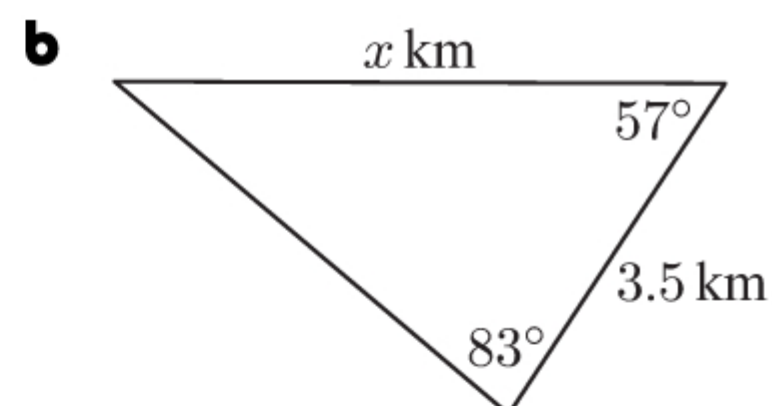
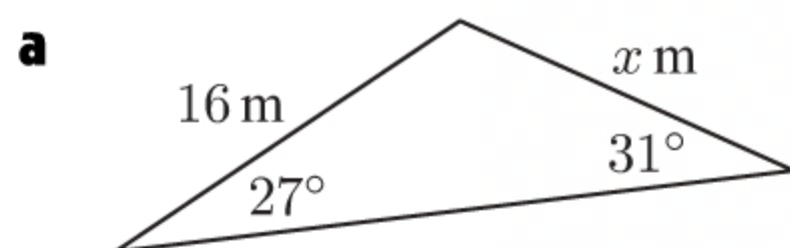


- 33** Find  $x$ :

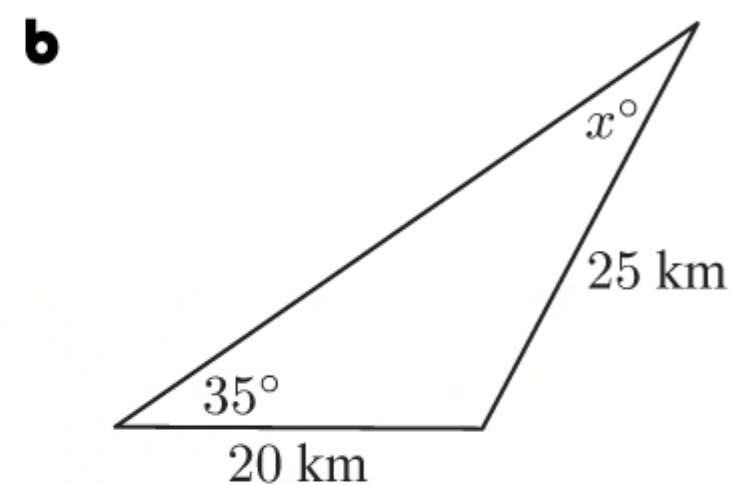
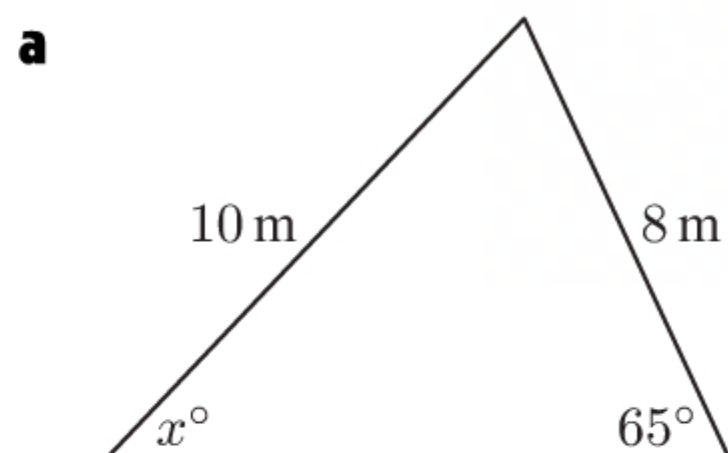




**34** Find the value of  $x$ :



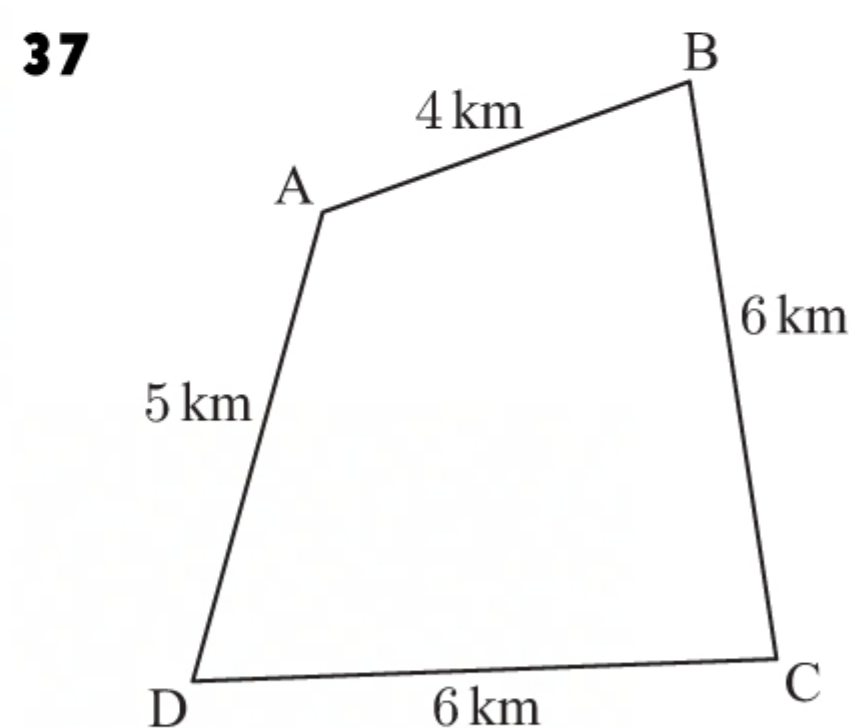
**35** Find the value of  $x$ :



**36** In triangle ABC,  $AB = 15$  cm,  $AC = 12$  cm, and  $\widehat{ABC}$  measures  $30^\circ$ .

**a** Find the two possible values of  $\widehat{ACB}$ .

**b** Given that  $\widehat{BAC}$  is acute, find its measure.



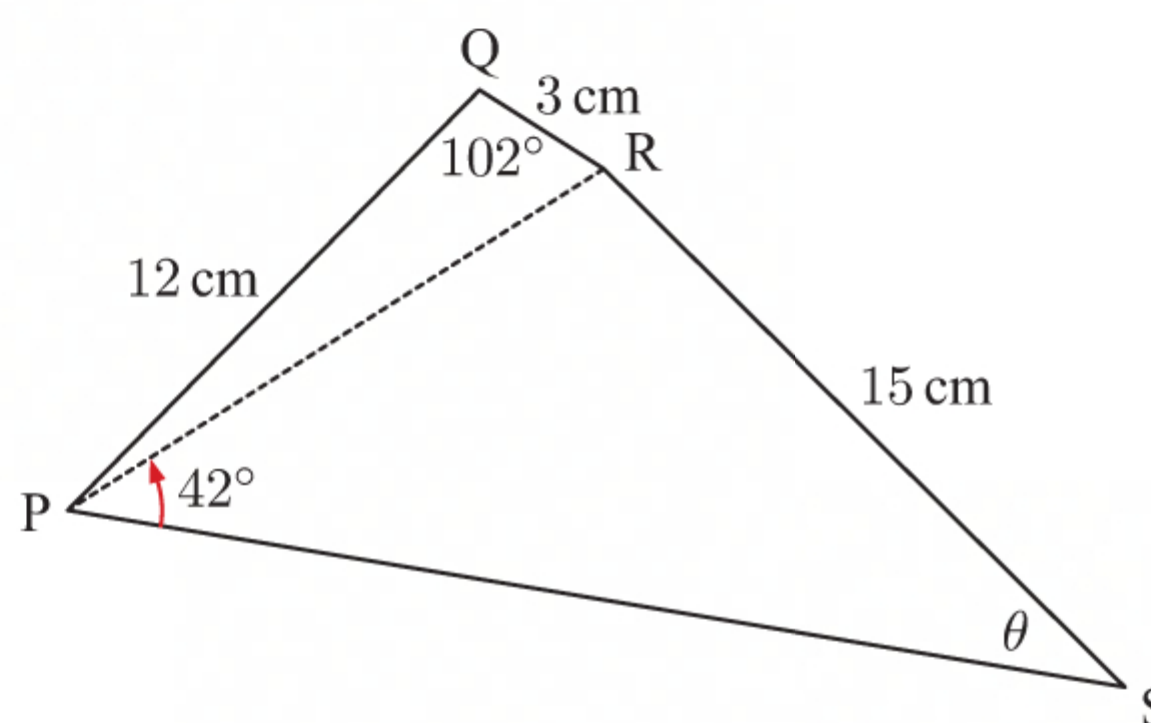
In quadrilateral ABCD, the diagonal [BD] has length 8 km.

Find the length of the diagonal [AC].

**38** Quadrilateral PQRS has the measurements shown.

**a** Find the length of [PR].

**b** Determine the measure of the angle marked  $\theta$ .



**39** Monument X is observed from two points B and C which are 330 m apart.  $\widehat{XBC}$  is  $63^\circ$  and  $\widehat{BCX}$  is  $75^\circ$ .

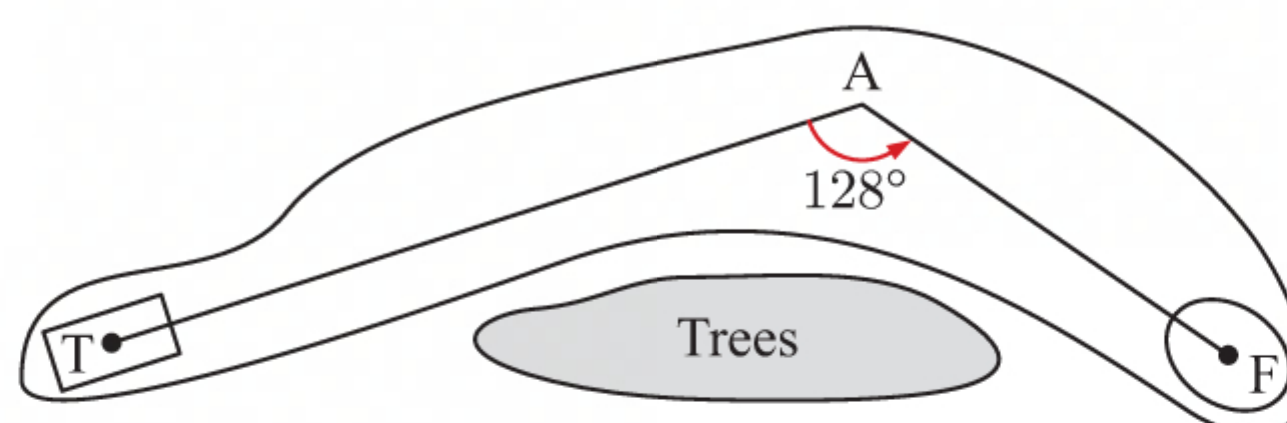
**a** Draw a neat, labelled diagram to illustrate this information.

**b** Find the distance between the monument and B.

**40** The 5th hole at the Flagstaff golf course has the layout shown. From T to A on the fairway, the distance is 240 m, and from A to F the distance is 135 m.

**a** Find the distance from T to F in a straight line.

**b** Find the measure of  $\widehat{ATF}$ .



**41** The diagram below shows the routes of two triangular orienteering courses ABC and DEF. The distance from E to F is 20% longer than the distance from A to C.

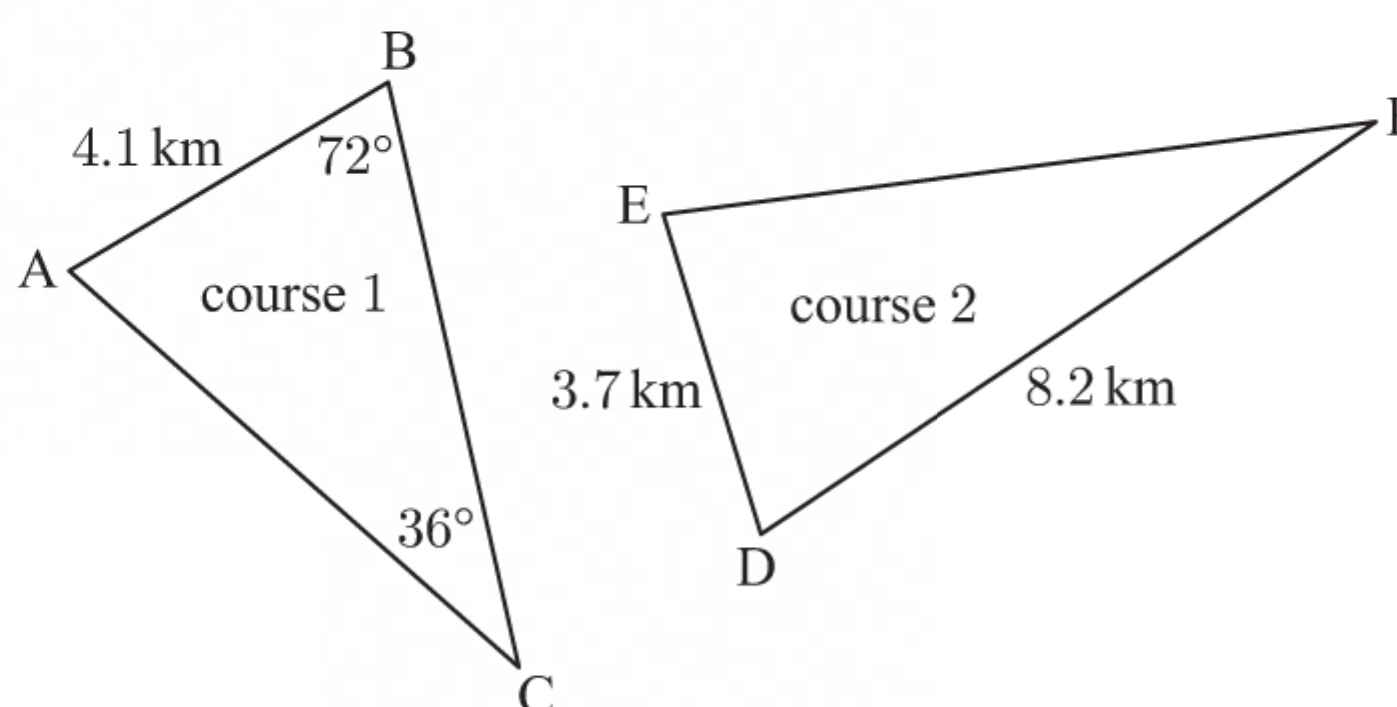
Find:

**a** the length of [EF]

**b** the measure of  $\widehat{DEF}$

**c** the total area covered by course 2

**d** the total length of course 1.

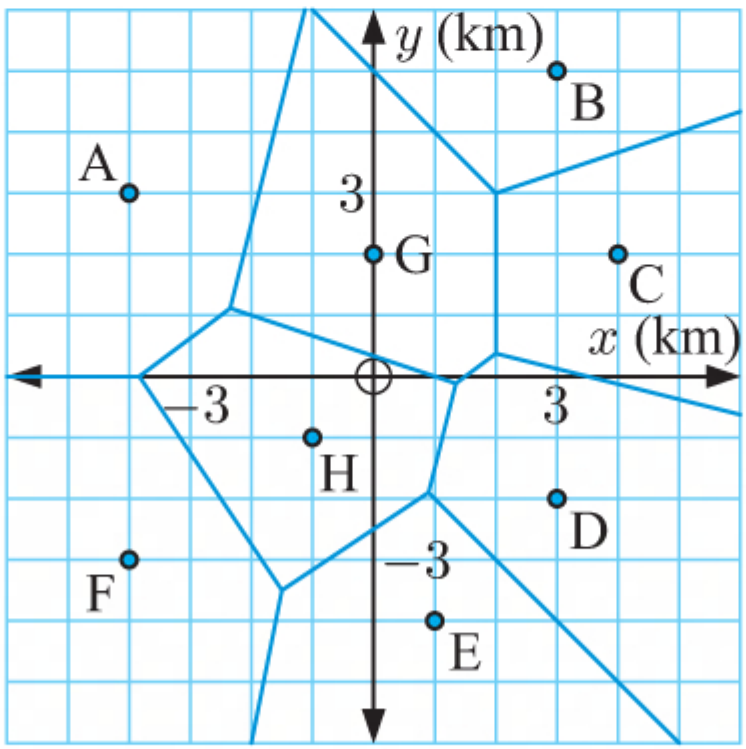




- 42** A boat travels 12 km on the bearing  $072^\circ$ , and then travels 18 km on the bearing  $141^\circ$ .
- Display this information on a diagram.
  - Find the distance and bearing of the boat from its starting point.
- 43** Let  $a = \sin 20^\circ$  and  $b = \tan 50^\circ$ . Write, in terms of  $a$  and  $b$ , expressions for:
- $\sin 160^\circ$
  - $\tan(-50^\circ)$
  - $\cos 70^\circ$
  - $\tan 20^\circ$
- 44** Given that  $\sin \theta = -\frac{1}{2}$  and  $\cos \theta = -\frac{\sqrt{3}}{2}$  where  $0^\circ < \theta < 360^\circ$ , find the exact value of:
- $\theta$
  - $\tan \theta$
  - $\tan 2\theta$
- 45** Find the exact value of:
- $\sin \frac{5\pi}{3}$
  - $\cos \frac{3\pi}{4}$
  - $\tan\left(-\frac{\pi}{3}\right)$
- 46** Without using a calculator, evaluate:
- $\sin \frac{\pi}{3} \cos \frac{\pi}{4}$
  - $2 \tan^2\left(\frac{2\pi}{3}\right) + 1$
  - $\frac{\cos \frac{5\pi}{6} \tan^2\left(\frac{3\pi}{4}\right)}{\sin\left(-\frac{\pi}{3}\right)}$
- 47** Find  $\theta$  if  $0 \leq \theta \leq 2\pi$  and:
- $\cos \theta = -\frac{1}{2}$
  - $\sin \theta = \frac{1}{\sqrt{2}}$
  - $\tan^2 \theta = \frac{1}{3}$
- 48** Find the possible exact values of:
- $\cos \theta$  if  $\sin \theta = \frac{4}{5}$
  - $\sin \theta$  if  $\cos \theta = -\frac{2}{7}$ .
- 49** Without using a calculator, find:
- $\sin \theta$  if  $\cos \theta = -\frac{1}{4}$  and  $\pi < \theta < \frac{3\pi}{2}$
  - $\cos \theta$  if  $\sin \theta = \frac{2}{3}$  and  $\frac{\pi}{2} < \theta < \pi$
  - $\tan \theta$  if  $\sin \theta = -\frac{5}{6}$  and  $\frac{3\pi}{2} < \theta < 2\pi$
  - $\tan \theta$  if  $\cos \theta = \frac{1}{3}$  and  $0 < \theta < \frac{\pi}{2}$ .
- 50** Find exact values for  $\cos \theta$  and  $\sin \theta$  given that:
- $\tan \theta = -\frac{1}{3}$  and  $\frac{\pi}{2} < \theta < \pi$
  - $\tan \theta = \frac{1}{\sqrt{2}}$  and  $\pi < \theta < \frac{3\pi}{2}$ .
- 51** Find all  $\theta$  such that  $0^\circ \leq \theta \leq 360^\circ$  and:
- $\cos \theta = -0.3$
  - $\sin \theta = -\frac{7}{9}$
  - $\tan \theta = -\frac{3}{\sqrt{5}}$
- 52** Solve for  $x$  if  $0 \leq x \leq 2\pi$ :
- $\sin x = 0.785$
  - $2 \cos x = 5 \sin x$
  - $\tan 3x = 0.9$
  - $4 \sin^2 x = \cos^2 x$
- 53** Solve for  $x$  on the domain  $-\pi \leq x \leq \pi$ :
- $7 \sin x = 3$
  - $3 \tan(2(x+1)) = 2$
  - $3 \cos^2 x - \sin(x+1) = 2 - x$
- 54** Find the exact solutions of these equations for  $0 \leq x \leq 2\pi$ :
- $\sqrt{2} \cos x + 1 = 0$
  - $\sin x = -\sqrt{3} \cos x$
- 55** The population of butterflies after  $t$  years is  $P(t) = 4500 + 500 \sin\left(\frac{2\pi}{7}(t-3)\right)$  for  $0 \leq t \leq 10$ .
- Find:
    - the initial population
    - the population after 3 years.
  - When is the population:
    - 4200
    - 4900?
  - During what time interval(s) does the population drop below 4300?
- 56** Find the equation of the perpendicular bisector of:
- A(3, 5) and B(7, 3)
  - M(7, 2) and N(1, -5)
- 57** A line segment has equation  $4x - 3y + 2 = 0$ . Its midpoint is (4, 6).
- State the gradient of:
    - the line segment
    - its perpendicular bisector.
  - State the equation of the perpendicular bisector. Write your answer in the form  $ax + by + d = 0$ .

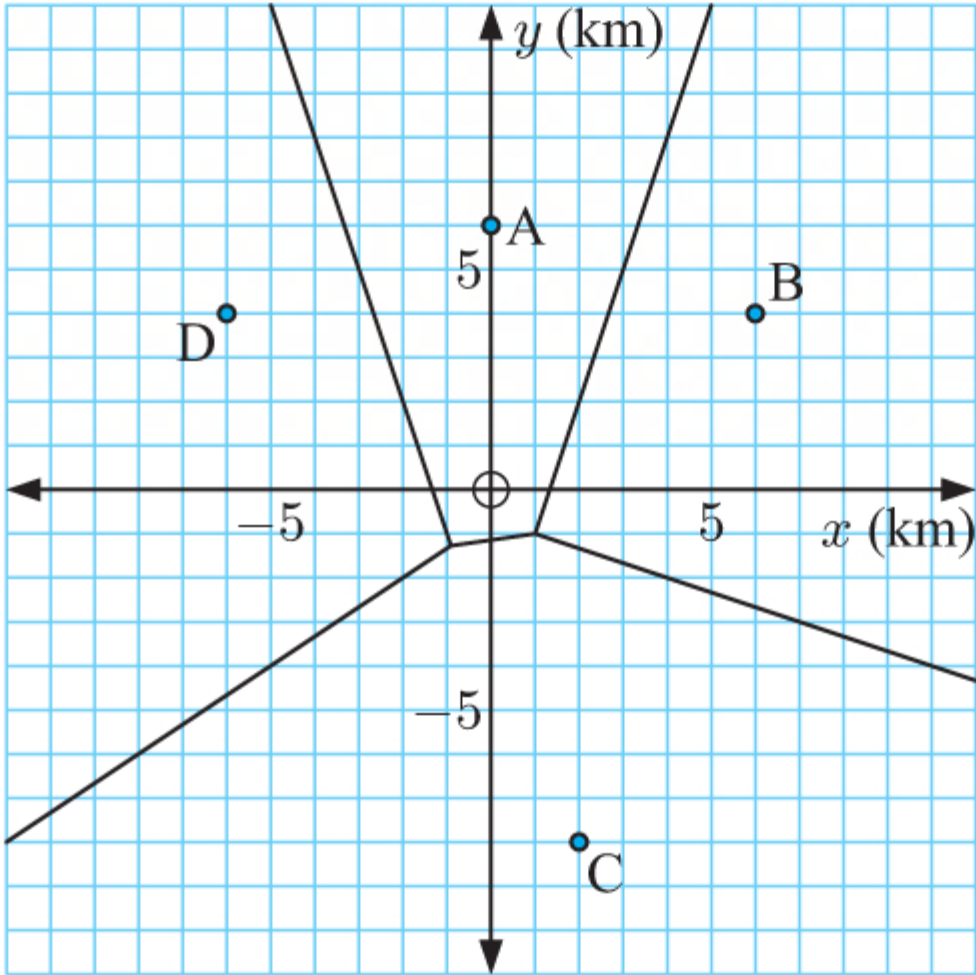


**58** This Voronoi diagram shows the bus stops in a particular suburb.  
Children living in the area walk to the nearest stop to catch the bus to school.

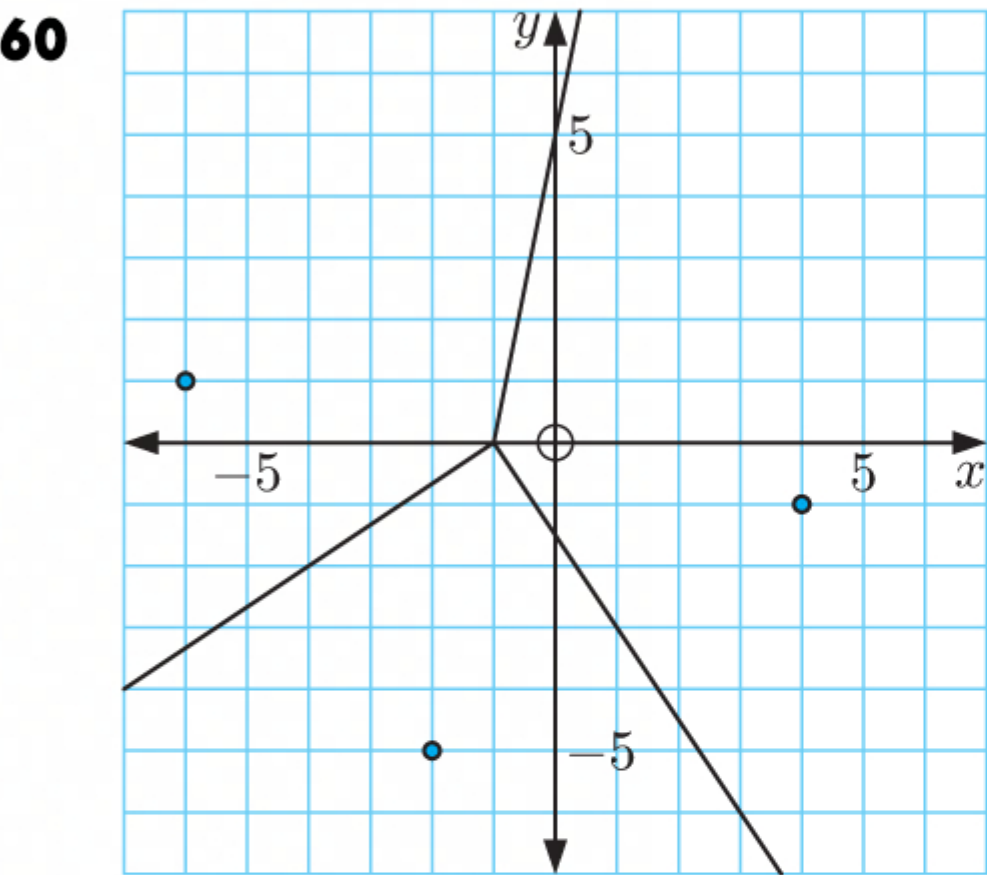


- a** Identify the nearest bus stop for a child living at:
- i**  $(0, -2)$       **ii**  $(3, 2)$       **iii**  $(-5, 5)$       **iv**  $(-1, -4)$
- b** Jerome is closest to bus stops B, C, and G.
- i** Where does Jerome live?
- ii** How far does Jerome need to walk to get to any of these bus stops?

**59** Terrence owns a fast food chain. The locations of his restaurants in a particular city are shown in the Voronoi diagram alongside.



- a** Find the closest restaurant for someone living at:
- i**  $(-2, 8)$       **ii**  $(5, -5)$       **iii**  $(-9, -6)$
- b** Terrence opens a new restaurant E at  $(2, 4)$ .
- i** Redraw the diagram to include the new restaurant.
- ii** Determine the area of the region whose closest restaurant is E.
- iii** What proportion of residents who are now closest to restaurant E were originally closest to restaurant B?

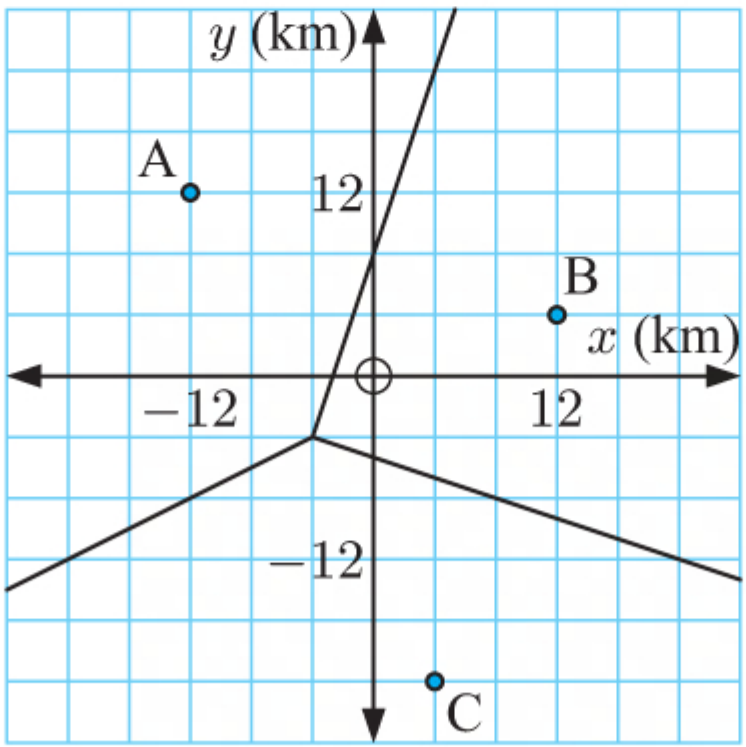


Consider the Voronoi diagram alongside.  
Redraw the diagram with a new site added at:

- a**  $(2, -5)$       **b** the vertex of the existing diagram.

**61** The internet speed is measured at 6 pm at three exchanges in a particular area.

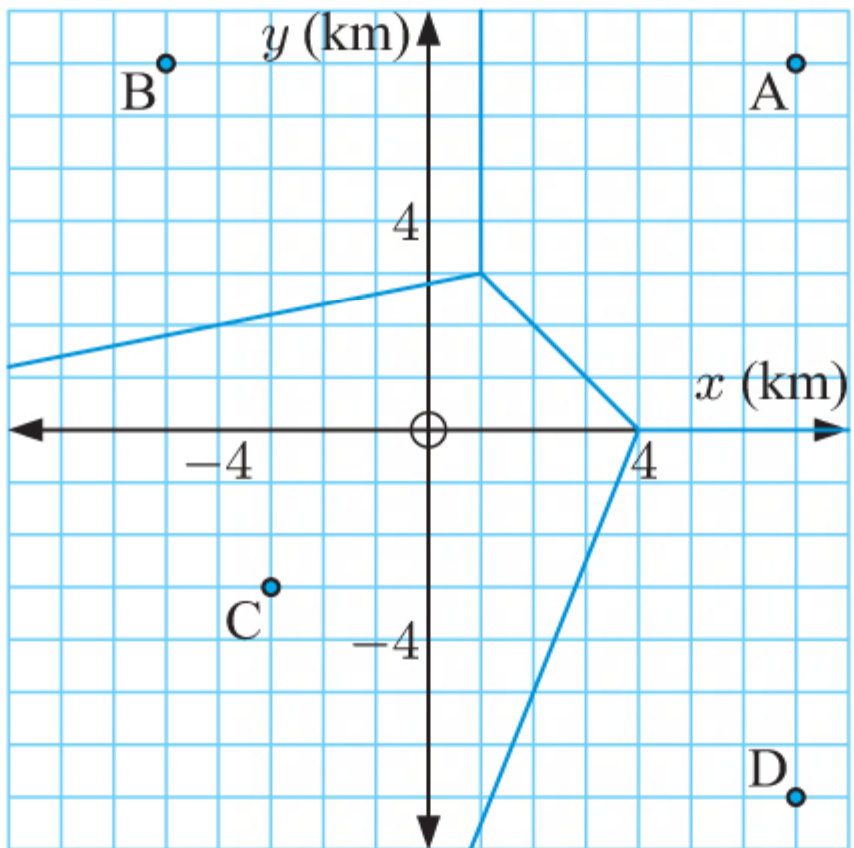
Location	Internet speed (Mbps)
A	44.3
B	42.7
C	45.9



Use nearest neighbour interpolation to estimate the internet speed at 6 pm at:

**a**  $(12, 16)$       **b**  $(-20, -12)$       **c**  $(-4, -4)$

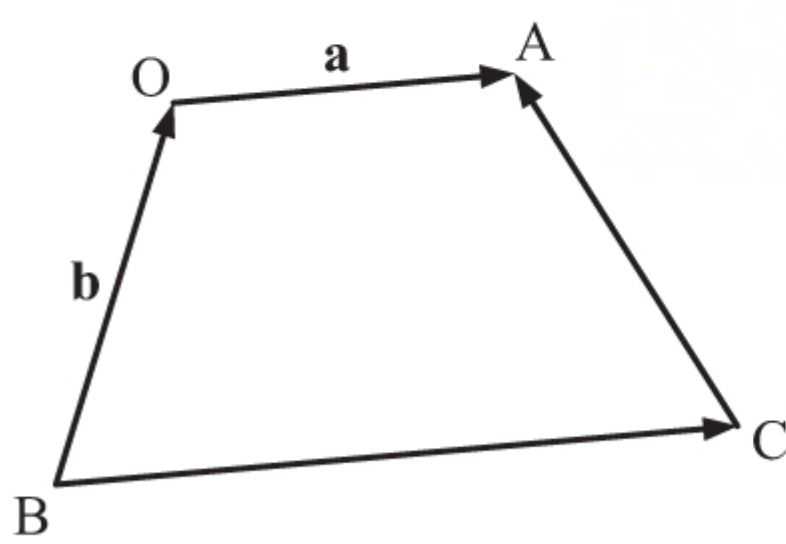
**62** This Voronoi diagram shows the locations of a particular bank's automatic teller machines (ATMs) around a city.



- After receiving customer feedback, the bank has decided to add a new ATM.  
The new ATM will be placed so that it is as far as possible from the existing ATMs.
- a** Where should the bank place the new ATM?
- b** Redraw the Voronoi diagram with the new ATM.
- c** Allan is currently at  $(-1, 1)$  and needs to withdraw money. How far does Allan need to walk to get to the nearest ATM?
- d** Write down an assumption made in **c**.



63



In the given figure,  $[BC]$  is parallel to  $[OA]$  and twice its length.

Write, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , vector expressions for:

**a**  $\overrightarrow{BC}$   
**d**  $\overrightarrow{OC}$

**b**  $\overrightarrow{CB}$   
**e**  $\overrightarrow{AC}$

**c**  $\overrightarrow{BA}$   
**f**  $\overrightarrow{CA}$

 64 Find  $k$  such that:

**a**  $-\frac{1}{3}\mathbf{i} + k\mathbf{j}$  is a unit vector

**b**  $\begin{pmatrix} 3 \\ k \\ k+2 \end{pmatrix}$  has magnitude  $\sqrt{61}$  units.

65 On grid paper, illustrate how to find the vector  $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$  where  $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ , and  $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .  
Check your answer algebraically.

66 If  $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ , and  $\mathbf{c} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ , find:

**a**  $\mathbf{c} - \mathbf{a}$

**b**  $\frac{1}{2}\mathbf{c} + 3\mathbf{a}$

**c**  $\mathbf{b} - 2\mathbf{c} - \mathbf{a}$

**d**  $|\mathbf{c} - 3\mathbf{a} + 2\mathbf{b}|$

67 Let  $\mathbf{a} = \mathbf{i} + \mathbf{j}$ ,  $\mathbf{b} = \mathbf{j} + \mathbf{k}$ , and  $\mathbf{c} = \mathbf{i} + \mathbf{k}$ . Find:

**a**  $\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

**b**  $|-5\mathbf{c}|$

**c**  $\frac{1}{|\mathbf{b}|}\mathbf{b}$

**d**  $|2\mathbf{a} - 3\mathbf{b} - \mathbf{c}|$

68 Consider vectors  $\mathbf{a} = 3\mathbf{i} - 6\mathbf{j}$  and  $\mathbf{b} = 7\mathbf{i} + 2\mathbf{j}$ .

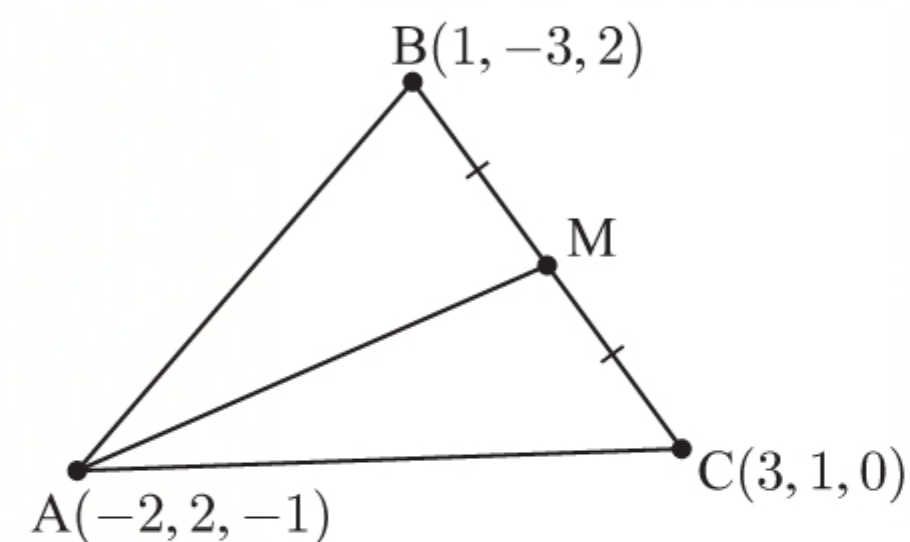
The point  $C(5, 22)$  can be located using the vector  $\overrightarrow{OC} = r\mathbf{a} + s\mathbf{b}$ . Find the values of  $r$  and  $s$ .

69 Consider the diagram alongside.

**a** Find the coordinates of  $M$ .

**b** Find vectors  $\overrightarrow{AB}$ ,  $\overrightarrow{AM}$ , and  $\overrightarrow{AC}$ .

**c** Verify that  $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$ .



70 Find the vector  $\mathbf{v}$  which has:

**a** the same direction as  $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$  and length 4 units

**b** the opposite direction to  $\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$  and length 3 units.

71 Find the velocity vector of an object moving at  $3 \text{ m s}^{-1}$  in the direction  $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ .

72 Find the coordinates of the point which is 6 units from  $(2, -1, 3)$  in the direction  $\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$ .

73 For  $\mathbf{p} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ ,  $\mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}$ , and  $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ , find:

**a**  $\mathbf{p} \cdot \mathbf{r}$

**b**  $\mathbf{q} \cdot (\mathbf{r} + \mathbf{p})$

**c**  $(2\mathbf{p} + \mathbf{q}) \cdot \mathbf{r}$

**d**  $|\mathbf{q}|^2$

**e**  $k$  such that  $k\mathbf{p} + \mathbf{q}$  is perpendicular to  $\mathbf{r}$ .

74 Find the angle between the vectors:

**a**  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$

**b**  $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$  and  $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$

75 Find the acute angle between two diagonals of the rectangular prism formed by the vectors  $2\mathbf{i}$ ,  $3\mathbf{j}$ , and  $5\mathbf{k}$ .

76 **a** Given  $\mathbf{a} \cdot \mathbf{b} < 0$ , what conclusion can you draw about the angle between  $\mathbf{a}$  and  $\mathbf{b}$ ?

**b** For the vectors  $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ , find:

**i**  $\mathbf{a} \cdot \mathbf{b}$

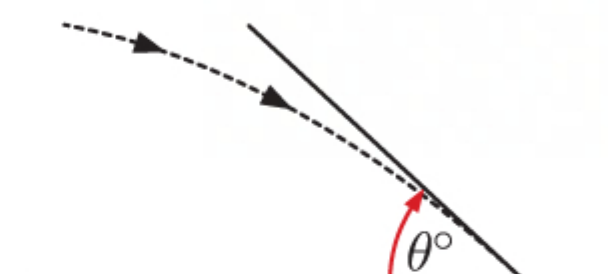
**ii** the angle between  $\mathbf{a}$  and  $\mathbf{b}$ , in degrees correct to one decimal place.



- 77** Suppose  $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ , and  $\mathbf{c} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$ . Find:
- a**  $\mathbf{a} \times \mathbf{b}$  **b**  $(\mathbf{b} \times \mathbf{c}) \cdot 2\mathbf{a}$ .
- 78** Suppose  $\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$  and  $\mathbf{b} = \mathbf{j} + 2\mathbf{k}$ .
- a** Find  $\mathbf{a} \times \mathbf{b}$ .
- b** Find a vector of length 5 units which is perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ .
- 79** Let  $\mathbf{a} = -4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$  and  $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$ . Given that  $|m\mathbf{a} \times \mathbf{b}| = 35$  for  $m \in \mathbb{R}$ , find the value of  $m$ .
- 80** Suppose  $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{a} \times \mathbf{b} = \mathbf{j} - 2\mathbf{k}$ , and the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $\frac{\pi}{4}$ . Find  $|\mathbf{b}|$ .
- 81** ABCD is a parallelogram with  $A(-1, 2, 3)$ ,  $B(0, 2, 4)$ , and  $C(1, 5, -1)$ .
- a** Find the coordinates of D. **b** Find the area of ABCD.
- 82** Let  $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{s} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ , and  $\mathbf{t} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$  be the position vectors of the points R, S, and T respectively. Find the area of triangle RST.
- 83** For the vectors  $\mathbf{a}$  and  $\mathbf{b}$ , find the component of  $\mathbf{a}$ :
- i** in the direction of  $\mathbf{b}$  **ii** perpendicular to  $\mathbf{b}$ .
- a**  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$  and  $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$  **b**  $\mathbf{a} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$  and  $\mathbf{b} = 5\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$
- 84** Describe each of the following lines using:
- i** a vector equation **ii** parametric equations.
- a** a line parallel to  $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  which passes through  $(2, -1, 3)$
- b** a line perpendicular to the  $YZ$ -plane which passes through  $(0, 1, 2)$ .
- 85** Suppose  $\mathbf{m} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$  and  $\mathbf{n} = -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$ . Find:
- a**  $|\mathbf{m} + \mathbf{n}|$
- b**  $\mathbf{m} \cdot \mathbf{n}$
- c** the vector equation of the line passing through  $(1, -1, 2)$  which is parallel to  $\mathbf{m}$ .
- 86** Write a vector equation for the line:
- a** parallel to  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  and passing through the point  $(5, 0, -2)$
- b** parallel to  $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$  and passing through the point  $(-2, 5, 4)$
- c** perpendicular to the  $XZ$ -plane and passing through the point  $(2, -4, 1)$ .
- 87** Object A moves with velocity  $\mathbf{v}_A = \begin{pmatrix} 20 \\ -5 \\ 7 \end{pmatrix}$ , and object B moves with velocity  $\mathbf{v}_B = \begin{pmatrix} k \\ -11 \\ 15 \end{pmatrix}$ ,  $k \in \mathbb{R}$ .
- Time is measured in seconds, and the distance units are metres.
- a** Given the objects are moving in perpendicular directions, find the value of  $k$ .
- b** Which object is moving faster?
- 88** Suppose A is  $(-1, 2, 1)$  and B is  $(0, 1, 3)$ .
- a** Find the equation of the line (AB) in the form  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ ,  $\lambda \in \mathbb{R}$ .
- b** Find the angle between (AB) and the line  $L$  defined by  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ ,  $\mu \in \mathbb{R}$ .
- 89** Line 1 has vector equation  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ,  $s \in \mathbb{R}$ .
- Line 2 has vector equation  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ ,  $t \in \mathbb{R}$ .
- Find the point where the two lines meet.

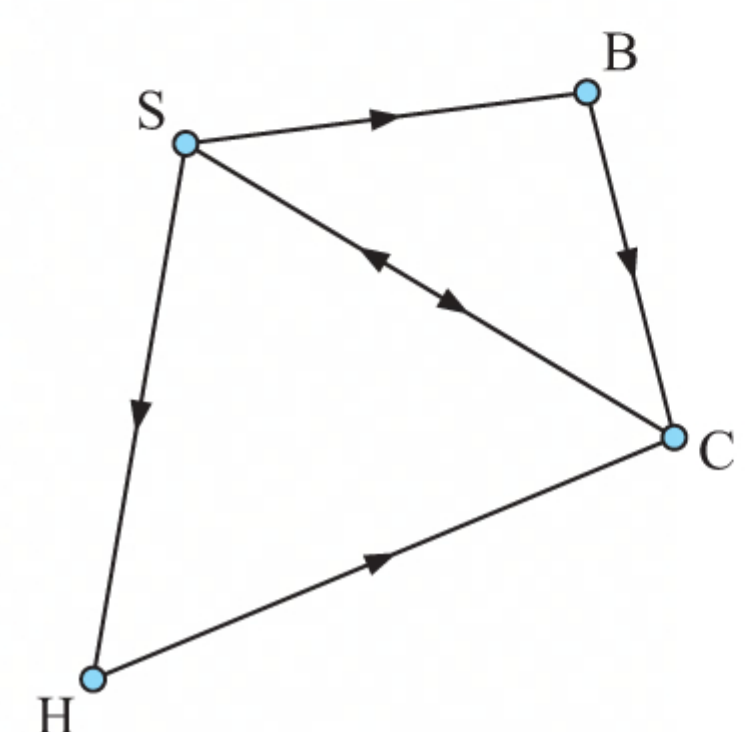
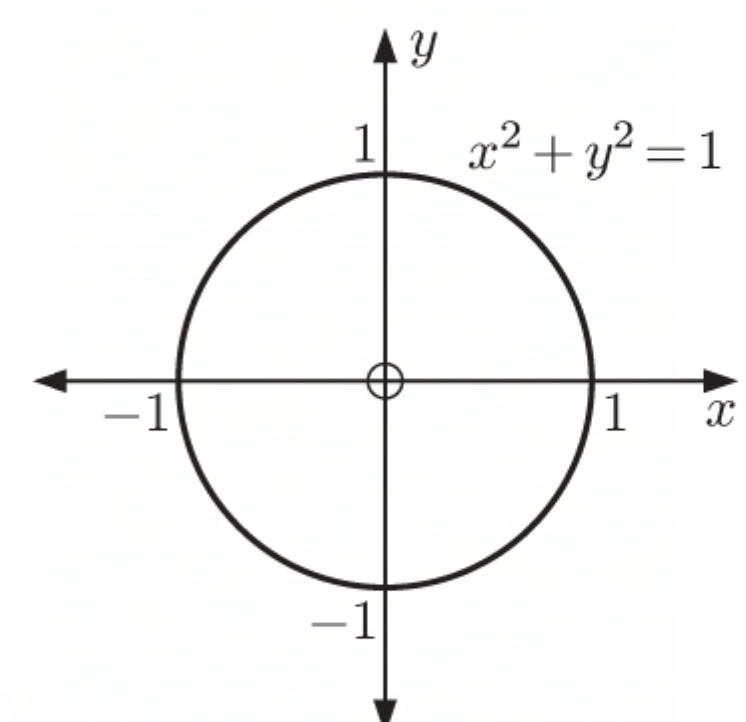


- 90** Toy car  $A$  is initially at  $(9, -3)$ , and is travelling at  $\sqrt{13} \text{ m s}^{-1}$  in the direction  $-2\mathbf{i} + 3\mathbf{j}$ .  
Toy car  $B$  is initially at  $(-1, 4)$ , and travels at a constant velocity for 3 seconds to  $(5, 7)$ .
- Write a vector equation for the position at time  $t$  seconds for each toy car.
  - Show that the paths of the cars intersect at  $(3, 6)$ .
  - Will the cars collide? Explain your answer.
- 91** Ian and Grant are swimming in the ocean. Initially, Ian is at  $(9, 7)$  and swims with velocity vector  $\begin{pmatrix} 0.6 \\ -0.5 \end{pmatrix}$ , and Grant is at  $(-3, 2)$  and swims with velocity vector  $\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$ .
- The distances are in metres, and the velocities are in metres per second.
- Find vector equations for the motion of each swimmer after  $t$  seconds.
  - Find the distance between Ian and Grant after:
    - 2 seconds
    - 10 seconds.
  - Find the shortest distance between the swimmers, and the time when this occurs.
- 92** Lines  $L_1$ ,  $L_2$ , and  $L_3$  are defined by:
- $$L_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}, \quad t \in \mathbb{R}$$
- $$L_2: \quad x = 5 - r, \quad y = -4 + 2r, \quad z = 1 + r, \quad r \in \mathbb{R}$$
- $$L_3: \quad x = 5 - 3s, \quad y = 5 + 4s, \quad z = 1 + 2s, \quad s \in \mathbb{R}$$
- Show that  $L_1$  is parallel to  $L_2$ .
  - Show that  $L_1$  and  $L_3$  intersect, and find the angle between them.
- 93** A particle  $P$  moves with position equations  $x(t) = 1 - 2t$ ,  $y(t) = t^3 + t + 2$  where  $0 \leq t \leq 5$  seconds and distances are in centimetres.
- Find the initial position of  $P$ .
  - Find the velocity vector of  $P$ .
  - Find the speed of  $P$ :
    - initially
    - after 5 seconds.
  - Find the maximum and minimum speed of  $P$  during its motion.
- 94** A car moves through a car park with velocity vector  $\begin{pmatrix} 2 \\ \sqrt{2t+3} \\ 1 \end{pmatrix}$ ,  $t \geq 0$ . After 3 seconds, the car is at  $(9, 4)$ . Time is measured in seconds and distance units are metres.
- Find the exact initial speed of the car.
  - Find the speed  $S$  of the car after  $t$  seconds.
  - Use technology to help sketch the graph of  $S$  against  $t$  for  $0 \leq t \leq 10$ .
  - Hence describe the speed of the car as  $t \rightarrow \infty$ .
  - Find the position equations of the car.
  - Find the position of the car:
    - initially
    - after 11 seconds.
- 95** A softball is thrown and its position at any time  $t \geq 0$  is given by  $x(t) = 13t$ ,  $y(t) = 1.4 + 14t - 4.9t^2$ , where  $y = 0$  is ground level. Time is measured in seconds and distance units are metres.
- How high was the ball above the ground initially?
  - How long was the ball in the air?
  - How far did the ball travel horizontally?
  - What was the maximum height reached?
  - Find the velocity vector of the ball.
    - Hence find the angle  $\theta^\circ$  at which the ball struck the ground.
  - Find the ball's:
    - initial speed
    - final speed.
- 96**
- Find the transformation matrix  $\mathbf{A}$  for an anticlockwise rotation about  $O$  through  $\frac{2\pi}{3}$ .
  - Find the image of the following points under this transformation:
    - $(-1, 2)$
    - $(1, -\sqrt{3})$



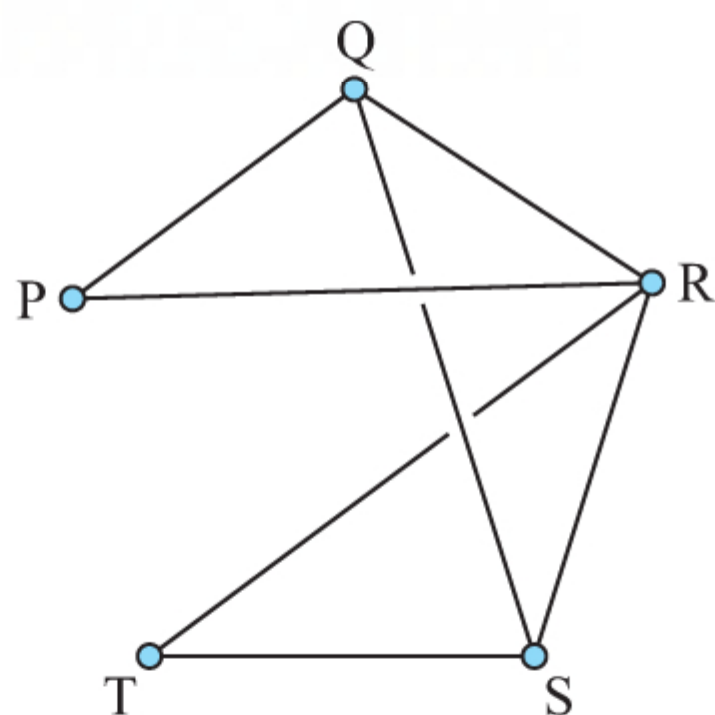


- 97** Consider the transformation with equations  $\begin{cases} x' = \frac{1}{2}(x + \sqrt{3}y) \\ y' = \frac{1}{2}(\sqrt{3}x - y) \end{cases}$
- Find the transformation matrix  $\mathbf{A}$ .
  - Hence determine the nature of the transformation.
  - Find the image of  $(\sqrt{6}, \sqrt{2})$  under this transformation. Comment on your result.
- 98**
- Write down the transformation matrix  $\mathbf{A}$  for a horizontal stretch with scale factor  $\frac{4}{3}$ .
  - Find the image of  $(12, -24)$  under this transformation.
  - Describe the transformation with transformation matrix  $\mathbf{A}^2$ .
- 99** Consider an enlargement with scale factor  $\frac{5}{2}$ .
- Write down the transformation matrix  $\mathbf{A}$ .
  - Hence find the image of  $(-2, 4)$  under the transformation.
  - Find  $\mathbf{A}^{-1}$  and describe the transformation  $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$ .
- 100**
- Find the transformation matrix  $\mathbf{A}$  for an anticlockwise rotation through  $\frac{\pi}{3}$  about O.
  - Write down the matrix equation for:
    - an anticlockwise rotation through  $\frac{\pi}{3}$  about O followed by a translation through  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
    - a translation through a vector  $\mathbf{b}$  followed by an anticlockwise rotation through  $\frac{\pi}{3}$  about O.
  - Suppose the transformations in **b** are equivalent.
    - Show that  $\mathbf{A}\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .
    - Hence find  $\mathbf{b}$ .
- 101** Let Transformation 1 be a reflection in the line  $y = -3x$ , and Transformation 2 be a clockwise rotation of  $\frac{\pi}{2}$  about  $(0, 0)$ .
- Find the transformation matrix for:
    - Transformation 1
    - Transformation 2.
  - Write down the transformation matrix for Transformation 1 followed by Transformation 2.
    - Find the single transformation which is equivalent to this composite transformation.
- 102** An object is enlarged with scale factor  $\frac{5}{3}$ , then translated 1 unit left and 2 units up.
- Find the transformation equation in the form  $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ .
  - The point  $(-6, 2)$  lies on the object. Find its image under this transformation.
  - The image of the object has area 60 units<sup>2</sup>. Find the area of the object.
- 103** Under an enlargement with scale factor  $k$ , followed by a stretch parallel to the  $x$ -axis with scale factor  $h$ , the point  $(4, 2)$  is transformed to  $(5, 10)$ .
- Write down the transformation matrix for this composite transformation in terms of  $k$  and  $h$ .
  - Find  $k$  and  $h$ .
  - The unit circle  $x^2 + y^2 = 1$  is transformed under this composite transformation. Find the area of its image.
- 104** In a small shopping centre, roadways connect the parking areas as shown in the diagram. B is the bakery, C is the chemist, H is the hardware store, and S is the supermarket.
- Write down the out degree of S. Interpret your answer in the context of the question.
  - Construct the adjacency matrix  $\mathbf{A}$  for the roadways.
  - Calculate:
    - $\mathbf{A}^2$
    - $\mathbf{A}^3$
  - What is the significance of:
    - the 0 in row 1 column 2 of  $\mathbf{A}^2$
    - the 2 in row 4 column 2 of  $\mathbf{A}^2$ ?
  - What is the least number of steps required to drive from the bakery to the hardware store? Explain your answer.





105



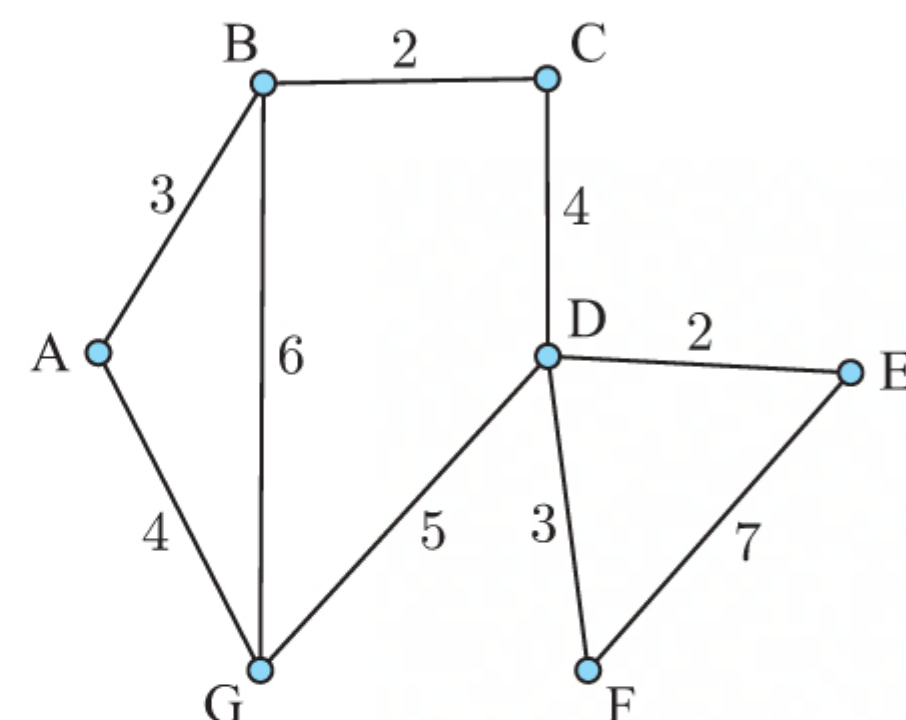
The points P, Q, R, S, and T are linked by paths as shown in the graph alongside.

- Is the graph connected? Explain your answer.
- Write down the degree of vertex R.
- Find the adjacency matrix  $A$  for this graph.
- Find  $A^2$ , and interpret the value in row 2, column 3 of  $A^2$ .
- Show that the graph is semi-Eulerian.
- Find an Eulerian trail.

106 An electrical network is planned to connect 7 cities in a particular region.

The possible connections and the estimated costs (in millions of dollars) are summarised in the graph alongside.

- Use Kruskal's algorithm to find the minimum spanning tree.
- Hence determine the minimum total cost to connect all 7 cities.



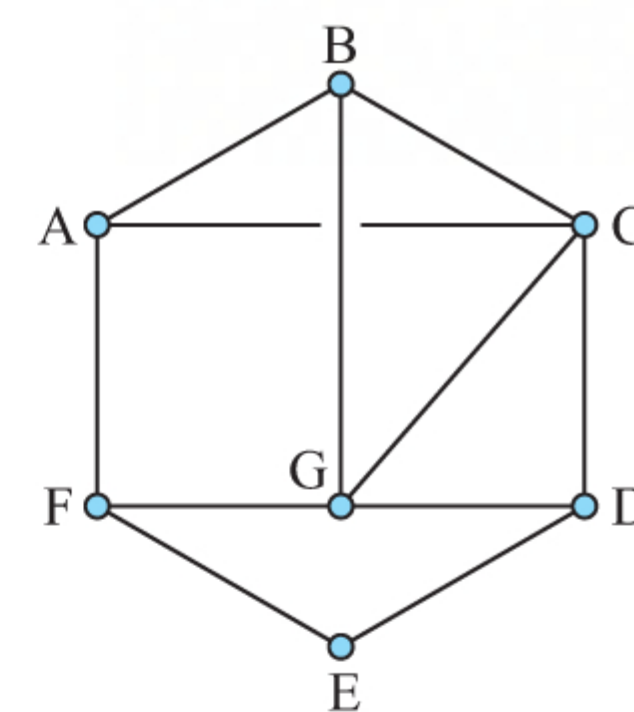
107 A group of small islands are to be connected by a series of bridges. This table shows the estimated cost, in thousands of dollars, of constructing a bridge between each pair of islands.

The islands should be connected in a way which minimises the total cost.

- Find the estimated cost of a bridge connecting islands B and E.
- Apply Prim's algorithm to this table to find the minimum spanning tree, and hence list the bridges which should be constructed to connect the islands.
- Find the total cost of constructing the bridges in this case.
- How much more expensive would it be to instead construct bridges between island B and every other island?

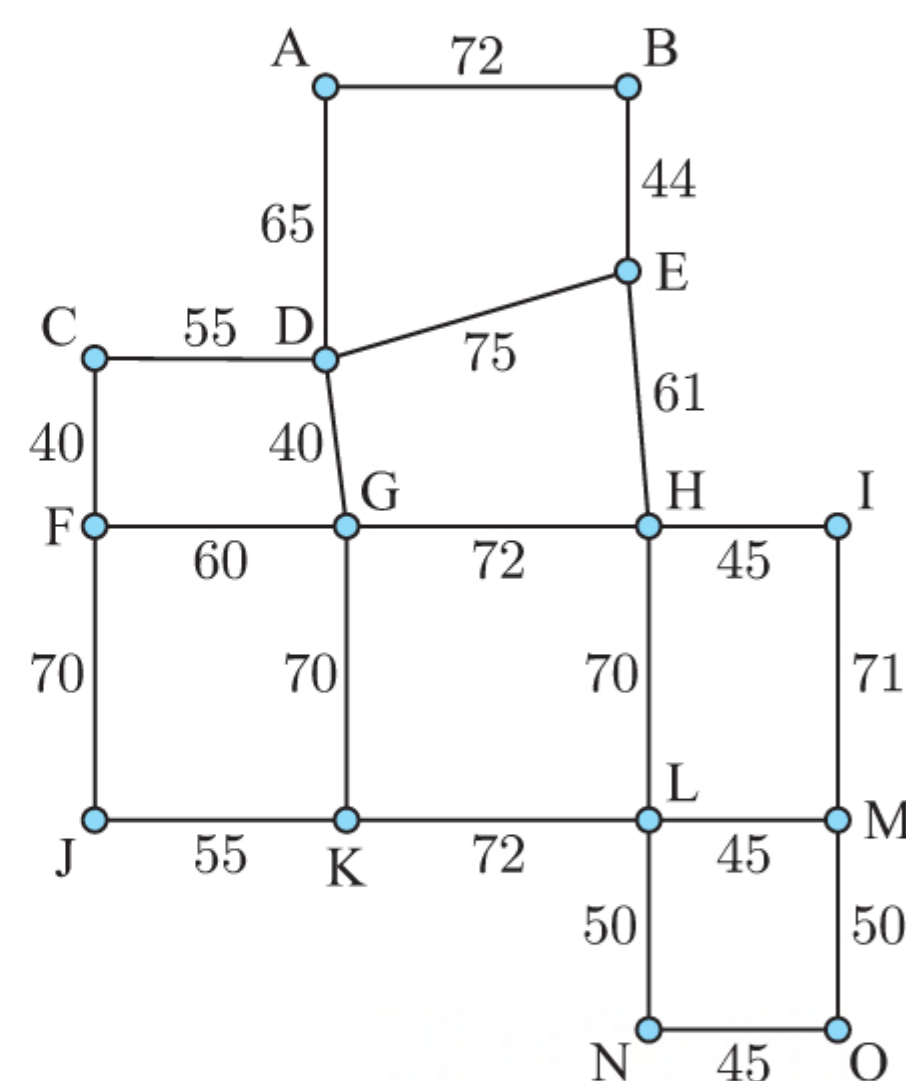
	A	B	C	D	E
A	—	650	800	300	340
B	650	—	410	470	750
C	800	410	—	450	500
D	300	470	450	—	420
E	340	750	500	420	—

- Show that this graph is neither Eulerian nor semi-Eulerian.
- Suppose another edge connecting vertices A and B is added to the graph.
  - Draw the resulting graph.
  - Is the graph semi-Eulerian? Justify your answer.
- One more edge is added to the graph in **b**, and the resulting graph is Eulerian.
  - Between which two vertices was the edge added?
  - Draw the resulting graph.
  - Find an Eulerian circuit.



109 Starting from D, a streetsweeper must traverse each road of this neighbourhood at least once, before returning to D. The distances shown are in metres.

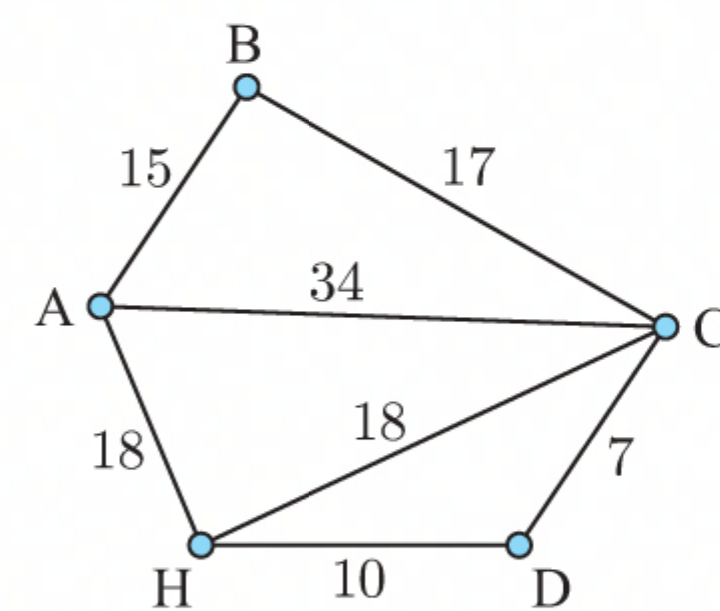
- Is this graph complete? Explain your answer.
- List the vertices of odd degree.
- Find the shortest distance the streetsweeper must travel, and state the roads the streetsweeper must traverse twice in this case.
- Given that the streetsweeper first traverses DA, find a possible route the streetsweeper can take with this shortest distance.





- 110** Carl is a research scientist who is currently at university H. Starting from H, he will visit universities A, B, C, and D, before returning to H.

Carl would like to complete the trip in the shortest time possible. The roads between the universities, and the times in minutes to travel along the roads, are shown in the diagram.



- Complete the graph with edges showing the shortest time to travel between universities.
- Use the nearest neighbour algorithm to find an upper bound for the time Carl will spend travelling.
- By deleting vertex A, use the deleted vertex algorithm to find a lower bound for the time Carl will spend travelling.
- Hence write down the quickest route Carl can take to visit the universities.



## TOPIC 4: STATISTICS AND PROBABILITY

### SAMPLING

We obtain data from a **sample** of a population when it is impractical to obtain data from the entire population.

You should know the four main categories of **error** that can arise from sampling:

- **Sampling errors** occur when a characteristic of a sample differs from that of the population.
- **Measurement errors** are inaccuracies in measurement during data collection.
- **Coverage errors** occur when a sample does not truly reflect the population.
- **Non-response errors** occur when a large number of people selected for a survey choose not to respond.

### SAMPLING METHODS

- In **simple random sampling**:
  - ▶ Each member of the population has the same chance of being selected in the sample.
  - ▶ Each set of  $n$  members of the population has the same chance of being selected as any other set of  $n$  members.
- In **systematic sampling**, the sample is created by selecting members of the population at regular intervals.
- In **convenience sampling**, members are chosen for the sample because they are easier to select or more likely to respond.
- In **stratified sampling** or **quota sampling**, the population is divided into subgroups, and the number of members sampled from each subgroup is proportional to the fraction of the population represented by that subgroup. If the members of each subgroup are randomly selected, the sample is a **stratified sample**. If the members are specifically chosen, the sample is a **quota sample**.

### TYPES OF DATA AND ITS REPRESENTATION

**Categorical data** refers to data which describes a particular quality or characteristic.

**Discrete data** can take any of a set of exact number values  $\{x_1, x_2, x_3, \dots\}$ . It is normally **counted**.

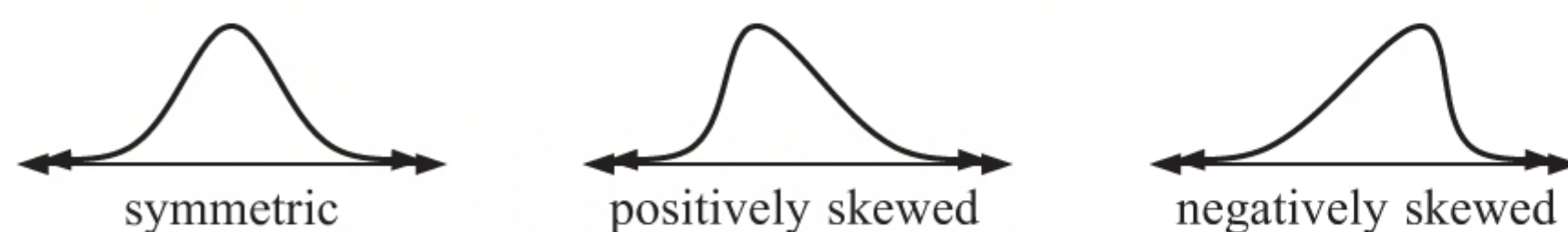
**Continuous data** can take any numerical value within a certain range. It is normally **measured**.

**Grouped data** is numerical data which is collected in groups or classes. The **modal class** is the class with the highest frequency.

A **column graph** is used to display discrete data and grouped data. The columns have spaces between them.

A **frequency histogram** is used to display continuous data. The classes are of equal width, and there are no spaces between the columns.

Data may be symmetric, positively skewed, or negatively skewed.



We use a **cumulative frequency graph** to display the cumulative frequency for each data value in a distribution. This enables us to read off the values at each percentile.

### MEASURING THE CENTRE OF DATA

The **mean** of a set of scores is their arithmetic average.

For a large population, the **population mean**  $\mu$  is generally unknown. The **sample mean**  $\bar{x}$  is used as an approximation for  $\mu$ .

For ungrouped data,  $\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$

For data in a frequency table,  $\bar{x} = \frac{\sum xf}{\sum f}$  where  $f$  is the frequency of each value.

For grouped data we can only estimate the mean. We use the **mid-interval value** within each group to represent all scores within that group.



The **median** is the middle value of an ordered data set.

- For an **odd number** of data, the median is one of the original data values.
- For an **even number** of data, the median is the average of the two middle values, and may not be in the original data set.

The **mode** is the most frequently occurring score. If there are two modes we say the data is **bimodal**. For continuous data we refer to a **modal class**.

## PERCENTILES

The  **$k$ th percentile** is the score  $a$  such that  $k\%$  of the scores are less than  $a$ .

The **lower quartile** ( $Q_1$ ) is the 25th percentile.

The **median** ( $Q_2$ ) is the 50th percentile.

The **upper quartile** ( $Q_3$ ) is the 75th percentile.

You should know how to generate a **cumulative frequency graph** and use it to estimate  $Q_1$ ,  $Q_2$ , and  $Q_3$ .

## MEASURING THE SPREAD OF DATA

The **range** is the difference between the maximum and the minimum data values.

The **interquartile range**  $IQR = Q_3 - Q_1$ .

The **variance**  $\sigma^2$  is the average of the squares of the distances from the mean.

The **standard deviation**  $\sigma$  is the square root of the variance.

You should be able to use technology to calculate standard deviation. If we are given the whole population we use the population standard deviation  $\sigma_x$ . If we are only given a sample from a larger population, we use the sample standard deviation  $s_x$ .

## OUTLIERS

**Outliers** are extraordinary data that are separated from the main body of the data. We test for outliers by calculating upper and lower boundaries:

- upper boundary =  $Q_3 + 1.5 \times IQR$
- lower boundary =  $Q_1 - 1.5 \times IQR$

Any data outside of these boundaries is considered an outlier.

## BOX AND WHISKER DIAGRAMS

A **box and whisker diagram** or **box plot** illustrates the **five-number summary** of a data set:

- minimum value
- $Q_1$
- median
- $Q_3$
- maximum value

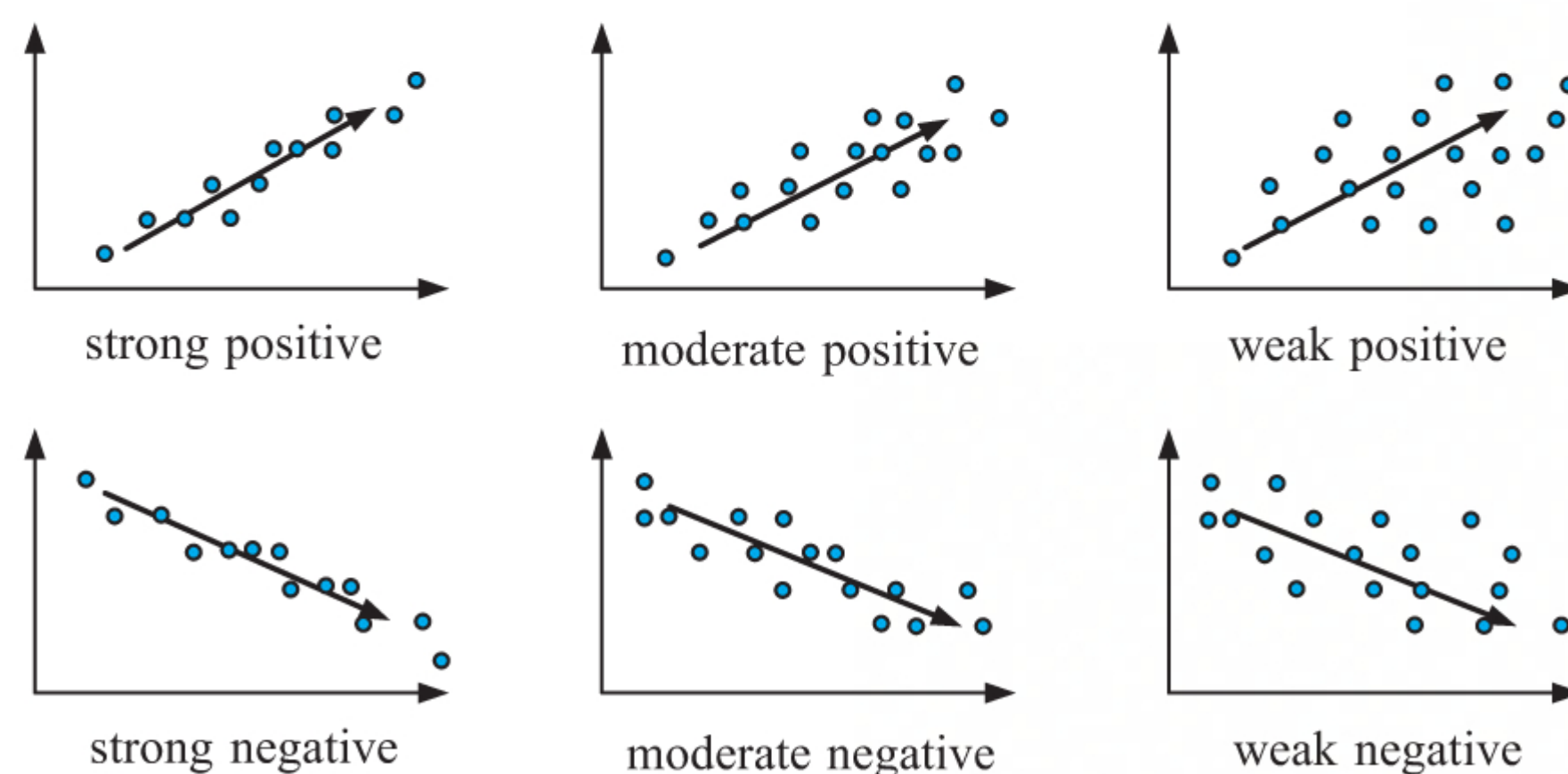


An outlier is indicated by an asterisk \*.

## BIVARIATE STATISTICS

**Correlation** refers to the relationship between two numerical variables.

We can use a **scatter diagram** to help identify **outliers** and to describe the correlation between variables. We consider **direction**, **strength**, and **linearity**.





If a change in one variable *causes* a change in the other variable then we say there is a **causal relationship** between them.

To measure the strength of the relationship between two variables, we use **Pearson's product-moment correlation coefficient**  $r$ .

The correlation coefficient lies in the range  $-1 \leq r \leq 1$ .

- The sign of  $r$  indicates the direction of correlation.
  - ▶ A positive value for  $r$  indicates the variables are positively correlated.
  - ▶ A negative value for  $r$  indicates the variables are negatively correlated.
- The size of  $r$  indicates the strength of correlation.
  - ▶ A value of  $r$  close to  $+1$  or  $-1$  indicates strong correlation between the variables.
  - ▶ A value of  $r$  close to zero indicates weak correlation between the variables.

## The coefficient of determination

To help describe the correlation between two variables, we can also calculate the **coefficient of determination**,  $r^2$ . This is simply the square of Pearson's product-moment correlation coefficient  $r$ , so the direction of correlation is eliminated.

If there is a causal relationship, then  $r^2$  indicates the degree to which change in the independent variable explains change in the dependent variable.

## Line of best fit

If two variables are linearly correlated, we can draw a line of best fit to illustrate their relationship.

We can draw a **line of best fit by eye**, which passes through the **mean point**  $(\bar{x}, \bar{y})$ , and which fits the trend of the data.

To get a more accurate line of best fit, we use a method called **linear regression**. The line obtained is called the **least squares regression line**. You should be able to find this line using your calculator.

When using a line of best fit to estimate values, **interpolation** is usually reliable, whereas **extrapolation** may not be.

**Spearman's rank correlation coefficient** of a bivariate data set is defined as the Pearson product-moment correlation coefficient of the variables' **ranks**. It is often used when the data is clearly non-linear, but has an upward or downward trend.

## Sum of squared residuals

When fitting a model to a set of data points, the **residual** of the  $i$ th data point  $(x_i, y_i)$  is  $r_i = y_i - \hat{y}_i$  where  $\hat{y}_i$  is the model's predicted value of  $y$  at  $x = x_i$ .

The **sum of squared residuals** is defined as  $SS_{\text{res}} = \sum_{i=1}^n r_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2$ .

## Non-linear regression

**Non-linear regression** is used to fit non-linear models to data by minimising  $SS_{\text{res}}$ .

When given several fitted models, you should be able to choose the most appropriate one by calculating  $SS_{\text{res}}$  for each model.

## STATISTICAL RELIABILITY AND VALIDITY

- **Statistical reliability** is a measure of how *consistently* a variable can be measured.

We can consider:

- ▶ **Test-retest reliability**, in which identical tests are performed at different times. We are interested in how consistent the measurement is over *time*.
- ▶ **Parallel forms reliability**, in which similar tests are performed. We are interested in how consistent the measurement is across different *versions*.
- **Statistical validity** considers how *accurately* a variable measures a particular aspect of a population.
  - ▶ **Content validity** considers how well the field of study or *content domain* is covered.
  - ▶ **Criterion validity** considers how well one variable predicts another valid variable, called the **criterion variable**.



## PROBABILITY

A **trial** occurs each time we perform an experiment.

The possible results from each trial of an experiment are called its **outcomes**.

The **sample space**  $U$  is the set of all possible outcomes of an experiment.

### Experimental probability

In many situations, we can only measure the probability of an event by experimentation.

experimental probability = relative frequency of event

### Theoretical probability

If all outcomes are equally likely, the probability of event  $A$  is  $P(A) = \frac{n(A)}{n(U)}$ .

For any event  $A$ ,  $0 \leq P(A) \leq 1$ .

For any event  $A$ ,  $A'$  is the event that  $A$  does not occur.  $A$  and  $A'$  are **complementary events**, and  $P(A) + P(A') = 1$ .

The event that both  $A$  **and**  $B$  occur is written  $A \cap B$ .

The event that  $A$  **or**  $B$  **or both** occur is written  $A \cup B$ .

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For **disjoint** or **mutually exclusive** events,  $P(A \cap B) = 0$ .

### Making predictions using probability

If there are  $n$  trials of an experiment, and an event has probability  $p$  of occurring in each of the trials, then the number of times we *expect* the event to occur is  $np$ .

### Independent events

Two events are **independent** if the occurrence of each of them does not affect the probability that the other occurs. An example of this is sampling **with replacement**.

For independent events  $A$  and  $B$ ,  $P(A \cap B) = P(A)P(B)$ .

### Dependent events

Two events are **dependent** if the occurrence of one of them *does* affect the probability that the other occurs. An example of this is sampling **without replacement**.

For dependent events  $A$  and  $B$ ,  $P(A \cap B) = P(A) \times P(B \text{ given that } A \text{ has occurred})$ .

### Conditional probability

For any two events  $A$  and  $B$ , “ $A \mid B$ ” represents the event “ $A$  given that  $B$  has occurred”, and  $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ .

## MARKOV CHAINS

A **Markov chain** is a model which describes how a sequence of random events evolves over time.

- Time is measured in **discrete** steps.
- The possible outcomes of each event are the **states** that the system could be in.
- The probabilities for the possible states of the system at the next time step depend *only* on the state at the current time step.

The **state matrix**  $\mathbf{s}_n$  shows the state of the system at time  $n$ . The state matrix may represent either:

- the probability or proportion of the population that are in each state, or
- the actual number from the population that are in the given state.

The **initial state matrix** is  $\mathbf{s}_0$ .

In a **transition matrix**  $\mathbf{T} = (t_{ij})$ :

- The columns represent the current state.
- The rows represent the next state.
- $t_{ij}$  is the probability of moving to state  $i$  from state  $j$ .  $t_{ij} = P(\text{next state is } i \mid \text{current state is } j)$

The state of the system at time  $n$  is given by  $\mathbf{s}_n = \mathbf{T}^n \mathbf{s}_0$ .



## Steady state

When there is very little or no change in the values of the state matrices from one time step to the next, we say the system has reached a **steady state**.

We can find the steady state of a Markov chain with transition matrix  $\mathbf{T}$  by either:

- Calculating  $\mathbf{s}_n = \mathbf{T}^n \mathbf{s}_0$  for large values of  $n$ .
- Finding  $\mathbf{s}$  such that  $\mathbf{T}\mathbf{s} = \mathbf{s}$  algebraically.

## DISCRETE RANDOM VARIABLES

A **random variable** represents the possible numerical outcomes of an experiment.

A **discrete random variable** can take any of a set of distinct values.

If  $X$  is a discrete random variable with possible values  $\{x_1, x_2, \dots, x_n\}$  and corresponding probabilities  $\{p_1, p_2, \dots, p_n\}$ , then:

- $0 \leq p_i \leq 1$  for all  $i = 1, \dots, n$
- $\sum_{i=1}^n p_i = p_1 + p_2 + \dots + p_n = 1$
- $\{p_1, p_2, \dots, p_n\}$  describes the **probability distribution** of  $X$ .

We can also describe the probability distribution of  $X$  using a **probability mass function**  $P(x) = P(X = x)$ .

The **expectation** of a discrete random variable  $X$  is  $E(X) = \mu = \sum_{i=1}^n x_i p_i$ .

A game where  $X$  is the gain to the player is said to be **fair** if  $E(X) = 0$ .

The **mode** is the data value  $x_i$  whose probability  $p_i$  is the highest.

The **variance** is  $\text{Var}(X) = \sigma^2$

$$\begin{aligned} &= E[(X - \mu)^2] \\ &= \sum (x_i - \mu)^2 p_i \\ &= \sum x_i^2 p_i - \mu^2 \\ &= E(X^2) - \mu^2 \\ &= E(X^2) - [E(X)]^2 \end{aligned}$$

The **standard deviation** is  $\sigma(X) = \sqrt{\text{Var}(X)}$ .

$E(aX + b) = aE(X) + b$  and  $\text{Var}(aX + b) = a^2 \text{Var}(X)$ .

## THE BINOMIAL DISTRIBUTION

In a **binomial experiment** there are two possible results: success and failure.

Suppose there are  $n$  independent trials of the same experiment with the probability of success being a constant  $p$  for each trial. If  $X$  represents the number of successes in the  $n$  trials, then  $X$  has a **binomial distribution**, and we write  $X \sim B(n, p)$ .

The **binomial probability mass function** is  $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$  where  $x = 0, 1, 2, \dots, n$ .

You should be able to use your calculator to find:

- $P(X = x)$  using the binomial probability distribution function
- $P(X \leq x)$  or  $P(X \geq x)$  using the binomial cumulative distribution function.

If  $X \sim B(n, p)$ , then:

- $E(X) = \mu = np$
- $\text{Var}(X) = np(1 - p)$
- $\sigma = \sqrt{\text{Var}(X)} = \sqrt{np(1 - p)}$



## THE POISSON DISTRIBUTION

The **Poisson distribution** arises when considering the number of occurrences within a certain interval (of time or space).

Suppose there are an average of  $\lambda$  occurrences in a given interval. If  $X$  represents the number of occurrences in a particular interval, then  $X$  has a Poisson distribution, and we write  $X \sim \text{Po}(\lambda)$ .

The probability mass function of  $X$  is  $P(x) = \text{P}(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$  for  $x = 0, 1, 2, \dots$

$E(X) = \mu = \lambda$  and  $\text{Var}(X) = \sigma^2 = \lambda$ .

## THE NORMAL DISTRIBUTION

If the random variable  $X$  has a normal distribution with mean  $\mu$  and variance  $\sigma^2$ , we write  $X \sim \text{N}(\mu, \sigma^2)$ .

The probability density function is  $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$  for  $x \in \mathbb{R}$ .

$f(x)$  is a bell-shaped curve which is symmetric about  $x = \mu$ .

It has the property that:

- $\approx 68\%$  of all scores lie between  $\mu - \sigma$  and  $\mu + \sigma$
- $\approx 95\%$  of all scores lie between  $\mu - 2\sigma$  and  $\mu + 2\sigma$
- $\approx 99.7\%$  of all scores lie between  $\mu - 3\sigma$  and  $\mu + 3\sigma$ .

You should be able to use your calculator to find normal probabilities for the situations:

- $\text{P}(X \leq a)$
- $\text{P}(X \geq a)$
- $\text{P}(a \leq X \leq b)$

You should also be able to use your calculator to find the scores corresponding to particular probabilities. These scores are known as **quantiles**.

## ESTIMATION AND CONFIDENCE INTERVALS

Consider a population with distribution  $X$ .

A **random sample** of size  $n$   $\{X_1, X_2, \dots, X_n\}$  is a set of independent observations from the **same** population.

- Each  $X_i$  is a random variable that has the same distribution as the population.
- $X_1, X_2, \dots, X_n$  are **independent random variables**.

### Linear combinations of random variables

A **linear combination** of the random variables  $X_1, X_2, \dots, X_n$  has the form  $a_1 X_1 + a_2 X_2 + \dots + a_n X_n$  where  $a_1, a_2, \dots, a_n$  are constants.

For  $n$  random variables  $X_1, X_2, \dots, X_n$ , and constants  $a_1, a_2, \dots, a_n$ :

$$\mathbf{1} \quad E\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i E(X_i)$$

$$\mathbf{2} \quad \text{If } X_1, X_2, \dots, X_n \text{ are all independent of one another, then } \text{Var}\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i^2 \text{Var}(X_i).$$

If  $X \sim \text{Po}(\lambda_1)$  and  $Y \sim \text{Po}(\lambda_2)$  are independent Poisson random variables, then  $(X + Y) \sim \text{Po}(\lambda_1 + \lambda_2)$ .

Any linear combination of independent normally distributed random variables is itself a normally distributed random variable.

For a random sample  $\{X_1, X_2, \dots, X_n\}$ :

- the sample mean is  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$
- the sample variance is  $S_{n-1}^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ .



## Properties of $\bar{X}_n$ and $S_{n-1}^2$

For a population with mean  $\mu$  and standard deviation  $\sigma$ :

- $E(\bar{X}_n) = \mu$ , so  $\bar{X}_n$  is an unbiased estimator for  $\mu$ .
- $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$  and  $\sigma(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$
- If the population is normally distributed, then  $\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$  for all  $n$ .
- $E(S_{n-1}^2) = \sigma^2$ , so  $S_{n-1}^2$  is an unbiased estimator for  $\sigma^2$ .

## The Central Limit Theorem

For any population, even one not normally distributed,  $\bar{X}_n$  is approximately normally distributed for sufficiently large  $n$ , with  $\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ .

You should use the “rule of thumb” of  $n \geq 30$  being sufficiently large.

## Confidence intervals

A **confidence interval** for a population mean  $\mu$ , is an interval of values between two limits, together with a percentage indicating our confidence that  $\mu$  lies in that interval.

### Confidence intervals for a population mean with known variance

The general confidence interval for  $\mu$  given a known population variance  $\sigma^2$  and a data set with sample mean  $\bar{x}$  is

$$\bar{x} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

where  $P(Z \geq z_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$  and  $Z \sim N(0, 1^2)$ .

The interval contains  $\mu$  with probability  $1 - \alpha$ , so the confidence level of the interval is  $(1 - \alpha) \times 100\%$ .

### Confidence intervals for a population mean with unknown variance

The general confidence interval for  $\mu$  given a data set with sample mean  $\bar{x}$  and sample standard deviation  $s$  is

$$\bar{x} - t_{n-1, \frac{\alpha}{2}} \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{n-1, \frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

where  $P(T \geq t_{n-1, \frac{\alpha}{2}}) = \frac{\alpha}{2}$  and  $T \sim t_{n-1}$  is the  $t$ -distribution with  $n - 1$  degrees of freedom (df).

The interval contains  $\mu$  with probability  $1 - \alpha$ , so the confidence level of the interval is  $(1 - \alpha) \times 100\%$ .

## HYPOTHESIS TESTING

### Terminology

- A **statistical hypothesis** is a claim about a population parameter.
- The **null hypothesis**  $H_0$  is a claim that the population parameter is *equal* to a particular value.
- The **alternative hypothesis**  $H_1$  is a claim that the population parameter is *different* to the value specified by  $H_0$ . For example, given the null hypothesis  $H_0: \mu = \mu_0$ , the alternative hypothesis could be:
  - ▶  $H_1: \mu > \mu_0$  (**one-tailed hypothesis**)
  - ▶  $H_1: \mu < \mu_0$  (**one-tailed hypothesis**)
  - ▶  $H_1: \mu \neq \mu_0$  (**two-tailed hypothesis**, as  $\mu \neq \mu_0$  could mean  $\mu > \mu_0$  or  $\mu < \mu_0$ ).
- A **Type I error** is when we make the mistake of rejecting  $H_0$  when  $H_0$  is in fact true.
- A **Type II error** is when we make the mistake of accepting  $H_0$  when  $H_0$  is in fact false.
- A **test statistic** is a random variable that summarises the information in a sample.
- The distribution of the test statistic under the assumptions of  $H_0$  is called the **null distribution**.



- The **p-value** of a test statistic is the probability of a result that is as or more “extreme” being observed if  $H_0$  is true.
- The **significance level**  $\alpha$  of a statistical hypothesis test is the largest  $p$ -value that would result in rejecting  $H_0$ . Any  $p$ -value less than or equal to  $\alpha$  results in  $H_0$  being rejected.
  - ▶ If a statistical hypothesis test has significance level  $\alpha$ , the probability of a Type I error is  $\alpha$ .
  - ▶ The significance level may be given as a decimal or a percentage.
- The **critical region**  $\mathcal{C}$  is the set of all values of the test statistic which result in  $H_0$  being rejected.
- The **acceptance region**  $\mathcal{A}$  is the set of all values of the test statistic which result in  $H_0$  being accepted.
- We can make a decision about  $H_0$  using the test statistic directly by comparing it to a **critical value**  $c$  which is the value in the critical region which has the largest  $p$ -value associated with it.

## General procedure

*Step 1:* Formulate **statistical hypotheses**.

*Step 2:* Choose a **significance level** for the test. This is a threshold for making a decision, like the confidence levels we saw previously.

*Step 3:* Use data from a sample to calculate a **test statistic**.

*Step 4:* Calculate a **p-value** for the test statistic. This is the probability of that test statistic occurring under the assumptions of one of the hypotheses.

*Step 5:* Make decisions about the hypotheses.

*Step 6:* Interpret the decision in the context of the problem.

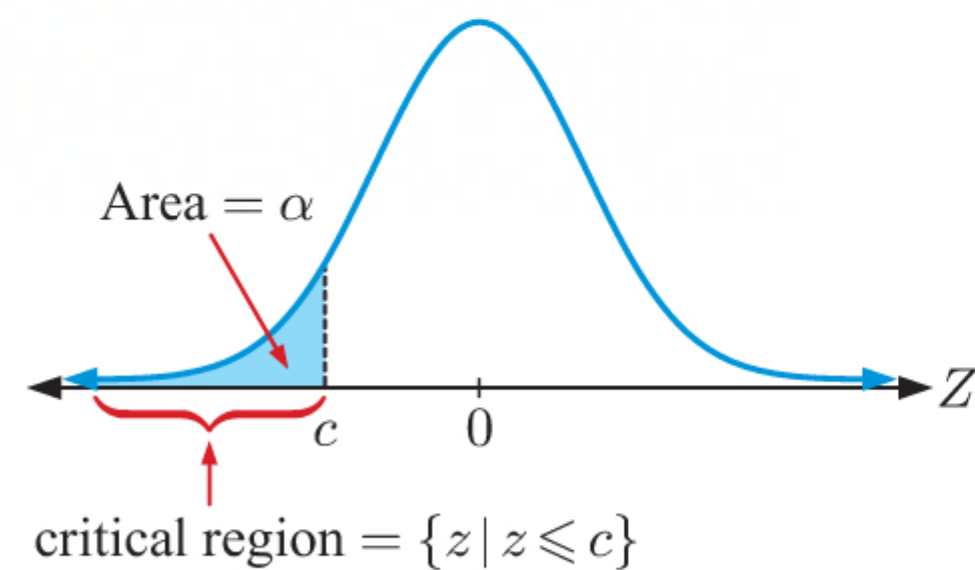
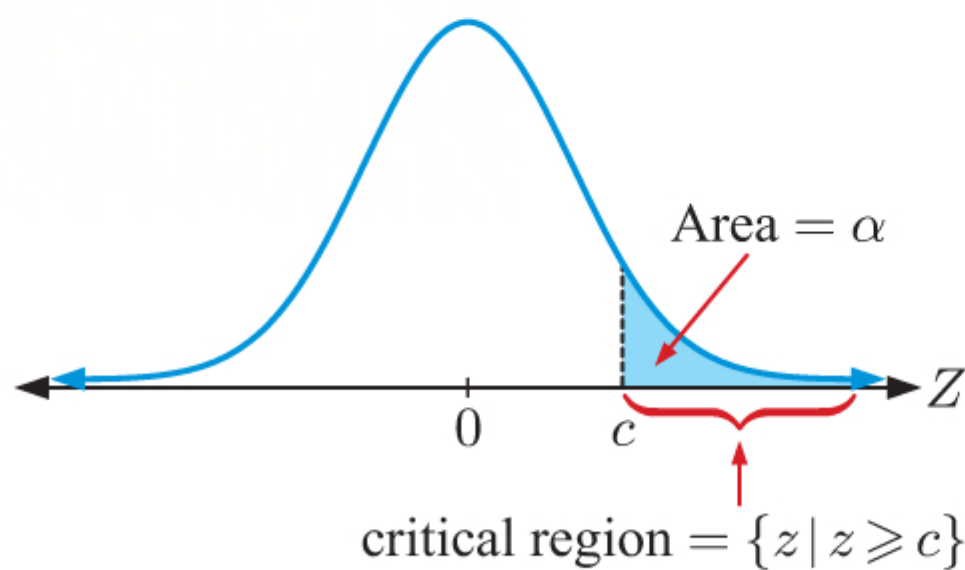
## The Z-test

The **Z-test** is used to test hypotheses about a population mean  $\mu$  when:

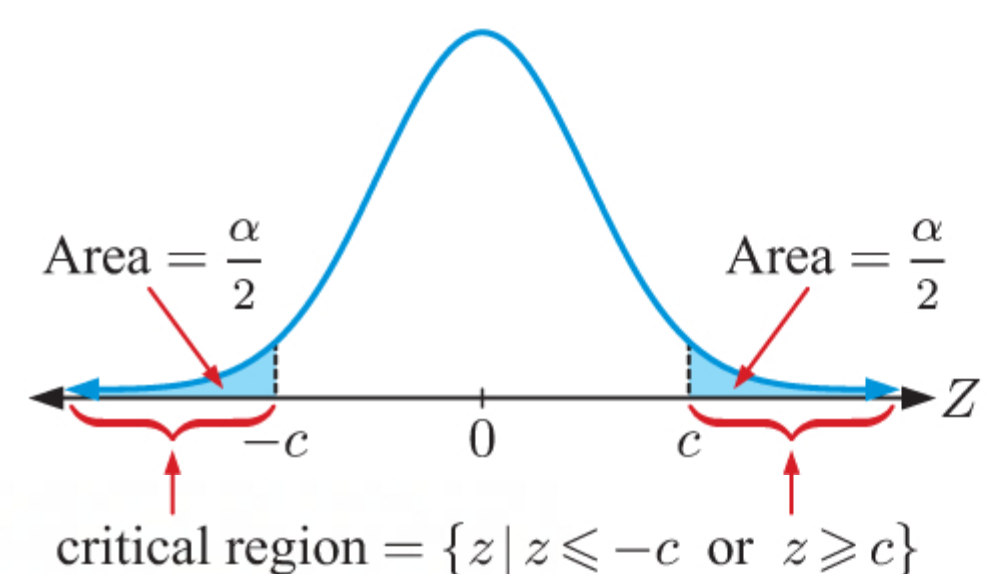
- the population is normally distributed
- the population variance  $\sigma^2$  is **known**.

For a Z-test of  $H_0: \mu = \mu_0$  (where the population standard deviation is  $\sigma$ ) using a sample of size  $n$  with sample mean  $\bar{x}$ :

- the **test statistic** is  $Z = \frac{\bar{X}_n - \mu_0}{\frac{\sigma}{\sqrt{n}}}$  which has **observed value**  $z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$
- the **null distribution** is  $Z \sim N(0, 1^2)$
- the **p-value** calculation depends on  $H_1$ :
  - ▶ If  $H_1: \mu > \mu_0$ ,  $p\text{-value} = P(Z \geq z)$ .
  - ▶ If  $H_1: \mu < \mu_0$ ,  $p\text{-value} = P(Z \leq z)$ .
  - ▶ If  $H_1: \mu \neq \mu_0$ ,  $p\text{-value} = 2 \times P(Z \geq |z|)$ .
- the **critical value(s)** and **critical region** depend on  $H_1$  and the significance level  $\alpha$ :
  - ▶ If  $H_1: \mu > \mu_0$ , the critical value  $c$  satisfies  $P(Z \geq c) = \alpha$ .
  - ▶ If  $H_1: \mu < \mu_0$ , the critical value  $c$  satisfies  $P(Z \leq c) = \alpha$ .



- ▶ If  $H_1: \mu \neq \mu_0$ , then the critical value  $c$  satisfies  $2 \times P(Z \geq |c|) = \alpha$   
 $\therefore P(Z \geq |c|) = \frac{\alpha}{2}$





## The one-sample $t$ -test

The  $t$ -test is used to test hypotheses about a population mean  $\mu$  when:

- the population is normally distributed
- the population variance is **unknown**.

For a  $t$ -test of  $H_0: \mu = \mu_0$  using a sample of size  $n$  with sample mean  $\bar{x}$  and sample standard deviation  $s$ :

- the **test statistic** is  $T = \frac{\bar{X}_n - \mu_0}{\frac{S_{n-1}}{\sqrt{n}}}$  which has **observed value**  $t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$
- the **null distribution** is  $T \sim t_{n-1}$
- the  $p$ -value calculation depends on  $H_1$ :
  - If  $H_1: \mu > \mu_0$ ,  $p\text{-value} = P(T \geq t)$ .
  - If  $H_1: \mu < \mu_0$ ,  $p\text{-value} = P(T \leq t)$ .
  - If  $H_1: \mu \neq \mu_0$ ,  $p\text{-value} = 2 \times P(T \geq |t|)$ .

## Paired $t$ -tests

A **paired  $t$ -test** is used to compare two sets of results for **one** sample. In other words, the data is **matched in pairs**.

To perform a paired  $t$ -test we:

- calculate the difference of each pair of data values  $d_i$
- perform a one-sample  $t$ -test for the mean of the differences  $\mu_D$  with null hypothesis  $H_0: \mu_0 = 0$ .

## The two-sample $t$ -test

The **two-sample  $t$ -test** is used to compare the means of **two** samples from different populations.

If the populations have means  $\mu_1$  and  $\mu_2$ , the null hypothesis has the form:

$$H_0: \mu_1 = \mu_2 \quad \text{or equivalently} \\ H_0: \mu_1 - \mu_2 = 0$$

You should be able to use technology to calculate the test statistic and  $p$ -value.

In this course you are expected to assume **equal variances** and hence use the **pooled two-sample  $t$ -test** on your calculator.

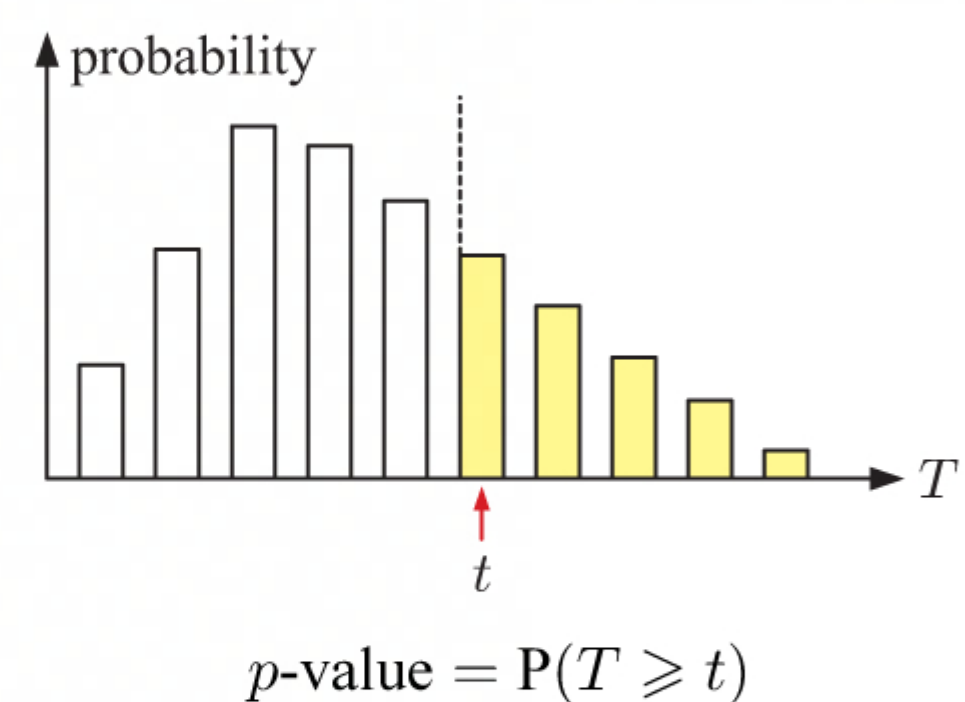
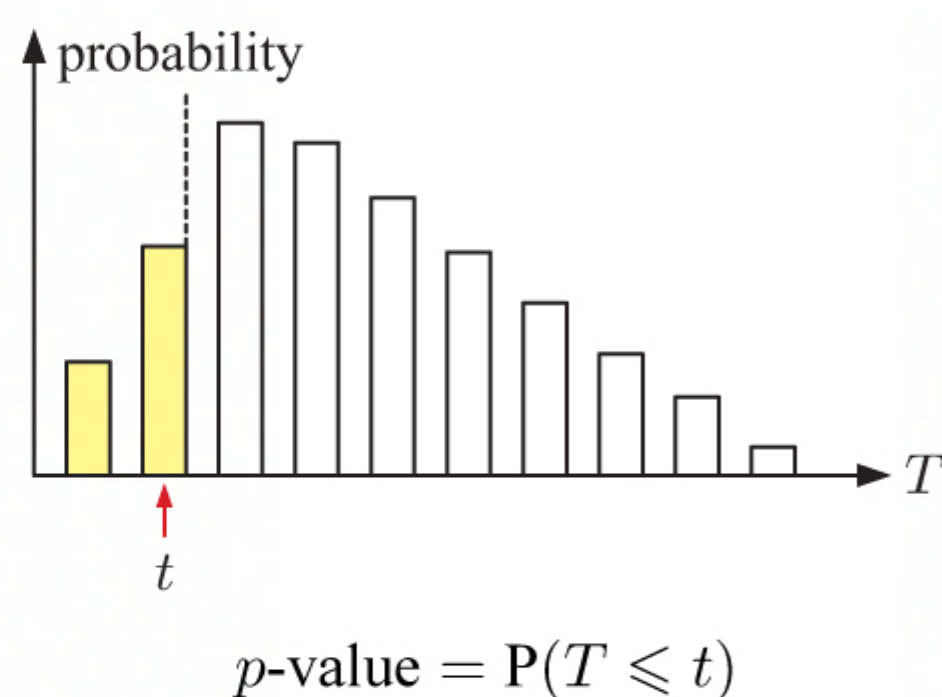
## Hypothesis tests for the mean of a Poisson population

Consider a statistical hypothesis test of  $H_0: \lambda = \lambda_0$  for a Poisson distributed population. Given a sample of size  $n$  with observed sample mean  $\bar{x}$ :

- the **test statistic** is  $T = n\bar{X}_n$  which has observed value  $t = n\bar{x}$
- the **null distribution** is  $T \sim \text{Po}(n\lambda_0)$ .

In this course we only consider  $p$ -value calculations for Poisson tests with a one-tailed alternative hypothesis  $H_1$ .

- If  $H_1: \lambda < \lambda_0$ , we use the **lower tail**.
- If  $H_1: \lambda > \lambda_0$ , we use the **upper tail**.



## Hypothesis tests for a population proportion

Consider a statistical hypothesis test of  $H_0: p = p_0$ , where  $p$  is the proportion of the population with a particular property. Given a sample of size  $n$ :

- the **test statistic** is  $X$ , the number of members in the sample with the property of interest
- the **null distribution** is  $X \sim B(n, p_0)$ .

Like the Poisson distribution, the binomial distribution is discrete. We calculate the  $p$ -value using the same principles.



Hypothesis tests for a population correlation coefficient

When we talk about the correlation between two variables in a population, we use the **population product-moment correlation coefficient**  $\rho$ .

If no relationship or association exists between two variables, then there is **no correlation** between them and  $\rho = 0$ . In a hypothesis test, this is equivalent to stating a null hypothesis  $H_0: \rho = 0$ .

If we are interested in detecting:

- **positive** correlation, we use  $H_1: \rho > 0$
- **negative** correlation, we use  $H_1: \rho < 0$
- **any** correlation, whether positive or negative, we use  $H_1: \rho \neq 0$ .

You should be able to use your calculator to conduct hypothesis tests about  $\rho$ .

Error probabilities and statistical power

significance level =  $\alpha = P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ true})$

$\beta = P(\text{Type II error}) = P(\text{Retain } H_0 \mid H_0 \text{ false})$

The **power** of a hypothesis test is defined as  $1 - \beta$ , the probability of (correctly) rejecting the null hypothesis  $H_0$  when  $H_0$  is false.

For a  $Z$ -test, a test for a Poisson population mean, and a test for a population proportion, you should be able to calculate:

- $\alpha$  given the critical region
- $\beta$  given the critical region and the value that the population parameter takes under the alternative hypothesis  $H_1$ .

The  $\chi^2$  goodness of fit test

The  $\chi^2$  goodness of fit test is used to determine whether a probability distribution fits a set of data.

Consider a scenario with  $k$  categories. Let  $p_i$  be the population proportion of individuals in category  $i$ , where  $p_1 + p_2 + \dots + p_k = 1$ .

The **hypotheses** in a  $\chi^2$  goodness of fit test have the form:

$$H_0: p_1 = p_{01}, p_2 = p_{02}, \dots, \text{ and } p_k = p_{0k}$$
$$H_1: \text{ at least one of } p_i \neq p_{0i}$$

where  $p_{0i}$  is the population proportion of category  $i$  under the null hypothesis.

You should also be able to calculate  $p_{01}, p_{02}, \dots, p_{0k}$  for special probability distributions including the binomial distribution, Poisson distribution, and normal distribution.

The **test statistic** for a  $\chi^2$  goodness of fit test is: 
$$\chi^2_{\text{calc}} = \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$$

where  $f_{\text{obs}}$  is an **observed** frequency  
 $f_{\text{exp}}$  is an **expected** frequency.

You should combine similar categories to ensure that no expected frequencies are less than 5.

**Degrees of freedom (df)** refers to the number of values that are “free to vary”.

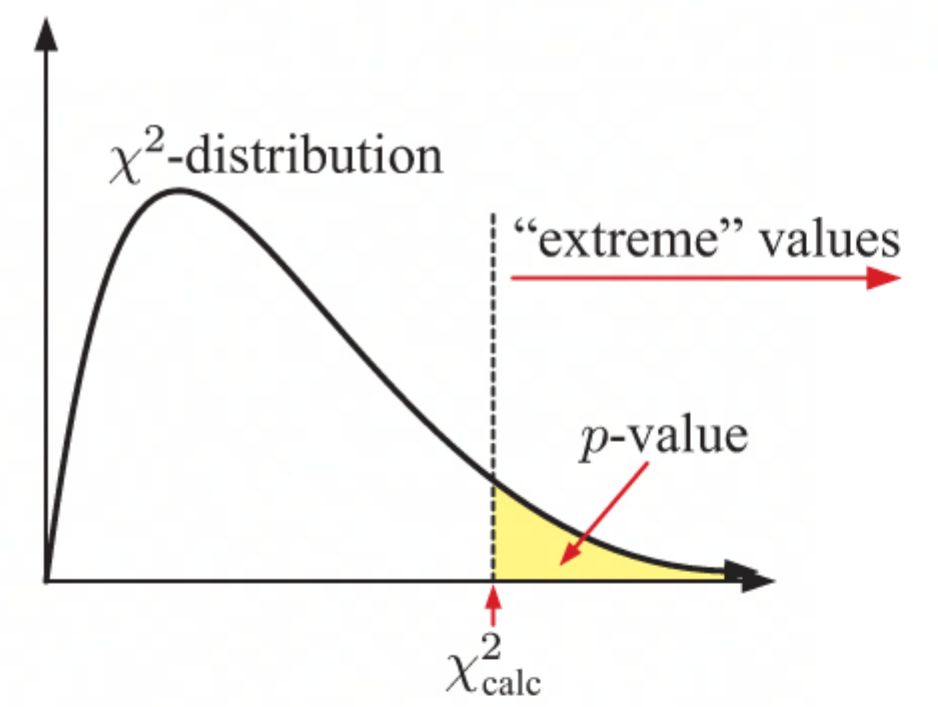
For a  $\chi^2$  goodness of fit test, **df = number of categories – number of estimated parameters – 1**.

This table gives the degrees of freedom of the goodness of fit test for various distributions when the data is sorted into  $k$  categories:

Distribution	Estimated parameters	df
Binomial $B(n, p)$	$p \approx \frac{\bar{x}}{n}$	$k - 2$
Poisson $Po(\lambda)$	$\lambda \approx \bar{x}$	$k - 2$
Normal $N(\mu, \sigma^2)$	$\sigma^2 \approx s^2$	$k - 2$
	$\mu \approx \bar{x}, \sigma^2 \approx s^2$	$k - 3$



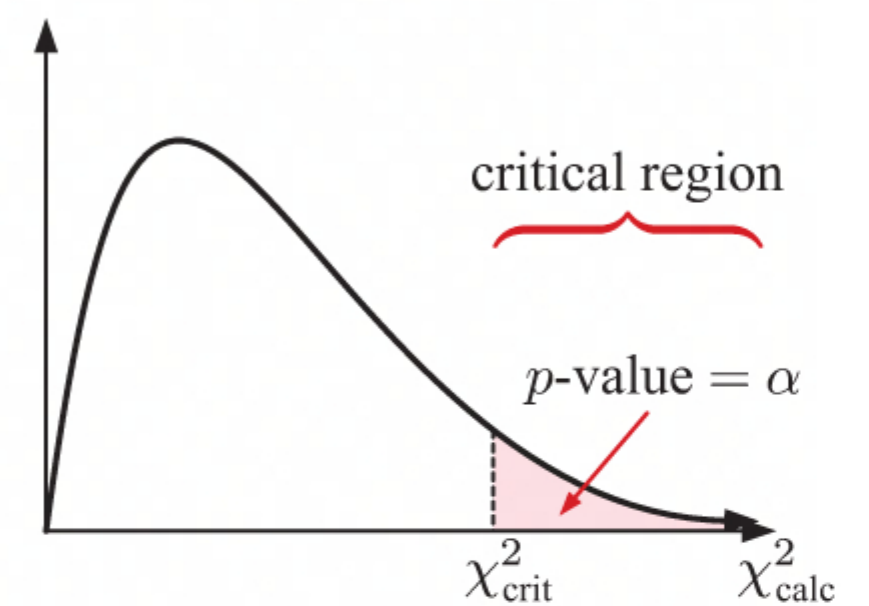
**p-value** = probability of observing a value greater than or equal to  $\chi^2_{\text{calc}}$ .



For a  $\chi^2$  goodness of fit test, we denote the critical value as  $\chi^2_{\text{crit}}$ .

Since we use the upper tail of the null distribution in calculating the  $p$ -value, the critical region is the set of values  $\geq \chi^2_{\text{crit}}$ .

The inequality  $\chi^2_{\text{calc}} \geq \chi^2_{\text{crit}}$  is called the **rejection inequality**.



## The $\chi^2$ test for independence

The  $\chi^2$  test for independence is used to determine if two variables in a **contingency table** are independent or not. It is a special case of the  $\chi^2$  goodness of fit test.

The hypotheses for the  $\chi^2$  test for independence are  $H_0$ : the variables are independent

$H_1$ : the variables are dependent

The test statistic for the  $\chi^2$  test for independence is calculated in a similar way to the  $\chi^2$  goodness of fit test. The expected frequency of each cell in the contingency table is given by  $f_{\text{exp}} = \frac{\text{row sum} \times \text{column sum}}{\text{total}}$ .

The  $p$ -value and critical value  $\chi^2_{\text{crit}}$  for the  $\chi^2$  test for independence are calculated in the same way as the  $\chi^2$  goodness of fit test.

For a contingency table which has  $r$  rows and  $c$  columns,  $df = (r - 1)(c - 1)$ .



SKILL BUILDER QUESTIONS

1 Gerard wants to estimate the average height of the 500 students at his school. He randomly selects a sample of 10 students, and uses a tape measure to find the height of each student.

Explain why this approach may produce a:      **a** coverage error      **b** measurement error.

2 The students at Hoylebury Middle School are to be surveyed on their attitudes on wearing school uniform. The numbers of students in each year level are shown.

	Boys	Girls
Year 8	135	140
Year 9	130	145
Year 10	125	130

- a**    **i** What are the advantages of surveying 50 students?
- ii** What are the disadvantages of surveying all students?
- b** A stratified sample system is used to select 50 students.
  - i** How many Year 8 boys will be selected?
  - ii** How many girls will be selected in total?
- c** Explain why a stratified sample is better than a random sample in this case.

3 Marie is organising a staff lunch in a large office building.

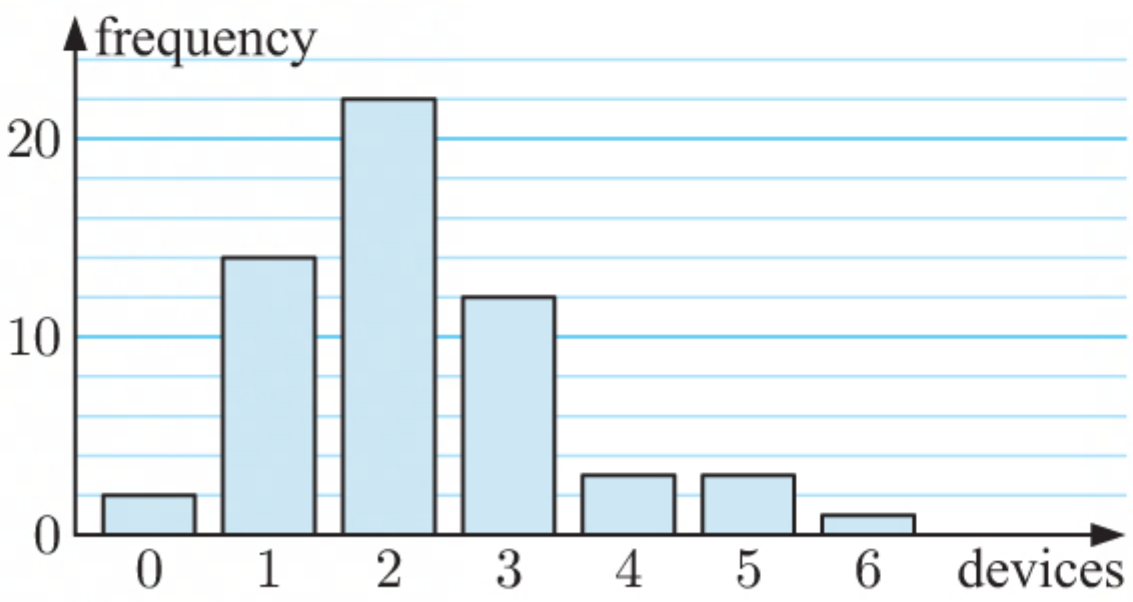
She asks the first 10 people to visit her office for their preferences, and then makes a decision.

- a** Explain why this is a convenience sample.
  - b** In what ways will Marie’s sample be biased?
  - c** Suggest a more appropriate sampling method that Marie should use.
- 4 A ticket inspector checks the tickets of every 20th passenger leaving a train terminal, starting from the 8th passenger.
- a** Identify the sampling method used. Explain your answer.
  - b** List the next six passengers to be checked.
  - c** Given that 5000 passengers left the terminal that day, find the number of passengers checked.
- 5 Consider the question “What is your job?”.
- a** List ways in which the question can be interpreted. Include any possible misinterpretations.
  - b** Rewrite the question so that it is more specific.
- 6 Consider the question “What is your annual income?”.
- a** Explain why this question is likely to produce:
    - i** a measurement error
    - ii** a non-response error.
  - b** List ways in which the question could be improved.

7 Classify each variable as categorical, discrete, or continuous:

- a** The number of houses on a particular street.
- b** The number of hours spent travelling on an airplane.
- c** The brand of laptop someone uses.

8 A random sample of people were asked “How many devices have you used to browse the internet in the last month?”. The results are displayed in the column graph.



- a** How many people were surveyed?
- b** Find the mode of the data.
- c** What percentage of people browsed the internet using 1 or 2 devices?
- d** Describe the distribution of the data.

9 Each student in a class writes down the total number of children in their families:

1   2   4   3   2   1   1   2   1   5   4   2  
2   1   3   2   3   4   1   2   1   2   2   1

- a** Explain why the data is discrete.
- b** Construct a frequency table to organise the data.
- c** Draw a column graph to display the data.
- d** Describe the distribution of the data.
- e** In what percentage of families are there 3 or more children?



- 10** The heights of a sample of emperor penguins were measured. The results are given in the table alongside.
- a** Explain why *height* is a continuous variable.
  - b** How many emperor penguins were measured?
  - c** Construct a frequency histogram to display the data.
  - d** Describe the distribution of the data.
  - e** What is the modal class? Explain what this means.

Height ( $h$ cm)	Frequency
$105 \leq h < 110$	3
$110 \leq h < 115$	5
$115 \leq h < 120$	14
$120 \leq h < 125$	19
$125 \leq h < 130$	8
$130 \leq h < 135$	1

- 11** The number of customers entering a convenience store each hour on a particular day were:
- 14, 23, 26, 34, 24, 18, 26, 16, 25

Without using technology, find the **a** mode **b** median **c** mean of the data.

- 12** An art gallery has added two new exhibits alongside their permanent collection. The number of tickets sold for each exhibit was counted every day for a month:

Exhibit A										Exhibit B									
42	49	55	48	62	81	91	50	60	59	59	51	60	44	57	90	98	50	62	55
47	73	84	89	55	59	35	42	51	83	44	62	75	99	57	57	49	53	71	70
75	28	30	19	39	45	69	65	27	32	68	32	33	24	47	43	61	42	52	46

- a** Find the:
    - i** mean number of visitors for each exhibit
    - ii** median number of visitors for each exhibit.
  - b** Which exhibit was more popular? Explain your answer.
- 13** **a** The mean of 7 integers is 14. In ascending order, the integers are 9, 10,  $a$ , 13,  $b$ , 16, 21.  
Find the values of  $a$  and  $b$ .
- b** In ascending order, a set of six numbers are: 1, 5, 9, 11, 16,  $p$ . The mean of the six numbers is the same as their median. Find  $p$ .
- 14** Miguel uses an application on his phone to find the amount of sleep he gets each night. The duration of his sleep, in hours, for the past 30 nights are:

7.5   6.8   7.8   6.3   8.6   9.1   7.1   5.8   7.7   7.3   7.7   7.4   11.5   7.1   7.4  
8.0   7.6   7.1   9.1   8.0   7.5   7.4   7.5   8.1   8.6   8.7   6.8   7.4   7.7   8.5

- a** Calculate the mean and the median of the data.
- b** Identify the outlier in this data set.
- c** The outlier was the result of a recording error.
  - i** Calculate the mean and the median of the data with the outlier removed.
  - ii** Which measure of centre is most affected if the outlier is removed?

- 15** The frequency table alongside shows the number of touchdowns scored by teams in a professional gridiron league after one round.

- a** For this data set, find the:
  - i** mean
  - ii** median
  - iii** mode.
- b** Construct a column graph for the data.
- c** Describe the distribution of the data.
- d** Which measure of centre is most appropriate for this data?

Number of touchdowns	Frequency
0	2
1	10
2	7
3	6
4	4
5	2
6	1



- 16 The frequency table alongside shows the number of cars owned by different families.
- a Add a column to the table showing the *cumulative frequency* values.
  - b For this data set, calculate the:
    - i mean
    - ii median
    - iii mode.

Number of cars	Frequency
0	78
1	117
2	69
3	18
4	2
Total	284

- 17 After seven netball matches, Kai has averaged 11 goals per game.
- a Find the value of  $a$ .
  - b How many goals will she need to score in the next game to improve her overall average to 12?

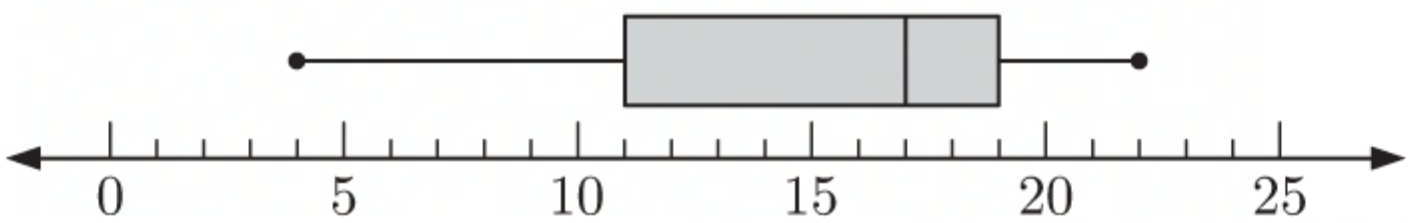
Score	7	9	$a$	13	16
Frequency	1	2	1	2	1

- 18 This table shows the weekly rent for a sample of studio apartments in Italy.
- a Estimate the mean weekly rent.
  - b Find the probability that the weekly rent for a randomly chosen studio apartment will be €140 or greater.

Weekly rent (€ $r$ )	Frequency
$80 \leq r < 100$	3
$100 \leq r < 120$	15
$120 \leq r < 140$	26
$140 \leq r < 160$	30
$160 \leq r < 180$	14
$180 \leq r < 200$	1

- 19 Cailan and Miles regularly play golf together, and have recorded their scores from their last 10 rounds:
- Cailan:* 84, 81, 86, 92, 85, 83, 80, 87, 90, 79  
*Miles:* 87, 85, 83, 90, 88, 82, 84, 84, 91, 82
- a Calculate the range and interquartile range for each data set.
  - b Which golfer had the lower:
    - i range
    - ii interquartile range?
  - c Which measure of spread is more appropriate for determining who is generally the more consistent golfer? Explain your answer.
- 20 Consider this data set: 16, 20, 10, 16, 4, 12, 23, 18, 17, 9, 18, 16, 31, 26, 18, 14, 12, 14, 15
- a Write the data set in order, and construct a five-number summary.
  - b Calculate the interquartile range.
  - c Calculate the upper and lower boundaries, and hence identify any outliers in the data set.
  - d Draw a box plot to represent the data.

- 21 A box plot has been drawn to show the heights of some petunia seedlings, in centimetres.



- State the:
- a minimum value
  - b maximum value
  - c median
  - d upper quartile
  - e lower quartile
  - f range
  - g interquartile range.

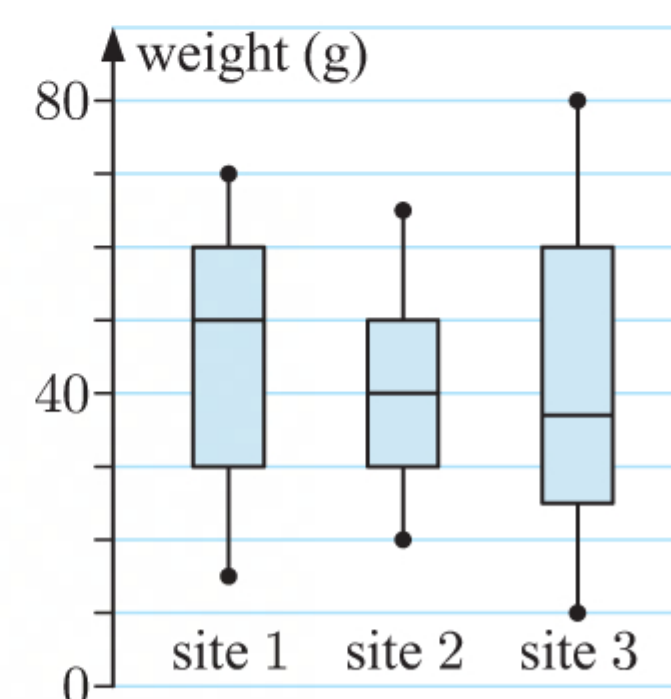
- 22 After a 10 minute run, a class of 24 students measured their pulse rates. The results were:

85 106 148 112 105 96 100 108 135 126 144 156  
98 108 112 128 148 140 120 123 133 144 118 125

- a Construct a five-number summary for the data.
- b Draw a box plot to represent the data.
- c Find the range and interquartile range.



- 23** These parallel box plots show the weights of particular species of fungi collected from 3 different sites in a forest.



- Write down the five-number summary for site 1.
- Which site has the greatest range of weights?
- At which site do the weights of fungi have the least variation?
- Which site has the highest median weight of fungi?
- Which site has the highest proportion of weights above 40 grams?

- 24** A soft drink distributor is testing a new recipe for one of their best selling drinks. A randomly selected group of people are asked to taste the old and new recipes, and to give each a score out of 10. The results are given below:

*Old recipe:* 7 8 7.5 9 7 6 7 8 9 7 8 8  
*New recipe:* 6 8 7 9 7.5 4 6.5 7 8 5 7.5 8.5

- Find the five-number summary for each data set.
- Draw a parallel box plot for the data.
- Do you think the distributor should adopt this new recipe for their drink? Explain your answer.

- 25** The heights of a random sample of trees in an apple orchard are summarised in the table alongside.

Height ( $h$ m)	Frequency
$7 \leq h < 8$	8
$8 \leq h < 9$	59
$9 \leq h < 10$	74
$10 \leq h < 11$	22
$11 \leq h < 12$	1

- Construct a cumulative frequency graph for the data.
- Estimate the median height.
- Estimate the interquartile range.
- Estimate the 90th percentile. Interpret your answer.

- 26** Anthony and Katherine are two musicians in an orchestra. They each recorded the number of hours they spent practising in the 10 days before a performance.

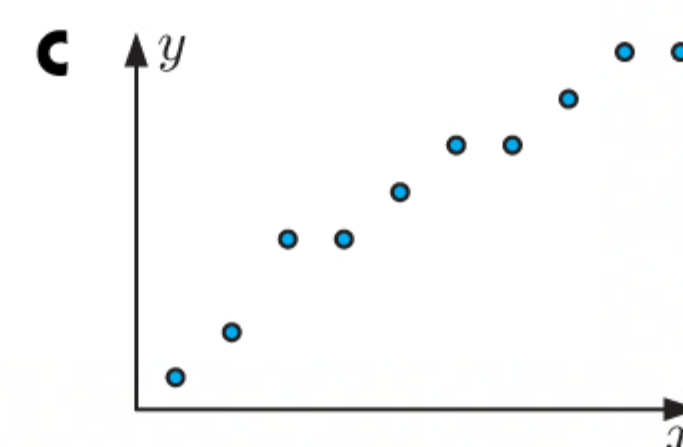
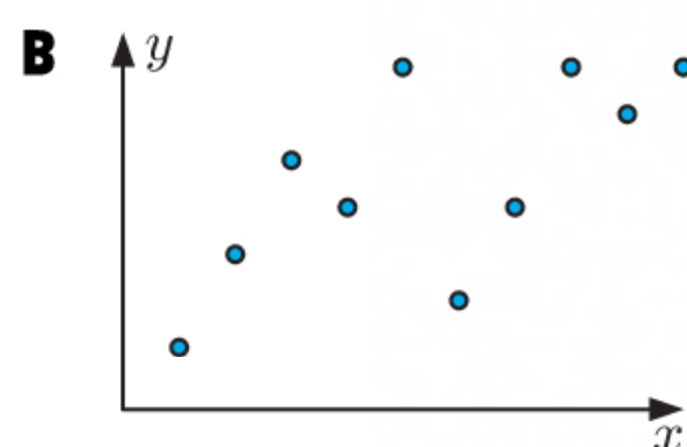
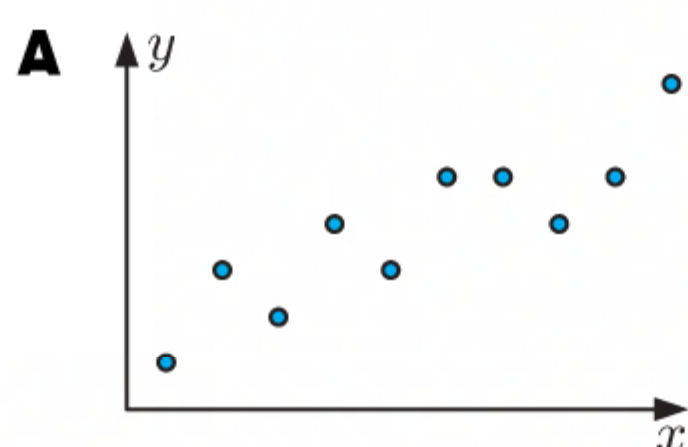
*Anthony:*  $1\frac{1}{2}$ , 2,  $2\frac{1}{2}$ , 4,  $4\frac{1}{2}$ , 3,  $3\frac{1}{2}$ , 5, 6, 6  
*Katherine:*  $3\frac{1}{2}$ , 4, 3, 3,  $3\frac{1}{2}$ , 4, 4,  $4\frac{1}{2}$ , 4

- Calculate the mean and standard deviation of each data set.
- Which person generally practised for longer?
- Which person practised more consistently?

- 27** This table shows the distribution of marks obtained on a logic test.  
 Use technology to find the mean and population standard deviation of the test scores.

Mark	3	4	5	6	7	8	9	10
Frequency	1	3	5	8	4	2	0	1

- 28** Consider the scatter diagrams below.



- For each scatter diagram, determine whether the association between  $x$  and  $y$  is positive, negative, or zero.
- Complete the table by matching each description with scatter diagram **A**, **B**, or **C**.

Strength of correlation	Scatter diagram
Weak	
Moderate	
Strong	



- 29** A journalist compares the scores given to two camera models by 6 online reviewers.

<i>Camera A</i>	8.5	8	9	7	8.5	7.5
<i>Camera B</i>	7	6	7.5	9	7.5	6

- Draw a scatter diagram of the data.
  - Identify the outlier in the data.
  - It was found that the outlier was a recording error, and was removed.
    - Describe the correlation between camera A's scores and camera B's scores.
    - Does an increase in camera A's scores cause an increase in camera B's scores? Explain your answer.
- 30** Ten students were given aptitude tests on language skills and mathematics. The table below shows the results:

<i>Language (x)</i>	12.5	15.0	10.5	12.0	9.5	10.5	15.5	10.0	14.0	12.0
<i>Mathematics (y)</i>	32	45	27	38	18	25	35	22	40	40

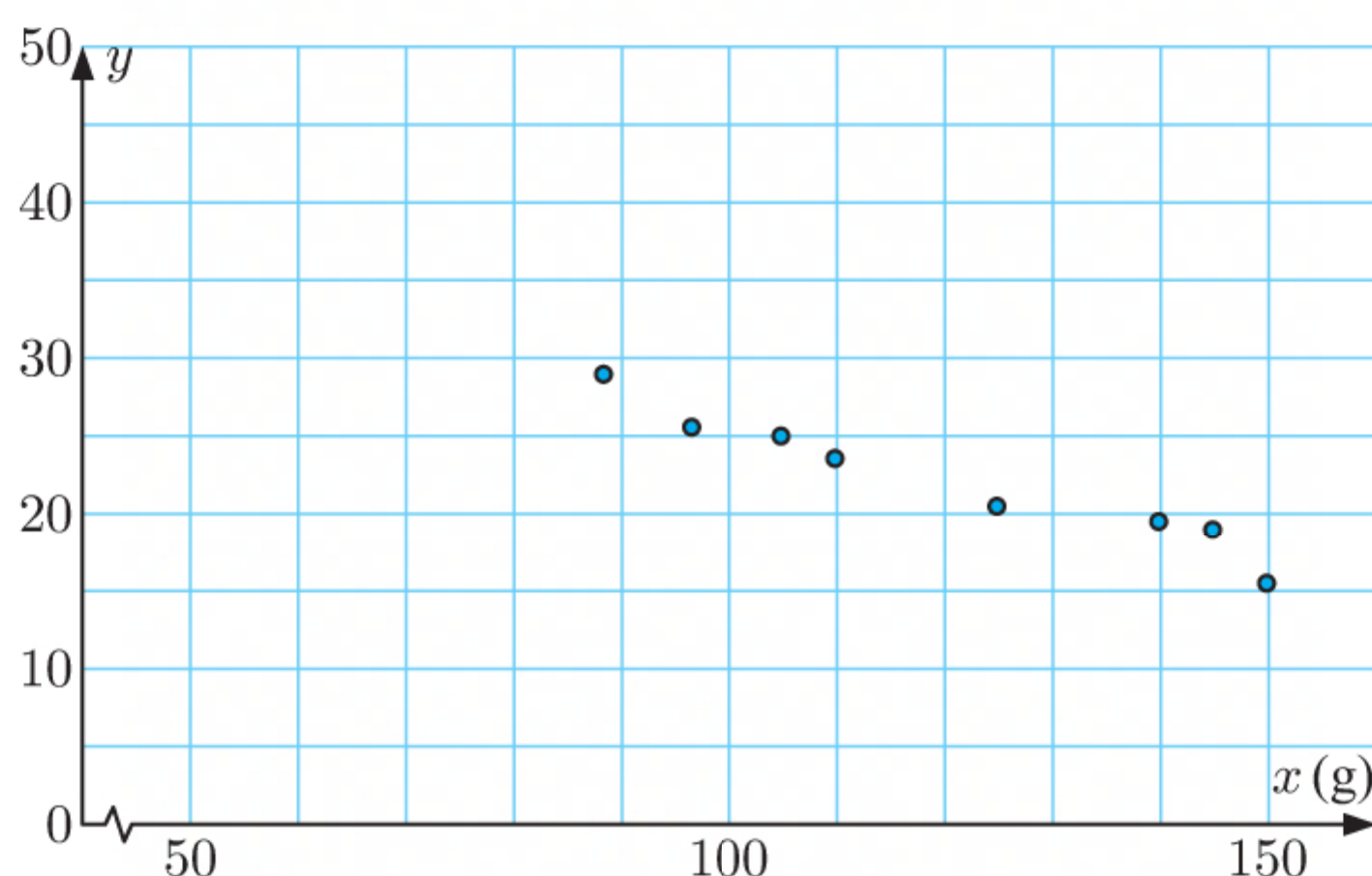
- Plot the data on a scatter diagram.
  - Find the correlation coefficient  $r$ .
  - Use your results to comment on the statement: "Those who do well in languages also do well in mathematics."
- 31** Consider the following data on farm production.

<i>Monthly rainfall (mm)</i>	5	10	15	20	25	30
<i>Crop yield (tonnes)</i>	14	21	29	31	30	28

- Calculate Pearson's correlation coefficient  $r$ .
  - What does the value of  $r$  suggest about the nature and strength of the relationship between monthly rainfall and crop yield?
  - Draw the scatter diagram for the data.
  - Explain why Pearson's correlation coefficient may not be an appropriate measure for the data.
- 32** For a group of cyclists, the association between *distance travelled* and *average speed* has correlation coefficient  $r \approx -0.6321$ . Find the coefficient of determination, and interpret its meaning.
- 33** In a sample of 2.5 kg bags of potatoes, the number of potatoes and their median weight is given below.

<i>Median weight (x g)</i>	88	97	105	110	125	140	145	150
<i>Number in bag (y)</i>	28	26	26	23	21	19	18	16

$$\bar{x} = 120 \text{ and } \bar{y} = 22.125.$$

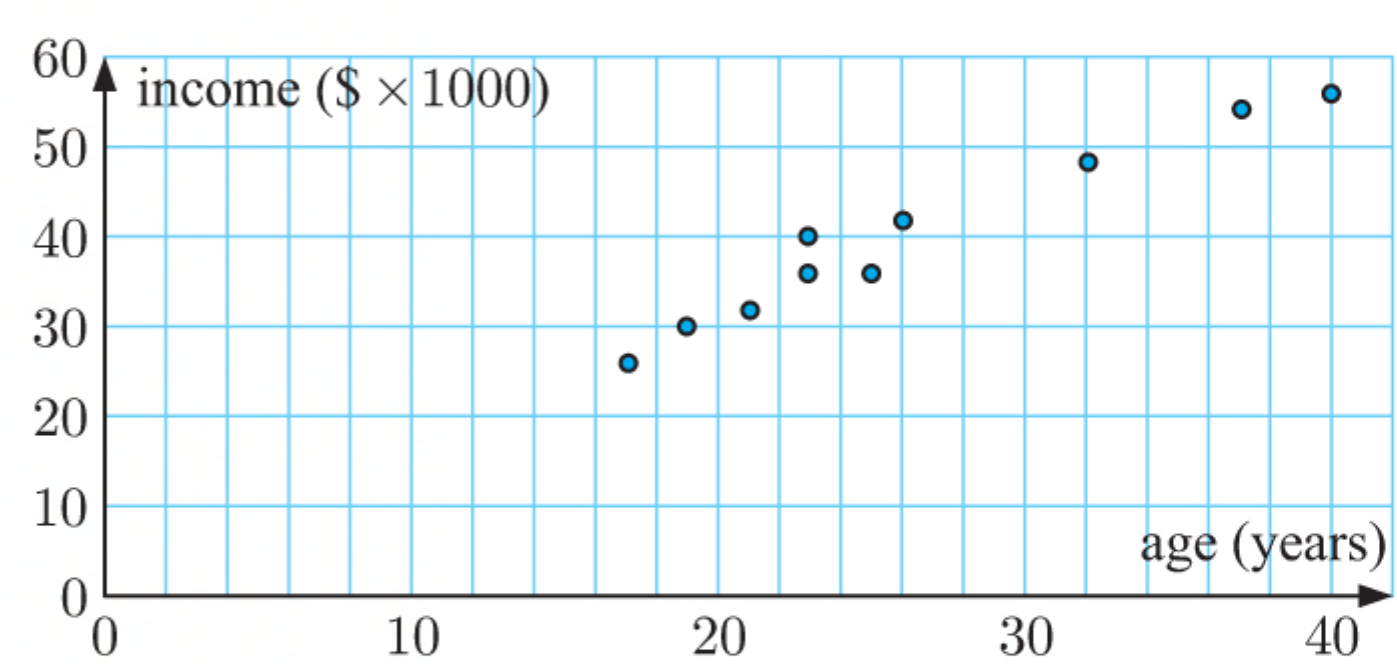


- Copy the scatter diagram, and draw a line of best fit by eye.
- Hence estimate the number of potatoes in a bag if the median weight is:
  - 100 grams
  - 70 grams.
- Which of the estimates in **b** is likely to be more reliable? Give a reason for your answer.



**34** This scatter diagram shows the age and annual income of 10 randomly chosen individuals. The mean age is 27 and the mean income is \$40 000.

- a** Describe the relationship between the age and annual income for these individuals.
- b** Do you think there is a causal relationship between the variables?  
Explain your answer.
- c** Draw a line of best fit by eye on the graph.
- d** Estimate the annual income for someone who is 30 years old. Comment on the reliability of your estimate.



**35** A jeweller measured the volume and mass of some samples of silver. He suspects that one of the samples might be fake. The results are listed in the table.

Sample	A	B	C	D	E	F	G	H	I	J	K	L
Volume ( $x \text{ cm}^3$ )	3	6	4	7	16	8	5	12	9	6	10	11
Mass ( $y \text{ g}$ )	40	95	50	160	285	130	65	210	155	90	170	190

- a** Draw a scatter diagram for this data.
  - b** Calculate Pearson’s product-moment correlation coefficient  $r$ .
  - c** Describe the relationship which appears to exist between the volume and mass of the samples of silver.
  - d** Do you agree with the jeweller that there is a fake sample?
  - e**
    - i** Remove the suspect value from the data and find the equation of the regression line for the remaining data.
    - ii** Use your equation to find the expected mass of the sample of silver with the same volume as the suspect sample.
- 36** 9 students sat a Mathematics examination. The number of hours that each of them studied and the results they obtained are shown in the table.

Study time ( $x \text{ h}$ )	7	6	3	16	15	11	18	32	20
Result ( $y \%$ )	56	42	25	80	65	60	85	96	90

- a** Write down the equation of the least squares regression line.
- b** Calculate the correlation coefficient  $r$ , and the coefficient of determination  $r^2$ .
- c** Describe the correlation between the variables.
- d** Do you think there is a causal relationship between the variables? Explain your answer.
- e** Tony’s score in the examination was 70%. Use the line of best fit to estimate how long he studied for.
- f** Interpret the  $y$ -intercept and the gradient of the equation of the line of best fit.

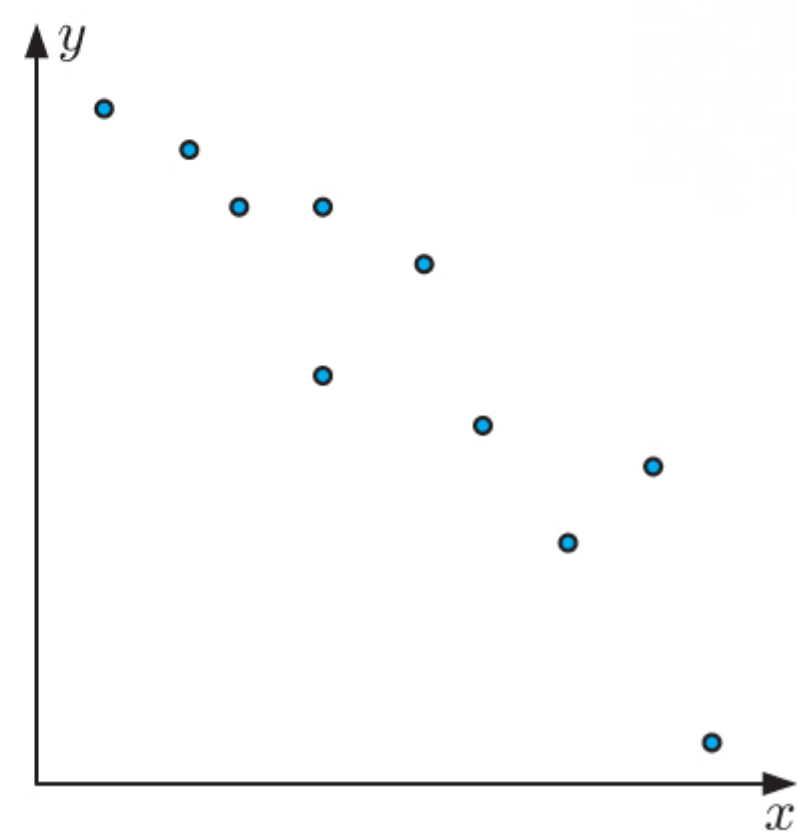
**37** The average height  $h$  (in mm) of grass  $t$  days after being mowed, is shown in the table below.

Time ( $t \text{ days}$ )	0	1	2	3	4	5	6	7	8	9
Height ( $h \text{ mm}$ )	5	5.7	5.7	6.2	6.8	7.1	8	8.3	9	9.3

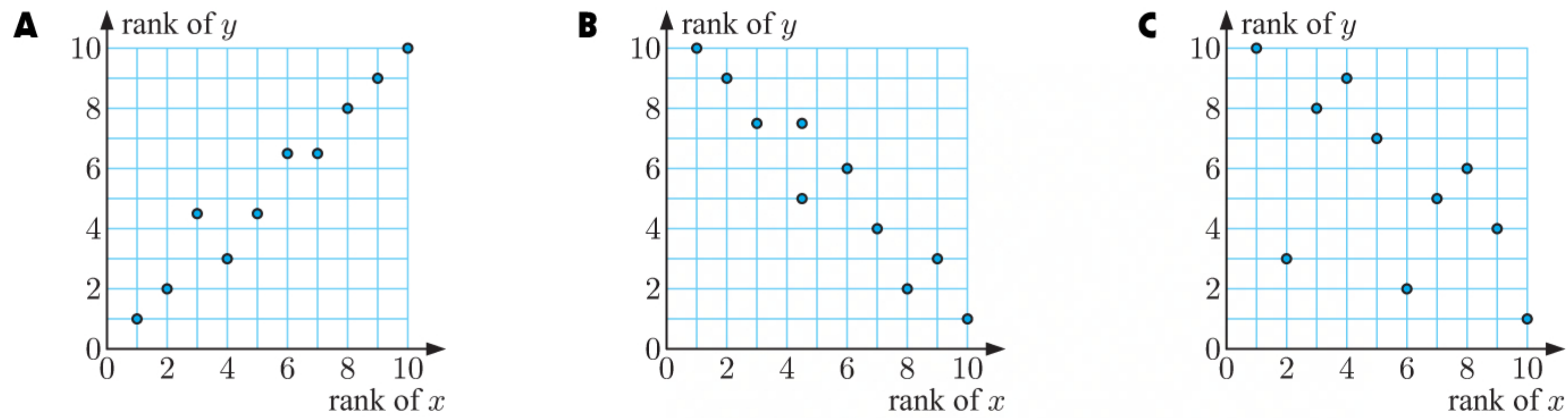
- a** Calculate Pearson’s product-moment correlation coefficient  $r$ .
- b** Explain the significance of the size and sign of  $r$ .
- c** The regression line for  $h$  against  $t$  is  $h \approx 0.4879t + 4.9145$ . Use this equation to estimate the:
  - i** height of the grass after 14 days
  - ii** time required for the grass height to reach 20 mm.



38 Consider the scatter diagram alongside.



a Which of the following scatter diagrams is the scatter diagram of the ranks?



b Hence identify the correct value of Spearman's rank correlation coefficient for this data:

- A**  $r_s \approx -0.612$       **B**  $r_s \approx 0.976$       **C**  $r_s \approx -0.960$

39 This table shows the *number of matches played* and the *highest score* of a cricketer in her last 12 seasons.

<i>Number of matches played</i> ( $x$ )	11	5	10	16	2	1	8	20	15	2	4	4
<i>Highest score</i> ( $y$ )	92	65	71	82	21	7	55	85	79	18	60	51

- a Draw a scatter diagram for the data.      b Calculate Pearson's coefficient  $r_p$ .  
c Find the ranks for each of the variables.      d Calculate Spearman's rank correlation coefficient  $r_s$ .  
e Describe the correlation between the variables.

40 The population of a city since 2000 is given alongside.

$T$ (years since 2000)	0	5	10	15
$P$ (population in millions)	28.5	31.8	36.3	41.2

- a Draw a scatter diagram of  $P$  against  $T$ .  
b i Find an exponential model connecting  $T$  and  $P$ .  
ii Fit a quadratic model of the form  $P = aT^2 + bT + c$  to the data.  
c i Calculate  $SS_{\text{res}}$  for each model in b.      ii Which model in b is a better fit for the data?

41 The data in the table below shows the mean daily petrol price in a city for the past 28 days.

<i>Day</i> ( $t$ )	1	2	3	4	5	6	7	8	9	10	11	12	13	14
<i>Price</i> ( $P$ cents per L)	135	135	138	144	147	149	151	150	149	143	140	135	134	130

<i>Day</i> ( $t$ )	15	16	17	18	19	20	21	22	23	24	25	26	27	28
<i>Price</i> ( $P$ cents per L)	129	128	131	135	136	143	144	148	148	150	151	145	142	137

We want to model the data with a trigonometric function of the form  $P = a \sin(b(t - c)) + d$ .

- a Draw a scatter diagram of the data.  
b Without using technology, estimate:  
i  $b$       ii  $a$       iii  $d$       iv  $c$   
c Use technology to find a sine model that best fits the data.  
d Calculate  $SS_{\text{res}}$  for the models in b and c. Comment on your answer.



- 42** The  $VO_2 \text{ max}$  value of a person is a measure of the maximum amount of oxygen the person can utilise during exercise. A researcher wants to test the validity of using an athlete's  $VO_2 \text{ max}$  to predict their performance in a long distance run. She selected 10 athletes, and recorded their  $VO_2 \text{ max}$  value and their time taken to run 10 km.

$VO_2 \text{ max}$ (mL/kg/min)	52.1	63.2	47.7	58.8	59.1	50.8	62.9	55.6	66.2	54.1
Time taken (minutes)	62.8	37.3	65.2	50.7	43.5	58.1	39.2	50.4	34.1	61.1

- a** Calculate the Pearson's product-moment correlation coefficient between the variables  $VO_2 \text{ max}$  and *time taken*.  
**b** Explain why criterion validity is being considered here, and identify the criterion variable.  
**c** Comment on the criterion validity of the predictor variable.  
**d** Does a high  $VO_2 \text{ max}$  value indicate that the athlete will perform better or worse in the 10 km run? Explain your answer.
- 43** For each of the following scenarios determine the type of reliability (test-retest or parallel forms) being considered.  
**a** Xavier records the time it takes him to brush his teeth every morning.  
**b** Abigail randomly chooses a set of 20 times tables for her students every Friday.  
**c** Rufus records the number of aces scored by a tennis player each match for one season.

- 44** Arthur records his time spent travelling to work over several months. Find, to 3 decimal places, the experimental probability that his next trip to work will last:

Time (min)	Frequency
35 - 39	10
40 - 44	46
45 - 49	43
50+	15

- a** 40 to 44 minutes  
**b** at least 50 minutes  
**c** between 35 and 49 minutes (inclusive).

- 45** An annual squash tournament groups players into 5 divisions according to their skill level.

The table shows the number of players at the tournament over 3 years.

Find the probability that a player:

- a** in the 2017 tournament played in division 1  
**b** in any of the past tournaments played in division 3  
**c** in the 2019 tournament did *not* play in division 2 or 4.

Division	2017	2018	2019
1	4	5	5
2	6	7	8
3	13	12	14
4	18	10	14
5	20	17	16
Total	61	51	57

- 46** A hospital recorded the age and gender of its 1020 melanoma patients over one year. The data is shown alongside.

- a** Complete the table.  
**b** Find the probability that a randomly selected melanoma patient was:  
**i** male **ii** female and younger than 40  
**iii** 60 or older, given they were female  
**iv** male, given they were 40 or older.

	< 40	40 - 59	$\geq 60$	Total
Male	56	127		
Female	75	113	230	
Total				1020

- 47** A die is rolled, and a square spinner with sectors 1, 2, 3, and 4 is spun.

- a** Draw a grid to illustrate the sample space of possible outcomes.  
**b** Use your grid to find the probability of getting:  
**i** two 1s **ii** two 5s **iii** a sum of 6  
**iv** a 2 and a 3 **v** a 2 or a 3 (or both) **vi** exactly one 4.

- 48** Suppose  $P(A) = 0.37$ ,  $P(B) = 0.41$ , and  $P(A \cup B) = 0.78$ .

- a** Find  $P(A \cap B)$ . **b** What can you say about  $A$  and  $B$ ?

- 49**  $A$  and  $B$  are mutually exclusive events. If  $P(B) = 0.3$  and  $P(A \cup B) = 0.55$ , find  $P(A)$ .

- 50** Given that  $P(A) = \frac{23}{50}$ ,  $P(B) = \frac{5}{7}$ , and  $P((A \cup B)') = \frac{1}{12}$ , find  $P(A \cap B)$ .



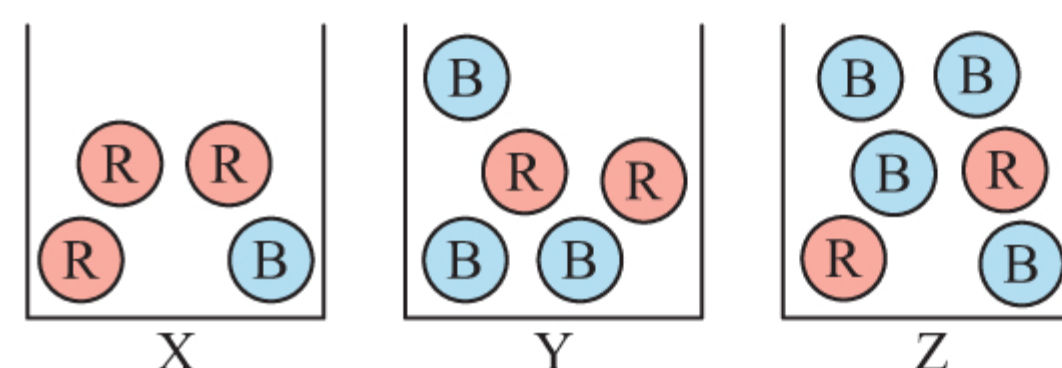
- 51** One ball is drawn from each of the boxes shown.

**a** Draw a tree diagram to illustrate the situation.

**b** Find the probability that:

- i** exactly two red balls are drawn
- ii** blue balls are drawn from boxes X and Z
- iii** at most one blue ball is drawn.

**c** Suppose an extra red ball is added to box Y. Which of the probabilities in **b** will be affected?

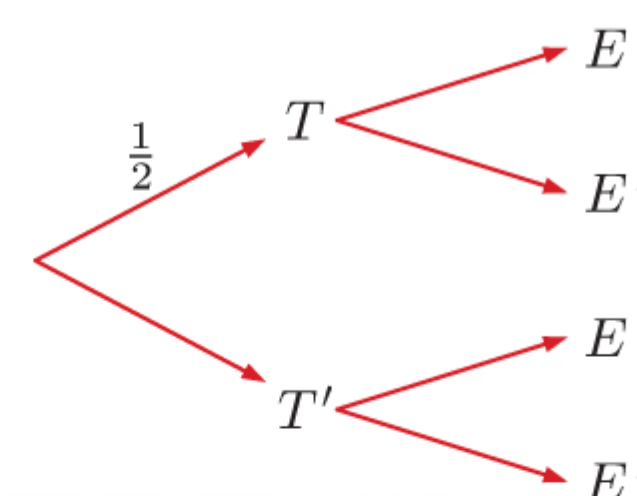


- 52** Suppose you toss a coin and roll a die simultaneously.

Let  $T$  represent a tail with the coin and  $E$  represent a 2 or a 5 with the die.

**a** Complete the tree diagram showing the probabilities of the different outcomes.

**b** Find: **i**  $P(T \cap E')$  **ii**  $P(T \cup E')$



- 53** Twins Tom and Harry are keen archers. The probability that Tom successfully hits a target is 0.7. The probability that Harry successfully hits a target is 0.6. Suppose they both shoot at a target. Find the probability that:

- a** only one of them is successful
- b** at least one of them is successful.

- 54** A bag contains seven purple tickets and three red tickets. Michelle draws two tickets from the bag without replacement.

**a** Illustrate on a tree diagram the possible outcomes and the probabilities for each draw.

**b** Find the probability that Michelle will select:

- i** at least one red ticket
- ii** one ticket of each colour
- iii** a purple ticket second.

- 55** When Nick goes shopping, there is a 70% chance that Donna will join him. When Donna joins Nick, there is probability 0.3 that he purchases a packet of potato chips. When Nick shops alone, this probability rises to 95%.

Find the probability that:

- a** when Nick goes shopping, he purchases potato chips
- b** Donna joined Nick, given that Nick purchased potato chips.

- 56** A box of chocolates contains 6 dark brown, 4 light brown, and 2 white truffles. Two truffles are selected from the box without replacement.

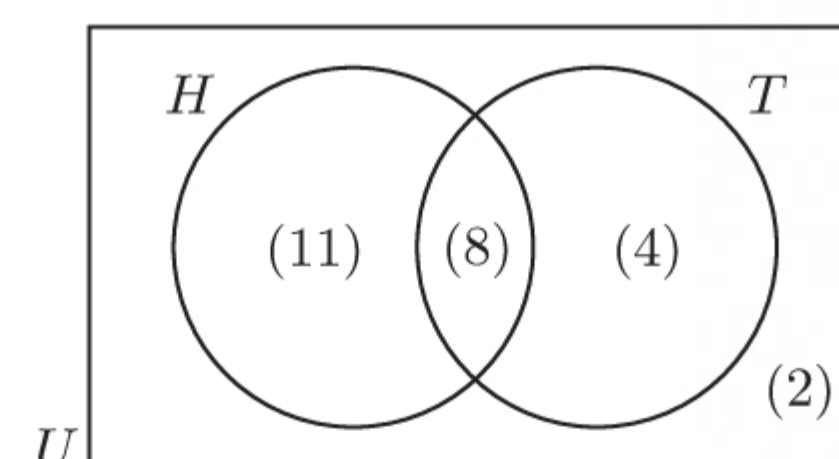
Find the probability of selecting:

- a** 2 white truffles
- b** different coloured truffles.

- 57** This Venn diagram illustrates the number of students in a particular class who play hockey ( $H$ ) and tennis ( $T$ ).

A student from the class is picked at random. Find the probability that the student:

- a** plays hockey
- b** does not play tennis
- c** plays at least one of the two sports
- d** plays tennis given that they play hockey.



- 58** 40% of students in a class own an orange highlighter, 20% own a blue highlighter, and 50% do not own either coloured highlighter.

**a** Draw a Venn diagram to describe the situation.

**b** Find the probability that a randomly selected student:

- i** owns a blue highlighter, given they own an orange highlighter
- ii** owns an orange highlighter, given they do not own a blue highlighter.

- 59** In a class of 30 students, 17 have brown hair, 12 have blue eyes, and 4 have neither brown hair nor blue eyes.

**a** Display this information in a Venn diagram.

**b** A student is randomly selected. Find the probability that the student:

- i** has blue eyes but not brown hair
- ii** has brown hair, given the student has blue eyes.



**60** In tennis, the probability of Roger getting a “first serve” in is  $\frac{7}{9}$ . How many “first serves” would you expect Roger to get in out of 180 attempts?

- 61** **a** If 3 coins are tossed, find the probability that two fall heads and the other falls tails.  
**b** Suppose 3 coins are tossed 400 times. On how many occasions would you expect to see exactly one tail?

**62** The manager of a football team has noticed the following trend amongst the team’s players over a long period.

		This week		
		Able to play	Injured	Suspended
Next week	Able to play	85%	40%	20%
	Injured	12%	60%	10%
	Suspended	3%	0%	70%

- a** Draw a transition diagram for this situation.  
**b** Write down the transition matrix  $\mathbf{T}$ .  
**c** State the meaning of the numbers in the first column of  $\mathbf{T}$ .  
**d** Find  $\mathbf{T}^2$ , and state the meaning of the numbers in the second column of  $\mathbf{T}^2$ .  
**e** Find the percentage of players currently able to play, who will not be able to play in two weeks’ time due to injury or suspension.  
**f** The football team has 40 players. At the start of the season, all players were able to play. How many players would the manager expect in each category:  
**i** after one week **ii** after two weeks?

**63** Alfred and Mae are candidates seeking to be elected at the next council election. Telephone polls of people living in the area have revealed the trend in voting patterns shown alongside.

		This week	
		Alfred	Mae
Next week	Alfred	80%	15%
	Mae	20%	85%

- a** Find the transition matrix  $\mathbf{T}$ .  
**b** Calculate  $\mathbf{T}^3$ , and explain what column 2 of  $\mathbf{T}^3$  means.  
**c** Currently the polls suggest that Mae is winning, with 70% of the vote. What percentage of the vote will Mae have in 4 weeks?  
**d** Find the exact steady state proportions for the system algebraically, and interpret your answer.

**64** Find  $k$  in each of these probability distributions:

**a**

$x$	0	1	2	3
$P(X = x)$	$k$	0.2	0.5	0.1

**b**

$x$	2	4	6
$P(X = x)$	$2k$	0.1	$0.6 - k$

**65** Find  $k$  for the following probability mass functions:

**a**  $P(x) = k(x + 3)$  for  $x = 0, 1, 2, 3, 4$

**b**  $P(x) = \frac{k^{x-3}}{x-1}$  for  $x = 3, 4, 5$

**66** Two fair dice are rolled. Let  $X$  be the difference between the numbers rolled.

- a** Explain why  $X$  is a discrete random variable. **b** State the possible values of  $X$ .  
**c** Find  $P(X = 3)$ .

**67** A discrete random variable  $X$  has probability mass function  $P(x) = \frac{a}{(x-3)^2}$  for  $x = 0, 1, 2$ .

- a** Find  $a$ . **b** Find  $P(X = 2)$ . **c** Find the mode and median of the distribution.

**68** A bag contains 3 red tickets and 2 blue tickets. Tickets are selected from the bag, without replacement, until at least one ticket of each colour is selected. Let  $X$  be the total number of tickets selected.

- a** State the possible values of  $X$ . **b** Find the probability distribution of  $X$ .  
**c** Find the mode of  $X$ . **d** Find the expected value of  $X$ .

**69** Consider the probability mass function defined by  $P(x) = P(X = x) = \frac{1}{24}(x + 6)$  for  $x = 1, 2, 3$ .

- a** Find  $P(x)$  for  $x = 1, 2$ , and 3. **b** Find the expected value of  $X$ .

**70** Each day, Russell drinks 0, 1, 2, 3, 4, or 5 cups of tea, with the probabilities shown.

Cups of tea	0	1	2	3	4	5
Probability	0.1	0.07	0.16	0.37	0.21	0.09

- a** Find the mode of the distribution.  
**b** On average, how many cups of tea does Russell drink per day?



**71** A random variable  $X$  has the probability mass function  $P(x) = \frac{x^2 + kx}{50}$  for  $x = 1, 2, 3, 4$ .

- a** Find  $k$ . **b** Find the mean of the distribution of  $X$ . **c** Find  $P(X \geq 2)$ .

**72** For the probability distribution alongside, find the:

- a** mean  $\mu$  **b** mode  
**c** variance  $\sigma^2$  **d** standard deviation  $\sigma$ .

$x$	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.4	0.2	0.1

**73**  $X$  is a random variable with mean 7 and standard deviation 2. Find  $E(Y)$  and  $\text{Var}(Y)$  for:

- a**  $Y = 4X + 3$  **b**  $Y = \frac{1}{2}(5 - X)$  **c**  $Y = \frac{2X - 1}{3}$

**74** Six questions are asked in a weekly quiz show. From past shows, the number of correctly answered questions  $X$  has  $E(X) = 3.5$  and  $\text{Var}(X) = 1.19$ .

Sara begins the quiz with 20 points, and scores 3 points for each correct answer.

Let  $Y$  be Sara's score after the quiz. Find:

- a**  $E(Y)$  **b**  $\text{Var}(Y)$  **c**  $\sigma(Y)$

**75**  $X$  has probability distribution:

$x$	1	2	3	4
$P(X = x)$	0.25	0.38	0.17	0.2

Find:

- a**  $E(X)$  **b**  $\text{Var}(X)$  **c**  $\sigma(X)$   
**d**  $E(X + 2)$  **e**  $\text{Var}(2 - 3X)$  **f**  $\sigma(2X - 10)$

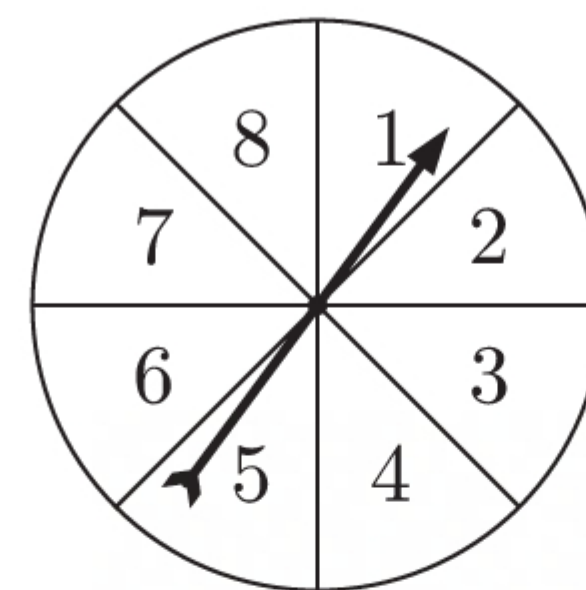
**76** A bag contains 1 blue ticket, 3 red tickets, and 8 yellow tickets. A player randomly selects a ticket from the bag, and receives \$40 for a blue ticket, \$20 for a red ticket, and \$5 for a yellow ticket.

- a** Calculate the expected return for one trial of this game.  
**b** Given that the game costs \$15 to play, explain why it would not be advisable to play this game.  
**c** Find the number of extra red tickets that should be added to the bag to make the game fair.

**77 a** Write down the first 4 rows of Pascal's triangle.

**b** The spinner alongside is spun 4 times. Find the probability of getting:

- i** exactly 3 numbers greater than 3  
**ii** at least 2 numbers greater than 5  
**iii** at most 1 number that is spelt with 3 letters.



**78** 80% of residents in a particular suburb oppose the construction of traffic lights at a particular intersection. A survey of 20 randomly selected residents is conducted.

Find the probability that:

- a** exactly 16 residents oppose the construction **b** 16 or more residents oppose the construction  
**c** between 10 and 15 residents oppose the construction **d** more than 8 residents support the construction.

**79** 5% of all items coming off a production line are defective. The manufacturer packages the items in boxes of six, and guarantees a refund if more than two items in a box are defective.

- a** On what percentage of boxes will the manufacturer have to pay a refund?  
**b** Patrick purchases 10 boxes. Find the probability that he will get a refund for exactly 1 box.

**80** A hundred seeds are planted in ten rows of ten seeds per row. Assuming that each seed independently germinates with probability  $\frac{1}{2}$ , find the probability that the row with the maximum number of germinations contains at least 8 seedlings.

**81** A company manufactures computer chips, and it is known that 3% of them are faulty. In a batch of 500 chips, find the probability that between 1 and 2 percent (inclusive) of them are faulty.



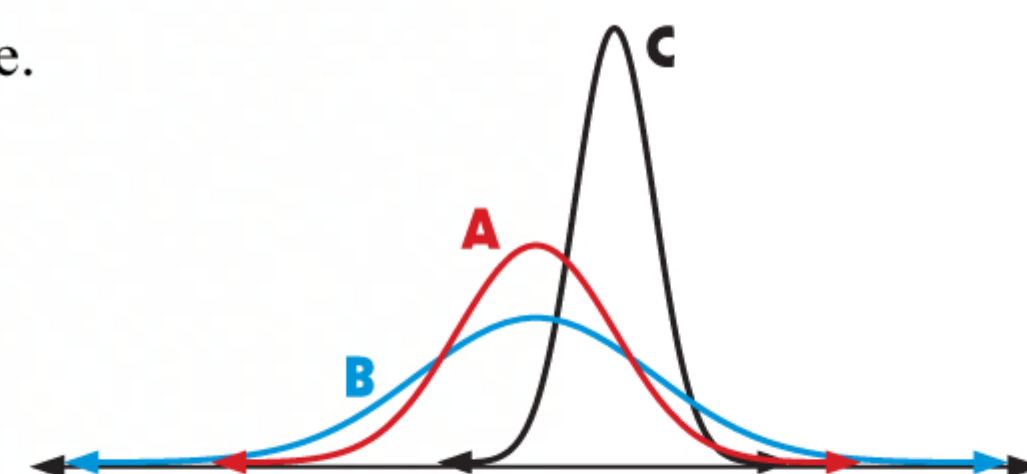
- 82** In a game, a player rolls a biased four-sided die. The probability of obtaining each possible score is shown in the table.

Score	1	2	3	4
Probability	$\frac{1}{12}$	$k$	$\frac{1}{4}$	$\frac{1}{3}$

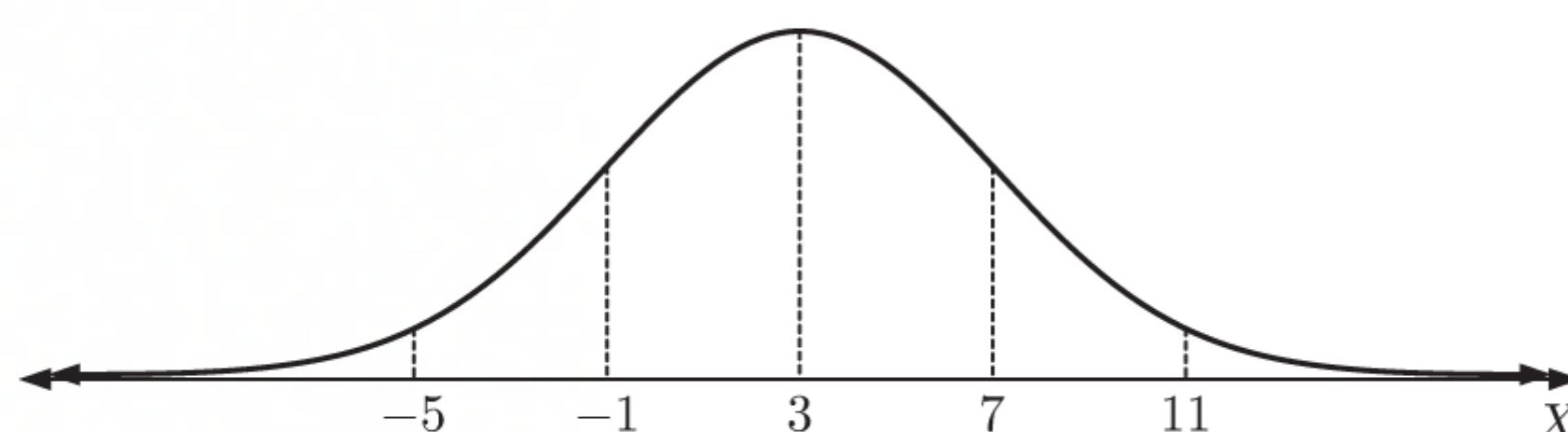
- a** Find the value of  $k$ .
- b** Let the random variable  $X$  denote the number of 2s that occur when the die is rolled 2400 times. Calculate the exact mean and standard deviation of  $X$ .
- 83** A multiple choice test consists of 30 questions with 5 answers to choose from. For each question, only one choice is correct. Let  $Y$  be the number of correct answers chosen if each answer is randomly guessed.
- a** Find the mean  $\mu$  and standard deviation  $\sigma$  of  $Y$ .      **b** Find  $P(Y = 20)$ .
- c** Find  $P(Y \geq \mu + 2\sigma)$ .
- 84** A typist makes on average 1 error per page. Suppose  $X$  is the number of errors made by the typist in typing a 12 page document.
- a** Is  $X$  a binomial or Poisson random variable?
- b** Find the mean and standard deviation of  $X$ .
- c** Find  $P(X = 10)$ .
- d** Find the probability the typist makes at least 10 errors in this document.
- 85** A Poisson variable  $X$  has standard deviation 3.1 .
- a** Find the mean of  $X$ .
- b** Find:
- i**  $P(X = 8)$       **ii**  $P(X \geq 11)$       **iii**  $P(X \geq 13 \mid X \geq 9)$
- 86** The average number of amoebas in 50 mL of pond water is 20. The number of amoebas in pond water follows a Poisson distribution.
- a** Find the probability that less than 6 amoebas are present in 10 mL of pond water.
- b** A researcher collected 10 mL of pond water each day for 20 days. Find the probability that the researcher collected less than 6 amoebas on at least 11 occasions.
- 87** Discuss whether the following variables are likely to be normally distributed. Sketch a graph to illustrate the possible distribution of each variable.
- a** the amount of sleep a person receives per night
- b** the number of lollies in a sample of 500 g bags of lollies
- c** the ages of people attending a high school musical.

- 88** Suppose  $X \sim N(\mu, \sigma^2)$ . Match each pair of parameters with the correct curve.

- a**  $\mu = 4, \sigma = 1$
- b**  $\mu = 2, \sigma = 2$
- c**  $\mu = 2, \sigma = 3$



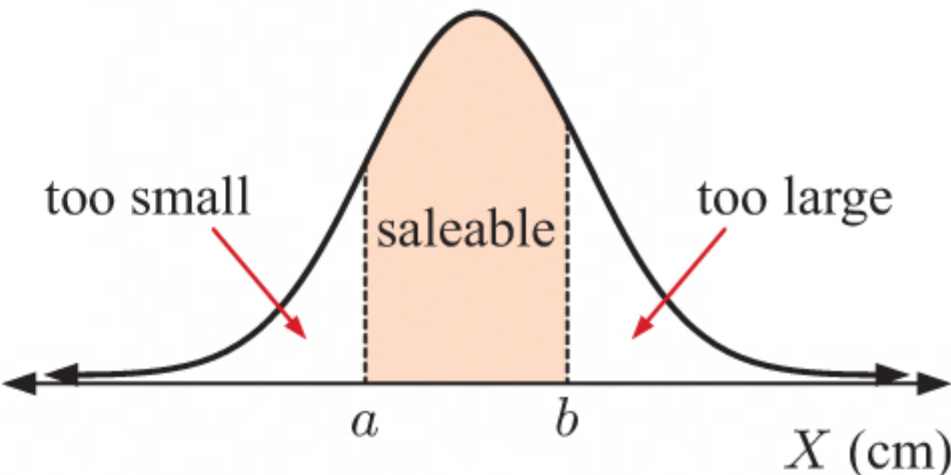
- 89** Consider the distribution curve of  $X \sim N(3, 4^2)$  shown:



Copy the above graph, and on the same set of axes sketch the distribution curve for:

- a**  $N(1, 4^2)$       **b**  $N(3, 2^2)$       **c**  $N(2, 64)$
- 90** Suppose a population is normally distributed with mean  $\mu = 30$  and standard deviation  $\sigma = 5$ . Copy and complete:
- a** Approximately 68% of the population lies between ..... and 35.
- b** Approximately 95% of the population lies between 20 and .....
- c** Approximately ..... of the population lies between 15 and 45.



- 91** Containers of a particular brand of ice cream have a capacity of 1050 mL. They are advertised as containing 1 litre of ice cream. The quantity of ice cream added to each container is normally distributed with mean 1020 mL and standard deviation 17 mL.
- Find the probability that the container has less than the advertised capacity.
  - Find the percentage of containers that overflow.
  - A sample of 75 containers are taken. Find the probability that at most three of the containers overflow.
- 92** The volume of drink dispensed by a coffee machine is normally distributed with mean 254 mL and standard deviation 2.3 mL.
- Find the probability that a randomly selected drink from the machine will have volume less than 254 mL.
  - Find the percentage of drinks dispensed by the machine which have volume between 252 mL and 256 mL.
  - A sample of 80 drinks is taken from the machine. Determine the number of drinks which will be expected to have volume at least two standard deviations above the mean.
  - The machine operator guarantees that at least 95% of drinks will have volume at least 250 mL.
    - Is the guarantee valid?
    - A technician adjusts the machine so the standard deviation is now 2.5 mL. What effect does this have on the operator's guarantee?
- 93** A machine fills bottles with tomato sauce. Each bottle is filled independently of all other bottles. The volume of sauce in each bottle is normally distributed with mean 500 mL and standard deviation 2.5 mL. Bottles are deemed to require extra sauce if the machine delivers less than 495 mL.
- Calculate the probability that a randomly selected bottle requires extra sauce.
  - From a sample of 200 bottles, calculate the probability that at least 8 bottles require extra sauce.
- 94** The time taken for a skier to complete a particular downhill run is normally distributed with mean 45 seconds and standard deviation 4 seconds.
- Find the probability that the skier completes:
    - one downhill run in under 40 seconds
    - two consecutive downhill runs in under 40 seconds each.
  - The skier completes a total of 60 independent runs. How many times would you expect the run to take between 44 seconds and 47 seconds?
- 95** The mean birth weight of babies in a population is normally distributed with mean 3.4 kg and standard deviation 300 grams.
- What proportion of babies in this population have birth weights:
    - in excess of 4 kg
    - between 3 kg and 4 kg?
  - A *low birth weight* corresponds to any newborn weighing in the lowest 10% of birth weights. State the weight below which a baby is classified as having a *low birth weight*.
- 96** The length of a zucchini is normally distributed with mean 24.3 cm and standard deviation 6.83 cm. A supermarket buying zucchinis in bulk finds that 15% of them are too small and 20% of them are too large for sale. The remainder, with lengths between  $a$  cm and  $b$  cm, are able to be sold.
- 
- Find  $a$  and  $b$ .
  - A zucchini is chosen at random. Find the probability that:
    - it is of saleable length
    - its length lies between 20 cm and 26 cm
    - its length is less than 24.3 cm.
- 97** Consider the continuous random variable  $X \sim N(44, 20)$ . Suppose  $P(X \leq m) = 0.65$  and  $P(m \leq X \leq n) = 0.2$ . Find  $n - m$ .
- 98** The times taken for the 200 runners in the school cross country event were normally distributed with mean 26 minutes and standard deviation 4 minutes.
- Estimate the number of runners who completed the course in:
    - less than 22 minutes
    - more than 27 minutes.
  - The fastest 40% of runners finished quicker than what time?



- 99** Alisdair earns an allowance for completing chores around the house. The number of chores  $X$  Alisdair completes in one week has the probability distribution:

$x$	0	1	2	3	4	5
$P(X = x)$	0.1	0.05	0.25	0.3	0.15	0.15

His parents give him \$5 each week, plus \$2 extra for each chore completed. Let  $Y$  be the amount in dollars Alisdair earns in one week.

- a** Write an expression for  $Y$  in terms of  $X$ .
- b**
  - i** Find the expected number of chores Alisdair is expected to complete in one week.
  - ii** Hence find the expected amount of money he is expected to earn in one week.
- c** Alisdair visits the corner store each weekday. On each visit, he has probability 0.4 of purchasing a \$1 chocolate bar with his allowance.

How much money can Alisdair expect to have left over each week?

- 100** Let  $X$  and  $Y$  be independent random variables.

- a** If  $E(3X - 2Y) = 4$  and  $E(2X + Y) = 5$ , find:
  - i**  $E(X)$  and  $E(Y)$
  - ii**  $E(2Y - X)$ .
- b** If  $\text{Var}(X - Y) = \frac{3}{4}$  and  $\text{Var}(3Y - X) = \frac{11}{4}$ , find:
  - i**  $\text{Var}(X)$  and  $\text{Var}(Y)$
  - ii**  $\text{Var}(7X + 5Y)$ .

- 101** Suppose  $X \sim \text{Po}(5)$  and  $Y \sim \text{Po}(2)$  are independent Poisson random variables.

- a** State the distribution of  $X + Y$ .
- b** Find  $P(X + Y \leq 1)$ .
- c** Find  $P(Y = 1 \mid X + Y \leq 1)$ .

- 102** The height of basketball players is normally distributed with mean 190 cm and standard deviation 5 cm.

A random sample of 5 players are chosen. Find the probability that:

- a** the combined height of the 5 players is at least 985 cm
- b** exactly 3 of the players are shorter than 185 cm.

- 103** The random variable  $X$  is normally distributed with mean 20 and standard deviation 2.

- a** For the sample means of size 9,  $\bar{X}_9$ , find the:
  - i** mean
  - ii** standard deviation.
- b** Is  $\bar{X}_9$  normally distributed? Explain your answer.
- c** Find  $P(\bar{X}_9 \geq 21)$ .

- 104** The mean amount of vitamin C in an orange is 53.2 mg with standard deviation 3.95 mg. For a sample of 70 oranges, find the probability that the average amount of vitamin C per orange lies between 53 mg and 54 mg.

- 105** The reaction times of 100 students are measured. The sample mean is 0.4 seconds. Assuming the population standard deviation is 0.1 seconds, find a 95% confidence interval for the population mean reaction time.

- 106** A sample of 40 cabbages are weighed, and the sample mean weight is 850 g. Assume the population standard deviation is 9.7 g.

- a** Find a 90% confidence interval for the population mean weight of cabbages.
- b** Find the width of the confidence interval.
- c** Describe how the width of the confidence interval will change if:
  - i** a higher confidence level is used
  - ii** a larger sample is taken.

- 107** The time it takes in seconds for 35 randomly chosen individuals to complete a sudoku puzzle is recorded:

348	365	376	354	356	329	352	339	360	338	371	337
345	328	381	345	351	332	367	343	349	363	329	351
335	316	354	335	346	350	353	348	355	346	380	

- a** Find the sample mean  $\bar{x}$  and the sample standard deviation  $s$  for the data.
- b** Hence construct a 95% confidence interval for the population mean completion time.



- 109** Consider the table alongside. What type of error is represented by:

**b** B?

	<i>Retain <math>H_0</math></i>	<i>Reject <math>H_0</math></i>
<i><math>H_0</math> true</i>		A
<i><math>H_0</math> false</i>	B	

- a** State the null and alternative hypotheses.

- c** Hence determine the validity of Gregory's claim.

- A sample of 500 tennis balls manufactured by a company was taken, and the mean diameter of the sample was  $\bar{x} = 6.548$  cm with sample standard deviation  $s = 0.173$  cm.

At the 5% level of significance, is there sufficient evidence to conclude that the tennis balls produced by this company do not meet the minimum international standard diameter?

- Before starting the program, each participant runs a 5 km course, and their time in minutes is recorded. After completing the program, the same 5 km course is run, and the times are recorded again. A random sample of 10 participants are chosen. Their results are shown in the table below.

<i>Before program</i>	29.4	33.6	27.2	25.3	35.7	35.2	29.8	37.8	40.1	34.1
<i>After program</i>	28.2	31.9	28.1	26.7	35.2	35.3	28.2	35.3	39.3	35.6

**113** Bao is a potato farmer. Between two harvests, he decides to change the fertiliser he uses. A sample of 100 potatoes is taken from each harvest, and the weights of each sample, in grams, are summarised below:

	<i>Sample mean</i>	<i>Sample standard deviation</i>
<i>Before change</i>	183	5.83
<i>After change</i>	184	2.35

**114** A local supermarket is looking to upgrade their store due to an increase in popularity. They will go ahead with the upgrade if, during peak shopping times, the number of customers entering the supermarket exceeds 200 per hour. The number of customers entering the supermarket per hour follows a Poisson distribution with unknown mean  $\lambda$ .

- b** In a 3 hour period during peak shopping time, 653 customers entered the supermarket. Test the hypotheses in **a** at a 5% significance level to determine whether the supermarket should be upgraded.

- a** Which hypothesis test should Erin carry out if she is interested in whether the coin is:

- ii biased *towards* heads?

- b** Erin suspects that the coin is biased towards heads. Conduct an appropriate hypothesis test with significance level  $\alpha = 0.05$  to determine whether Erin's suspicion is justified.



- 116** This table shows the weight and volume of parcels delivered by a courier in one day.

<i>Weight (x kg)</i>	2.5	0.7	1.2	1.4	2.9	0.8	1.6	1.8
<i>Volume (<math>y \times 1000 \text{ cm}^3</math>)</i>	32	15	13	25	30	21	26	16

A hypothesis test is to be conducted to determine whether the variables are positively correlated at the 2% level of significance.

- a** State the hypotheses to be tested. **b** Calculate the test statistic and  $p$ -value.
- c** State the outcome of the test.
- 117** A stationery company sells rolls of tape advertised as 20.3 m in length. The true lengths of the rolls of tape are normally distributed with standard deviation 13.5 cm.

Simon suspects the mean length of tape is not 20.3 m. To test his claim, he uses a sample of 20 rolls of tape, and a 3% level of significance.

- a** State the probability of making a Type I error.
- b** Given that the true mean length is 20.2 m, find:
- i** the probability of a Type II error **ii** the power of the test.

- 118** The number of service faults  $X$  in a game of tennis follows a Poisson distribution with unknown mean  $\lambda$ .

Dominic wants to test the hypothesis  $H_0: \lambda = 3$  against  $H_1: \lambda > 3$ .

A random sample  $\{x_1, \dots, x_{15}\}$  of 15 independent games of tennis is taken.

The decision rule is: accept  $H_0$  if  $\sum_{i=1}^{15} x_i < 55$ , otherwise reject it.

- a** Find the critical region for  $t = \sum_{i=1}^{15} x_i$ .
- b** Define a Type I error for this scenario, and calculate  $P(\text{Type I error})$ .
- c** The true mean  $\lambda = 3.5$ . Define a Type II error for this scenario, and calculate  $P(\text{Type II error})$ .
- 119** Brianna provides cookies for her colleagues to have at morning tea. Of the cookies she provides, 35% are choc-chip, 25% are oatmeal, 20% are shortbread, and the rest are butter.

<i>Type of cookie</i>	<i>Number eaten</i>
choc-chip	789
oatmeal	542
shortbread	423
butter	389
<i>Total</i>	2143

The number of each type of cookie eaten over the past month are shown alongside.

Conduct an appropriate hypothesis test with a 1% level of significance to determine whether Brianna should change the proportion of cookies she provides.

- 120** The heights of 200 Year 12 males is shown in the table alongside.

It is suspected that the data is normally distributed with  $\mu = 174 \text{ cm}$  and  $\sigma = 6 \text{ cm}$ .

A  $\chi^2$  goodness of fit test will be performed at the 10% level of significance.

- a** State the hypotheses to be tested.
- b** Construct a table of expected frequencies.
- c** Calculate the  $\chi^2$  value.
- d** Given that  $\chi_{\text{crit}}^2 = 9.24$  for this test, determine the outcome of this test.

<i>Height (H cm)</i>	<i>Frequency</i>
$160 \leq H < 165$	18
$165 \leq H < 170$	33
$170 \leq H < 175$	59
$175 \leq H < 180$	61
$180 \leq H < 185$	18
$185 \leq H < 190$	11



**121** A software company is developing a game consisting of four challenges. To determine the difficulty of the game, 200 randomly selected participants are invited to play the game. The number of challenges successfully completed by each participant is recorded.

Challenges completed	Number of participants
0	22
1	63
2	67
3	45
4	3

A  $\chi^2$  goodness of fit test is to be performed to determine whether the data is binomially distributed.

- a** Find the mean number of challenges completed by the participants.
  - b** Hence estimate the proportion  $p$  for the binomial distribution.
  - c** Conduct the test at the 5% level of significance to determine whether the data is binomially distributed.
- 122** A survey of people found the following preferences for flavoured milk:

		Preferred milk flavour			
		Chocolate	Coffee	Strawberry	Caramel
Age	Adult	26	30	15	12
	Child	40	12	15	10

It is claimed that the preferred flavour is independent of age.

- a** Write suitable null and alternative hypotheses for a  $\chi^2$  test for independence.
  - b** Find the value of the test statistic  $\chi^2_{\text{calc}}$ .
  - c** Given  $\chi^2_{\text{crit}} = 7.81$ , is there evidence at a 5% significance level to support the claim?
- 123** An experiment was conducted in an orchard to determine whether a fertiliser affected the yield of three varieties of oranges in the same way. The number of oranges per tree was calculated and the results were as follows:

	Variety A	Variety B	Variety C	Sum
Fertiliser	65	48	75	188
No fertiliser	40	54	58	152
Sum	105	102	133	340

The expected frequency table below shows some of the expected yields of each variety of orange, assuming the effect of the fertiliser is independent of the variety of orange.

	Variety A	Variety B	Variety C
Fertiliser			73.5
No fertiliser		45.6	59.5

- a** Write down a suitable null hypothesis  $H_0$  and alternative hypothesis  $H_1$  to test this independence.
- b** Copy and complete the expected frequency table.
- c** Write down the value of the test statistic  $\chi^2_{\text{calc}}$ .
- d** Find the number of degrees of freedom.
- e** Find the  $p$ -value.
- f** At the 5% significance level, is the effect of the fertiliser independent of the variety of orange?



# TOPIC 5: CALCULUS

## LIMITS

If  $f(x)$  can be made as close as we like to some real number  $A$  by making  $x$  sufficiently close to  $a$ , we say that  $f(x)$  has a **limit** of  $A$  as  $x$  approaches  $a$ , and we write  $\lim_{x \rightarrow a} f(x) = A$ .

We say that as  $x$  approaches  $a$ ,  $f(x)$  **converges** to  $A$ .

## RATES OF CHANGE

The **instantaneous rate of change** of a variable at a particular instant is given by the **gradient of the tangent** to the graph at that point.

$\frac{dy}{dx}$  gives the rate of change in  $y$  with respect to  $x$ .

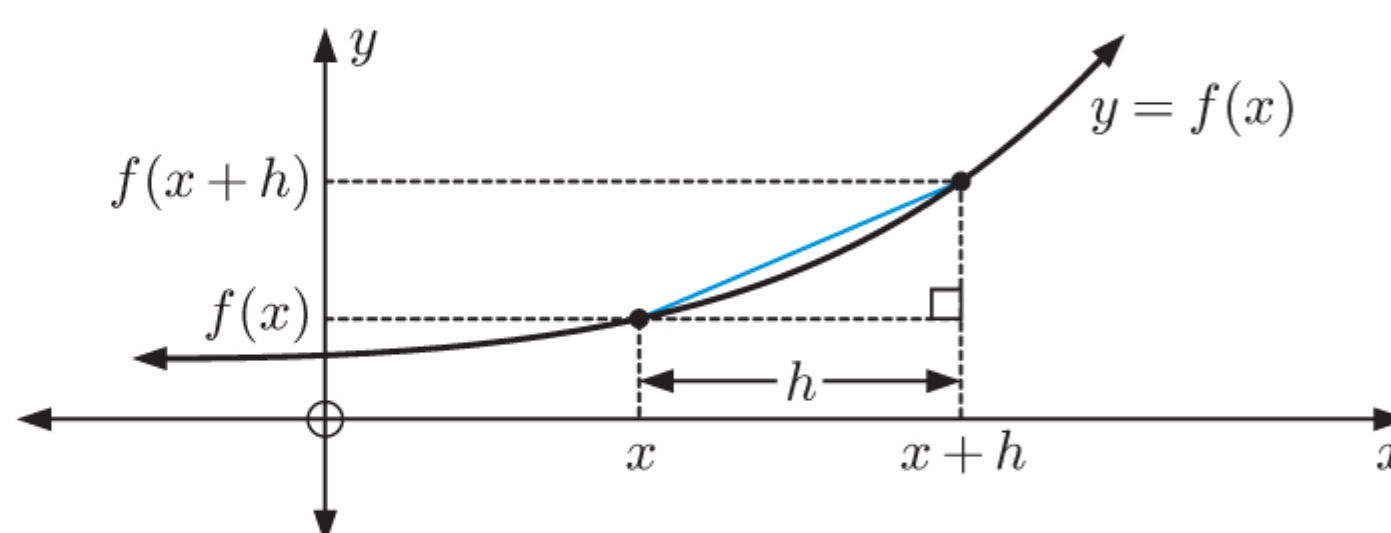
If  $\frac{dy}{dx}$  is positive, then as  $x$  increases,  $y$  also increases.

If  $\frac{dy}{dx}$  is negative, then as  $x$  increases,  $y$  decreases.

## DIFFERENTIATION

The **gradient function** or **derivative function**  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  provides:

- the rate of change of  $f$  with respect to  $x$
- the gradient of the tangent to  $y = f(x)$  for any value of  $x$ .



## RULES OF DIFFERENTIATION

$f(x)$	$f'(x)$	Name of rule
$c$	$0$	exponentials logarithms
$x^n$	$nx^{n-1}$	
$e^{f(x)}$	$e^{f(x)} f'(x)$	
$\ln f(x)$	$\frac{f'(x)}{f(x)}$	
$\sin x$	$\cos x$	trigonometric functions
$\cos x$	$-\sin x$	
$\tan x$	$\frac{1}{\cos^2 x}$	

$f(x)$	$f'(x)$	Name of rule
$cu(x)$	$cu'(x)$	addition rule product rule quotient rule
$u(x) + v(x)$	$u'(x) + v'(x)$	
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	
$\frac{u(x)}{v(x)}$	$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$	

## Chain rule

If  $y = f(u)$  where  $u = u(x)$  then  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ .



SECOND DERIVATIVES

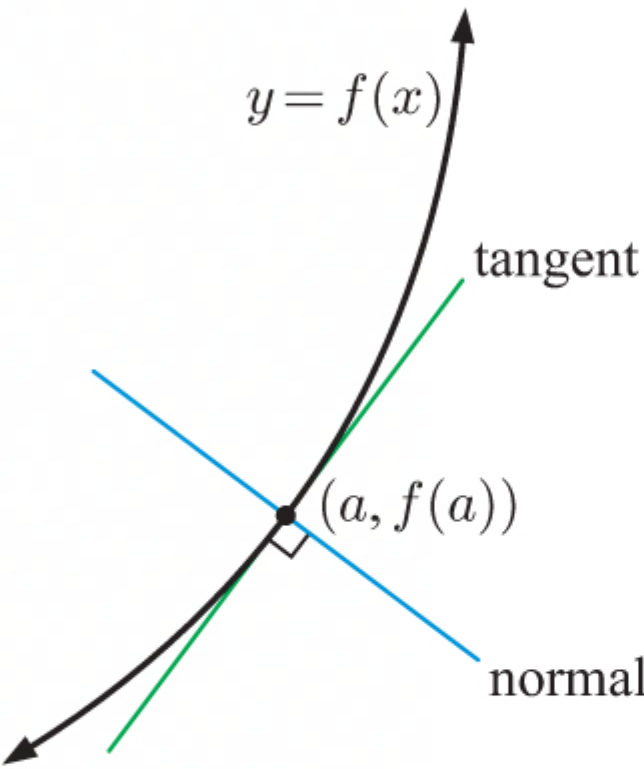
The second derivative of  $y = f(x)$  is written  $f''(x)$  or  $\frac{d^2y}{dx^2}$ .

PROPERTIES OF CURVES

Tangents and normals

For the curve  $y = f(x)$ :

- The gradient of the tangent at  $x = a$  is  $f'(a)$ .
- The equation of the tangent at  $x = a$  is  $y = f'(a)(x - a) + f(a)$ .
- The gradient of the normal at  $x = a$  is  $-\frac{1}{f'(a)}$ .
- The equation of the normal at  $x = a$  is  $y = -\frac{1}{f'(a)}(x - a) + f(a)$ .



Increasing and decreasing functions

$f(x)$  is **increasing** on an interval  $S \Leftrightarrow f(a) \leq f(b)$  for all  $a, b \in S$  such that  $a < b$ .

$f(x)$  is **decreasing** on  $S \Leftrightarrow f(a) \geq f(b)$  for all  $a, b \in S$  such that  $a < b$ .

For most functions:

- $f(x)$  is increasing on  $S \Leftrightarrow f'(x) \geq 0$  for all  $x$  in  $S$ .
- $f(x)$  is decreasing on  $S \Leftrightarrow f'(x) \leq 0$  for all  $x$  in  $S$ .

Stationary points

A **stationary point** of a function is a point such that  $f'(x) = 0$ .

You should be able to identify and explain the significance of local and global maxima and minima, as well as stationary and non-stationary inflections.

Stationary point where $f'(a) = 0$	Sign diagram of $f'(x)$ near $x = a$	Shape of curve near $x = a$
local maximum		
local minimum		
stationary inflection	 or 	

Shape

If  $f''(x) \leq 0$  for all  $x \in S$ , the curve is **concave down** on the interval  $S$ .

If  $f''(x) \geq 0$  for all  $x \in S$ , the curve is **concave up** on the interval  $S$ .

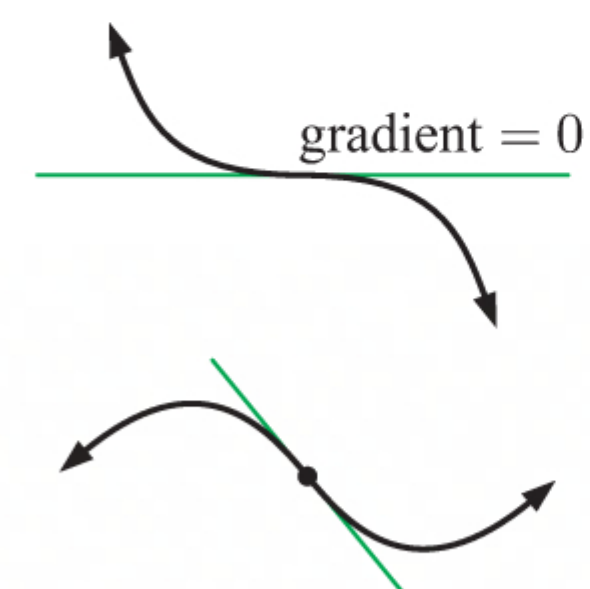




There is a **point of inflection** at  $x = a$  if  $f''(a) = 0$  **and** the sign of  $f''(x)$  changes on either side of  $x = a$ . It corresponds to a change in shape of the curve.



If  $f'(a) = 0$ , the point of inflection is a **stationary inflection**: the tangent at  $x = a$  is horizontal.



If  $f'(a) \neq 0$ , the point of inflection is a **non-stationary inflection**: the tangent at  $x = a$  is *not* horizontal.

## OPTIMISATION PROBLEMS

It is important to remember that a local minimum or maximum does not always give the minimum or maximum value of a function in a particular domain. You must check for other turning points in the domain, and the values of the function at the end points of the domain.

### Optimisation problem solving method

- Step 1:* Draw a large, clear diagram of the situation.
- Step 2:* Construct a **formula** with the variable to be optimised as the subject. It should be written in terms of one convenient variable, for example  $x$ . You should write down what domain restrictions there are on  $x$ .
- Step 3:* Find the **first derivative** and find the value(s) of  $x$  which make the first derivative **zero**.
- Step 4:* For each stationary point, use the **sign diagram test** or **second derivative test** to determine whether you have a local maximum or local minimum.
- Step 5:* Identify the optimal solution, also considering end points where appropriate.
- Step 6:* Write your answer in a sentence, making sure you specifically answer the question.

## RELATED RATES

If the variables  $x$  and  $y$  are related, then  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  are **related rates**.

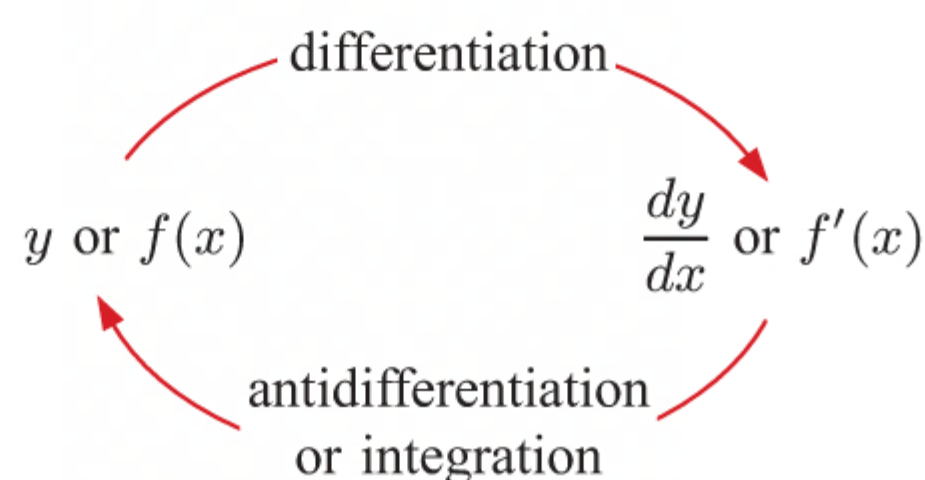
To solve problems involving related rates, we:

- Write an equation connecting the variables.
- Use the chain rule to differentiate the equation with respect to time  $t$ .
- Substitute the values for the *particular case* corresponding to some instant in time, and solve to find the required unknown.

## INTEGRATION

**Antidifferentiation** or **integration** is the reverse process of differentiation.

The **antiderivative** or **integral** of  $f(x)$  is the simplest function  $F(x)$  such that  $F'(x) = f(x)$ .





Techniques for integration

When integrating, we use the rules for differentiation in reverse. Do not forget to include the **constant of integration**.

Function	Integral
$k$	$kx + c$
$x^n$	$\frac{x^{n+1}}{n+1} + c, \quad n \neq -1$
$e^x$	$e^x + c$
$\frac{1}{x}$	$\ln x  + c$
$e^{ax+b}$	$\frac{1}{a}e^{ax+b} + c, \quad a \neq 0$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c, \quad a \neq 0, \quad n \neq -1$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln ax+b , \quad a \neq 0$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c, \quad a \neq 0$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c, \quad a \neq 0$
$\frac{1}{\cos^2(ax+b)}$	$\frac{1}{a}\tan(ax+b) + c, \quad a \neq 0$

Integration by substitution

$$\int f(u) \frac{du}{dx} dx = \int f(u) du$$

When using substitution to evaluate a definite integral, make sure you change the limits of integration to correspond to the new variable.

DEFINITE INTEGRALS

Fundamental Theorem of Calculus

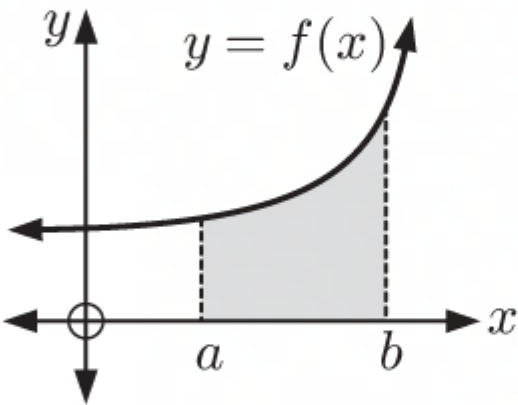
For a continuous function  $f(x)$  with antiderivative  $F(x)$ ,  $\int_a^b f(x) dx = F(b) - F(a)$ .

Properties of definite integrals

- $\int_a^a f(x) dx = 0$
  - $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
  - $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- $\int_b^a f(x) dx = -\int_a^b f(x) dx$
  - $\int_a^b cf(x) dx = c \int_a^b f(x) dx$

Area under a curve

If  $f(x)$  is a continuous *positive* function on the interval  $a \leq x \leq b$ , then  $\int_a^b f(x) dx$  is the area under the curve between  $x = a$  and  $x = b$ .

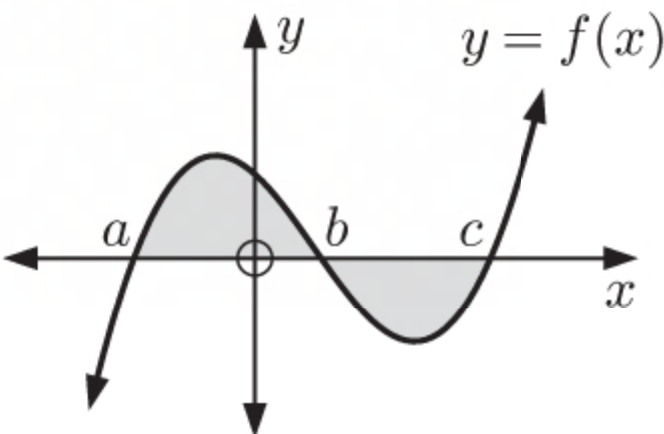


To find the total area enclosed by  $y = f(x)$  and the  $x$ -axis between  $x = a$  and  $x = b$ , we need to be careful about where  $f(x) < 0$ .

On an interval  $c \leq x \leq d$  where  $f(x) < 0$ , the area is  $-\int_c^d f(x) dx$ .

For example:

The total shaded area =  $\int_a^b f(x) dx - \int_b^c f(x) dx$   
 $\neq \int_a^c f(x) dx$ .



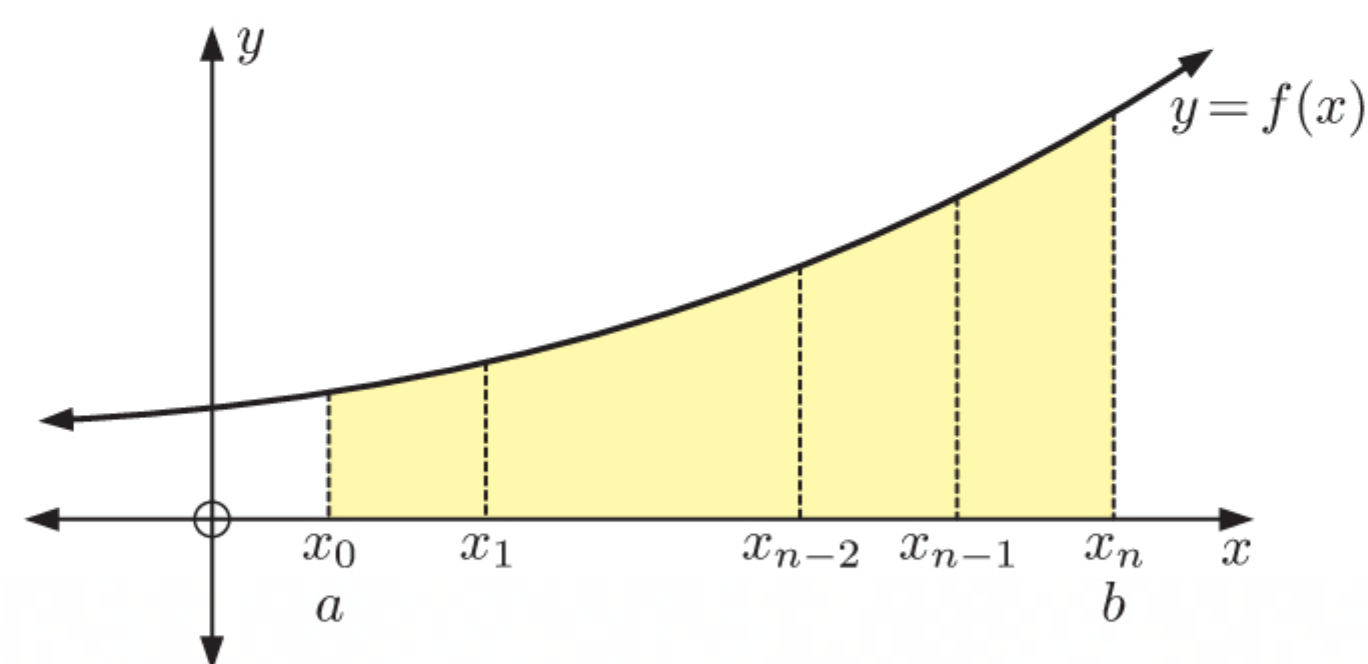


## The trapezoidal rule

Suppose we divide the interval  $a \leq x \leq b$  into  $n$  subintervals of equal width  $h = \frac{b-a}{n}$ .

The shaded area  $\int_a^b f(x) dx$  is approximated by

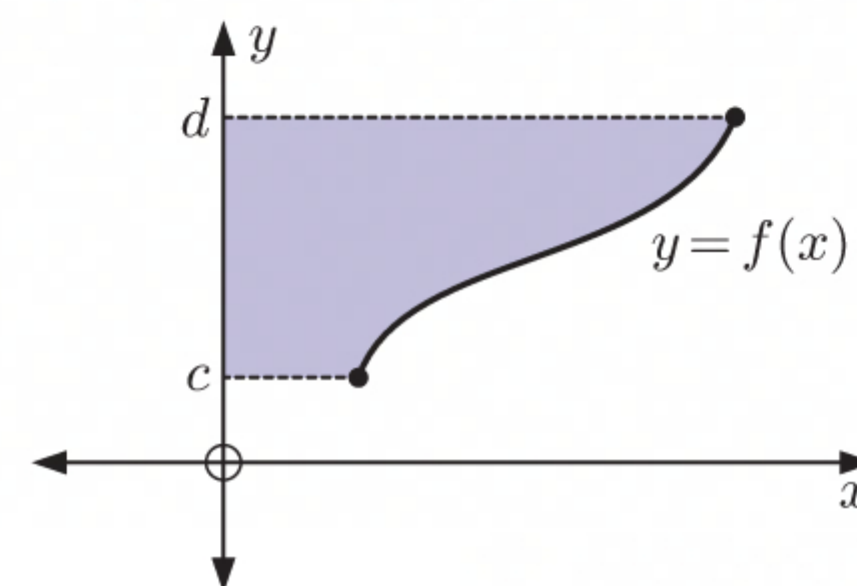
$$\int_a^b f(x) dx \approx \frac{h}{2} \left( f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) \right)$$



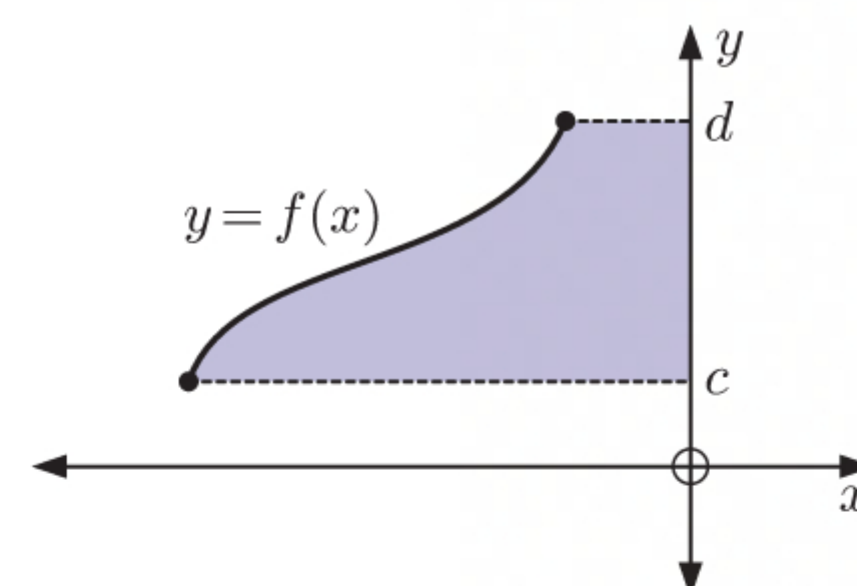
## The area between a curve and the y-axis

Consider an invertible function  $f(x)$ .

- If  $x = f^{-1}(y) > 0$  for  $c \leq y \leq d$ , shaded area =  $\int_c^d f^{-1}(y) dy$



- If  $x = f^{-1}(y) < 0$  for  $c \leq y \leq d$ , shaded area =  $-\int_c^d f^{-1}(y) dy$

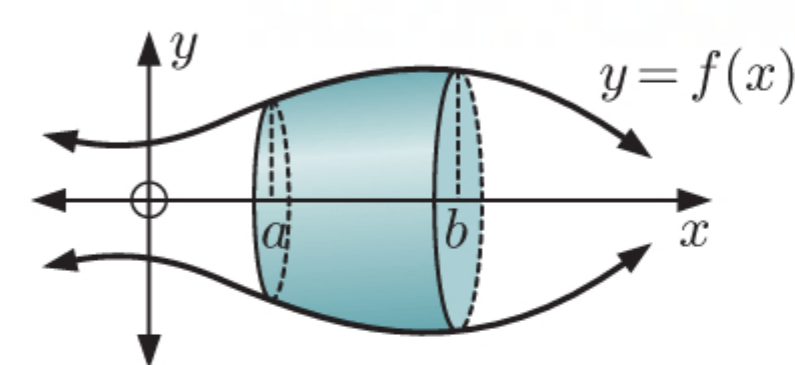


## Solids of revolution

- When the region enclosed by  $y = f(x)$ , the  $x$ -axis, and the vertical lines  $x = a$  and  $x = b$  is revolved through  $2\pi$  about the  $x$ -axis to generate a solid, the volume of the solid is given by

$$V = \pi \int_a^b [f(x)]^2 dx$$

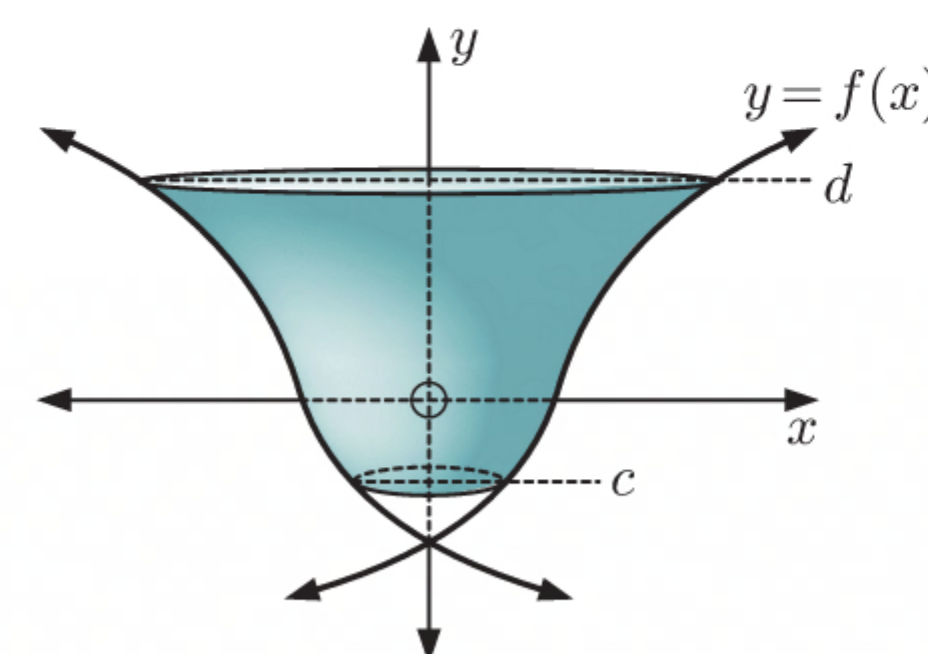
or  $\pi \int_a^b y^2 dx$



- When the region enclosed by the invertible function  $y = f(x)$ , the  $y$ -axis, and the horizontal lines  $y = c$  and  $y = d$  is revolved through  $2\pi$  about the  $y$ -axis to generate a solid, the volume of the solid is given by

$$V = \pi \int_c^d [f^{-1}(y)]^2 dy$$

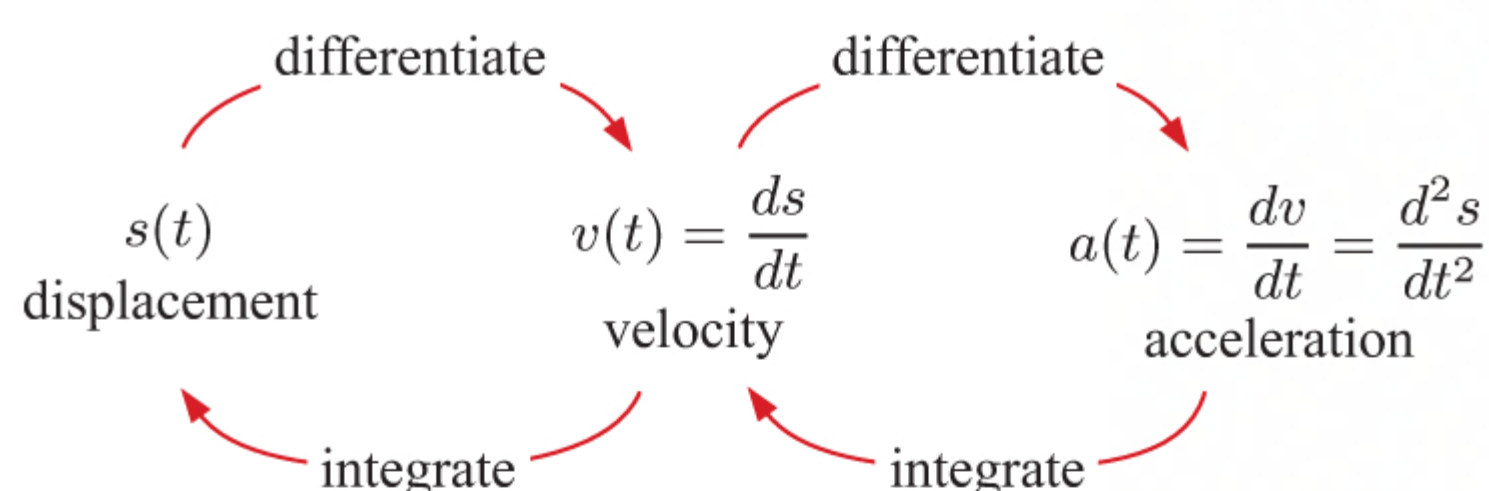
or  $\pi \int_c^d x^2 dy$ .



## KINEMATICS

Suppose an object moves along a straight line.

Its position relative to the origin at time  $t$  is given by a displacement function  $s(t)$ . Its instantaneous velocity is given by  $v(t) = s'(t)$ , and its instantaneous acceleration by  $a(t) = v'(t) = s''(t)$ .





You should understand the physical meaning of the different signs of displacement, velocity, and acceleration.

Displacement:

$s(t)$	Interpretation
$= 0$	The object is at O
$> 0$	The object is to the right of O
$< 0$	The object is to the left of O

Velocity:

$v(t)$	Interpretation
$= 0$	The object is instantaneously at rest
$> 0$	The object is moving to the right
$< 0$	The object is moving to the left

Acceleration:

$a(t)$	Interpretation
$> 0$	The velocity of the object is increasing
$< 0$	The velocity of the object is decreasing
$= 0$	The velocity of the object may be at a maximum or a minimum

Speed

The **speed** at any instant is the magnitude of the object’s velocity. If  $S(t)$  represents the speed then  $S = |v|$ .

If the signs of  $v(t)$  and  $a(t)$  are the same then the speed of the object is increasing.

If the signs of  $v(t)$  and  $a(t)$  are different then the speed of the object is decreasing.

Displacement and distance travelled

For the time interval  $t_1 \leq t \leq t_2$ :

- displacement  $= s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t) \, dt$
- total distance travelled  $= \int_{t_1}^{t_2} |v(t)| \, dt$ .

Velocity and acceleration in terms of displacement

If we are given an object’s velocity in terms of its displacement  $s$ , we can write its acceleration in terms of  $s$  using the formula  $a = v \frac{dv}{ds}$ .

DIFFERENTIAL EQUATIONS

A **differential equation** is an equation involving a derivative of a function.

**Euler’s method** allows us to approximate the solution curve to the differential equation  $\frac{dy}{dx} = f(x, y)$  with particular solution passing through the point  $(x_0, y_0)$ .

At each stage we perform the iterative procedure  $\begin{cases} x_i = x_{i-1} + h \\ y_i = y_{i-1} + hf(x_{i-1}, y_{i-1}) \end{cases}$  where  $h$  is the step size.

To approximate  $y(x_n)$  where  $x_n = x_0 + nh$ , we perform the procedure  $n$  times.

You should be able to use and interpret **slope fields** for differential equations of the form  $\frac{dy}{dx} = f(x, y)$ .

**Separable differential equations** are differential equations of the form  $\frac{dy}{dx} = f(x)g(y)$ .

To solve these equations, we rearrange the equation and integrate both sides with respect to  $x$  to obtain the form  $\int \frac{1}{g(y)} \, dy = \int f(x) \, dx$ . The variables are now separated, so we can find the two integrals separately and solve the equation.

Coupled differential equations

Differential equations which need to be solved simultaneously are said to be **coupled**.

For the coupled system of differential equations  $\begin{cases} \frac{dx}{dt} = f_1(x, y) \\ \frac{dy}{dt} = f_2(x, y) \end{cases}$ ,

the **trajectory vector** at any point  $(x, y)$  is given by  $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ .



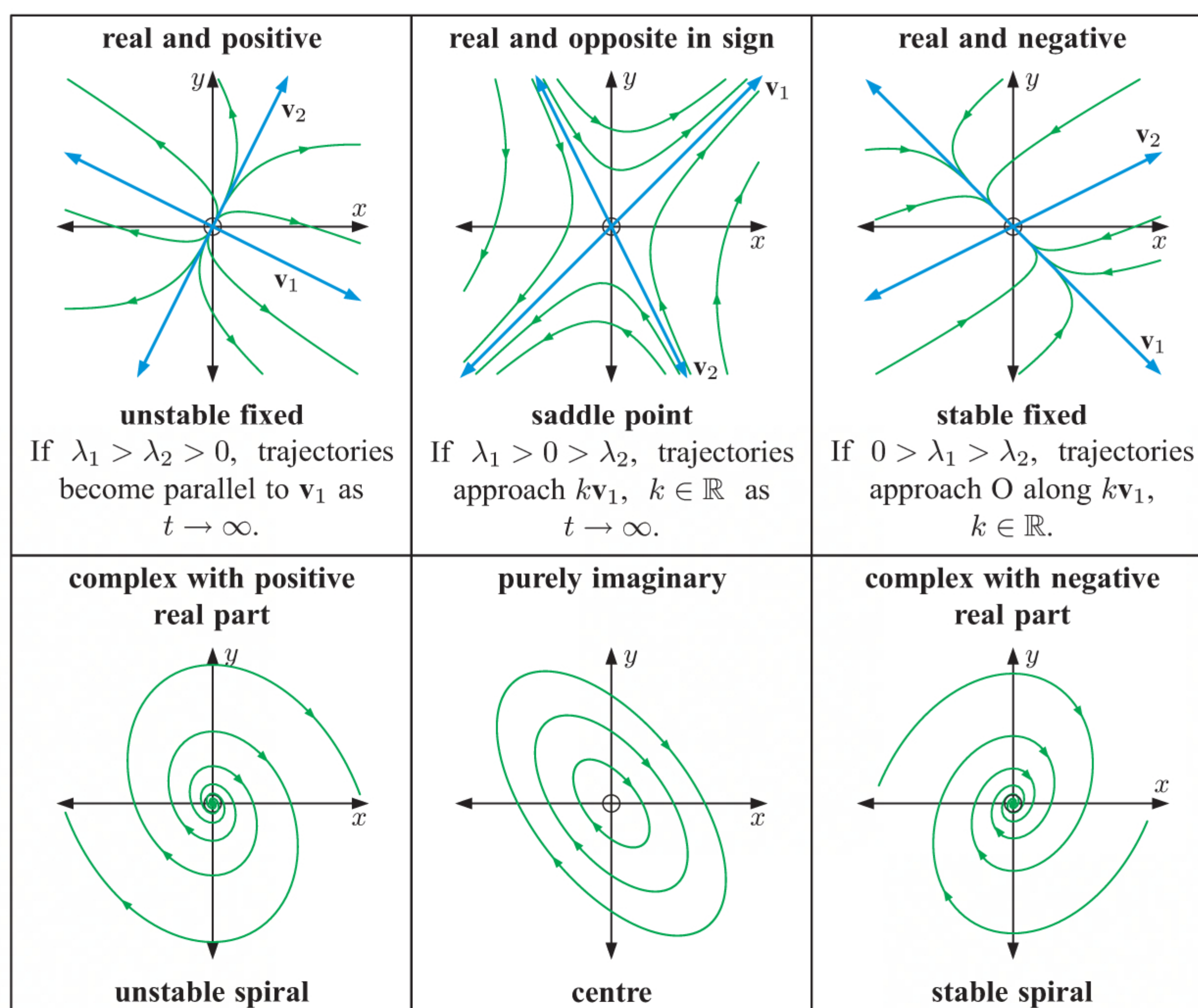
The **phase portrait** for the system is constructed by drawing the trajectory vector at every point on a grid. You should be able to draw a solution curve on a phase portrait from a given initial point.

An **equilibrium point** occurs when  $\frac{dx}{dt} = \frac{dy}{dt} = 0$  simultaneously.

**Coupled linear differential equations** have the form 
$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$$
. This can be written more concisely as  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ ,

where  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is the **matrix of coefficients**.

For coupled linear differential equations, the only equilibrium point is  $(0, 0)$ . The behaviour of the system around the equilibrium point is determined by the eigenvalues of  $\mathbf{A}$ .



If  $\mathbf{A}$  has real eigenvalues  $\lambda_1, \lambda_2$  with corresponding eigenvectors  $\mathbf{v}_1, \mathbf{v}_2$ , the general solution to the system is  $\mathbf{x} = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2$ . If an initial point is given, we can determine  $A$  and  $B$  and hence find a particular solution.

### Euler's method for coupled equations

Euler's method allows us to approximate the solution curve to the coupled equations 
$$\begin{cases} \frac{dx}{dt} = f_1(x, y, t) \\ \frac{dy}{dt} = f_2(x, y, t) \end{cases}$$
 with initial point  $(x_0, y_0)$  at time  $t_0$ .

At each stage we perform the iterative procedure 
$$\begin{cases} t_i = t_{i-1} + h \\ x_i = x_{i-1} + hf_1(x_{i-1}, y_{i-1}, t_{i-1}) \\ y_i = y_{i-1} + hf_2(x_{i-1}, y_{i-1}, t_{i-1}) \end{cases}$$
 where  $h$  is the step size.

### Second order differential equations

A **second order differential equation** is a differential equation which involves a second derivative, for example

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 3x = 0.$$

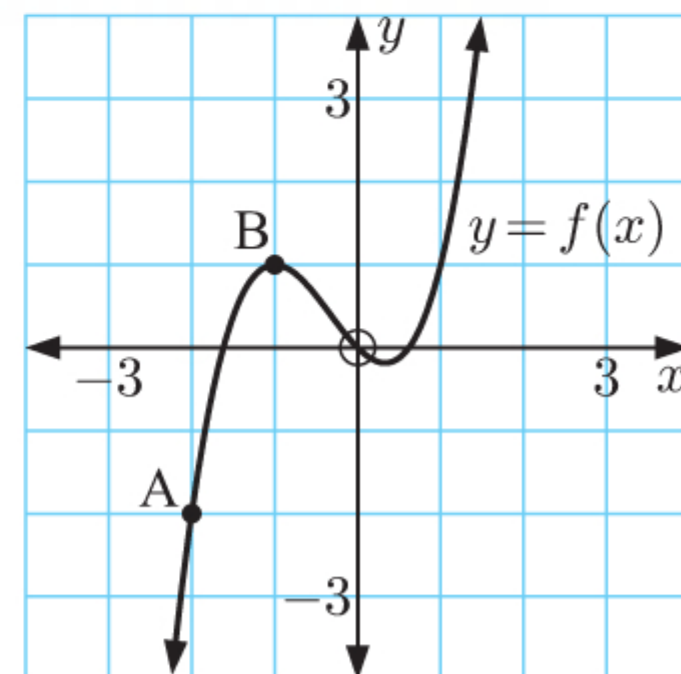
You should be able to transform a second order differential equation into a system of coupled first order differential equations.



## SKILL BUILDER QUESTIONS

- 1 Consider the graph of the function  $y = f(x)$  alongside.

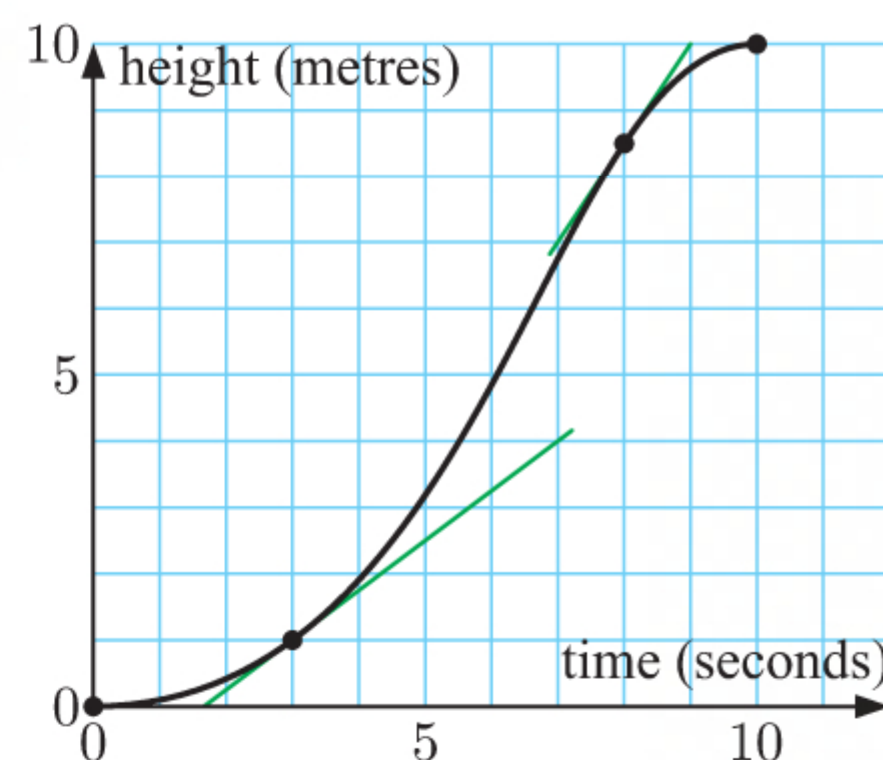
Find the average rate of change in  $f(x)$  from A to B.



- 2 This graph shows the height of an elevator above ground level.

Use the tangents drawn to find the elevator's instantaneous speed after:

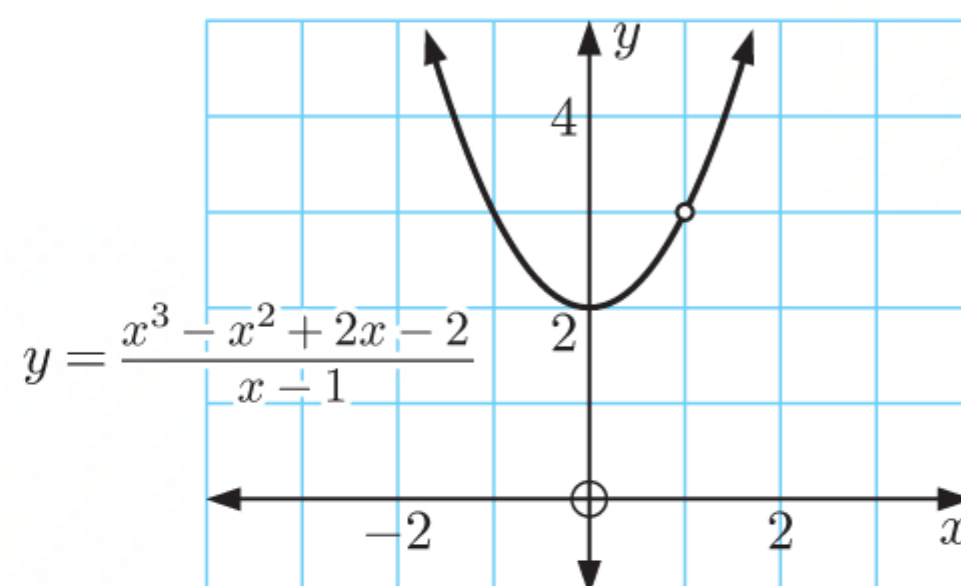
- a 3 seconds                                      b 8 seconds.



- 3 The graph of  $f(x) = \frac{x^3 - x^2 + 2x - 2}{x - 1}$  is shown alongside.

- a Explain why the function is undefined at  $x = 1$ .

- b Use the graph to find  $\lim_{x \rightarrow 1} f(x)$ .



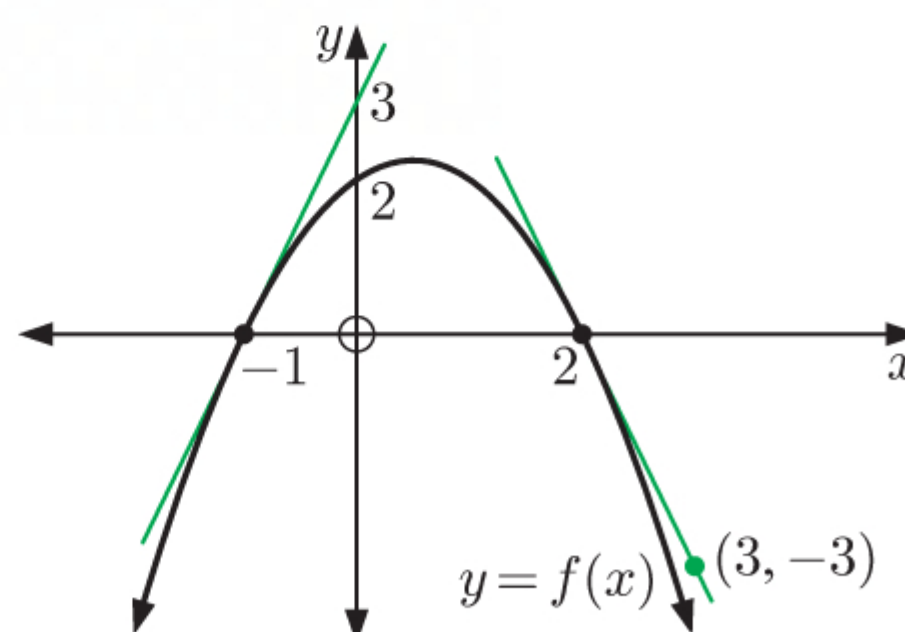
- 4 a Copy and complete this table of values:

- b Hence predict the value of  $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$ .

$x$	50	100	200	500	1000
$\frac{\ln x}{x}$					

- 5 For the given graph, find:

- a  $f'(-1)$                                       b  $f'(2)$



- 6 Differentiate with respect to  $x$ :

a  $6 - 3x + 2x^2$

b  $\frac{1}{2}x^2 + 3x - 5$

c  $\frac{1}{x} + \frac{1}{x^2}$

d  $\frac{2x^2 + x + 1}{x}$

- 7 Using the rules of differentiation, find  $\frac{dy}{dx}$  if  $y$  is:

a  $5x^3 - 3x^2 + 4x + 7$

b  $\frac{3}{\sqrt{x}} - 2\sqrt{x}$

c  $\frac{2x - x^2}{\sqrt{x}}$

- 8 Using the rules of differentiation, find  $\frac{dy}{dx}$  if  $y$  is:

a  $(x^2 - 3x)^5$

b  $\frac{3}{(x^2 + 3)^3}$

c  $\sqrt{x^2 - 3x}$

- 9 Find constants  $a$  and  $b$  such that:

a  $f(x) = ax^2 + bx^3$ ,  $f'(1) = -5$ , and  $f'(-1) = -13$

b  $f(x) = ax + \frac{b}{x^2}$ ,  $f(1) = 8$ , and  $f'(1) = -7$ .

- 10 Find the gradient of the tangent to:

a  $y = 3x - 2x^2$  at  $x = 4$

b  $y = \frac{x^2 + 4x - 1}{x^2}$  at  $(1, 4)$ .



- 11** The tangent to  $f(x) = ax^2 + bx - 7$  at the point  $(-1, -10)$  has gradient 1. Find  $a$  and  $b$ .
- 12** Find the coordinates of the point(s) on:
- a**  $y = 3x^2 + 5x + 1$  where the tangent has gradient 11
- b**  $f(x) = \frac{1}{2}x^3 - 4x - 2$  where the tangent has gradient  $\frac{1}{2}$ .
- 13** The gradient function of  $f(x) = (ax + b)^c$  is  $f'(x) = 81x^2 + 108x + 36$ . Find the constants  $a$ ,  $b$ , and  $c$ .
- 14** Find  $\frac{dy}{dx}$  for:
- a**  $y = x^2\sqrt{x^2 + 2x}$       **b**  $y = \sqrt{x}(2x + 3)^4$       **c**  $y = (2x + 1)^3(x - 5)^2$
- 15** Suppose  $y = (x - 2)^2(2x - 1)$ . For what values of  $x$  does  $\frac{dy}{dx} = 36$ ?
- 16** Find the gradient of the tangent to:
- a**  $y = \frac{x^3}{x^2 - 1}$  at  $x = 2$       **b**  $y = \frac{\sqrt{x}}{2x + 5}$  at  $x = 4$
- 17** Find  $f'(t)$  if:
- a**  $f(t) = 20te^{-0.1t}$       **b**  $f(t) = \frac{100}{1 + 7e^{-\frac{t}{4}}}$       **c**  $f(t) = \frac{t + 9}{e^t}$
- 18** Given  $f(x) = e^{ax+2} + x^2$  and  $f(2) = f'(2)$ , find  $a$ .
- 19** Find  $f'(x)$  if:
- a**  $f(x) = e^x \ln x$       **b**  $f(x) = \ln(2x + 3)$       **c**  $f(x) = [\ln(x^2 + 5)]^2$
- 20** Differentiate with respect to  $x$ :
- a**  $3 \sin(x - 4)$       **b**  $12x - 2 \cos \frac{x}{3}$       **c**  $x^2 \sin 3x$       **d**  $(\sin x)e^{\cos x}$
- 21** Find  $\frac{dy}{dx}$  for:
- a**  $y = \tan 2x$       **b**  $y = \tan(3x - 4)$       **c**  $y = \tan^2 x$
- 22** Find  $f'(x)$  for:
- a**  $f(x) = \sqrt{\sin(2x + 1)}$       **b**  $f(x) = \cos \frac{x}{2} \sin \frac{x}{3}$       **c**  $f(x) = \ln\left(\frac{\sin x}{x}\right)$
- 23** Find the gradient of the tangent to  $f(x) = \cos^4 x$  at the point where  $x = \frac{3\pi}{4}$ .
- 24** Find  $\frac{d^2y}{dx^2}$  for:
- a**  $y = \frac{3}{x^2}$       **b**  $y = 2x^3 + 3x^2 + 2$       **c**  $y = \frac{x + 3}{6 - x}$
- 25** Given  $f(x) = \ln(\cos x)$ , find:
- a**  $f\left(\frac{\pi}{4}\right)$       **b**  $f'\left(\frac{\pi}{4}\right)$       **c**  $f''\left(\frac{\pi}{4}\right)$ .
- 26** Find the equation of the tangent to:
- a**  $y = x^2 + 2x - 5$  at  $x = 1$       **b**  $y = 3 - \frac{2}{x}$  at  $x = -2$ .
- 27** Let  $g(x) = -x \cos x$ .
- a** Find  $g'(x)$ .      **b** Find the equation of the tangent to the graph  $y = g(x)$  at the point where  $x = \frac{\pi}{3}$ .
- 28** Find the equation of the tangent to  $f(x) = \ln(2x + 3)$  at the point where  $x = 2$ .
- 29** Let  $f(x) = -x^2 + 4x$ .
- a** Find  $f'(x)$ .      **b** Find the equation of the tangent to  $y = f(x)$  at the point where  $x = k$ .  
**c** Suppose this tangent has positive gradient and passes through  $(4, 9)$ . Find the value of  $k$ .
- 30** Find where the tangent to  $y = x^3 + 2x + 1$  at the point where  $x = -1$ , meets the curve again.
- 31** Consider the curve  $y = \frac{a}{x} - x^2 + 1$  where  $a \in \mathbb{R}$ . The gradient of the tangent to the curve is  $-5$  when  $x = 2$ .
- a** Find the value of  $a$ .      **b** Determine the equation of the tangent to the curve at  $x = 2$ .
- 32** Find the equation of the normal to:
- a**  $f(x) = x^3 - 2x$  at  $x = 1$       **b**  $f(x) = \frac{3}{x} - \frac{6}{x^2}$  at  $x = 2$ .



**33** The normal  $L_1$  to the function  $f(x) = ax^2 + bx$  at  $x = 2$  has equation  $x + 3y = -4$ .

**a** Show that: **i**  $4a + 2b = -2$  **ii**  $4a + b = 3$

**b** Hence find  $a$  and  $b$ .

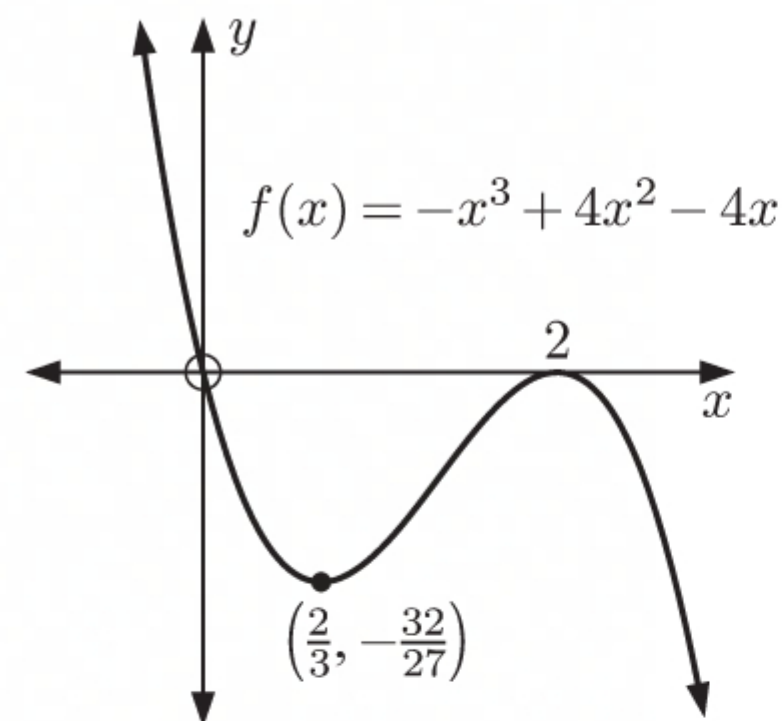
**c** The tangent  $L_2$  to  $y = f(x)$  at the point Q is parallel to  $L_1$ . Find the coordinates of Q.

**34** The graph of  $f(x) = -x^3 + 4x^2 - 4x$  is shown alongside.

**a** Use the graph to write down the intervals where the function is:

**i** increasing **ii** decreasing.

**b** Check your answer by finding  $f'(x)$ .



**35** Find the intervals where  $f(x)$  is increasing or decreasing:

**a**  $f(x) = 5 - 3x$

**b**  $f(x) = 2x^2 - 7x + 6$

**c**  $f(x) = -\frac{1}{x}$

**d**  $f(x) = 2x^3 - 9x^2 + 7x + 6$

**36** **a** Show that if  $f(x) = \ln\left(\frac{1-2x}{x^2+2}\right)$ , then  $f'(x) = \frac{2(x-2)(x+1)}{(1-2x)(x^2+2)}$ .

**b** On what intervals is  $f(x)$  decreasing?

**37** For each of the following functions, find and classify all stationary points.

**a**  $f(x) = x^3 - x^2$

**b**  $f(x) = x^4 - 2x^3 + 4x^2 - 8$

**c**  $f(x) = 2x + \frac{6}{x}$

**38**  $f(x) = 2x^3 + ax + b$  has a stationary point at  $(1, 1)$ .

**a** Find the values of  $a$  and  $b$ .

**b** Find the position and nature of all stationary points.

**39** Find the greatest and least values of:

**a**  $f(x) = x^3 - 2x^2$  for  $-1 \leq x \leq 1$

**b**  $f(x) = x^2 - \frac{27}{x}$  for  $-6 \leq x \leq -1$

**c**  $f(x) = x^3 - 6x^2 + 12x - 10$  for  $0 \leq x \leq 5$ .

**40** Find the exact coordinates and nature of the stationary points of:

**a**  $y = xe^{-x}$

**b**  $y = \frac{x-3}{x^2-5}$

**41** Consider the function  $f(x) = \frac{e^{3x}}{kx}$ ,  $k \neq 0$ .

**a** Find the  $x$ -coordinate of the stationary point.

**b** For what values of  $k$  is the stationary point: **i** a local minimum **ii** a local maximum?

**c** Given that the stationary point has  $y$ -coordinate  $-\frac{e}{2}$ , find  $k$  and determine the nature of the stationary point.

**d** State the location and nature of the stationary point of  $g(x) = -f(2x)$ .

**42** Consider  $g(x) = 3 - 2\cos 2x$ .

**a** Find  $g'(x)$ .

**b** Sketch  $y = g'(x)$  for  $-\pi \leq x \leq \pi$ .

**c** Write down the number of solutions to  $g'(x) = 0$  for  $-\pi \leq x \leq \pi$ .

**d** Mark a point M on the sketch in **b** where  $g'(x) = 0$  and  $g''(x) > 0$ .

**43** For each of the following functions, determine the intervals on which the function is:

**i** increasing

**ii** decreasing

**iii** concave upwards

**iv** concave downwards.

**a**  $f(x) = x^2 + 3x + 5$

**b**  $f(x) = e^{-x^2}$

**c**  $f(x) = x \ln(x^2)$

**44** Let  $f(x) = xe^{1-2x^2}$ .

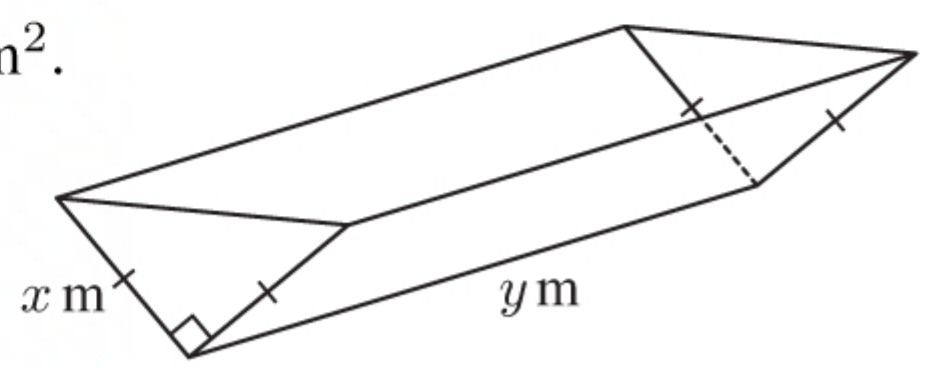
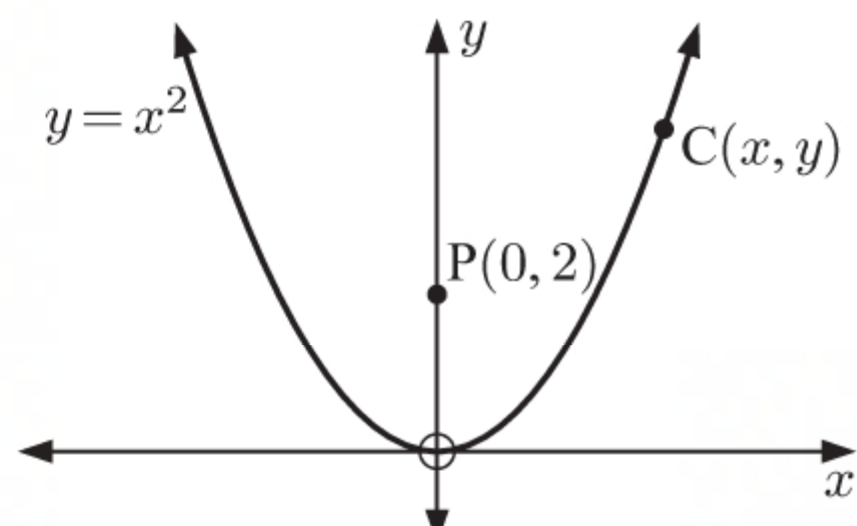
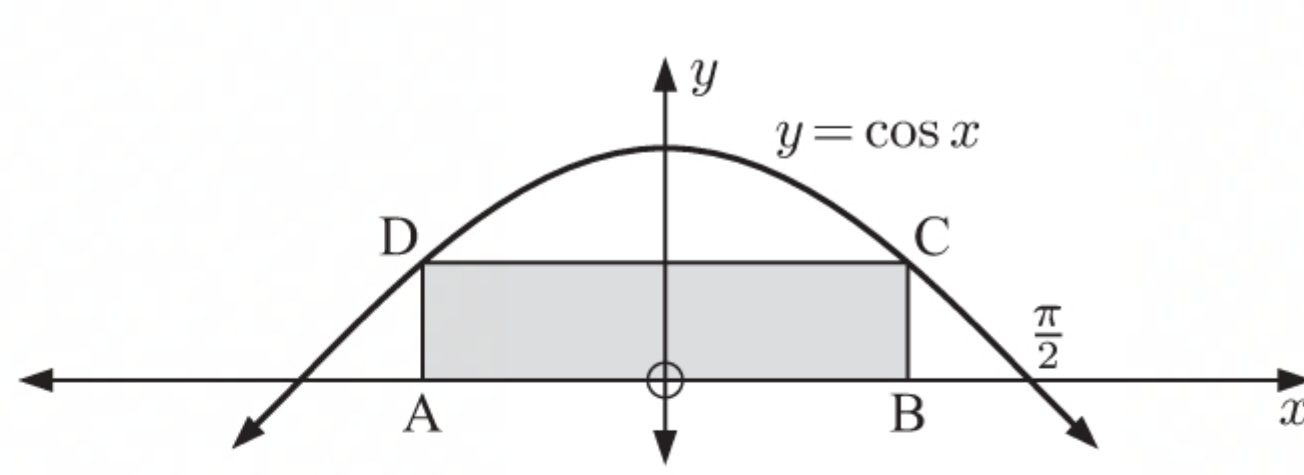
**a** Find  $f'(x)$  and  $f''(x)$ .

**b** Find the exact coordinates of the stationary points of the function and determine their nature.

**c** Find the exact  $x$ -coordinates of the inflection points of the function.

**d** Use technology to help sketch the function, showing the information you have found.

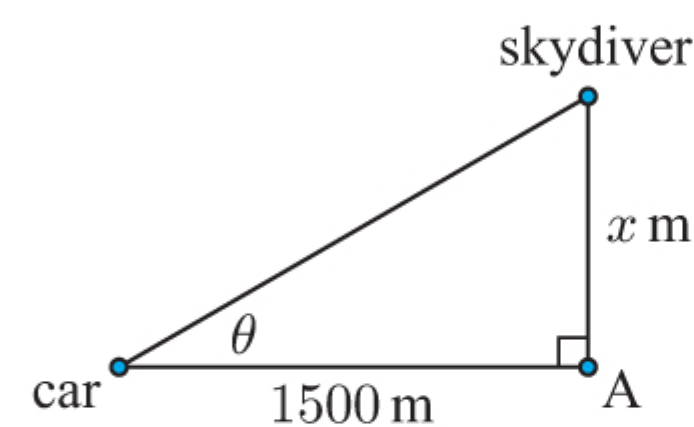


- 45** The cost of producing  $x$  bracelets is modelled by the function  $C(x) = -0.2x^2 + 4x + 10$  dollars for  $0 \leq x \leq 10$ .
- Calculate  $C(5)$  and explain what this represents.
  - Differentiate  $C$  with respect to  $x$ .
  - State the units that  $C'(x)$  is measured in.
  - Find  $C'(5)$ , and interpret your answer.
  - Determine the maximum value of  $C$  on  $0 \leq x \leq 10$ .
- 46** The tip of a clock's minute hand is  $H(t) = 10 \cos\left(\frac{\pi}{30}t\right) + 200$  cm above ground level, where  $t$  is the time in minutes after 2 pm.
- Find the height of the minute hand's tip at:
    - 2:30 pm
    - 2:45 pm
    - 2:51 pm.
  - Find  $H'(7)$  and interpret your answer.
  - Find, to the nearest minute, the time(s) between 2 pm and 3 pm at which the height of the minute hand's tip is increasing at 1 cm per minute.
- 47** The weight of a radioactive substance after  $t$  days is given by the function  $W(t) = 100e^{-\frac{t}{20}}$  grams,  $t \geq 0$ .
- Find the initial amount of radioactive substance present.
  - Find the time necessary for half of the mass to decay.
  - Find  $W'(t)$ , and interpret its sign.
  - Find  $W'(3)$ , and interpret your answer.
  - Discuss  $W$  as  $t$  increases.
- 48** Terry wants to fence off a rectangular garden plot of area  $48 \text{ m}^2$ . Three sides will be fenced with strong wire mesh costing \$18 per metre, and the remaining side will be fenced with corrugated iron costing \$30 per metre.
- By letting  $x$  be the length in metres of the side fenced with corrugated iron, show that the cost of fencing is  $C = 48\left(\frac{36}{x} + x\right)$  dollars.
  - Find the dimensions of the garden plot which will minimise the cost of fencing.
- 49** A man standing on a cliff above the ocean throws a ball high in the air. The height of the ball above the water  $t$  seconds after release is given by  $h(t) = 100 + 32t - 4t^2$  m.
- Find  $h'(t)$ .
  - Find the maximum height above water reached by the ball.
- 50** The diagram alongside shows an open trough. The total outside surface area is  $27 \text{ m}^2$ .
- Show that  $x^2 + 2xy = 27$ .
  - Find an expression for the volume  $V$  of the trough in terms of  $x$  only.
  - Hence show that the volume of the trough is maximised if  $x = y = 3$ .
- 
- 51** A comet travels in an orbit which can be described by the equation  $y = x^2$  as shown in the diagram.
- Show that the distance of the comet at  $C(x, y)$  from an observer at the point  $P(0, 2)$  is given by  $S(x) = \sqrt{x^4 - 3x^2 + 4}$ .
  - Find  $\frac{d}{dx}(S^2)$ , and hence find the values of  $x$  which minimise  $S^2$ .
  - Find the shortest and the greatest distance between the comet and the observer for  $-2 \leq x \leq 2$ .
- 
- 52** Rectangle ABCD is inscribed under one arch of  $y = \cos x$ . Suppose the point C has  $x$ -coordinate  $x$ . Find the coordinates of C such that ABCD has maximum area.
- 
- 53** A ball of ice cream with initial radius 8 cm takes 5 minutes to melt. During this time, its radius decreases at a constant rate.
- Given the ice cream has radius  $r$  cm after  $t$  minutes, explain why  $\frac{dr}{dt} = -1.6$ .
  - Find the rate of change of volume of the ice cream ball 2.5 minutes after it begins to melt.
  - Find the average rate of change of volume for the last 4 minutes of melting time.



- 54** A skydiver jumps from a plane and descends towards A at a constant rate of  $50 \text{ m s}^{-1}$ . A car is parked 1500 m from A.

Find the rate at which the angle of elevation  $\theta$  from the car to the skydiver is changing when  $\theta = \frac{\pi}{6}$ .



- 55** Consider the function  $f(x) = x^3 - 6x^2 + 11x + 7$ .

A table of values for  $f(x)$  is shown alongside.

$i$	$x_i$	$f(x_i)$
0	1	13
1	1.2	13.288
2	1.4	13.384
3	1.6	13.336
4	1.8	13.192
5	2	13

- a** Using the trapezoidal rule with 5 subintervals, estimate the area between  $y = f(x)$  and the  $x$ -axis for  $1 \leq x \leq 2$ .
- b** Find  $\int_1^2 f(x) dx$  and interpret your answer.
- c** Hence find the percentage error in your estimate in **a**.

- 56** Find the antiderivative of:

**a**  $2x$

**b**  $\frac{x^2}{3}$

**c**  $\frac{3}{x^2}$

- 57** Differentiate  $\frac{2}{x^2} - 3x$ , and hence find  $\int \left( \frac{8}{x^3} + 6 \right) dx$ .

- 58** Find:

**a**  $\int -3 dx$

**b**  $\int \left( \frac{3}{x^2} + 2x^3 - 4 \right) dx$

**c**  $\int \left( \frac{1}{x} + 2x \right)^2 dx$

- 59** Find  $y$  if:

**a**  $\frac{dy}{dx} = 4x$

**b**  $\frac{dy}{dx} = x^2 + \frac{1}{2}x + \frac{1}{3}$

**c**  $\frac{dy}{dx} = \frac{3x^4 + 5}{x^3}$

- 60** Integrate with respect to  $x$ :

**a**  $x\sqrt{x} - 5 \cos x$

**b**  $\sin x + \frac{1}{\sqrt[3]{x}}$

- 61** Find:

**a**  $\int (x - 3)^2 dx$

**b**  $\int \frac{x^2 + 3x + 5}{\sqrt[3]{x}} dx$

- 62** Find  $f(x)$  given that  $f'(x) = 4x - 3x^2$  and  $f(3) = -2$ .

- 63** Find  $f(x)$  given that:

**a**  $f''(x) = e^x + 2x - 1$ ,  $f'(0) = 4$ ,  $f(0) = 1$

**b**  $f''(x) = 2 + \sin x$ ,  $f'(\pi) = 1$ ,  $f(\frac{\pi}{2}) = \frac{\pi^2}{4}$

**c**  $f''(x) = \frac{2}{\sqrt{x}} + 3x$ ,  $f(1) = -\frac{19}{3}$ ,  $f(4) = \frac{64}{3}$

- 64** Find:

**a**  $\int (3x - 5)^3 dx$

**b**  $\int \frac{2}{\sqrt{4-x}} dx$

**c**  $\int (e^{2x} + 3e^{-x+2}) dx$

- 65** Integrate with respect to  $x$ :

**a**  $2 \sin(x - 3) + e^{3x}$

**b**  $\frac{2}{5x - 1}$

**c**  $\cos(5 - 7x)$

**d**  $\frac{12}{\sqrt[3]{x}}$

**e**  $(e^x - 4)^2$

**f**  $5 \sin x + \frac{3}{\cos^2 x}$

- 66** Find  $f(x)$  given  $f'(x) = \frac{4}{5-x}$  and  $f(4) = 6$ .

- 67** Find:

**a**  $\int 3x^2(5 + x^3)^4 dx$

**b**  $\int x e^{x^2+2} dx$

**c**  $\int \frac{e^{\frac{1}{x}}}{x^2} dx$

**d**  $\int \frac{2(\ln x)^2}{x} dx$

- 68** Integrate with respect to  $x$ :

**a**  $\sqrt{x^2 + 3x - 1} (2x + 3)$

**b**  $\frac{e^x + 2}{e^x + 2x}$

**c**  $\frac{6 - 8x}{2x^2 - 3x + 2}$

- 69** Find:

**a**  $\int_0^1 (x^2 + x) dx$

**b**  $\int_{-2}^1 (x^3 - 2x^2 - 4x + 9) dx$

**c**  $\int_3^5 \left( \frac{8}{x^2} + 3x \right) dx$



**70** Find:

**a**  $\int_{-1}^1 3x^2 dx$

**b**  $\int_2^3 \frac{5x^3 - 2x}{x^5} dx$

**c**  $\int_{-3}^0 (1 - 3x)^2 dx$

**71** Use technology to evaluate, correct to 4 significant figures:

**a**  $\int_0^3 2^x dx$

**b**  $\int_{-1.5}^{2.4} \frac{5x + 20}{x + 6} dx$

**c**  $\int_{-2}^{-1} \ln(-x) dx$

**d**  $\int_3^7 2\sqrt{x}e^{-2x} dx$

**72** Find  $\int_0^{\frac{\pi}{2}} (\sin 3x + 5 \cos x) dx$ .

**73** Find  $a$  given that  $\int_a^{2a} \sqrt{x} dx = 2$ .

**74** If  $y = x\sqrt{4-x}$ , find  $\frac{dy}{dx}$  and simplify your answer. Hence evaluate  $\int_0^2 \frac{8-3x}{\sqrt{4-x}} dx$ .

**75** Find:

**a**  $\int_1^5 \frac{2x^3 + 1}{x^2} dx$

**b**  $\int_{-1}^1 e^x (2 - 3e^{-x})^2 dx$

**c**  $\int_0^2 \frac{3}{5-2x} dx$

**d**  $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{4}{\cos^2(\frac{x}{2})} dx$

**76** Find the exact value of:

**a**  $\int_3^5 \frac{x}{x^2 - 8} dx$

**b**  $\int_0^2 x\sqrt{x^2 + 1} dx$

**c**  $\int_1^4 \frac{3e^{\sqrt{x}}}{\sqrt{x}} dx$

**77** For a continuous function  $f(x)$  defined on the interval  $a \leq x \leq b$ , the length of the curve can be found using

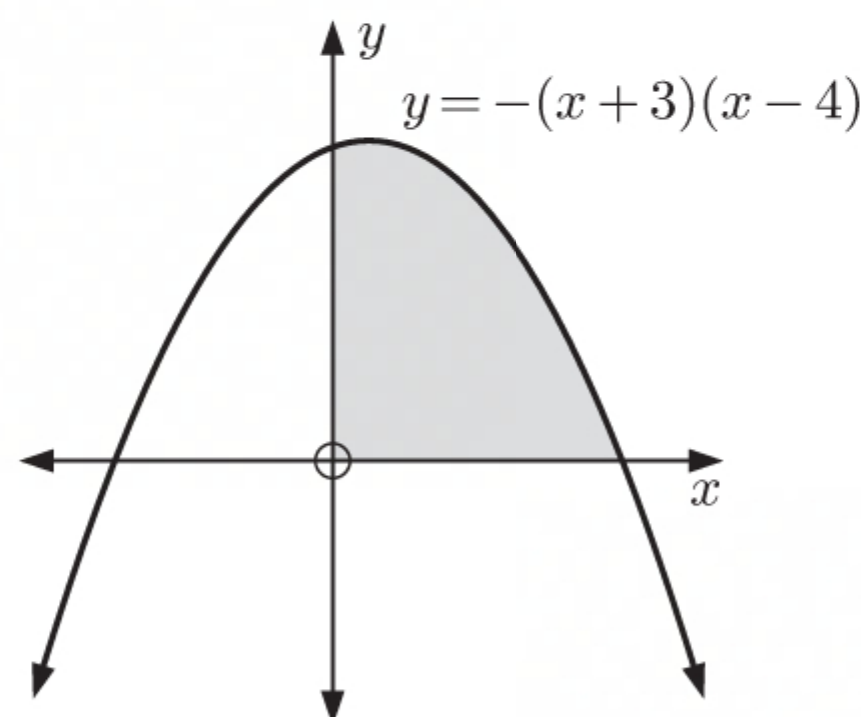
$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$ . Find the length of:

**a**  $f(x) = x^2$  on the interval  $0 \leq x \leq 1$

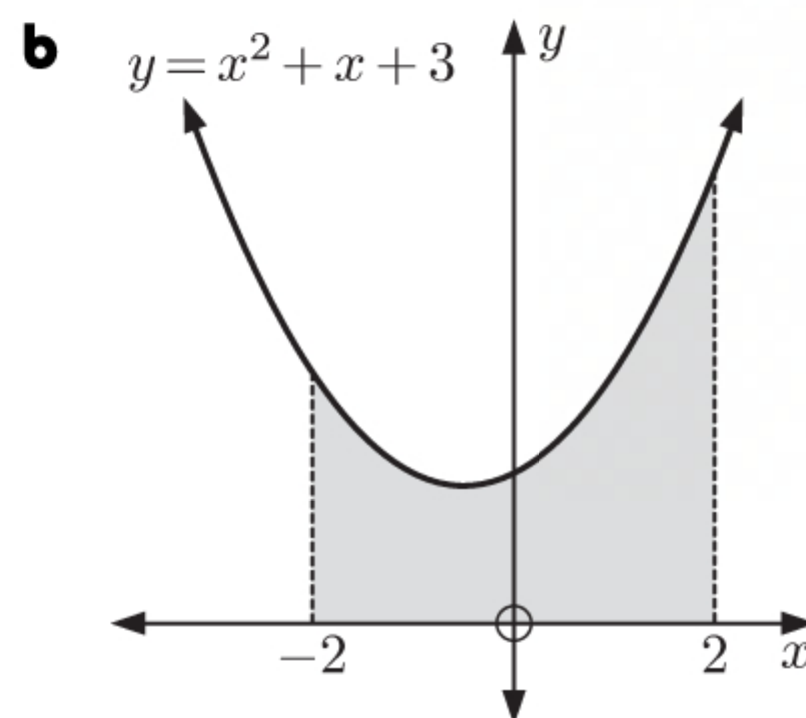
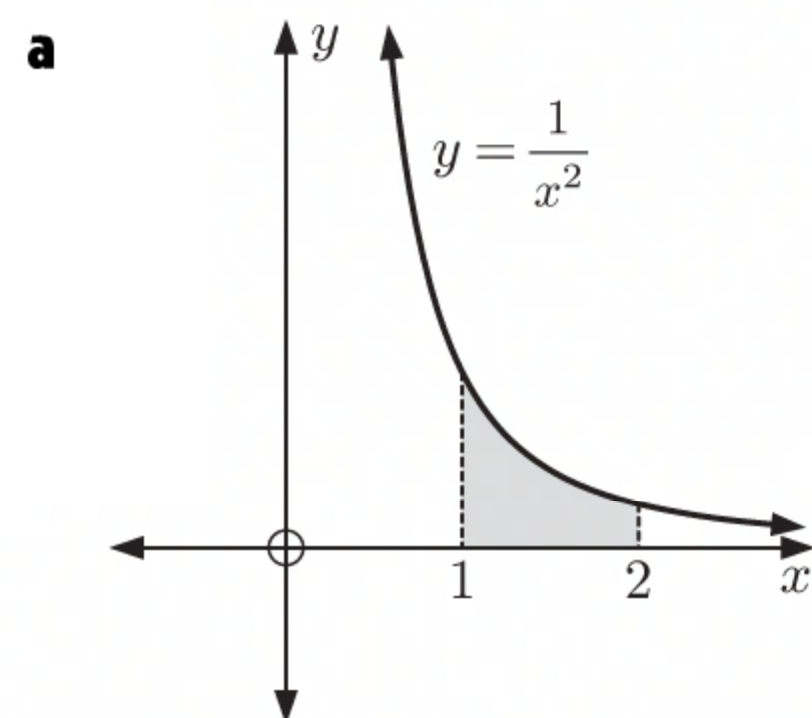
**b**  $f(x) = \sin x$  on the interval  $0 \leq x \leq \pi$ .

**78 a** Write an integral to represent the shaded area.

**b** Hence find the shaded area.



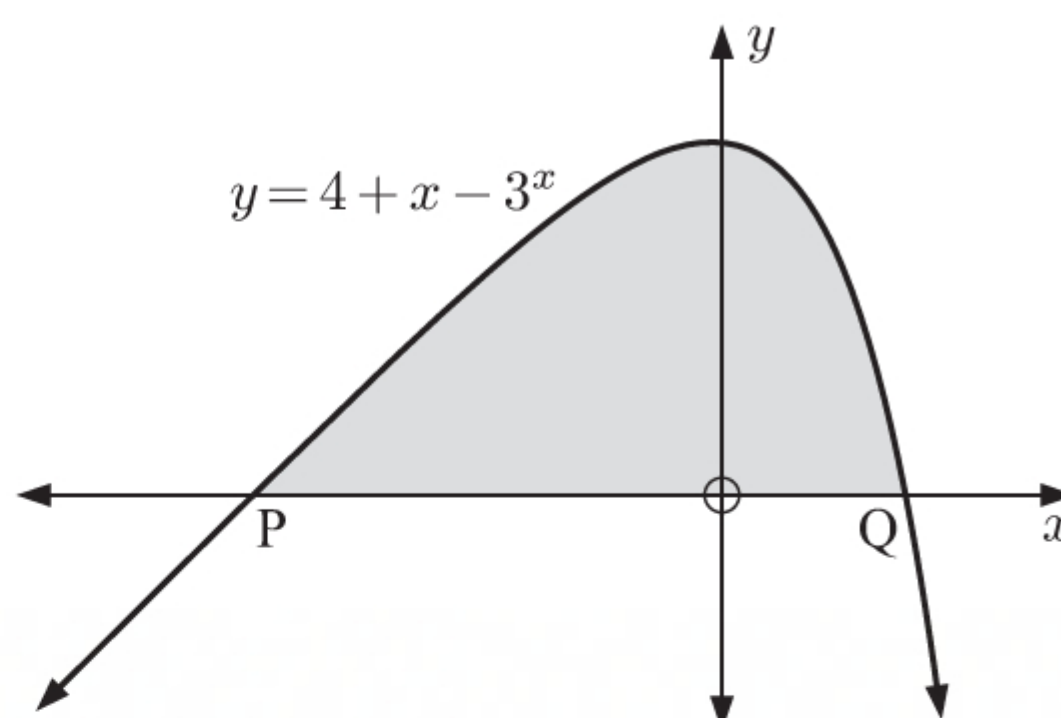
**79** Find the shaded area:



**80** Use technology to find:

**a** the coordinates of P and Q

**b** the shaded area.





**81** Find the area of the region bounded by:

**a**  $y = x^2 + 2x + 2$ , the  $x$ -axis,  $x = 0$ , and  $x = 3$

**c**  $y = (2x + 5)^2$ , the  $x$ -axis, and  $x = -4$ .

**b**  $y = 4 - x^2$  and the  $x$ -axis

**82** Find the area of the region bounded by:

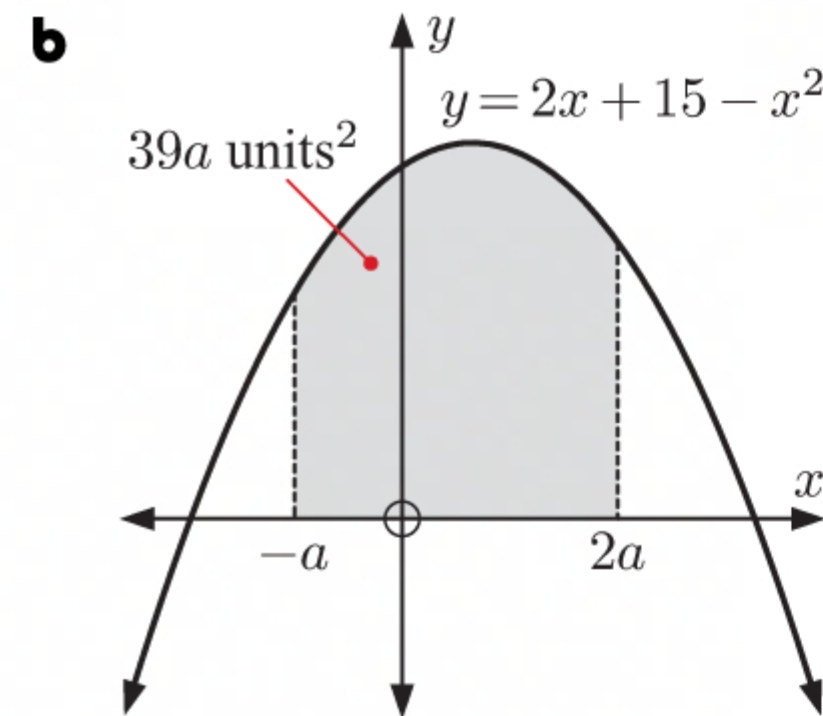
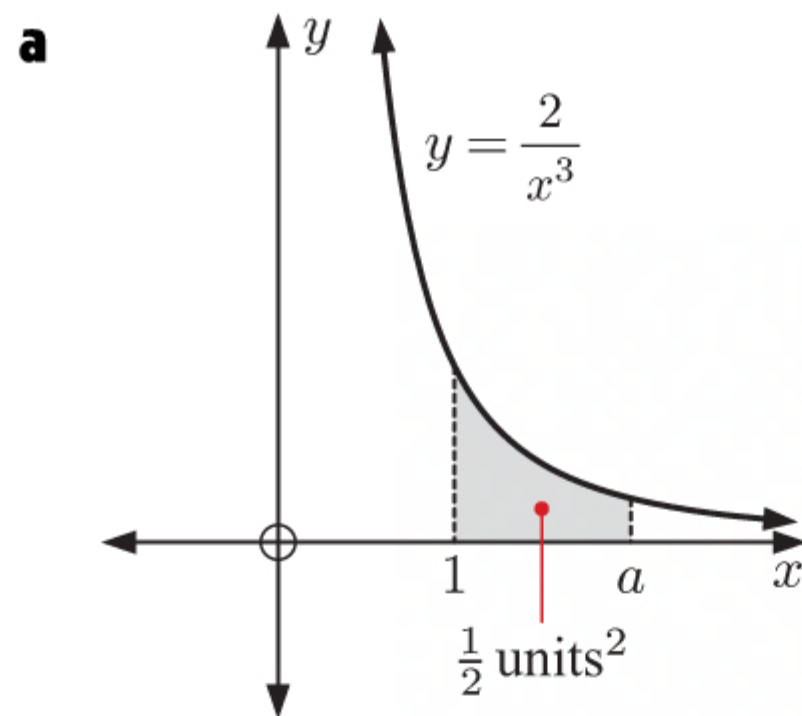
**a**  $y = x^2 + x$ , the  $x$ -axis,  $x = 2$ , and  $x = 4$

**c**  $y = \frac{1}{\sqrt{x+2}}$ , the  $x$ -axis,  $x = 2$ , and  $x = 7$

**b**  $y = \sin 2x$ , the  $x$ -axis,  $x = \frac{\pi}{4}$ , and  $x = \frac{\pi}{2}$

**d**  $y = e^{-3x}$ , the  $x$ -axis,  $x = 0$ , and  $x = 1$ .

**83** Find the exact value of  $a$ :



**84** The function  $f(x) = x(x^2 - k)$  is graphed alongside.

**a** Find the value of  $k$ .

**b** Find the coordinates of P.

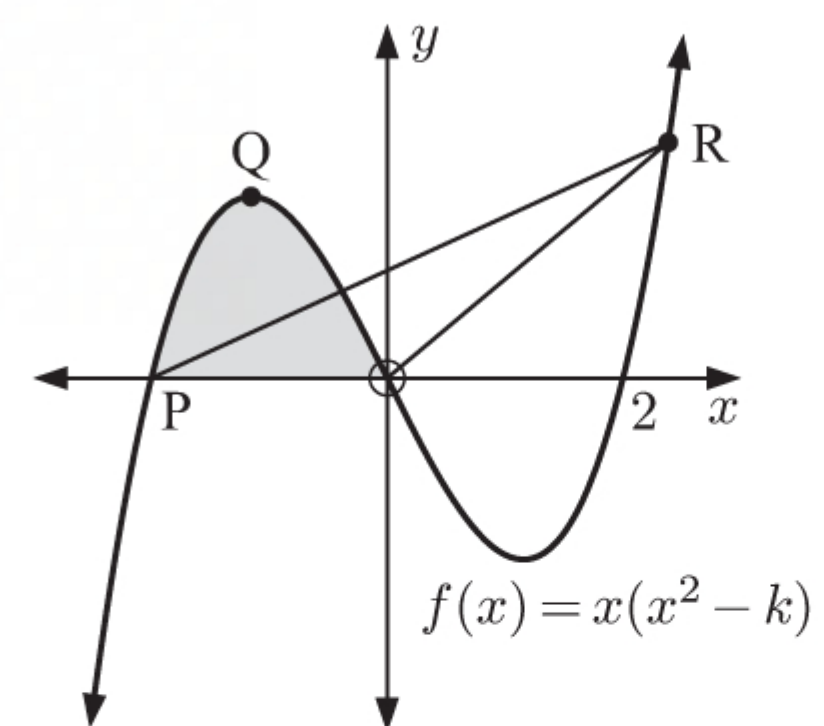
**c** Find  $f'(x)$ .

**d** Hence find the exact  $x$ -coordinate of the local maximum Q.

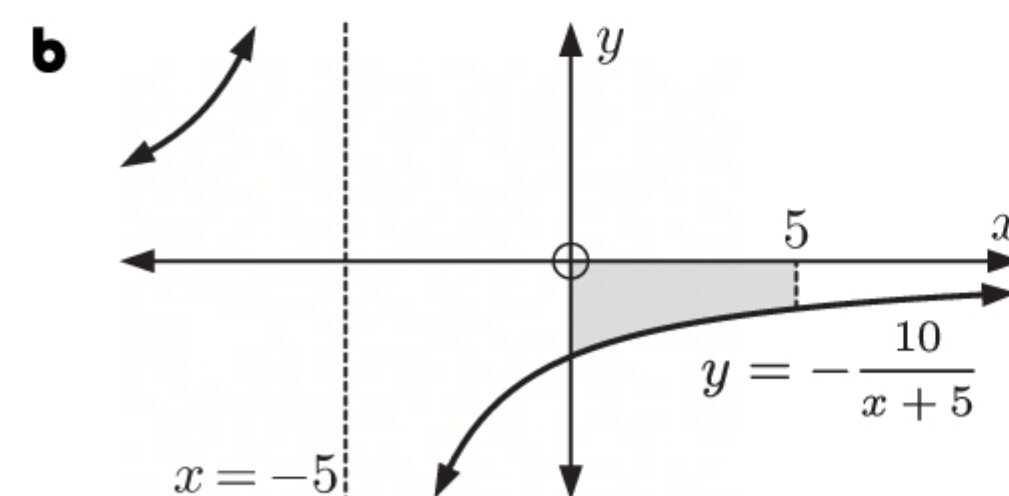
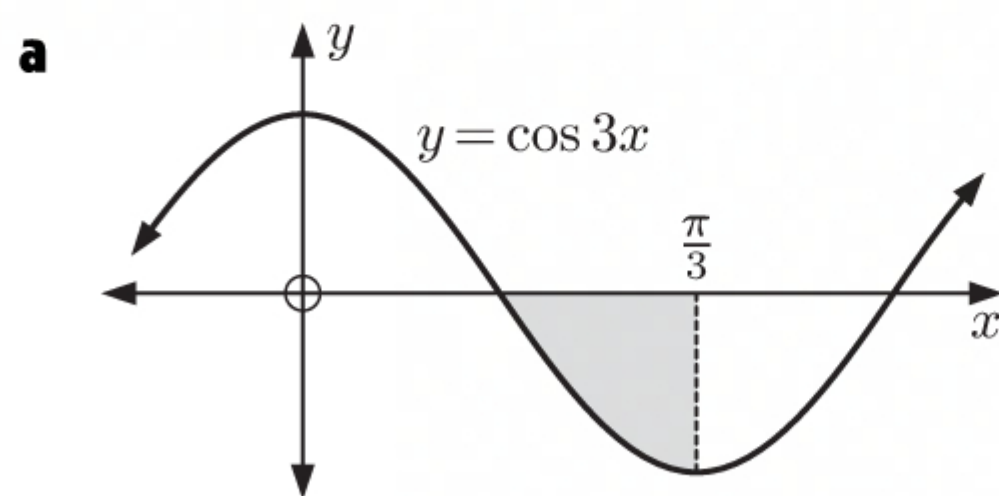
**e** Find  $\int f(x) dx$ .

**f** Hence find the shaded area.

**g** The point R lies on the graph of  $y = f(x)$  such that the area of triangle POR is equal to the shaded area. Find the coordinates of R.



**85** Find the shaded area:



**86** **a** Use technology to help sketch the graph of  $f(x) = \sqrt{x} - \frac{8}{x^2} - 2$ .

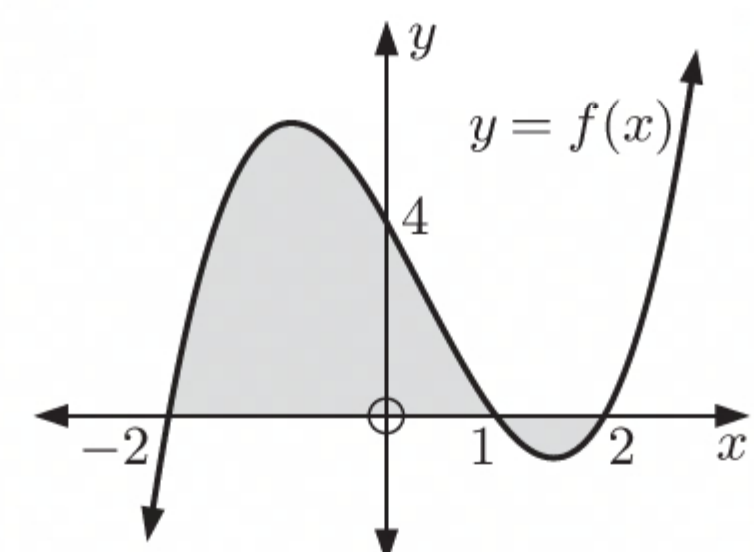
**b** Find the exact area of the region bounded by  $y = f(x)$ , the  $x$ -axis,  $x = 1$ , and  $x = 4$ .

**87** Consider the graph of  $f(x) = x^3 - x^2 - 4x + 4$ .

**a** Find  $\int_{-2}^2 f(x) dx$ .

**b** Explain why the value obtained in **a** does not represent the shaded area.

**c** Find the shaded area.



**88** Suppose  $f(x) = \frac{5}{x^2 + 2x - 9}$ . Use technology to:

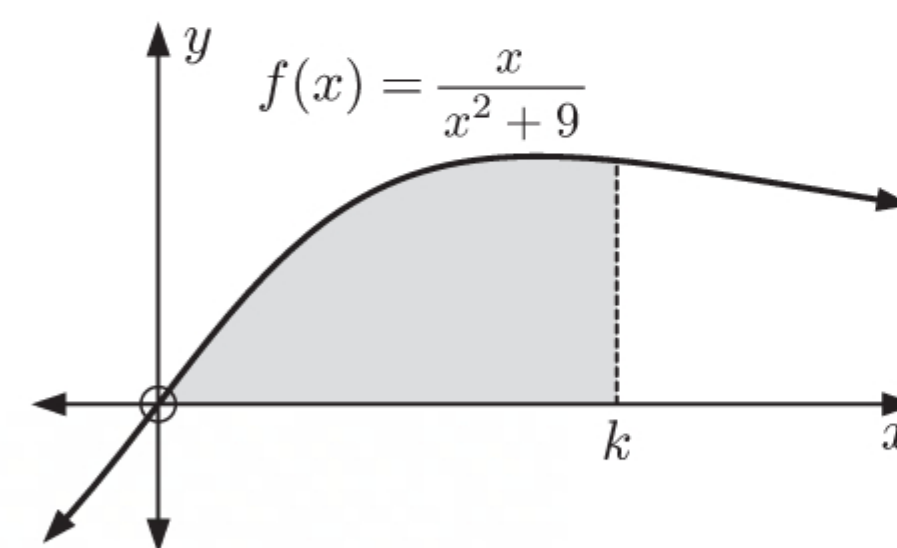
**a** sketch the graph of  $y = f(x)$

**b** find the area of the region bounded by  $y = f(x)$ , the axes, and  $x = 1$ .



**89** Find  $\frac{d}{dx}(\ln(x^2 + 9))$ .

The area of the shaded region is  $\ln 3$  units<sup>2</sup>. Find the exact value of  $k$ .

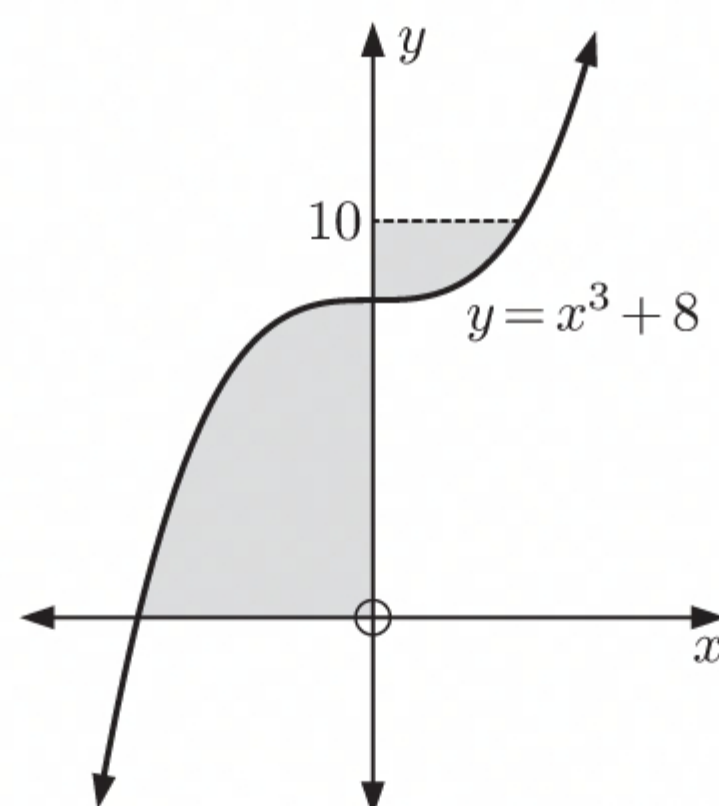


**90** Find the exact area of the region bounded by:

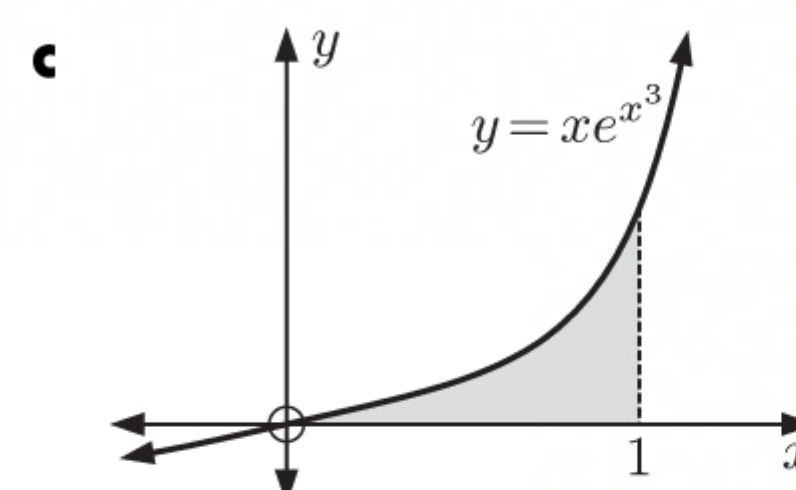
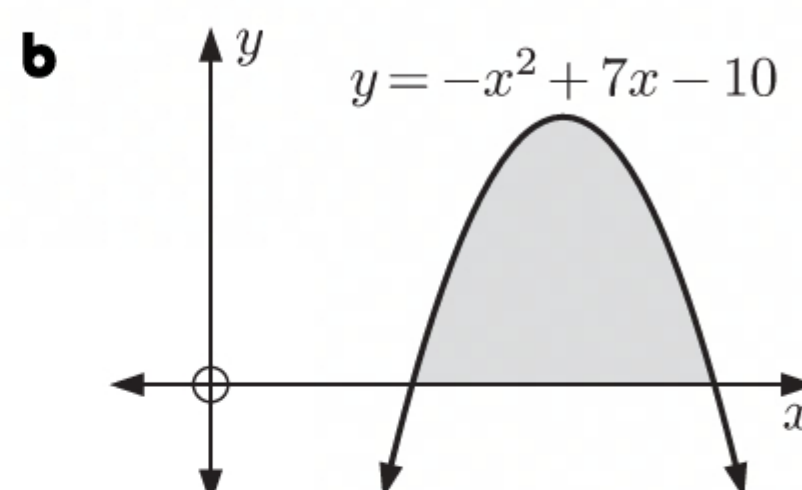
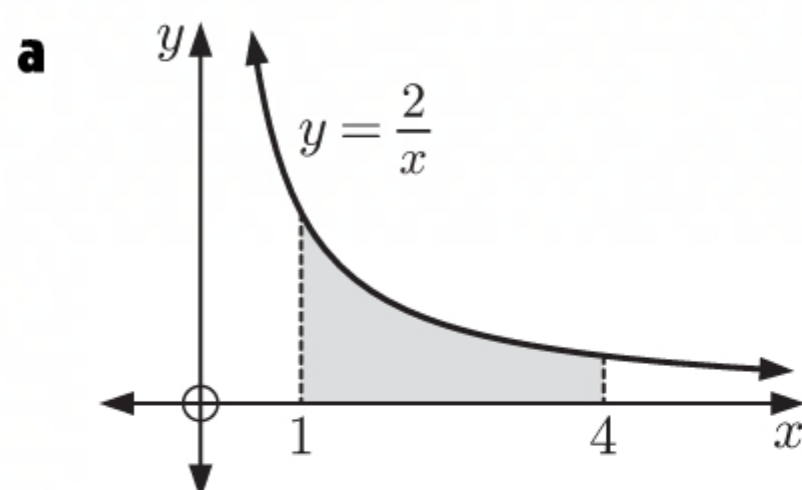
**a**  $y = \sqrt{9 - x}$ , the  $y$ -axis, and the lines  $y = 1$  and  $y = 2$

**b**  $y = -\frac{3}{2x}$ , the  $y$ -axis, and the lines  $y = 2$  and  $y = 5$

**91** Find the shaded area:



**92** Find the volume of revolution when the shaded region is revolved through  $2\pi$  about the  $x$ -axis.



**93** Find the exact volume of the solid formed when the region enclosed by  $y = \frac{1}{\cos x}$ , the  $x$ -axis,  $x = \frac{\pi}{6}$ , and  $x = \frac{\pi}{3}$ , is rotated about the  $x$ -axis.

**94** Find the exact volume of the solid formed when the region enclosed by  $y = \ln x$ , the axes, and the line  $y = \ln 3$  is rotated about the  $y$ -axis.

**95** Find the exact volume of revolution when the relation  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ,  $x \geq 0$  is rotated  $2\pi$  about the  $y$ -axis.

**96** The rate at which a tree grows  $t$  years after planting is given by  $G(t) = \frac{2.5}{t+1}$  metres per year.

**a** Explain why the tree is always growing taller.

**b** Evaluate the following integrals, and interpret their meaning: **i**  $\int_0^5 G(t) dt$  **ii**  $\int_5^{10} G(t) dt$

**c** After 15 years, the tree is struck by lightning and is cut down. How much did the tree grow over its lifetime?

**97** A particle is moving in a straight line with velocity given by  $v(t) = t^3 - 3t^2e^{0.05t}$ , where  $t \geq 0$  is in seconds, and distance units are in metres.

Use technology to find:

**a** the greatest speed reached by the particle in the first 4 seconds of motion

**b** the total distance travelled by the particle in the first 4 seconds of motion.

**98** A particle moves in a straight line with displacement function  $s(t) = 12t - 3t^3 + 1$  cm, where  $t \geq 0$  is in seconds.

**a** Find the velocity and acceleration functions for the particle's movement.

**b** Find the speed of the particle after:

**i** 1 second

**ii** 2 seconds.

**c** When is the particle's:

**i** velocity decreasing

**ii** speed decreasing?

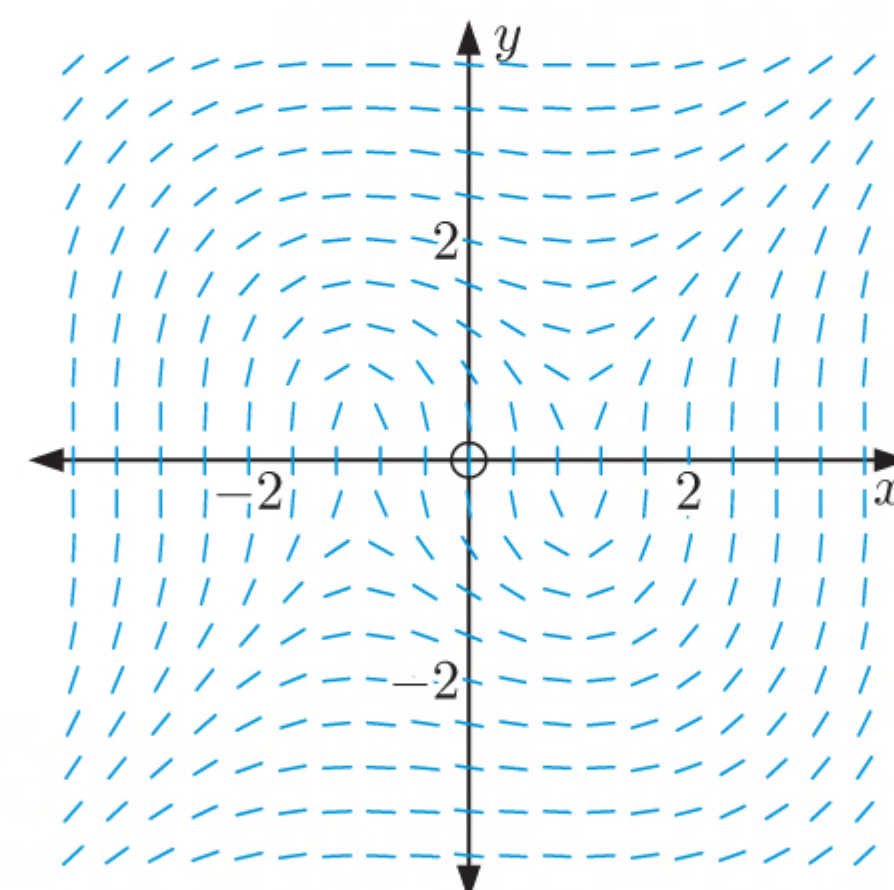


- 99** A particle moves from rest along a straight line. Its velocity is given by  $v = 2\sqrt{t} - t \text{ m s}^{-1}$ , where  $t \geq 0$  is the time in seconds.
- Find the speed of the particle after 5 seconds.
  - Find the acceleration function of the particle.
  - Show that the particle changes direction after 4 seconds.
  - Find the total distance travelled in the first 9 seconds.
- 100** The velocity of a truck  $t$  seconds after applying its brakes is  $v = \frac{20}{\sqrt{2t+1}} \text{ m s}^{-1}$ ,  $0 \leq t \leq 10$ .
- Find the speed of the truck when the brakes are applied.
  - Find the acceleration function.
  - At what time does the truck have acceleration  $-2.5 \text{ m s}^{-2}$ ?
  - Find the distance travelled by the truck in the first 10 seconds after applying the brakes.
- 101** A particle with displacement  $s \text{ m}$  moves with velocity  $v = 2s + \frac{1}{s} \text{ m s}^{-1}$ . The particle is initially 0.5 m to the right of the origin.
- Find the acceleration of the particle in terms of  $s$ .
  - Find the initial velocity and acceleration of the particle.
  - Is the speed of the particle increasing or decreasing initially? Explain your answer.
  - Given  $s \geq 0.5$ , find the displacement of the particle when:
    - its velocity is  $8.25 \text{ m s}^{-1}$
    - its velocity is equal to its acceleration.
- 102** Consider the differential equation  $\frac{dy}{dx} = e^x - 2x$  with  $y(0) = 1$ .
- Estimate  $y(1)$  by applying Euler's method with:
    - $h = 0.5$  for two steps
    - $h = 0.25$  for four steps.
  - Find  $y(1)$  exactly using the Fundamental Theorem of Calculus. Comment on your results.
- 103** Use Euler's method with step size 0.2 to estimate  $y(2)$  given  $\frac{dy}{dx} = x - y$ ,  $y(1) = 2$ .
- 104** Find the particular solution to:
- $\frac{dy}{dx} = 3x + 7x^3$  given  $y(0) = 5$
  - $\frac{dy}{dx} = \cos 2x - 3e^x$  given  $y(0) = -2$
  - $\frac{dy}{dx} = \frac{2 \sin 3x}{4 + \cos 3x}$  given  $y(\frac{\pi}{2}) = 2 \ln 2$
- 105** Find the general solution to the following differential equations:
- $\frac{dy}{dt} = \frac{\cos t}{\sqrt{1 - \sin t}}$
  - $4x \frac{dN}{dx} = \left(2x - \frac{3}{x}\right)^2$
  - $\frac{5}{s} + e^{-2s} - \frac{dQ}{ds} = 0$
- 106** Solve the following differential equations:
- $\frac{dy}{dx} = xy^2$
  - $\frac{dy}{dx} = 5\sqrt{y}$
  - $\frac{dy}{dx} = \frac{xy}{x^2 + 1}$
- 107** Find the particular solution to:
- $\frac{dP}{dz} = -3P^2z$  given  $P = 1$  when  $z = 2$
  - $\frac{dy}{dx} = x + \frac{1}{3}xy$  given  $y = 2$  when  $x = 1$ .
- 108** Solve the differential equation  $\frac{dy}{dx} = \frac{xy}{x-1}$  given that  $y = 2$  when  $x = 2$ .
- 109** Find the particular solution to  $\frac{dy}{dx} = \frac{2x}{\sin y}$  given that  $y(0) = \frac{\pi}{6}$ . Find the values of  $x$  for which the solution is defined.
- 110** An object moves in a resisting medium such that its velocity  $v$ , decreases at a rate  $\frac{dv}{dt} = -kv$ , where  $k$  is a positive constant. The initial velocity of  $100 \text{ m s}^{-1}$  is reduced to  $40 \text{ m s}^{-1}$  in 2 seconds.
- Show that  $k = \frac{1}{2} \ln\left(\frac{5}{2}\right)$ .
  - Find the distance the object travels in the medium in the first 2 seconds.



**111** The slope field for  $\frac{dy}{dx} = \frac{x^2 - 1}{y^2}$  is shown.

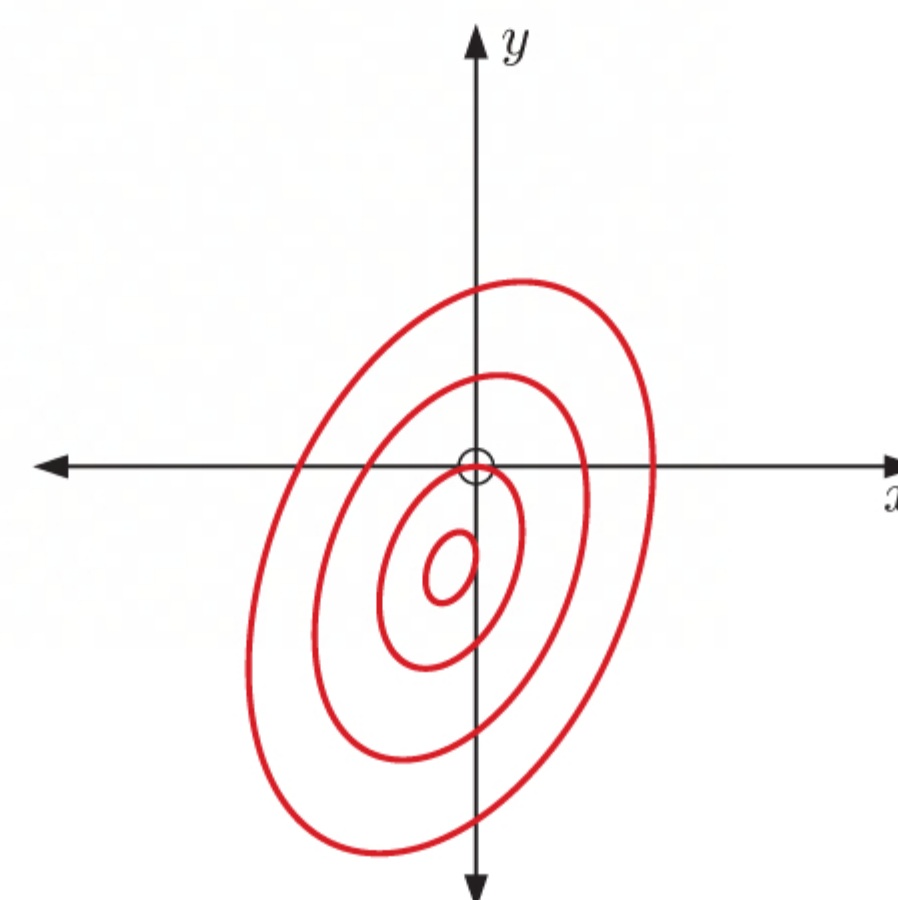
- Find the gradient of the tangent to the solution curve at  $(0, 1)$ .
- Sketch the solution curve which passes through  $(0, 1)$ .
- Find the equation of the solution curve drawn in **b**.



**112** The diagram alongside shows solution curves for the coupled system

$$\begin{cases} \dot{x} = 2y - x + 2 \\ \dot{y} = y - 4x \end{cases}$$

- Locate and describe the equilibrium point.
- Do the solution curves rotate clockwise or anticlockwise? Explain your answer.



**113** The system  $\begin{cases} \dot{x} = 3x + 2y \\ \dot{y} = x + 4y \end{cases}$  can be written in the form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  where  $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$  has eigenvalues 2, 5 with corresponding eigenvectors  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  respectively.

- Find the general solution to the system.
- Given the initial point  $(1, -1)$ , find:
  - $\dot{\mathbf{x}}$  when  $t = 0$
  - the particular solution to the system.
- Describe the equilibrium point at  $(0, 0)$ .
- Sketch the phase portrait, including the particular solution.
- Discuss the behaviour of the system in the long term.

**114** Consider the system  $\begin{cases} \frac{dx}{dt} = x - 6y \\ \frac{dy}{dt} = 6x - y \end{cases}$  with initial point  $(1, 0)$ .

- Find the eigenvalues of  $\begin{pmatrix} 1 & -6 \\ 6 & -1 \end{pmatrix}$ , and hence explain why the equilibrium point of the system is a centre.
- Find the initial trajectory vector.
- Do the solution curves rotate clockwise or anticlockwise?
- Sketch the phase portrait including the trajectory from the initial point.

**115** Consider the second order differential equation  $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 4x = 0$ .

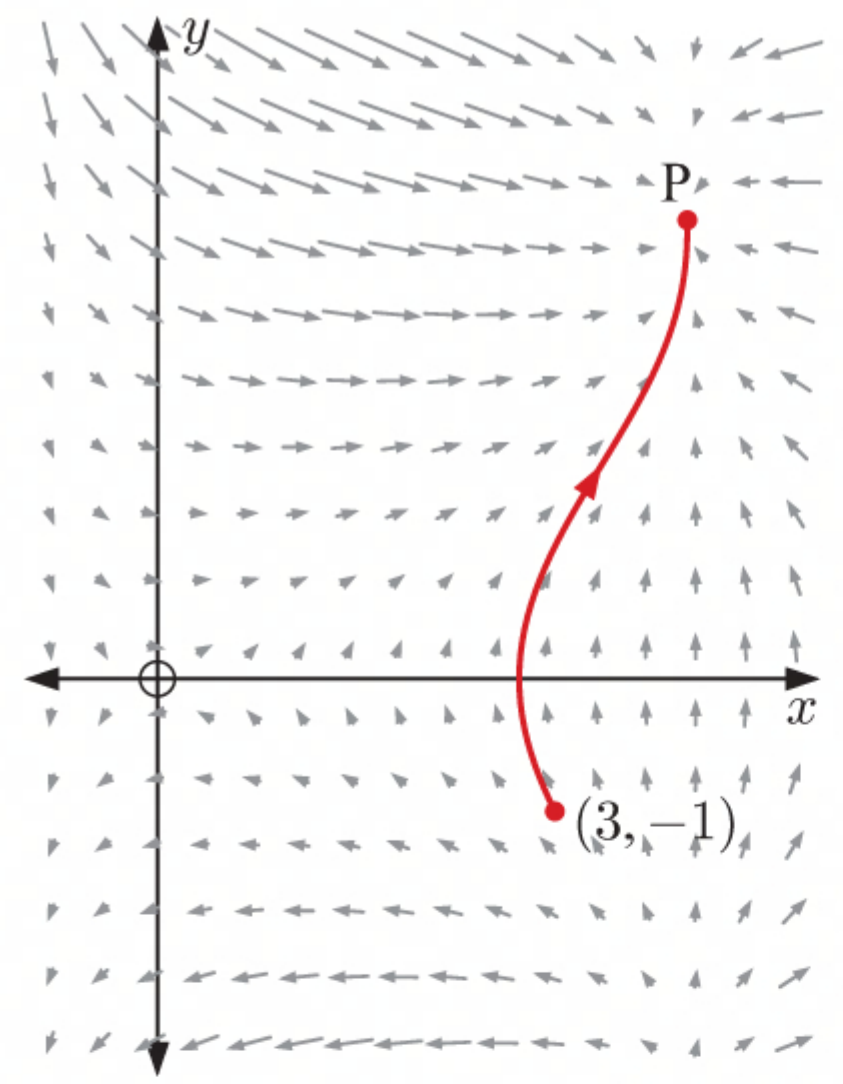
- Write this as a coupled system of first order differential equations.
- Write the system in the matrix form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ .
- Find the eigenvalues and eigenvectors of  $\mathbf{A}$ .
- Describe the equilibrium point of the system.
- Initially,  $x = -3$  and  $\frac{dx}{dt} = -2$ .
  - Find the particular solution for  $\mathbf{x}$ .
  - Describe the behaviour of  $x$  as  $t \rightarrow \infty$ .



**116** This phase portrait shows the solution curve for the coupled system

$$\begin{cases} \dot{x} = 3xy + 4y - x^2y \\ \dot{y} = 3x - y^2 \end{cases}, \quad x_0 = 3, \quad y_0 = -1.$$

- a** Find the initial trajectory for the solution curve.
- b** Use Euler's method with step length 0.1 to approximate:
  - i** the position of the solution curve when  $t = 0.5$
  - ii** the coordinates of the equilibrium point P.
- c** Find the exact coordinates of P, and hence determine the accuracy of your approximation in **b ii**.



**117** A pond centred at  $(0, 0)$  has radius 1 unit.

A cat, initially at  $(1, 0)$ , runs anticlockwise around the edge of the pond at 1 unit per second.

A dog is in the pond, pursuing the cat, and is always swimming directly towards the cat.

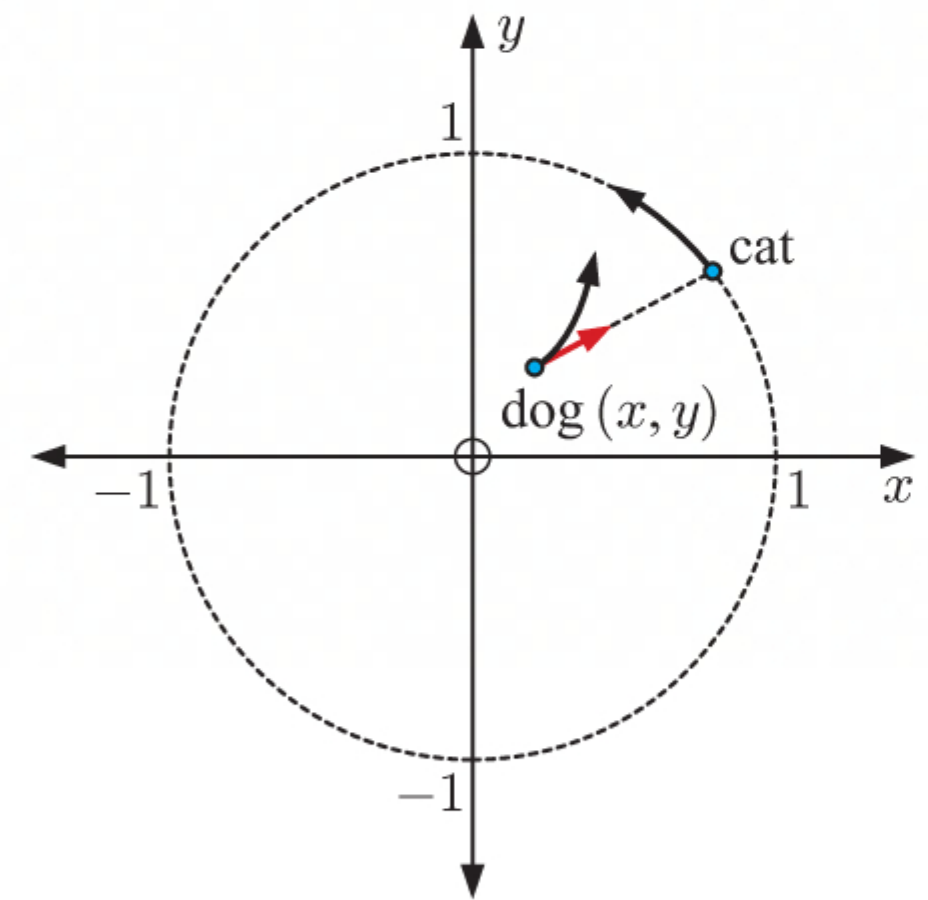
- a** Explain why the position of the cat at time  $t$  is  $(\cos t, \sin t)$ .
- b** Suppose the dog swims at  $D$  units per second.

Explain why the position  $(x, y)$  of the dog at time  $t$  satisfies

$$\begin{cases} \dot{x} = \frac{\cos t - x}{\sqrt{(\cos t - x)^2 + (\sin t - y)^2}} D \\ \dot{y} = \frac{\sin t - y}{\sqrt{(\cos t - x)^2 + (\sin t - y)^2}} D. \end{cases}$$

- c** The dog is initially at  $(0, 0)$ , and swims at half the speed of the cat, so  $D = 0.5$ .
  - i** Use Euler's method with step length 0.5 to estimate the position of the dog each second for the first 20 seconds.
  - ii** Find the distance of the dog from the centre of the pond after:
 

<b>(1)</b> 18 seconds	<b>(2)</b> 19 seconds	<b>(3)</b> 20 seconds.
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  - iii** Hence predict the behaviour of the dog in the long term.



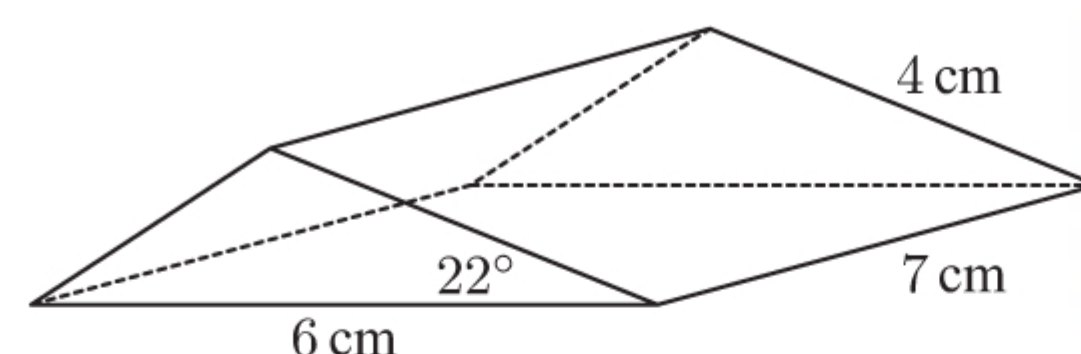


# Mixed questions

## MIXED QUESTIONS SET 1

- 1** In this triangular prism, the side lengths are rounded to the nearest centimetre, and the angle is rounded to the nearest degree.

Find the boundary values for the prism's volume.



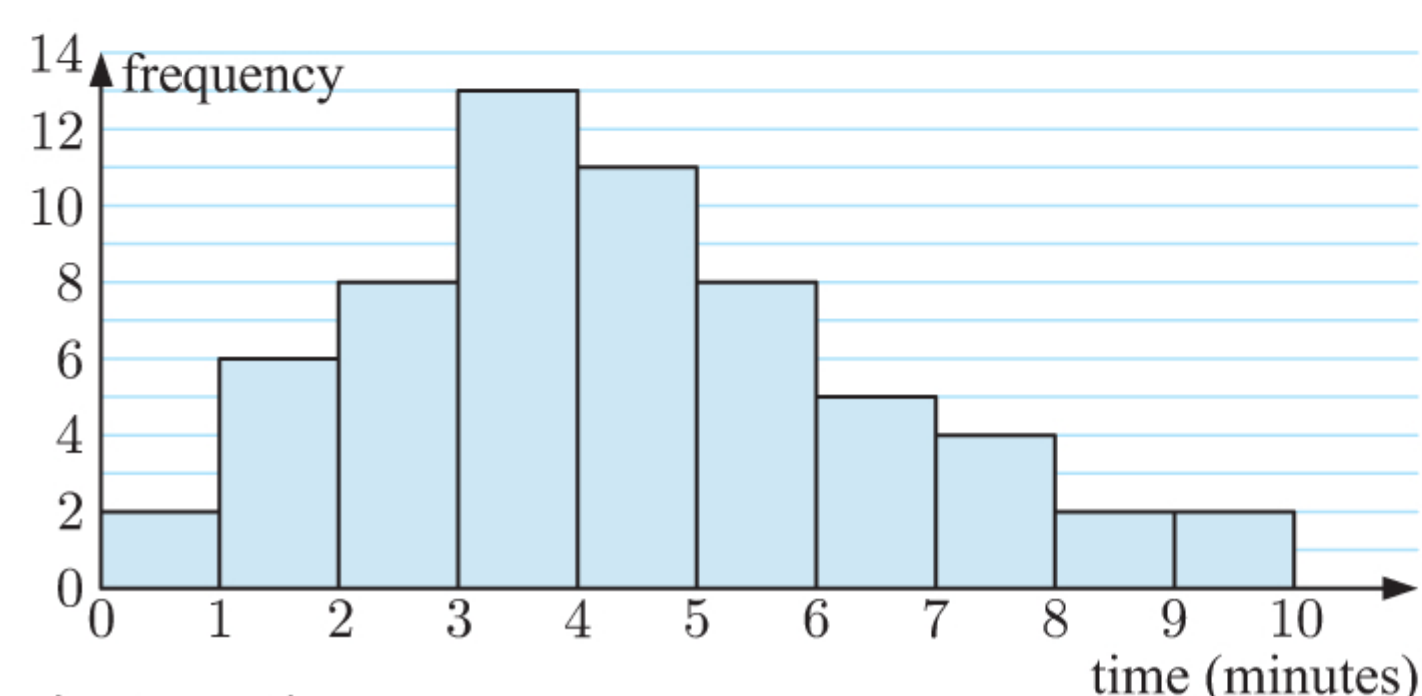
- 2** The height  $H$  of a small toy aeroplane,  $t$  seconds after it is thrown from the top of a building, is given by the function  $H(t) = 80 - 5t^2$  metres, where  $t \geq 0$ .

- Find the initial height of the toy aeroplane.
- Determine the time it takes for the toy aeroplane to hit the ground.
- Find  $H'(2)$ , and explain what this value means.

- 3** The value of a car decreases by 10% each year. After 3 years its value is \$26 244.

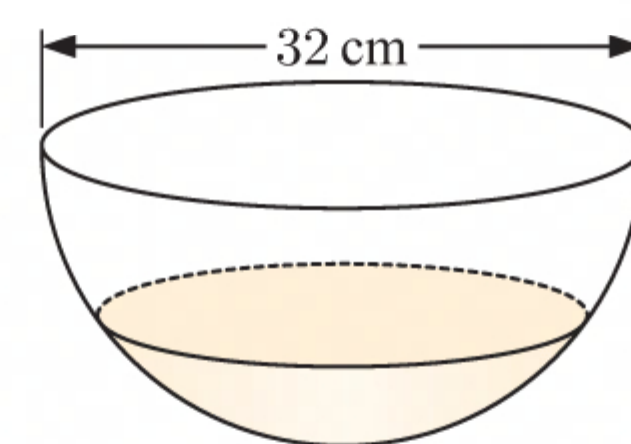
- Find the original value  $u_0$  of the car.
- Write a geometric sequence to describe the value of the car  $u_n$  after  $n$  years.
- In what year will the value of the car fall below \$10 000?

- 4** Before selecting a new mobile phone plan, George reviews the duration of calls he made over the last 3 months. George produced the histogram alongside to illustrate the data he collected.



- Write down the modal class.
- Organise the data into a frequency table.
- Estimate the mean length of a phone call.
- Estimate the probability that George's next call will last 6 minutes or longer.

- 5** A hemispherical mixing bowl has dimensions shown.



- Find the capacity of the bowl.
- Suppose the bowl is 20% full with cake batter.
  - How many litres of cake batter does it contain?
  - The cake batter is poured into a cylindrical cake tin with diameter 25 cm. How high will it reach up the tin?

- 6** The lengths and weights of a zoo's pygmy shrews were recorded during the annual health check.

Length (mm)	95	83	91	82	75	62	79	63	81	69	94	88	72	77
Weight (g)	5.4	4.5	5.0	4.1	3.7	2.6	4.5	3.1	4.7	3.7	5.1	4.8	3.6	4.2

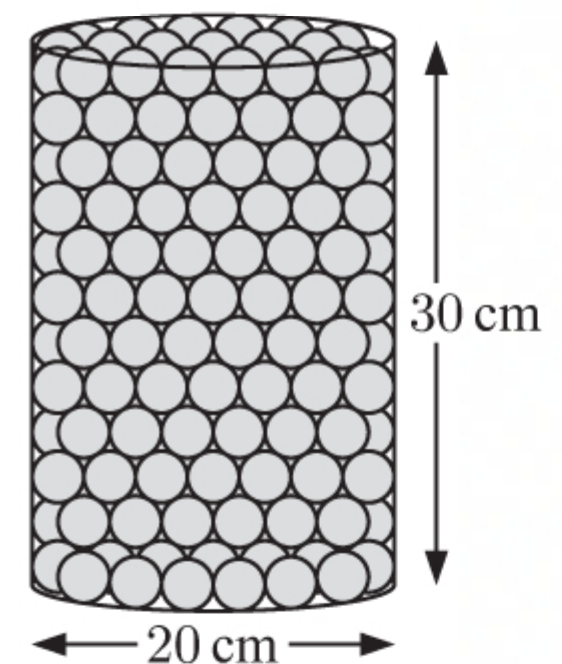
- Draw a scatter diagram for this data.
- Calculate Pearson's product-moment correlation coefficient  $r$  for the data.
- Hence describe the correlation between these two variables.
- Find the equation of the least squares regression line.
- Hence predict the weight of a pygmy shrew with length:
  - 110 mm
  - 70 mm
- Which of your predictions in **e** is more likely to be reliable? Explain your answer.



- 7** A particular airline has routes between four cities A, B, C, and D. Direct flights are only possible from A to B, A to D, B to A, B to D, C to B, and from D to C.
- Draw a graph to illustrate the possible flights between the cities.
  - Find the adjacency matrix  $\mathbf{A}$  for your graph.
  - Calculate  $\mathbf{A} + \mathbf{A}^2$  and hence show that it is not possible to travel between any two cities in at most 2 flights.
  - Calculate  $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$  and hence show that it is possible to travel between any two cities in at most 3 flights.
  - The airline adds direct flights from C to D, and D to A. Is it now possible to travel between any two cities in at most 2 flights? Explain your answer.
- 8** Let  $f(x) = \ln(x\sqrt{1-2x})$ .
- State the domain of the function.
  - Show that  $f'(x) = \frac{1-3x}{x(1-2x)}$ .
  - At what point(s) on the graph of  $y = f(x)$  does the normal have gradient  $-\frac{6}{5}$ ?
- 9** Consider the differential equation  $\frac{dy}{dx} = \frac{x}{1+x^2}$  with  $y(0) = 2$ .
- Estimate  $y(1)$  using Euler's method with step size 0.2.
  - Use technology to apply Euler's method with step size 0.005 for 200 steps.
  - Find the exact value of  $y(1)$  using the Fundamental Theorem of Calculus. Comment on your results.
- 10** The monthly water consumption of households in a suburb is normally distributed with standard deviation 5 kL. Last year the mean monthly water consumption was 27.3 kL.
- Roy would like to determine whether water consumption in the suburb has changed this year.
- State the hypotheses that should be tested.
  - For this test, at a 1% significance level, find:
    - the critical values
    - the critical region
    - the acceptance region.
  - Roy sampled 50 houses in the suburb, and the mean monthly water consumption for the sample was 29.6 kL.
    - Calculate the test statistic.
    - Is there sufficient evidence to conclude that water consumption has changed this year? Explain your answer.

## MIXED QUESTIONS SET 2

- 1** Charlie and Charlotte are on a road trip in Australia. They travel 36 km north-west from Wollongong to Picton, then 210 km south-west from Picton to Canberra.
- How far is Canberra from Wollongong?
  - Find the bearing of Wollongong from Canberra.
- 2** A curve has gradient function  $f'(x) = \frac{a}{x^2} + bx^2$  where  $a$  and  $b$  are constants. Find  $f(x)$  given that  $f(-1) = -7$ ,  $f(1) = 7$ , and  $f(2) = 26$ .
- 3** A jar 20 cm wide and 30 cm high is filled with marbles.
- To estimate the number of marbles in the jar, Julie assumes that the jar is a perfect cylinder, and the marbles occupy 60% of the jar's volume.
- Use Julie's assumptions to construct a model for the number  $N$  of marbles with radius  $r$  cm in the jar.
  - Julie measures the radius of a marble in the jar to be 1.5 cm. Use Julie's model to estimate the number of marbles in the jar.
  - The marbles actually occupy about 64% of the jar's volume. Do you think the estimate in **b** is an overestimate or an underestimate? Explain your answer.
  - Given that there are actually 426 marbles in the jar, find the percentage error in the estimate in **b**.
- 4** The heights  $X$  of maize plants two months after planting are normally distributed with mean 40 cm and standard deviation 6.8 cm.
- Find:
    - $P(X < 25)$
    - $a$  such that  $P(X < 25) = P(X > a)$ .
  - Six maize plants are randomly chosen. Find the probability that exactly four of them are more than 35 cm high.





- 5 A random variable  $X$  has the following distribution table:

$x$	-2	0	3	5
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	$k$	$\frac{1}{12}$

- a Is the random variable  $X$  discrete or continuous?  
 b Find  $k$ .  
 c Find the mode and median of  $X$ .  
 d Find  $E(X)$ ,  $\text{Var}(X)$ , and the standard deviation of  $X$ .
- 6 This table shows the echo signal strength received from a radar detecting an object  $d$  km away.

Distance ( $d$ km)	5	10	20
Signal strength ( $S$ units)	40	2.5	0.156 25

The signal strength is inversely proportional to the fourth power of the distance.

- a Find the model connecting  $S$  and  $d$ .  
 b Find the signal strength for an object 18 km away.  
 c Find the percentage decrease in signal strength if the distance is tripled.
- 7 From an airport, aeroplane A takes off with direction vector  $\begin{pmatrix} 10 \\ 18 \\ 3 \end{pmatrix}$ , and aeroplane B takes off with direction vector  $\begin{pmatrix} 15 \\ k \\ 4 \end{pmatrix}$ ,  $k \in \mathbb{R}$ .

- a Given that the paths of aeroplanes A and B are perpendicular, find the value of  $k$ .  
 b Aeroplane C takes off with direction vector  $12\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$ .

Find the angle between the paths of aeroplanes B and C.

- 8 The variables  $x$ ,  $y$ , and  $z$  are related by the model  $z = x^a y^b$ , where  $a$  and  $b$  are constants.

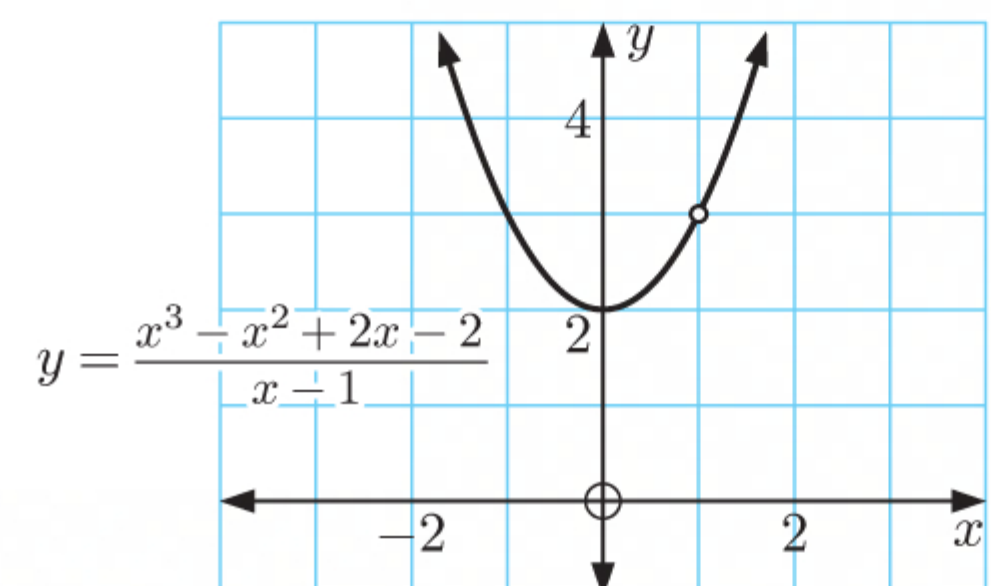
It is known that:

- when  $x = 1.5$  and  $y = 2.3$ ,  $z = 7.48$
- when  $x = 1.1$  and  $y = 0.8$ ,  $z = 0.983$ .

Find  $a$  and  $b$ , giving your answers correct to 3 significant figures.

- 9 The graph of  $f(x) = \frac{x^3 - x^2 + 2x - 2}{x - 1}$  is shown alongside.

- a Explain why the function is undefined at  $x = 1$ .  
 b Use the graph to find  $\lim_{x \rightarrow 1} f(x)$ .



- 10 A real estate company records the number of houses sold each day for 100 days. The results are shown in the table alongside.

Number of houses sold	0	1	2	3	4	5	6
Frequency	10	28	33	13	6	7	3

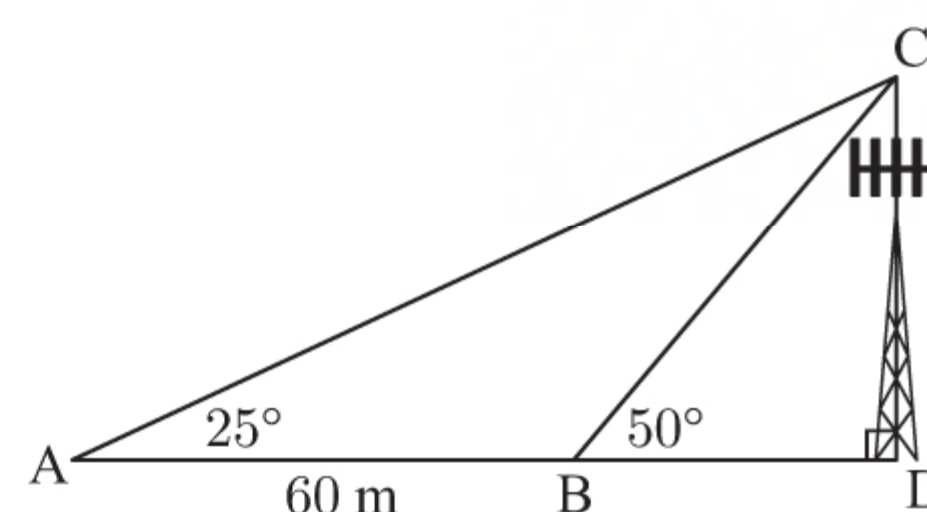
It is suspected that the data follows a Poisson distribution. A  $\chi^2$  goodness of fit test will be performed at a 10% significance level.

- a Use the data to find the mean number of houses sold each day.  
 b State the null hypothesis for the test.  
 c Assuming the null hypothesis is true, construct a table of expected frequencies.  
 d State the number of degrees of freedom for the test.  
 e Calculate the test statistic and the  $p$ -value.  
 f Explain whether a Poisson distribution is an appropriate model for the data.



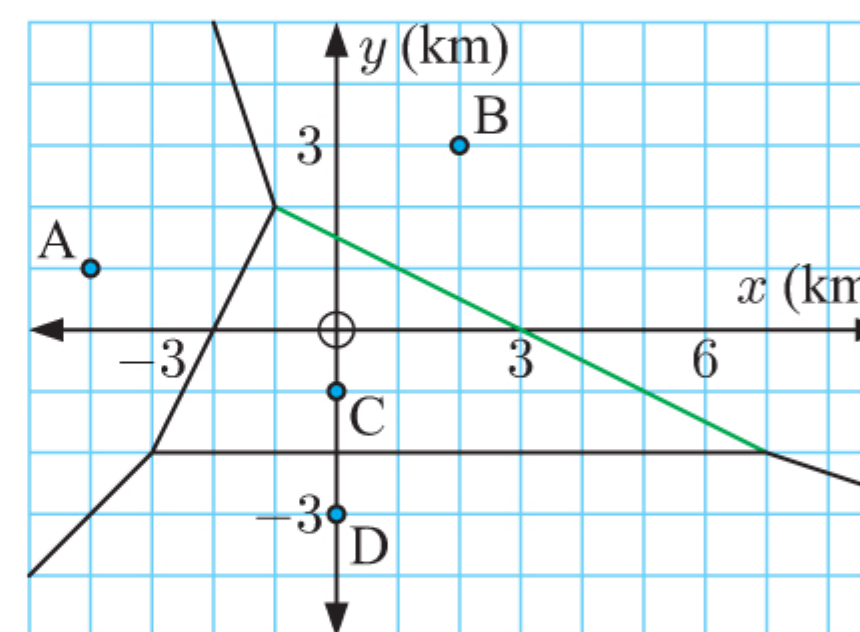
# MIXED QUESTIONS SET 3

- 1** Lin wants to calculate the height of a mobile phone tower. He measures the angle of elevation from point A to the top of the tower C, then moves 60 m closer to point B and takes a second measurement. The information is given in the diagram alongside.

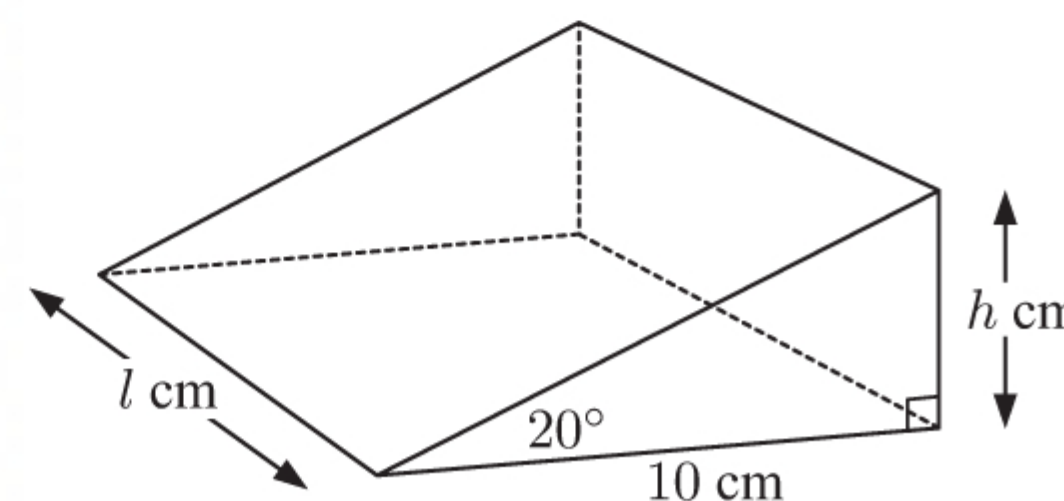


- a Calculate the measure of  $\widehat{ACB}$ .
  - b Determine the height of the tower.
- 2** Consider the arithmetic sequence with  $u_5 = 18$  and  $u_8 = 39$ .
- a Find the first term  $u_1$  and common difference  $d$ .
  - b Find the 12th term of the sequence.
  - c Find the sum of the first 10 terms of the corresponding arithmetic series.
- 3** Consider the function  $f(x) = ax^3 - bx^2$ . The line  $y = x - 6$  is a tangent to  $y = f(x)$  at  $x = 3$ .
- a Find the constants  $a$  and  $b$ .
  - b Find the point where the tangent meets  $y = f(x)$  again.
  - c Graph  $y = f(x)$  and  $y = x - 6$  on the same set of axes.

- 4** Consider this Voronoi diagram for the supermarkets  $A(-4, 1)$ ,  $B(2, 3)$ ,  $C(0, -1)$  and  $D(0, -3)$ .



- a Identify the supermarket which is closest to:
    - i  $(2, 2)$
    - ii  $(-5, -3)$
    - iii  $(6, -4)$
  - b Find the equation of the green edge. Write your answer in the form  $ax + by + d = 0$ .
  - c Find the area of the cell containing supermarket C.
  - d Jennifer's home is equally closest to supermarkets A, C, and D. State the location of Jennifer's home.
- 5** A manufacturer produces wooden door-stops with the shape of the triangular prism shown.
- a Calculate the height  $h$  correct to 4 significant figures.
  - b Determine the area of the triangular end of the prism.
  - c The volume of the door-stop is  $60 \text{ cm}^3$ . Determine its length  $l$ .
  - d Calculate the total surface area of each door-stop. Give your answer correct to 3 significant figures.
- 6** To test the effectiveness of a new mattress, ten people recorded how long they slept on one night with their old mattress, and on another night with the new mattress. The results, in hours, are shown below.



Old mattress	7.2	6.9	6.5	7.1	7.8	8.3	7.1	6.7	6.2	7.4
New mattress	7.6	7.5	6.2	7.4	8.2	8.1	7.4	7.5	6.9	7.5

- A paired  $t$ -test is performed at a 5% level to determine whether the new mattress increases sleep times.
- a State the null and alternative hypotheses for this test.
  - b Calculate the test statistic and  $p$ -value.
  - c State the conclusion of the test.
- 7** A target shot into the air has position equations  $x_1(t) = 0.6 + 5t$ ,  $y_1(t) = 10 + 5t - 4.9t^2$ ,  $t \geq 0$ . Time is measured in seconds and the distance units are metres.
- a Find the flight time of the target.
  - b Find the horizontal distance travelled by the target.
  - c A second target is shot 1 second after the first target from the same position with the same initial velocity
    - i Write down the position equations of the second target  $x_2(t)$  and  $y_2(t)$ , for  $t \geq 1$ .
    - ii Show that the straight line distance  $D$  between the two targets at time  $t$  satisfies  $D^2 = 96.04t^2 - 194.04t + 123.01$ .
    - iii Hence find the shortest distance between the two targets while both targets are in the air, and the time that it occurs.



- 8 Suppose  $f'(x) = \sqrt{4x+5}$  and that  $f(0) = -\frac{\sqrt{5}}{6}$ .
- a For what values of  $x$  is  $f'(x)$  defined?      b Find  $f(x)$ .
- 9 The number of vehicles which pass through a particular toll gate each minute has a Poisson distribution with mean 10.
- a Over a 10 minute interval, calculate the probability that:
- i less than 80 vehicles pass through the toll gate
- ii between 90 and 120 vehicles (inclusive) pass through the toll gate.
- b Vehicles passing through this toll gate must pay a \$0.75 toll.
- Let  $Y$  be the total value of tolls collected over a 10 minute interval. Calculate:      i  $E(Y)$       ii  $\text{Var}(Y)$ .
- 10 The average distances of the planets from the sun and the lengths of their orbits around the sun are given in the following table.

Planet	Average distance from sun ( $d$ millions of km)	Length of orbit around the sun ( $t$ days)
Mercury	57.9	88
Venus	108.2	225
Earth	149.6	365
Mars	227.9	687
Jupiter	778.3	4329
Saturn	1427.0	10 753
Uranus	2870.0	30 660
Neptune	4497.0	60 150

- a Draw a scatter diagram of  $\ln t$  against  $\ln d$ .
- b Explain why  $t$  and  $d$  are related by a power model.
- c Find the power model connecting  $d$  and  $t$ .
- d The average distance of the dwarf planet Pluto from the sun is  $5.907 \times 10^9$  km. Estimate the length of Pluto's orbit around the sun.

## MIXED QUESTIONS SET 4

- 1 The management of a large shopping centre chain sent a survey team to one of its suburban shopping centres. Between 10 am and 3 pm on a very busy Thursday, 100 people in the main mall were asked the following multiple choice question:
- “At which type of shopping centre do you prefer to shop?”*
- A** suburban      **B** central city      **C** equally preferred      **D** neither      **E** no opinion
- a Give *two* reasons why this survey is likely to contain a coverage error.
- b The results were: suburban 33%, central city 8%, equally preferred 51%, neither 4%, no opinion 4%
- Management concluded that *“more than four times as many people prefer suburban shopping to the central city”*. Explain why this conclusion is unreasonable.
- 2 The current in an electrical circuit  $t$  milliseconds after it is switched off is given by  $I(t) = 40e^{-0.1t}$  amps.
- a What current was flowing in the circuit initially?
- b What current was flowing in the circuit after 100 milliseconds?
- c Sketch  $I(t)$  and  $I = 1$  on the same set of axes.
- d How long did it take for the current to fall to 1 amp?
- 3 Soraya borrowed £7000 to go on a holiday. The bank charged an interest rate of  $r\%$  p.a. compounded monthly. She will repay the loan with monthly repayments of £220 for 3 years.
- a Find  $r$ .      b Find the total interest charged on the loan.
- c Suppose Soraya chose to repay the loan over 4 years instead of 3 years.
- i Find Soraya's monthly repayment.      ii How much *extra* interest will Soraya pay?



- 4 The number of people at a music festival  $t$  hours after 12 pm is modelled using the function  $N(t) = at^3 + bt^2 + ct + d$ . There were 2000 people at the festival at 12 pm. The number of people at the festival was increasing at 200 people per hour at 6 pm, and decreasing at 500 people per hour at 8 pm. The festival closed at 10 pm.
- a State *four* conditions for  $N(t)$  and  $N'(t)$  you can deduce from the information given.
  - b Hence find the function  $N(t)$ .
  - c Predict the maximum number of people at the festival, and the time when this occurred.

- 5 A tinned food company examined a sample of its tins of corn and tins of pineapple for defects. The results are summarised in the table alongside.

	Defective	Not defective
Corn	37	581
Pineapple	24	617

- a How many tins were included in the sample?
  - b Estimate the probability that the next randomly selected tin:
    - i is not defective
    - ii is a defective tin of pineapple
    - iii is defective, given it is a tin of corn.
- 6 Monica is a police officer. She wants to investigate whether house break-ins in her city are equally likely to occur on each day of the week.

She compiles the following data for 140 randomly chosen house break-ins over the past year.

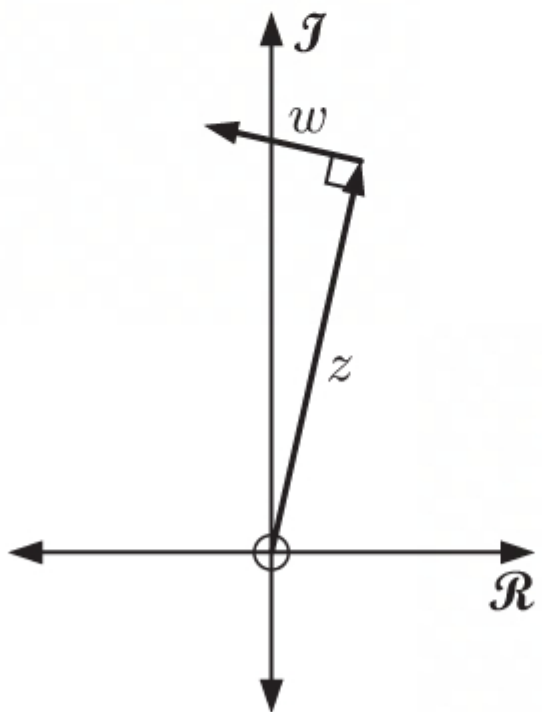
Day of the week	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Number of break-ins	15	11	17	18	27	29	23

- Monica will perform an appropriate  $\chi^2$  test at a 5% significance level. The critical value of  $\chi^2$  for this test is 12.59.
- a Write down the hypotheses that Monica should test.
  - b Assuming the null hypothesis is true, how many break-ins from Monica's sample would be expected to occur on each day?
  - c Calculate the test statistic  $\chi^2_{\text{calc}}$ .
  - d Hence determine whether the break-ins in this city are equally likely to occur on each day of the week.

- 7 On the Argand diagram alongside,  $|z| = 5$ ,  $|w| = 2$ , and  $\arg z = \frac{3\pi}{7}$ .

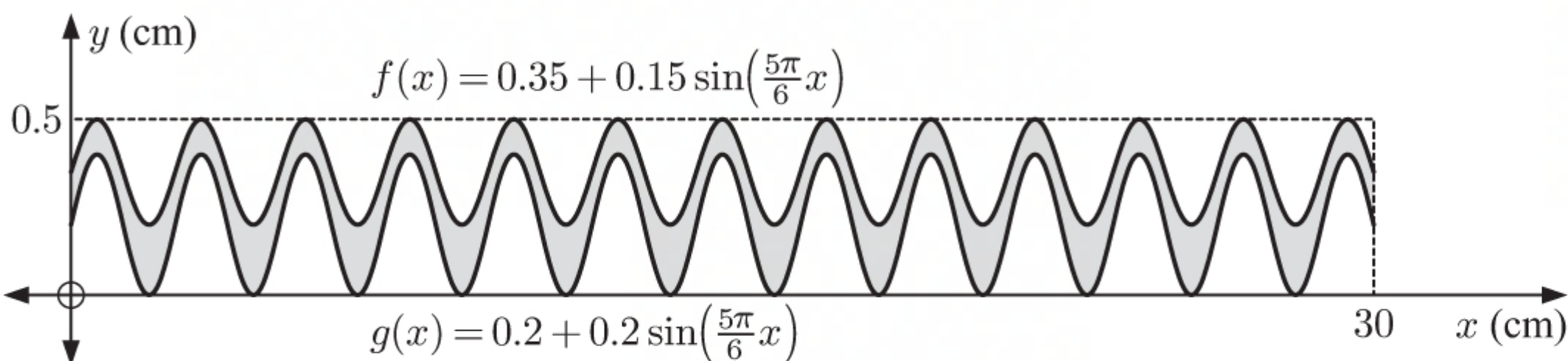
Find:

- a  $\arg w$
- b  $|z + w|$
- c  $\arg(z + w)$



- 8 The position of a car turning a corner is given by the equations  $x(t) = 20 \cos(\frac{\pi}{6}(t - 3))$ ,  $y(t) = -20 \sin(\frac{\pi}{6}(t - 3))$ ,  $0 \leq t \leq 3$ . Time is measured in seconds and the distance units are metres.
- a Find the position of the car:
    - i initially
    - ii after 3 seconds.
  - b Find the velocity vector of the car.
  - c Hence calculate the car's speed after  $t$  seconds.
  - d Show that  $x^2 + y^2 = 400$ . Hence describe the path of the car.
  - e Sketch the path of the car for  $0 \leq t \leq 3$ .
  - f Use geometric facts to find the distance that the car travelled in the first 3 seconds.
  - g Use f to calculate the car's speed. Compare your answer to c.

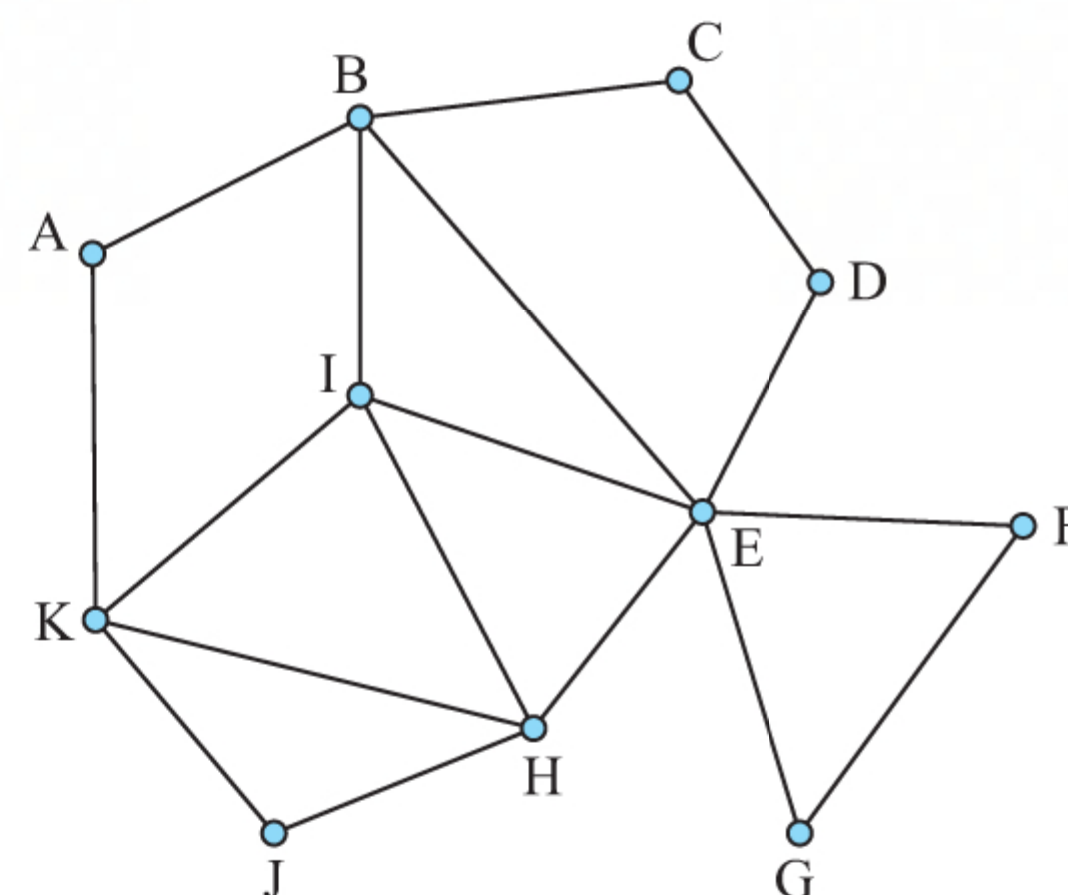
- 9 The cross-section of a 1 m long strip of cardboard is shown below.



Find the volume of the cardboard.



- 10** This graph shows the tracks visitors can walk along in a bird park.
- List the edges adjacent to BC.
  - Find a path of length 5 which starts at B and finishes at D.
  - Explain why the graph is Eulerian.
  - Find an Eulerian circuit which starts and finishes at A.
  - Suppose three of the tracks have closed down, but the graph is still Eulerian.  
It is known that one of the closed tracks is EH.
- Find the other two closed tracks.
  - Find an Eulerian circuit which starts and finishes at A in this case.



## MIXED QUESTIONS SET 5

- The fluoride concentration of lakes in a particular region was found to be  $3 \times 10^{-4}$  g per litre.
  - One lake has  $5.6 \times 10^8$  litres of water. Find the amount of fluoride in the lake, giving your answer in scientific notation.
  - Another lake contains  $4.13 \times 10^7$  g of fluoride. Find the volume of the lake.
- A graph of the quadratic  $y = ax^2 + bx + c$  is shown alongside, including the vertex V and  $y$ -intercept.
  - Determine the value of  $c$ .
  - Use the axis of symmetry to write an equation involving  $a$  and  $b$ .
  - Use the point  $(1, 7)$  to write another equation involving  $a$  and  $b$ .
  - Find  $a$  and  $b$ .
- A game is played in which the wheel shown is first spun by the player, and then by the game operator. The player wins \$ $a$  if their spin is higher than the operator's. It costs \$ $k$  to play the game. Find the relationship between  $a$  and  $k$  so that the game is fair.
- The lengths of adult fish of a certain species are normally distributed with mean 40 cm and standard deviation 5 cm.
  - Find the probability that a randomly chosen adult fish of this species is:
    - longer than 45 cm
    - between 35 cm and 50 cm long.
  - Determine the minimum length of the longest 10% of this species of fish.
  - A randomly selected fish is shorter than 48 cm. Find the probability that it is between 40 cm and 44 cm long.
- George is describing the dimensions of his triangular garden patch to his gardener. He tells the gardener that, in the triangle PQR,  $\widehat{PRQ} = 40^\circ$ ,  $PR = 2.4$  m, and  $PQ = 1.6$  m.
  - Show that there are two possible measures of  $\widehat{PQR}$ .
  - Sketch triangle PQR for each case.
  - In each case, find:
    - the measure of  $\widehat{QPR}$
    - the perimeter of the garden.
- A parabola passes through the points  $(-\pi, 0)$ ,  $(\pi, 0)$ , and  $(0, \alpha)$ . The area between the parabola and the  $x$ -axis is 4 units<sup>2</sup>. Calculate the possible values of  $\alpha$ .



7 A teacher wants to test his students’ spelling skills. He gives his students a test in which they must spell 30 words. One month later, he gives his students the same test with the same words.

Student	A	B	C	D	E	F	G	H	I	J	K	L
Test 1	25	21	18	27	19	22	21	26	14	17	20	16
Test 2	26	19	21	28	17	24	22	29	18	17	24	19

- a Explain why test-retest reliability is being considered in this case.

b Calculate Pearson’s product-moment correlation coefficient between the *test 1* and *test 2* results.

c Comment on the reliability of the test.

d List any factors which might have affected the test-retest reliability of the test.

e Describe how the test could be changed to consider parallel forms reliability.
- 8 Line  $L_1$  has equation  $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}, t \in \mathbb{R}$ . Line  $L_2$  cuts the  $X$ -axis at 3 and the  $Y$ -axis at  $-5$ .
- a Find the point where  $L_1$  meets the  $XY$ -plane.

b Find parametric equations for  $L_2$ .

c The line  $L_3$  is perpendicular to both  $L_1$  and  $L_2$ , and passes through  $(-2, 5, 1)$ . Find the equation of  $L_3$  in parametric form.
- 9 A cyclist travels with velocity  $v = \frac{3}{2}(s + 1)^{\frac{1}{3}} \text{ m s}^{-1}$ , where  $s$  is the displacement of the cyclist in metres.
- a Find the cyclist’s velocity when she has displacement 7 m.

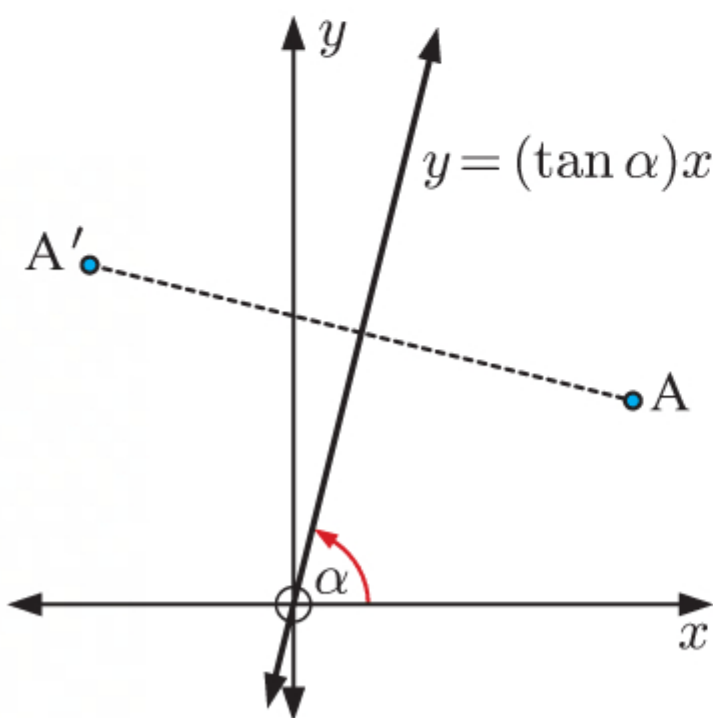
b Find the cyclist’s acceleration in terms of  $s$ .

c Given that the cyclist initially has displacement 0 m, solve the differential equation  $v = \frac{3}{2}(s + 1)^{\frac{1}{3}}$  to write  $s$  in terms of  $t$ .

d Find the displacement, velocity, and acceleration of the cyclist after 8 seconds.
- 10 Under a reflection in the line  $y = (\tan \alpha)x$ ,  $A(5, 3)$  is mapped to  $A'(-3, 5)$ .
- a Show that  $\tan \alpha = 4$ .

b Find the transformation matrix  $\mathbf{A}$  for this reflection.

c Find the image of  $(2, 2)$  under this reflection.



MIXED QUESTIONS SET 6

1 The table shows the amount of petrol remaining in a motorbike’s fuel tank and the number of kilometres travelled. The capacity of the tank is 10 litres.

Remaining fuel ( $x$ litres)	10	8	6	4	2	1
Distance ( $y$ km)	0	90	190	260	330	370

- a Plot this data on a scatter diagram.

b Find the equation of the regression line for  $y$  against  $x$ .

c Interpret the  $y$ -intercept of the regression line.

d The motorbike has travelled 220 km since its tank was refilled.

i Use your equation to estimate the amount of fuel left in the tank.

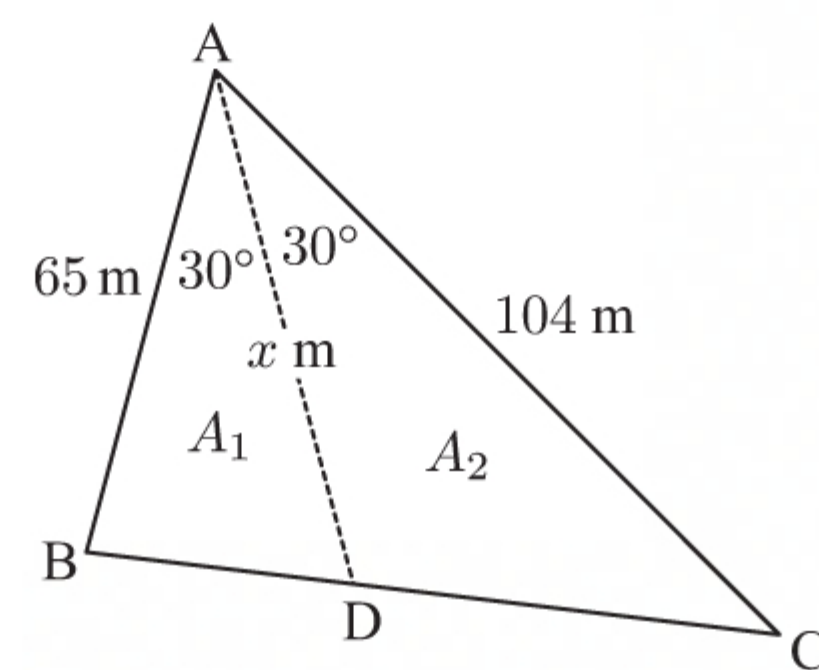
ii Find the average distance travelled per litre over the 220 km.
- 2 The probability of Mark waking up early is 0.8 . If he wakes up early, he will pack lunch with probability 0.6 . If he does not wake up early, he will pack lunch with probability 0.15 .
- a Display the sample space of possible outcomes on a tree diagram.

b Hence determine the probability that Mark will pack lunch today.



- 3** A farmer owns a triangular field ABC.

D is the point on [BC] such that [AD] bisects  $\widehat{BAC}$ . The farmer divides the field into two parts  $A_1$  and  $A_2$  by constructing a straight fence [AD] of length  $x$  m.



- Use the cosine rule to calculate the length of [BC].
- Find the total area of the field.
- Find, in terms of  $x$ , the area of:   
 i  $A_1$       ii  $A_2$ .
- Hence find  $x$ .

- 4** Let  $D(t)$  m be the distance between two stunt motorcyclists  $t$  seconds after they start riding. It is known that the motorcyclists were initially 42 m apart, and that  $D'(t) = 0.8t - 8$ .

- Find  $D(t)$ .
- Find the distance between the motorcyclists after 2 seconds.
- Find the minimum distance between the motorcyclists, and the time at which it occurs.

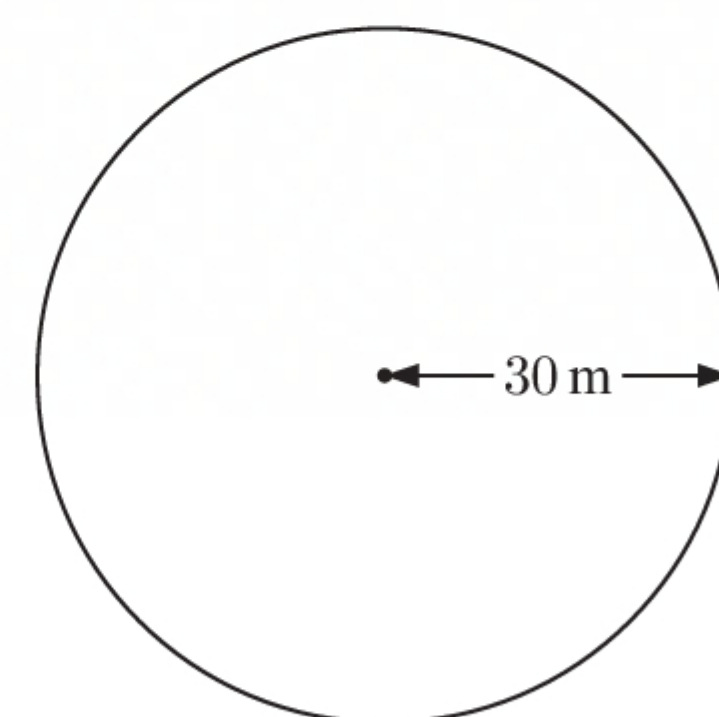
- 5** Melinda bought a car valued for £55 000. She borrowed the money for the car over 7 years, with interest charged at 9.25% p.a. compounded monthly.

- Calculate her monthly repayments.
- Calculate the outstanding debt after two and a half years.
- The car depreciates at 15% p.a. After 7 years, Melinda sold the car at its depreciated value. Find the total cost of the car to Melinda, taking depreciation and interest into account.

- 6** A gardener has been asked to perform maintenance on Globe Park, a circular lawn with radius 30 m.

This involves:

- mowing the interior of the lawn
- using a line trimmer to tidy the perimeter of the lawn.



From previous experience, the gardener knows that:

- a circular lawn with radius 10 m takes 30 minutes to maintain .... (1)
- a circular lawn with radius 20 m takes 110 minutes to maintain. .... (2)

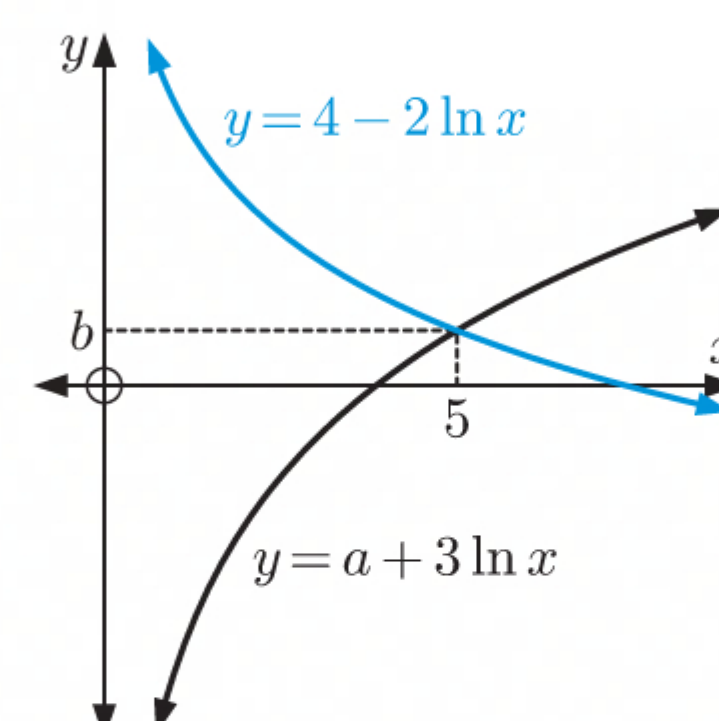
The gardener believes the time to maintain a circular lawn with radius  $r$  m can be modelled by  $T = ar + b$  minutes, where  $a$  and  $b$  are constants.

- Assuming the gardener's model is correct, write two equations connecting  $a$  and  $b$ .
- Hence find  $a$  and  $b$ .
- Explain why this model is not appropriate for small values of  $r$ .
- Use this model to predict how long it will take to maintain Globe Park.
- Given that it actually took 230 minutes to maintain Globe Park, find the percentage error in the prediction in **d**.
- The gardener's friend suggests a model of the form  $T = pr^2 + qr$ , where  $p$  and  $q$  are constants.
  - Explain why a model of this form is reasonable.
  - Use the information from (1) and (2) to find  $p$  and  $q$  for this model.
  - Is this model better at predicting the time taken to maintain Globe Park? Explain your answer.

- 7** The graphs of  $y = a + 3 \ln x$  and  $y = 4 - 2 \ln x$  are shown alongside.

Find in the form  $\ln k$  where  $k \in \mathbb{R}$ :

- $b$
- $a$





- 8 At the end of the semester, a university tutor is interested in how effective her teaching methods were in a particular course. A random sample of 10 of her students are selected. The number of tutorials attended, and the final mark, of each student, was recorded.

<i>Tutorials attended</i>	6	9	10	7	9	2	10	3	8	4
<i>Final mark</i>	80	86	89	93	90	62	93	72	85	77

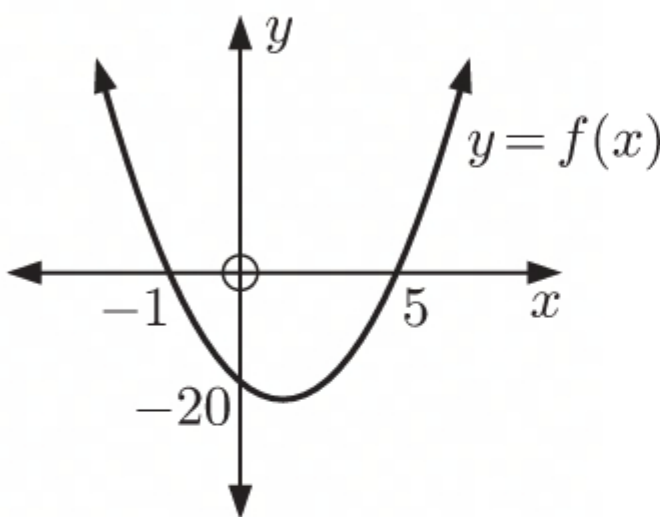
- The tutor carries out a hypothesis test to determine whether there is a positive correlation between *tutorials attended* and *final mark* at a 1% level of significance.
- a State the null and alternative hypotheses.
- b Use technology to calculate:      i the test statistic      ii the  $p$ -value.
- c Hence determine whether there is a positive correlation between *tutorials attended* and *final mark*.
- d Do you think there is a causal relationship between the variables? Explain your answer.
- 9 a Find  $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ .
- b Hence write down an eigenvalue and eigenvector of  $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ .
- c Find the characteristic polynomial for  $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ .
- d Hence find the remaining eigenvalue and corresponding eigenvector of  $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ .
- 10 The velocity of a boat travelling in a straight line after  $t$  seconds is given by  $v(t) = 30 - 20e^{-0.2t}$  m s<sup>-1</sup>.
- a Find the boat's:      i initial velocity      ii velocity after 2 seconds.
- b How long does it take for the boat's velocity to reach 20 m s<sup>-1</sup>? Give your answer correct to two decimal places.
- c What happens to  $v(t)$  as  $t \rightarrow \infty$ ?
- d Calculate  $v'(t)$  and show that the acceleration is always positive.
- e Graph  $v(t)$  against  $t$ , showing the information from a to c.
- f How far did the boat travel before its velocity reached 20 m s<sup>-1</sup>?

MIXED QUESTIONS SET 7

- 1 The data below are the recent sale prices, in thousands of dollars, of houses in two neighbourhoods.

<i>Neighbourhood A:</i>	275	281	320	265	305	258	310	430	285
	290	297	345	195	230	269	300	258	273
<i>Neighbourhood B:</i>	325	300	412	370	297	505	340	333	290
	428	305	520	360	410	275	320	431	410

- a Is the data discrete or continuous?      b Use technology to find the five-number summary for each data set.
- c Display the data in a parallel box plot.      d Compare and comment on the distributions of each data set.
- 2 Bags of rice are sold at a Jakartan wholesale market. The price per bag,  $P$ , if  $b$  bags are bought is shown alongside.
- |              |        |        |        |        |        |
|--------------|--------|--------|--------|--------|--------|
| $b$ (bags)   | 30     | 35     | 40     | 45     | 50     |
| $P$ (rupiah) | 38 000 | 36 000 | 34 000 | 32 000 | 30 000 |
- a Determine the function  $P(b)$ .      b Hence predict the total cost of purchasing 60 bags of rice.
- c Do you think this model can be used to predict the cost of 150 bags of rice? Explain your answer.
- 3 The function  $f$  can be written in the form  $f(x) = a(x - p)(x - q)$  where  $p > q$ .
- a Write down the values of  $p$  and  $q$ .
- b Find  $a$ .
- c Write down the equation of the axis of symmetry.





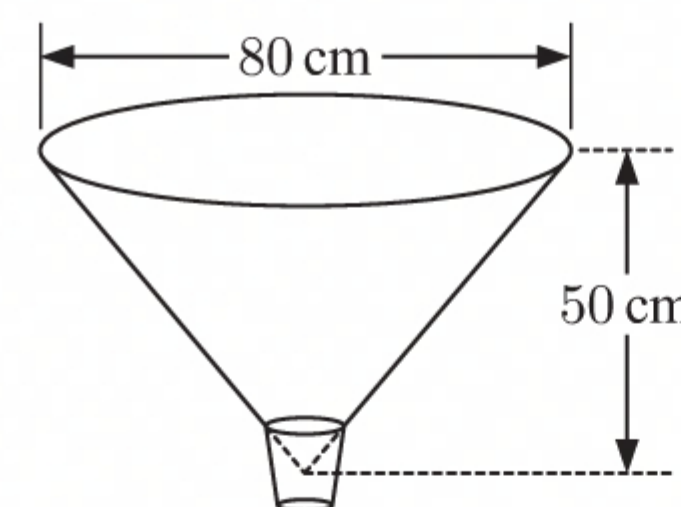
- 4 Trains A and B are 10 km apart, and are approaching the same train station.

Train A is 8 km from the train station on the bearing  $071^\circ$ . Train B is on the bearing  $296^\circ$  from the train station.

- Display this information on a diagram.
- Find the bearing of train B from train A.
- Train B is travelling at an average speed of  $7 \text{ m s}^{-1}$ . Find, to the nearest second, the time it will take for train B to reach the train station.

- 5 A conical funnel is 80 cm wide and 50 cm high.

- Estimate the capacity of the funnel in mL. Write your answer in the form  $a \times 10^k$ , where  $1 \leq a < 10$  and  $k \in \mathbb{Z}$ .
- The funnel is half full with liquid, and its contents are poured into a cylindrical tube 20 cm wide. How high up the tube will the liquid reach?



- 6 In a video game, the playable zone is initially a circle of radius 120 m.

The radius of the playable zone after  $t$  seconds is given by  $r = 120 - 0.1t$  m.

- Find the area  $A$  of the playable zone in terms of  $t$ .
- Find  $\frac{dA}{dt}$ .
- At what time is the area of the playable zone decreasing at  $70 \text{ m}^2$  per second?

- 7 Two year 7 students are selected each week to hoist the flag before the start of class. Year 7 has been divided into 2 classes: class A has 30 students, and class B has 27 students.

- Find the probability that, in any given week, the two selected students selected are in the same class.
- Over the course of 20 weeks, how many times do you expect that the two selected students are in the same class?

- 8 Meteor A travels according to  $\mathbf{r}_1 = \begin{pmatrix} 556 \\ -154 \\ -2313 \end{pmatrix} + \begin{pmatrix} 8 \\ 12 \\ 24 \end{pmatrix} t$ , and

meteor B travels according to  $\mathbf{r}_2 = \begin{pmatrix} 3796 \\ -1594 \\ 5607 \end{pmatrix} + \begin{pmatrix} -10 \\ 20 \\ -20 \end{pmatrix} t$ .

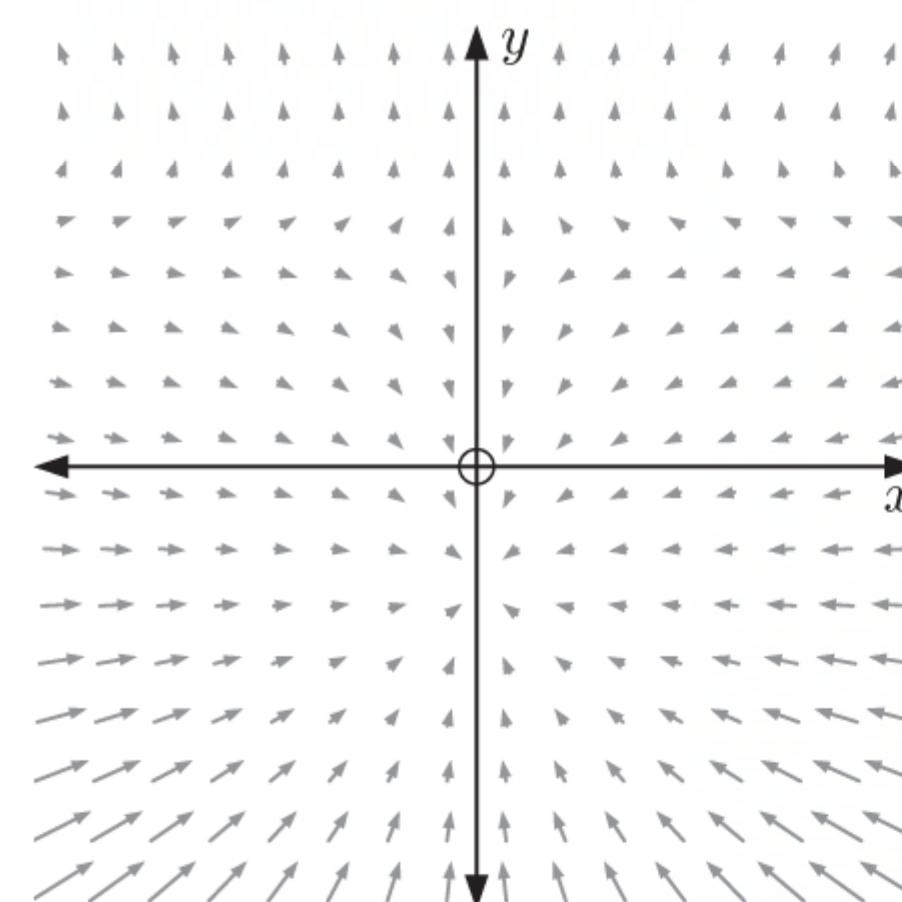
The time  $t \geq 0$  is in seconds, and the distance units are in kilometres.

- Find the speed of each meteor.
- Find the point where the paths of the two meteors intersect.
- Will the meteors collide at the intersection point? Explain your answer.

- 9 The phase portrait for the coupled system 
$$\begin{cases} \frac{dx}{dt} = xy - 3x \\ \frac{dy}{dt} = y^2 - y - 2 \end{cases}$$

is shown alongside.

- Locate and describe the equilibrium points.
- Illustrate the solution curve starting at  $(3, 1)$ .



- 10 This table shows the number of days of rainfall received each week by a city over 10 years (520 weeks).

- Show that the city received rainfall on 30% of the days over this period.
- A test is performed at a 5% level to determine whether the data is binomial with  $p = 0.3$ .
  - What type of test should be performed?
  - State the hypotheses to be tested.
  - Calculate the test statistic and  $p$ -value.
  - Hence determine whether the binomial model is suitable for the data.

Number of days	Frequency
0	45
1	137
2	169
3	101
4	41
5	16
6	7
7	4

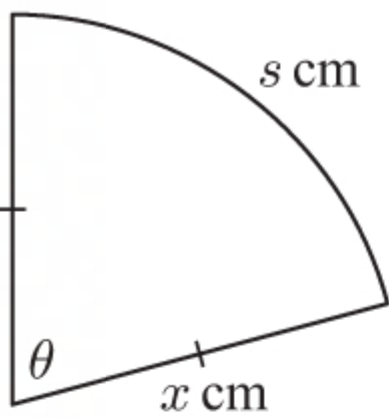


MIXED QUESTIONS SET 8

- 1 A block of land is for sale for €225 000 with a 10% deposit required. Finance can be arranged over ten years at 5.99% p.a. interest compounded quarterly.
- a Calculate the quarterly repayments necessary.
  - b How much interest will be charged on the loan over the ten years?
  - c Assuming inflation averages 3.5% per year, calculate the expected value of the block after ten years.

- 2 The probability of rain falling on any day in Dunedin is 0.4. Suppose two consecutive days are considered.
- a Determine the probability of:
    - i rain on both days
    - ii no rain on exactly one day.
  - b Given that rain fell on at least one day, find the probability of rain on the second day.

- 3 A 40 cm piece of wire is bent to form a sector of a circle with radius  $x$  cm.
- a Write  $\theta$  in terms of  $x$ .
  - b Show that the area of the sector is given by  $A = 20x - x^2$  cm<sup>2</sup>.
  - c Find  $x$  and  $\theta$  for which  $A$  is a maximum.



- 4 Before it is turned on, a refrigerator has an internal temperature of 27°C. Three hours later it has cooled to 6°C.
- The internal temperature  $T$  (in °C) of the refrigerator  $t$  hours after being turned on is given by the function  $T(t) = A \times B^{-t} + 3$ , where  $A$  and  $B$  are constants.
- a Determine the value of:
    - i  $A$
    - ii  $B$ .
  - b Find the internal temperature of the refrigerator 5 hours after being turned on.
  - c Write down the minimum temperature that the refrigerator could be expected to reach.

- 5 A company claims that their *Mega* speakers have a longer battery life than their rival's *Micro* speakers.
- To test this claim, the battery lives of 12 *Mega* speakers and 10 *Micro* speakers were measured in hours.

*Mega:* 22.4, 23.5, 24.1, 22.3, 23.4, 22.9, 22.7, 21.4, 20.9, 22.1, 23.8, 22.9  
*Micro:* 20.8, 21.2, 22.1, 20.7, 21.4, 22.2, 21.7, 23.5, 21.5, 22.5

- A two-sample  $t$ -test is performed at a 10% level of significance to determine whether the company's claim is valid.
- a Write down the null and alternative hypotheses for this test.
  - b Calculate the test statistic and  $p$ -value.
  - c Determine whether the company's claim is valid.

- 6 The masses of six of Jupiter's moons are shown alongside.
- a Scale these values using the function  $y = \log x$ .
  - b Display your results on a logarithmic scale.
  - c A value of 14.0 is obtained after scaling the mass of the moon Autonoe. Find the mass of Autonoe.

Moon	Mass (kg)
Carpo	$4.50 \times 10^{13}$
Leda	$5.68 \times 10^{15}$
Lysithea	$6.29 \times 10^{16}$
Amalthea	$2.08 \times 10^{18}$
Io	$8.93 \times 10^{22}$
Ganymede	$1.48 \times 10^{23}$

- 7 The following table lists the ages of contestants in a game, and the times they took to complete a task.

Age ( $x$ years)	28	40	21	38	30	26	18	32	25	29	20	24
Time ( $y$ min)	20	32	15	40	26	25	19	28	21	25	16	22

- a Find the value of the coefficient of determination  $r^2$ , and explain what this value means.
- b
  - i Write down the equation of the linear regression line in the form  $y = mx + c$ .
  - ii Interpret the coefficient  $m$  in this equation.



- 8 Trisha rides an escalator which leaves from  $(3, 1, 0)$ . She moves in the direction  $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$  at  $0.5 \text{ m s}^{-1}$ .
- Find Trisha's velocity vector.
  - Find Trisha's position after 10 seconds.
  - The end of the escalator has  $X$ -coordinate 0. Find the length of the escalator.
  - At what angle to the horizontal does the escalator travel?
- 9 The points  $(2, 4)$ ,  $(2, -6)$ , and  $(-1, 3)$  lie on a circle with equation  $x^2 + y^2 + ax + by + c = 0$ .
- Write three equations in the unknowns  $a$ ,  $b$ , and  $c$ .
  - Write the system of equations in matrix form.
  - Hence find the values of  $a$ ,  $b$ , and  $c$ .

- 10 This table shows the velocity  $V \text{ m s}^{-1}$  of a skydiver  $t$  seconds after jumping from a plane.

$t$ (seconds)	2	4	6
$V$ ( $\text{m s}^{-1}$ )	20	33	40

It is believed that  $V$  and  $t$  are connected by a model of the form  $V = \alpha(1 - e^{\beta t})$ , where  $\alpha$  and  $\beta$  are constants.

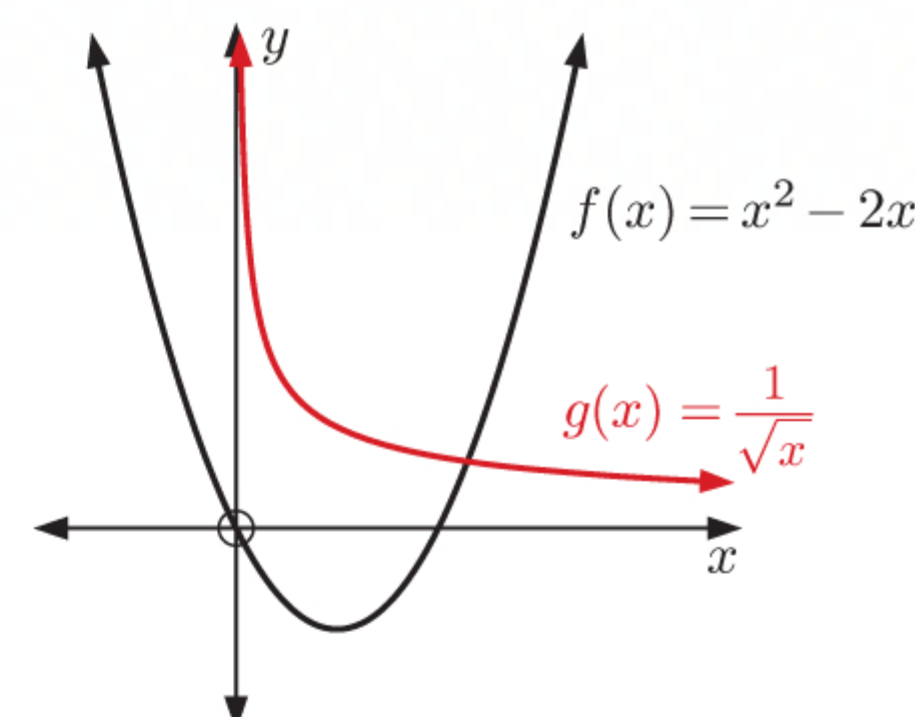
Lucinda suggests the model  $V = 50(1 - e^{-0.2t})$ .

Lewis suggests the model  $V = 55(1 - e^{-0.3t})$ .

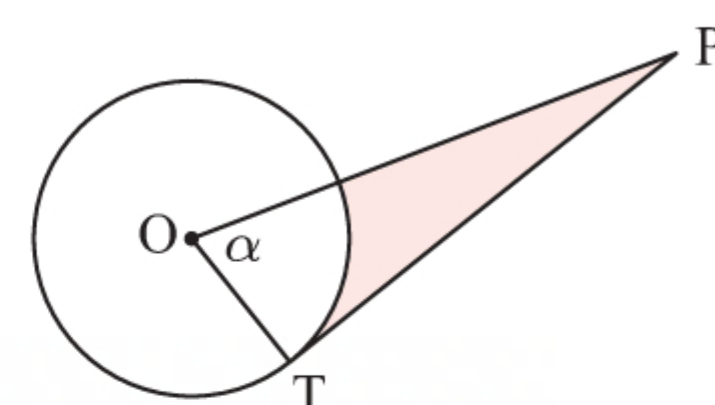
- Calculate the sum of square residuals for each model.
- Whose model is better? Explain your answer.

## MIXED QUESTIONS SET 9

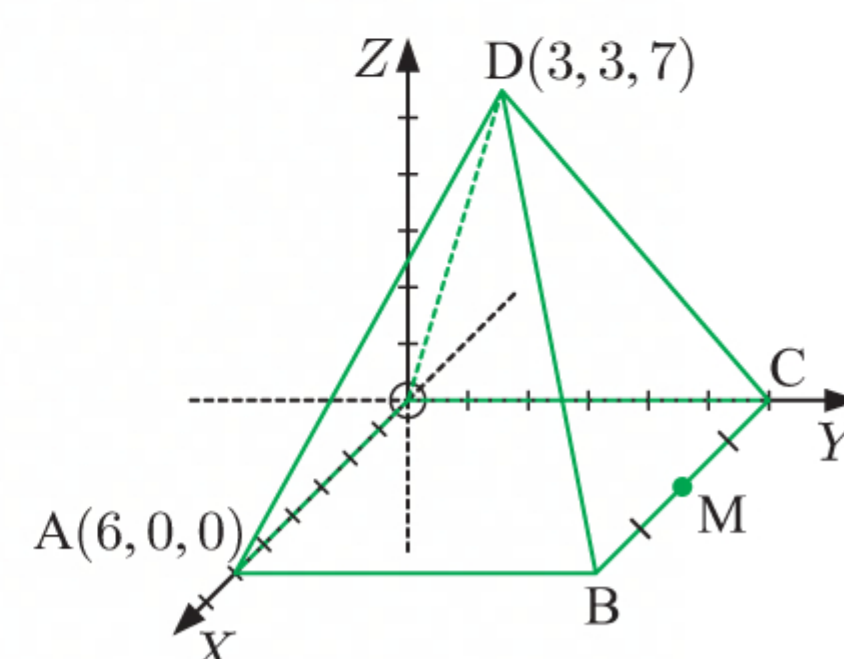
- 1 The graphs of  $f(x) = x^2 - 2x$  and  $g(x) = \frac{1}{\sqrt{x}}$  are shown alongside.
- Find  $f(1)$  and  $g(1)$ .
  - Explain why  $g$  is invertible but  $f$  is not.
  - Find  $x$  such that  $g^{-1}(x) = 4$ .



- 2 [PT] is a tangent to the given circle. The circle has radius 9 cm and  $OP = 30$  cm. Find:
- $\alpha$
  - the area of the shaded region.



- 3 Twins Pierre and Francesca were each given \$100 on their 10th birthday. They immediately put their money into their individual money boxes. Each week throughout the next year they added a portion of their weekly pocket money. Pierre added \$10 each week. Francesca added 50 cents the first week, \$1 the next, \$1.50 the next, and so on, adding an extra 50 cents each subsequent week.
- How much did Francesca add to her money box in the last week before her 11th birthday?
  - Find the total amount that each child had added to his or her money box after 8 weeks.
  - Who had more money in their money box after one year? Explain your answer.
- 4 Consider the square-based pyramid alongside. Find:
- the coordinates of  $B$  and  $C$
  - the volume of the pyramid
  - the coordinates of  $M$
  - the surface area of the pyramid.





5 This table shows the surface area and fish population of eight lakes in a particular region.

Lake	A	B	C	D	E	F	G	H
Surface area ( $x$ hectares)	25	10	35	16	19	27	14	16
Population ( $y$ )	5620	840	6125	1280	1805	3645	980	1110

- a Find Pearson’s product-moment correlation coefficient  $r_p$ .
- b Copy and complete this table of ranks:
- | Lake        | A | B | C | D | E | F | G | H |
|-------------|---|---|---|---|---|---|---|---|
| rank of $x$ |   | 1 | 8 |   |   |   |   |   |
| rank of $y$ |   | 1 |   |   |   |   | 2 |   |
- c Calculate Spearman’s rank correlation coefficient,  $r_s$ .
- d Use  $r_p$  and  $r_s$  to describe the relationship between  $x$  and  $y$ .
- e Suppose that, due to a recording error, the population of lake D was 1180 instead of 1280. Explain why this does not affect the value of  $r_s$ .

6 Gaetano and Jillian go fishing in a lake.

The number of fish Gaetano catches follows a Poisson distribution with rate 5 fish per hour.  
The number of fish Jillian catches follows a Poisson distribution with rate 7 fish per hour.  
Find the probability that they will catch at least 40 fish between them in 3 hours.

7 In a busy harbour, the time difference between successive high tides is 12.3 hours. The water level varies by 2.4 metres between high and low tide. The first high tide of the day is 4.7 metres, occurring at 1 am.

- a Find a cosine model for the height  $H$  of the tide  $t$  hours after midnight.
- b Sketch a graph of the water level in the harbour for  $0 \leq t \leq 24$ .
- 8 A transport department claims that 90% of their buses run on time.

Angela thinks that the proportion is less than 90%, so she decides to test this claim. She takes a sample of 50 bus trips, and if 40 or fewer of them run on time, she will reject the transport department’s claim.

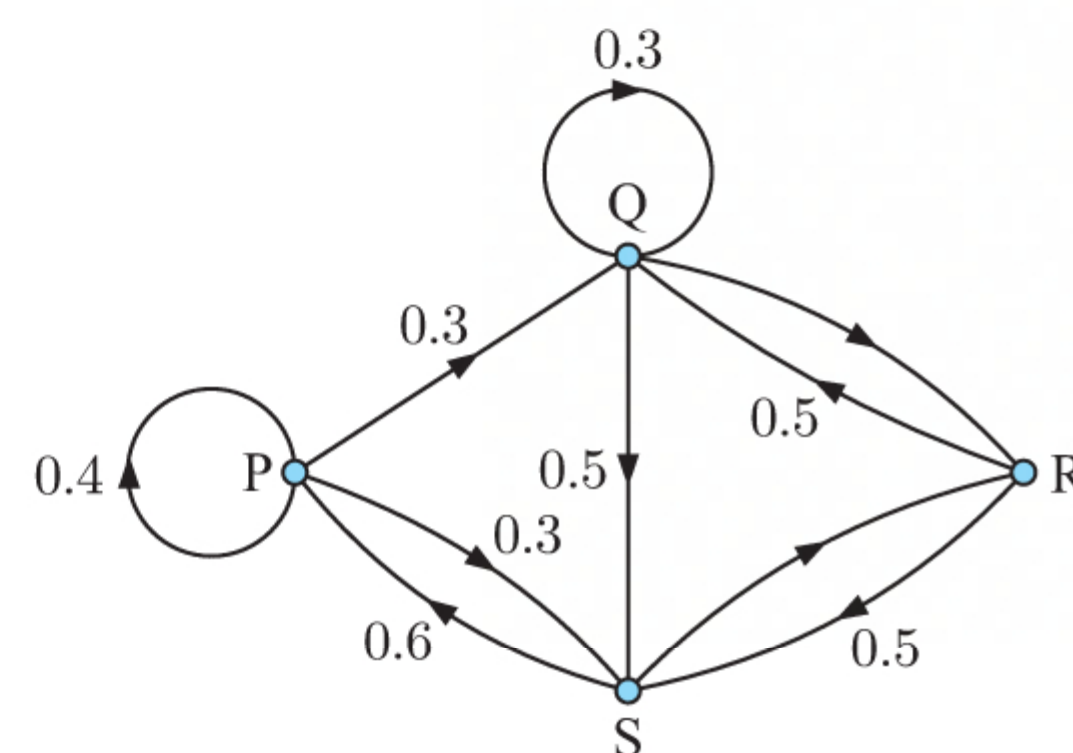
- a State a suitable null and alternative hypothesis for Angela’s test.
- b Find the probability of a Type I error.
- c The actual proportion of buses that run on time is 80%.  
Find the probability of a Type II error.
- 9 Consider the system 
$$\begin{cases} \dot{x} = -x - 2y \\ \dot{y} = x - 4y \end{cases}.$$
- a Find the eigenvalues and corresponding eigenvectors of  $\begin{pmatrix} -1 & -2 \\ 1 & -4 \end{pmatrix}$ .
- b Describe the equilibrium point of the system.
- c Given the initial point  $(-3, 0)$ , find:
- i  $\dot{x}$  when  $t = 0$

ii the particular solution to the system.
- d The minimum  $y$ -value of the particular solution curve occurs at point P, at time  $t = t^*$ .
- i Show that  $e^{t^*} = \frac{3}{2}$ .

ii Hence find the exact coordinates of P.
- e Sketch the phase portrait, including the particular solution.
- f Discuss the behaviour of the system in the long term.



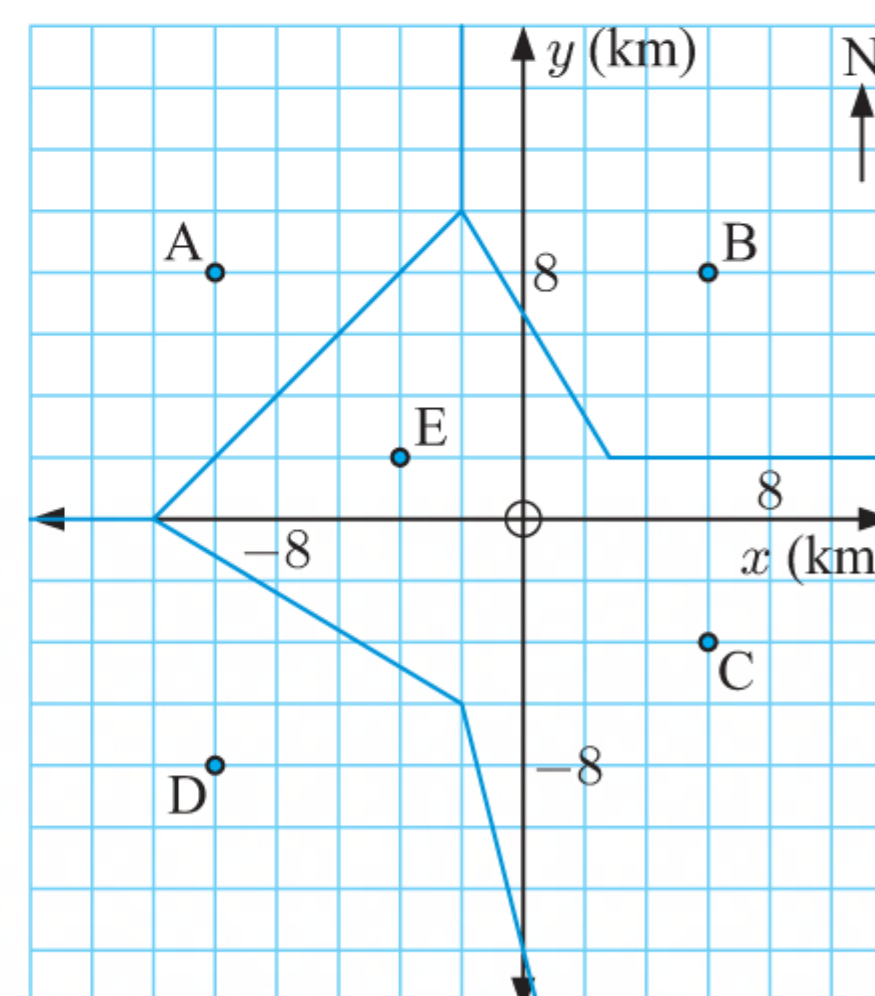
- 10** Clyde the cat moves between houses P, Q, R, and S in his neighbourhood. The graph alongside shows the probabilities with which Clyde moves between the houses each night.



- Complete the graph.
- Write down the in degree of S, and interpret your answer.
- Construct a transition matrix  $\mathbf{T}$  for the graph.
- Calculate  $\mathbf{T}^2$  and  $\mathbf{T}^3$ , and interpret the values in row 3 of  $\mathbf{T}^3$ .
- Suppose Clyde is at P on Monday night.
  - Find the probability that he will be at R on Thursday night, and find the routes Clyde could have taken for this to occur.
  - Given that Clyde is at P on Wednesday night, find the probability he was also at P on Tuesday night.

## MIXED QUESTIONS SET 10

- Suppose  $f'(x) = (x^2 + 2)^2$  and that  $f(1) = \frac{8}{15}$ . Find  $f(x)$ .
- A university club committee holds weekly meetings. Each committee member has a 70% chance of attending a given meeting. A meeting can only go ahead if at least 10 committee members are present.
  - If the club has 15 committee members, what percentage of meetings will go ahead?
  - Find the smallest number of committee members required to ensure that at least 90% of the meetings will go ahead.
- This incomplete Voronoi diagram shows petrol stations A, B, C, D, and E in a city.
  - Find the equation of the missing edge. Give your answer in the form  $ax + by + d = 0$ ,  $a, b, d \in \mathbb{Z}$ .
  - In the context of the question, explain the significance of cell D.
  - Riley is currently equally closest to stations C and E, and is due south of O.
    - Find Riley's location.
    - Riley's car has 6 km of petrol left. Will Riley be able to drive to a petrol station before his car runs out of petrol?



- 4** The distance travelled by two similar toy cars after rolling down a slope was measured 40 times each. The measurements were rounded to the nearest tenth of a metre.

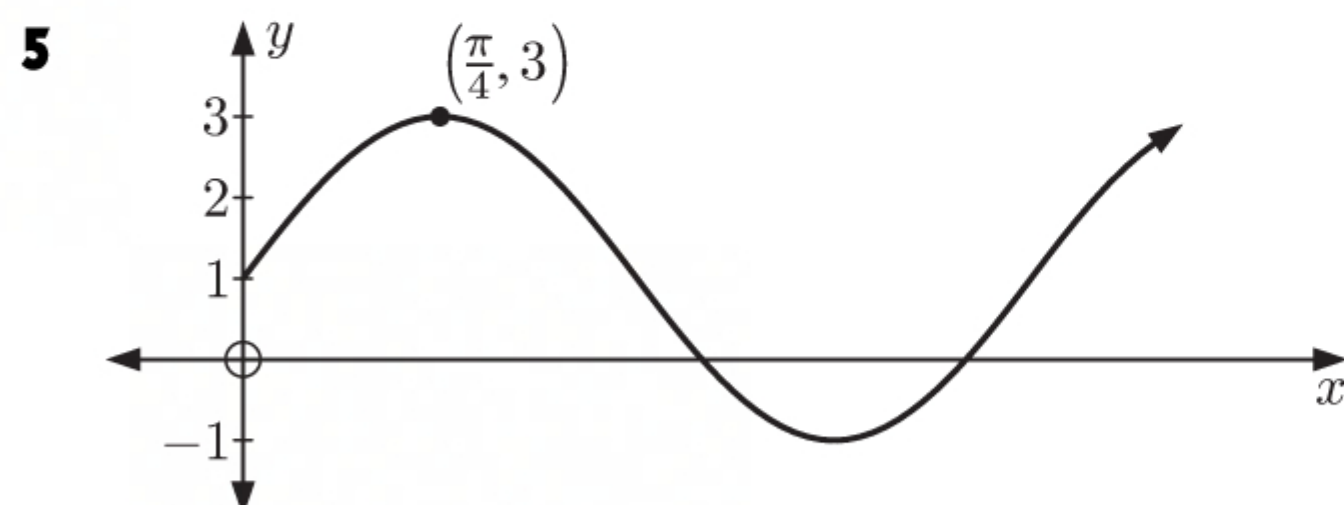
<b>Red car</b>	3.6	4.6	5.6	6.4	4.2	5.3	6.1	4.5
	5.4	4.6	3.9	6.2	5.8	4.5	5.4	6.1
	4.5	5.6	5.7	4.8	3.9	5.6	6.1	5.9
	4.1	5.3	4.2	6.2	7.4	5.4	5.8	4.5
	3.9	5.4	5.7	4.8	5.4	5.7	6.1	6.4

<b>Blue car</b>	Number of rolls	40
	Median distance	4.8 m
	Shortest distance	3.2 m
	Longest distance	6.7 m
	$Q_1$ Lower quartile	4.1 m
	$Q_3$ Upper quartile	5.4 m

- Complete this table of cumulative frequencies for the red car data.
- Draw the cumulative frequency graph for the distance travelled by the red car.
- Use the graph to find the following statistics for the red car:
  - median distance
  - lower quartile
  - upper quartile
- Draw a parallel box and whisker diagram to display the data for both cars.
- Compare the statistics for distance travelled by the two toy cars. Is it reasonable to assume that the same machine manufactured these two toys? Explain your answer.

Distance (m)	Cumulative frequency
$3.5 \leq d < 4$	
$4 \leq d < 4.5$	
$4.5 \leq d < 5$	
$5 \leq d < 5.5$	
$5.5 \leq d < 6$	
$6 \leq d < 6.5$	
$6.5 \leq d < 7$	
$7 \leq d < 7.5$	





Find the equation of the sine function shown in the graph.

- 6 Suppose  $\mathbf{i}$  represents a 1 km displacement due east and  $\mathbf{j}$  represents 1 km displacement due north. A lighthouse is located at the point  $(0, 10)$ . A ship is moving in a straight line with parametric equations  $x = 3 - 2t$ ,  $y = 3t + 1$ ,  $t \geq 0$ , where  $t$  is the number of hours after 8:30 am.

- What was the position of the ship at 8:30 am?
  - Find the ship's:   
 i velocity vector   
 ii speed.
  - Find the distance between the ship and the lighthouse at 10:30 am.
  - Find the time when the ship is directly west of the lighthouse.
  - Find the time when the ship is closest to the lighthouse, and the distance between the ship and the lighthouse at this time.
- 7
- Find the sum to infinity of the infinite geometric series  $1 + 0.6 + (0.6)^2 + (0.6)^3 + \dots$
  - When a ball is dropped from a height of 1 m, on each bounce it returns to 60% of the height it reached previously. Find the total distance travelled by the ball until it stops bouncing.
- 8 In acoustics, the intensity of sound is measured in **decibels** (dB). The **sound intensity level** (SIL) is given by  $L = 10 \log\left(\frac{I}{I_0}\right)$  dB, where  $I$  is the sound intensity, and  $I_0 = 10^{-12} \text{ w/m}^2$  is the reference sound intensity.
- Find, in dB, the SIL of a snare drum with sound intensity  $3 \times 10^{-2} \text{ w/m}^2$ .
  - A lawn mower has SIL 85 dB. Find its sound intensity in  $\text{w/m}^2$ , giving your answer in the form  $a \times 10^k$ ,  $1 \leq a < 10$ ,  $k \in \mathbb{Z}$ .

- 9 This table shows the monthly unemployment rates in France, Denmark, and the United Kingdom during 2018.

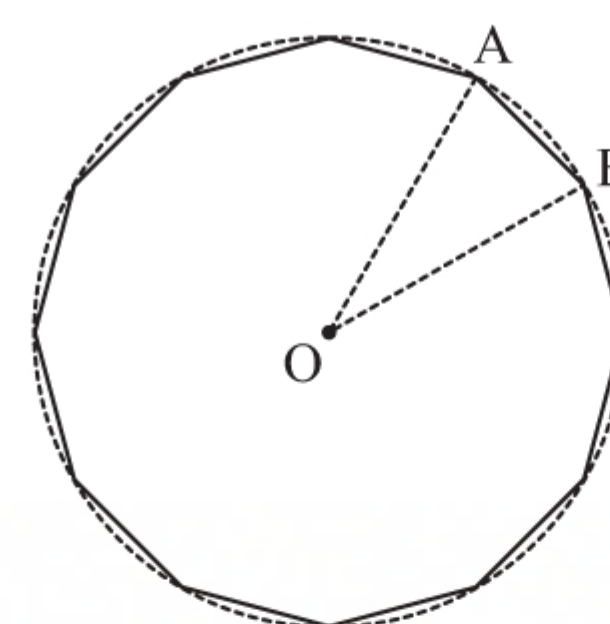
France (%)	9.2	9.3	9.2	9.2	9.1	9.0	9.0	9.0	9.0	8.9	8.9	8.9
Denmark (%)	5.1	5.2	5.0	5.3	5.2	5.1	5.1	4.9	4.9	4.9	5.3	5.1
United Kingdom (%)	4.2	4.2	4.2	4.0	3.9	4.0	4.0	4.0	4.0	3.9	3.9	3.8

Conduct a hypothesis test at a 5% significance level to determine whether there is a correlation between the unemployment rates of:

- France and Denmark
  - France and the United Kingdom.
- 10 A box contains 4 blue balls and  $n$  red balls. When two balls are drawn from the box without replacement, the probability that both are red is  $\frac{1}{3}$ . Find  $n$ .

## MIXED QUESTIONS SET 11

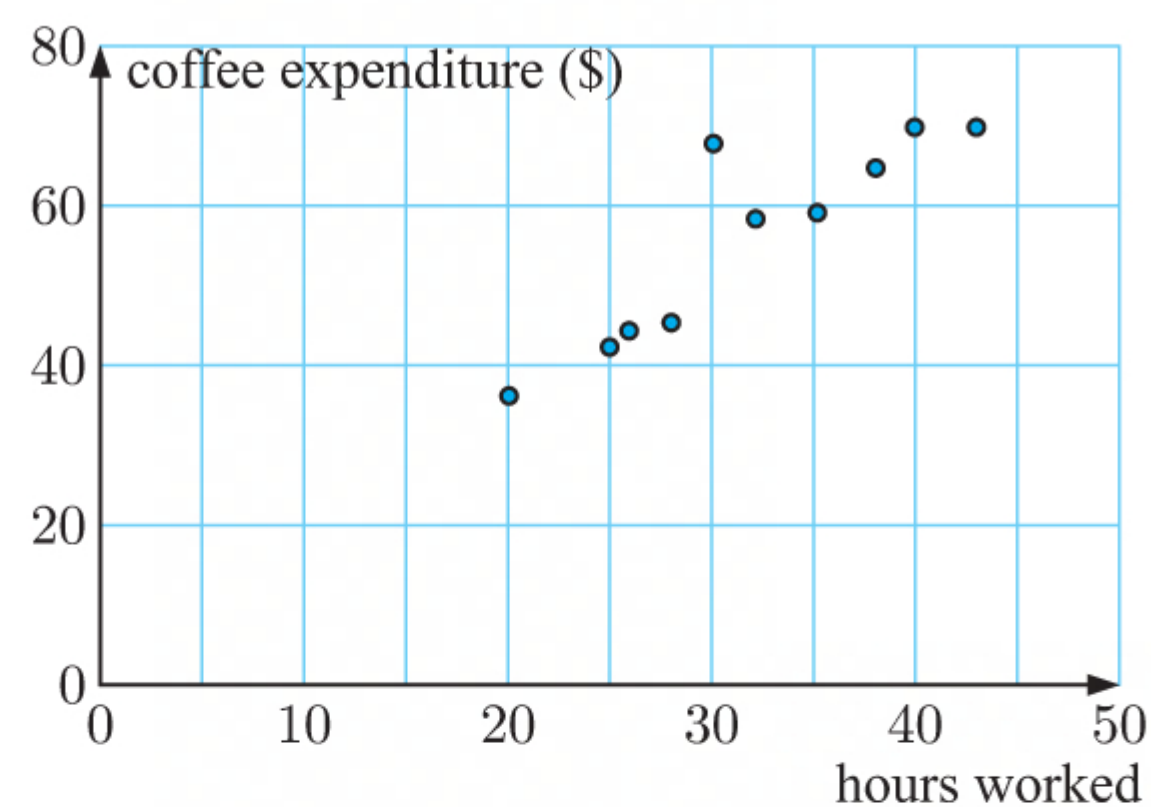
- 1 Let  $f(x) = 3 - 4^{-x}$ .
- Points  $A(2, p)$  and  $B(-2, q)$  lie on  $y = f(x)$ . Determine  $p$  and  $q$ .
  - For the graph of  $y = f(x)$ , determine the:   
 i  $y$ -intercept   
 ii equation of the horizontal asymptote.
  - Sketch the graph of  $y = f(x)$ , showing all details from above.
  - Write down the range of  $f(x)$ .
- 2 A regular dodecagon (12-sided polygon) is inscribed in a circle of radius 6 cm. Points A and B are adjacent vertices of the dodecagon, and both lie on the circle.
- Deduce that  $\angle AOB = 30^\circ$ .
  - Determine the area of the dodecagon.





- 3** This scatter diagram displays the amount James spends on coffee in the cafeteria against the number of hours he works in the week.

- James worked an average of 32 hours, and his average expenditure was \$56 per week. Plot the mean point  $P(32, 56)$  on the graph.
- Draw a line of best fit by eye which passes through  $P$ .
- Use this line to predict the amount James will spend on coffee if he works a 35 hour week.
- Describe the nature and strength of the linear relationship between the variables. Comment on whether the prediction in **c** is reliable.

 PRINTABLE  
GRAPH


- 4** A manufacturer states that the contents of its cereal boxes weigh an average of 320 g. A random sample of 24 boxes was weighed, with the following results recorded in grams:

312	320	326	330	326	322	326	330	331	315	323	316
315	325	311	320	308	325	320	332	316	309	314	324

- Organise the data using a frequency table, with the class intervals  $305 \leq w < 310$ ,  $310 \leq w < 315$ , and so on.
- Draw a frequency histogram to display the data.
- Describe the distribution of the data.
- Find the modal class of this data.
- Calculate the mean of the data. How does it compare to the manufacturer's claim?

- 5** Suppose  $\mathbf{P} = \begin{pmatrix} -1 & 4 & -1 \\ 2 & -1 & 5 \\ 1 & 1 & 2 \end{pmatrix}$ .

- a** Find  $\mathbf{P}^{-1}$ .

- b** Hence solve the system of equations 
$$\begin{cases} -x + 4y - z = -3 \\ 2x - y + 5z = -10 \\ x + y + 2z = -5 \end{cases}$$

- 6** Each day, Daniel completes 3 crossword puzzles and 2 word search puzzles.

The time  $X$  Daniel takes to complete a crossword puzzle has mean 15 minutes and standard deviation 5 minutes.

The time  $Y$  Daniel takes to complete a word search puzzle has mean 10 minutes and standard deviation 2 minutes, and is independent of  $X$ .

Let  $Z$  be the total time Daniel spends on puzzles in a day.

- Write an expression for  $Z$  in terms of  $X$  and  $Y$ .
- Find the mean and standard deviation of  $Z$ .

- 7** The temperature inside Pam's caravan  $t$  hours after 6 am is given by the function  $T(t) = 24 + 5 \sin\left(\frac{\pi}{12}(t - 6)\right)^\circ\text{C}$ .

- Sketch the graph of  $T$  against  $t$  for  $0 \leq t \leq 24$ .
- Find the temperature inside Pam's caravan at: **i** 2 pm **ii** 9 pm.
- Find the maximum temperature inside Pam's caravan, and the time at which it occurs.

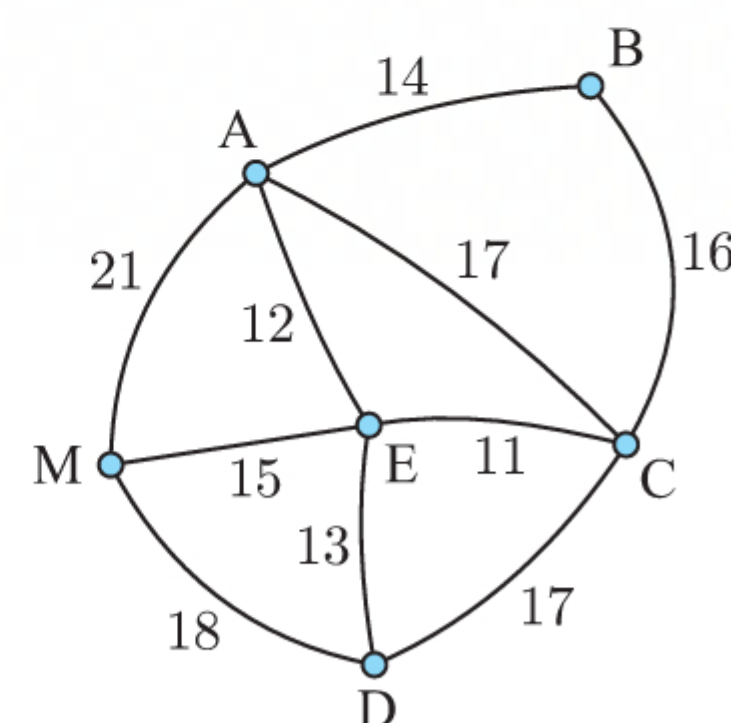
- 8** **a** Find the exact area of the region bounded by  $y = \frac{1}{\sqrt{x}}$ , the  $x$ -axis, and the lines  $x = 1$  and  $x = 9$ .

- b** If the area in **a** is rotated  $360^\circ$  about the  $x$ -axis, what is the volume of the resulting solid?

- 9** A cycling course is being planned in a rural town. The organisers would like the course to cover *all* the roads in the diagram, and start and finish at the mountain M.

All distances shown are in kilometres.

- i** Solve the Chinese Postman Problem for this graph to find the shortest possible distance for the course.
- ii** State a possible route with this shortest distance.
- The road MD has been closed due to a landslide. Find the shortest distance for the course, and a possible route in this case.





- 10** Let  $x$  be the area of a field covered by weed x, and  $y$  be the area of the field covered by weed y. The areas change according to the differential equations

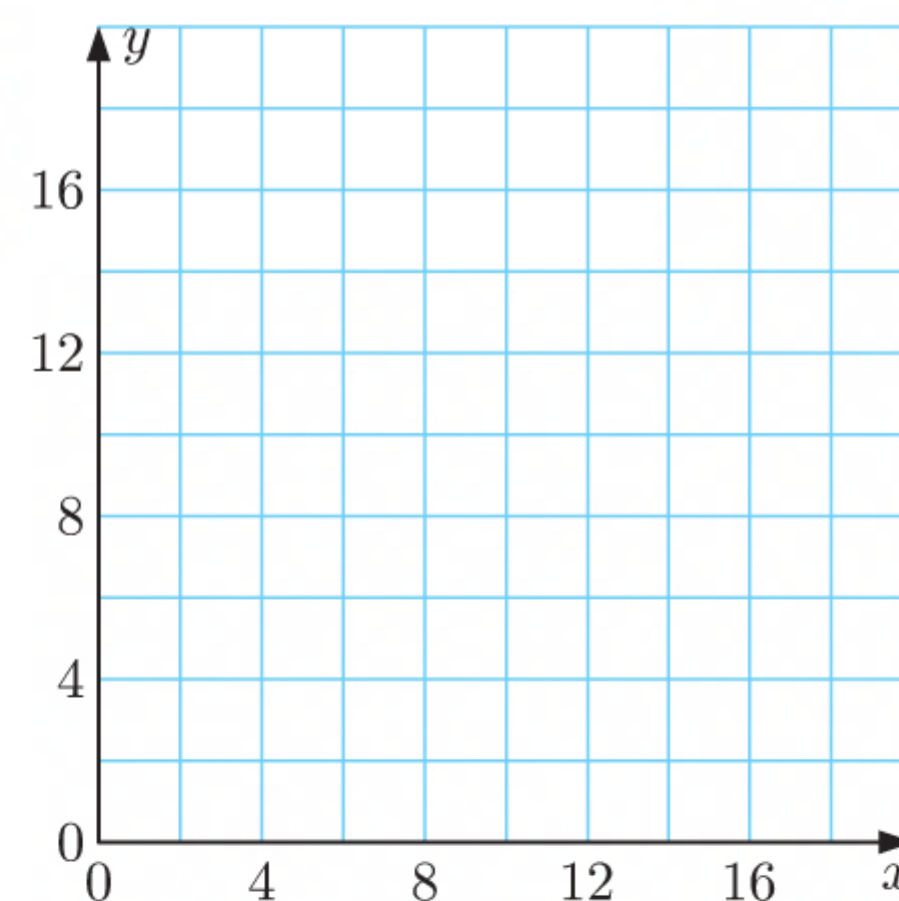
$$\frac{dx}{dt} = 4x + 3y$$

$$\frac{dy}{dt} = 2x - y$$

The matrix  $\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$  has eigenvectors  $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ .

Initially,  $x = 1 \text{ cm}^2$  and  $y = 12 \text{ cm}^2$ .

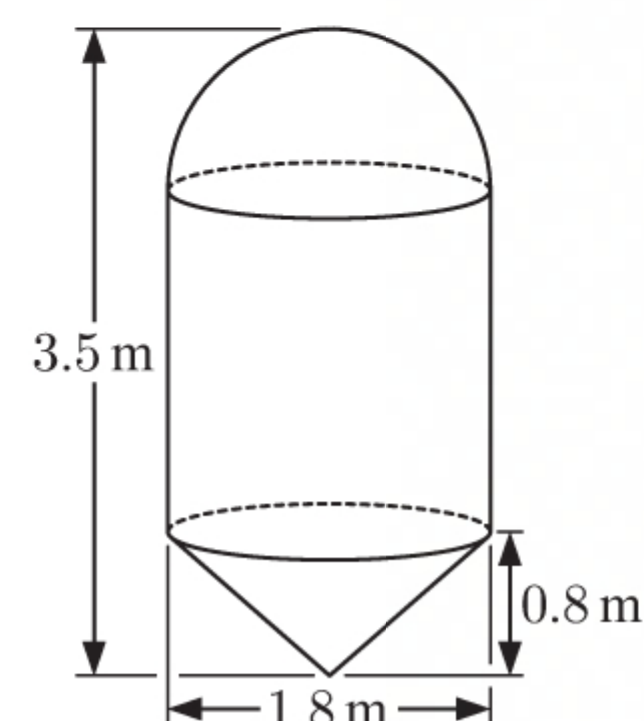
- Find the value of  $\frac{dy}{dx}$  when  $t = 0$ .
- Find the eigenvalues of  $\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$  corresponding to the given eigenvectors.
- Sketch the trajectory for the areas of the two types of weed on a grid like the one alongside. Include any asymptotic behaviour.



## MIXED QUESTIONS SET 12

- 1** A silo is made out of sheet metal using a hemisphere, a cylinder, and a cone.

- Find the height of the cylinder.
- Find the total amount of sheet metal used.
- Find the capacity of the silo in kL.



- 2** Let  $P(x)$  be the number of prime numbers less than or equal to  $x$ . It is known that  $P(x) \approx \frac{x}{\ln x}$  for large values of  $x$ .
- Estimate the number of prime numbers less than or equal to 1 000 000.
  - Given that there are actually 78 498 prime numbers less than or equal to 1 000 000, calculate the percentage error in your estimate in **a** to 1 decimal place.
  - Find  $x$  such that  $P(x) \approx 2000$ , and interpret your answer.

- 3** When an object falls from rest, the *distance* it has travelled is directly proportional to the square of the *time taken*.

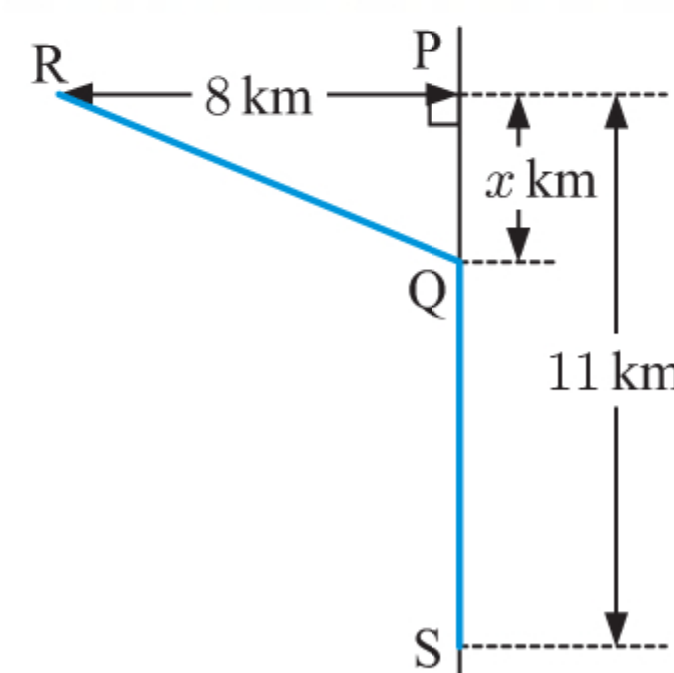
An object dropped from rest travels 19.6 m in 2 seconds.

- How far will the object travel in 3 seconds?
- How long will it take for the object to fall 100 m?

- 4** An offshore oil rig is at point R, 8 km from a straight shore. The point P is on the shore directly opposite the rig. A refinery is on the shore at S which is 11 km from P.

A pipeline is to be constructed under the sea from R to reach the shore at the point Q. From Q a pipeline is to be taken overland to S. The cost of the pipeline is \$5 million per km under the sea and \$3 million per km overland.

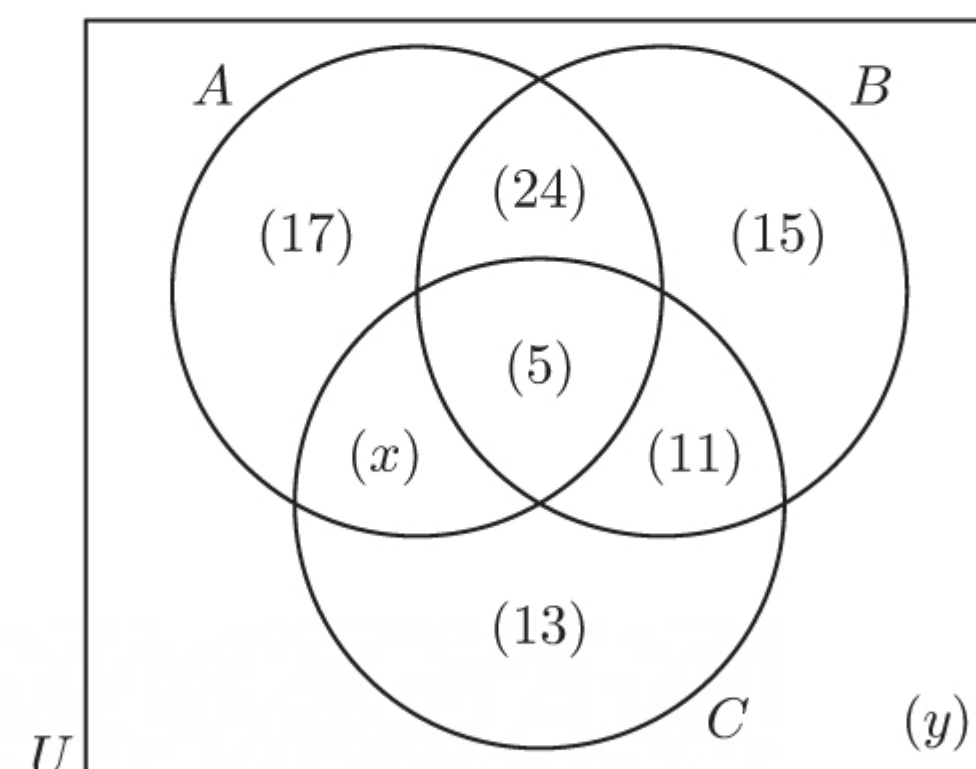
- If Q is  $x$  km from P, show that the cost to construct the pipeline from R to S is  $C(x) = 5\sqrt{x^2 + 64} + 33 - 3x$  million dollars.
- Find the minimum cost of the pipeline.





- 5** Stig is 32 years old. He has \$97 000 in a savings fund that earns 4.9% p.a. interest compounding quarterly. He will make quarterly contributions to the fund so that he can retire at age 55 years with \$1 000 000.
- How much should Stig contribute to his fund each quarter?
  - How much interest is generated in the fund after age 32?
  - After retiring at age 55 years, Stig rolls his \$1 000 000 in an annuity account earning 6.5% p.a. compounding monthly. How much can he withdraw per month if he wants his money to last for 30 years?
  - Stig is used to living on \$2700 per month at age 32 and inflation has averaged 3.7% p.a. Will his standard of living be maintained at the time of his retirement? Explain your answer.

- 6** 100 diners at a restaurant were given a set three-course meal. After the meal, the diners were asked whether they liked each of the courses. The results are summarised alongside.



- Given that 48 people liked course  $A$ , find  $x$  and  $y$ .
- Which course was the most popular?
- Find the probability that a randomly selected diner liked:
  - all of the courses
  - course  $B$ , but not course  $C$
  - exactly two courses, given that the diner liked course  $C$
  - none of the courses, given that the diner disliked course  $B$ .

- 7** Suppose a transformation has equations 
$$\begin{cases} x' = \frac{1}{\sqrt{5}}x - \frac{2}{\sqrt{5}}y \\ y' = \frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y \end{cases}.$$

- Show that this transformation is a rotation.
  - Find the angle of rotation.
  - Find the image of  $(2, 1)$  under this rotation.
- 8**
- Differentiate  $y = x \ln x$ ,  $x > 0$  with respect to  $x$ .
  - A management consultant uses a computer to schedule machines in a factory. The time  $T$  taken by the computer increases with the number of machines,  $N$ , according to  $T' = \frac{1}{20}(1 + \ln N)$ .

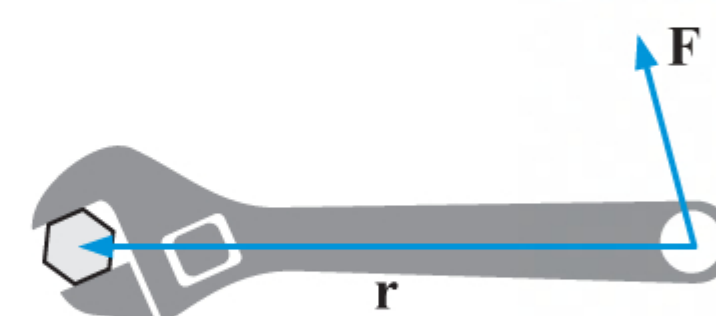
Given that the program takes 10 seconds to schedule 50 machines, determine, to the nearest second, the time required to schedule 100 machines.

- 9** The lengths of baguettes sold at a bakery are normally distributed with mean  $\mu$  cm and standard deviation 3 cm. To test the bakery's claim  $H_0: \mu = 25$  against  $H_1: \mu < 25$ , a sample of 15 baguettes is selected and measured.  $H_0$  will be accepted if the sample mean  $\bar{x} > 24$ , and rejected if  $\bar{x} \leq 24$ .

- Find the probability of a Type I error  $\alpha$  for this decision rule.
- Given that the true value of the mean is 24.5 cm, find the probability of a Type II error  $\beta$ .

- 10** When a force  $\mathbf{F}$  is applied to a wrench at a point with position vector  $\mathbf{r}$  from the bolt, the rotational force or *torque* applied to the bolt is given by  $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ .

Suppose the force  $\mathbf{F}^* = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$  N is applied at the point with position  $\mathbf{r}^* = 0.2\mathbf{i} - 0.15\mathbf{j}$  m from the bolt.



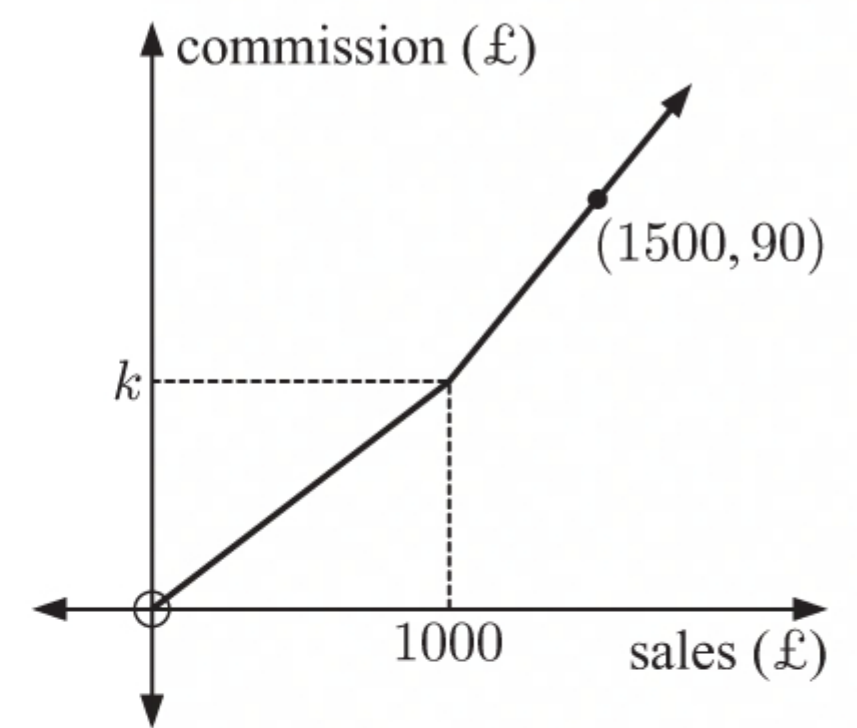
- Find, in Nm, the torque applied to the bolt.
- Find the magnitude of the torque applied to the bolt.
- Find the *maximum* magnitude of the torque that can be applied to the bolt by applying a force with magnitude  $|\mathbf{F}^*|$  from the position vector  $\mathbf{r}^*$ .



# MIXED QUESTIONS SET 13

- 1** Each day, a salesperson makes 5% commission on sales up to £1000, and a higher rate of commission on sales above £1000.

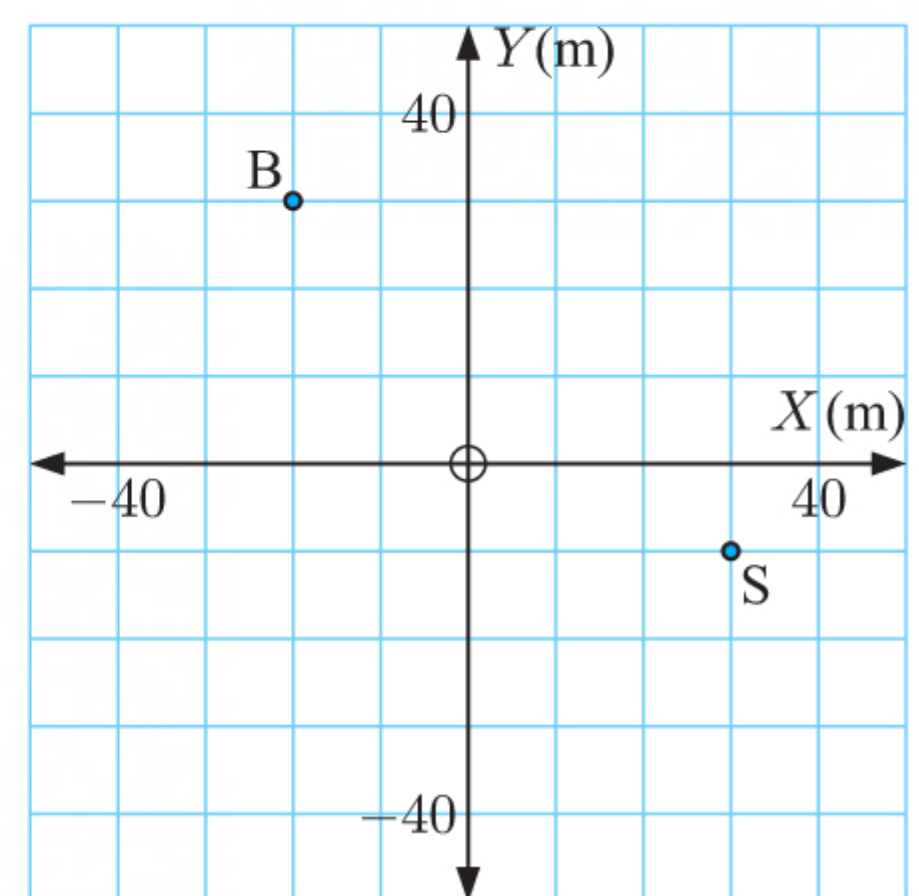
- Find the value of  $k$ .
- Find the higher rate of commission on sales above £1000.
- Find the commission earned when £1800 is made in sales.
- Find the value in sales needed to earn £150 commission.



- 2** Cynthia invested \$2000 in an account that pays 4.4% p.a. interest compounded quarterly for 5 years.
- Find the final value of the investment.
  - How much interest did Cynthia earn?
  - Given that inflation averages 2.5% p.a. over the investment period, find the real value of the investment.

- 3** This grid shows the position of a boat B and a shipwreck S.

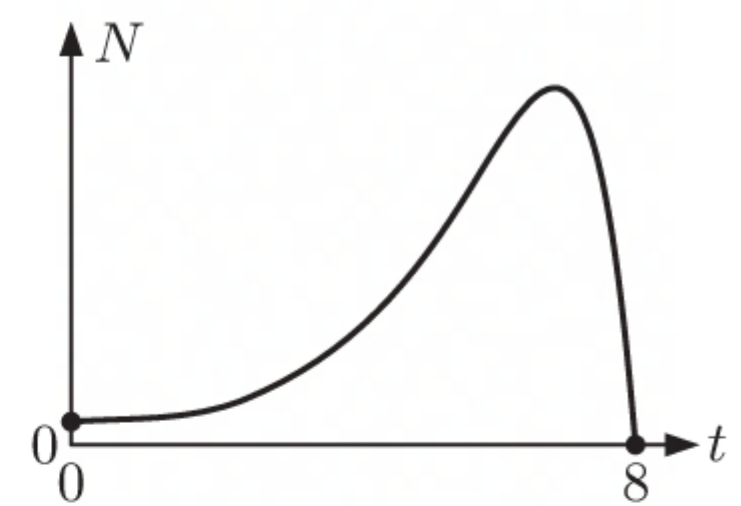
The boat's anchor is directly below the boat, 50 m below sea level. The shipwreck is 40 m below sea level. Suppose sea level has  $Z$ -coordinate 0.



- Find the 3-dimensional coordinates of:
    - the anchor
    - the shipwreck.
  - A diver swims from the boat to the shipwreck. How far does the diver swim?
  - Find:
    - the angle of depression from the boat to the shipwreck
    - the angle of elevation from the anchor to the shipwreck.
- 4** Suppose  $f$  and  $g$  are functions such that  $g(x) = 3f\left(\frac{1}{2}x\right)$ .
- What transformations are needed to map  $y = f(x)$  onto  $y = g(x)$ ?
  - Given that  $(-6, 3)$  lies on  $y = f(x)$ , find the coordinates of the corresponding point on  $y = g(x)$ .
  - Given that  $(4, -9)$  lies on  $y = g(x)$ , find the coordinates of the corresponding point on  $y = f(x)$ .

- 5** The number of bacteria found in a sample of human tissue  $t$  hours after infection occurred, is modelled by the function  $N = (8 - t)e^{t-6}$  million,  $0 \leq t \leq 8$ .

The graph of this function is shown alongside.



- Show that  $\frac{dN}{dt} = (7 - t)e^{t-6}$ .
  - Find the coordinates of the:
    - turning point
    - point of inflection
    - $t$ -intercept.
  - Use these coordinates to state:
    - the time when all the bacteria are dead
    - the maximum number of bacteria reached in the sample
    - the time at which the rate of increase of the bacteria is a maximum.
  - Copy the graph and indicate clearly which points on the graph represent your answers to **c**.
- 6** Two ships A and B have paths defined by the equations

$$\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

respectively, where distances are in kilometres and  $t$  is the time in hours.

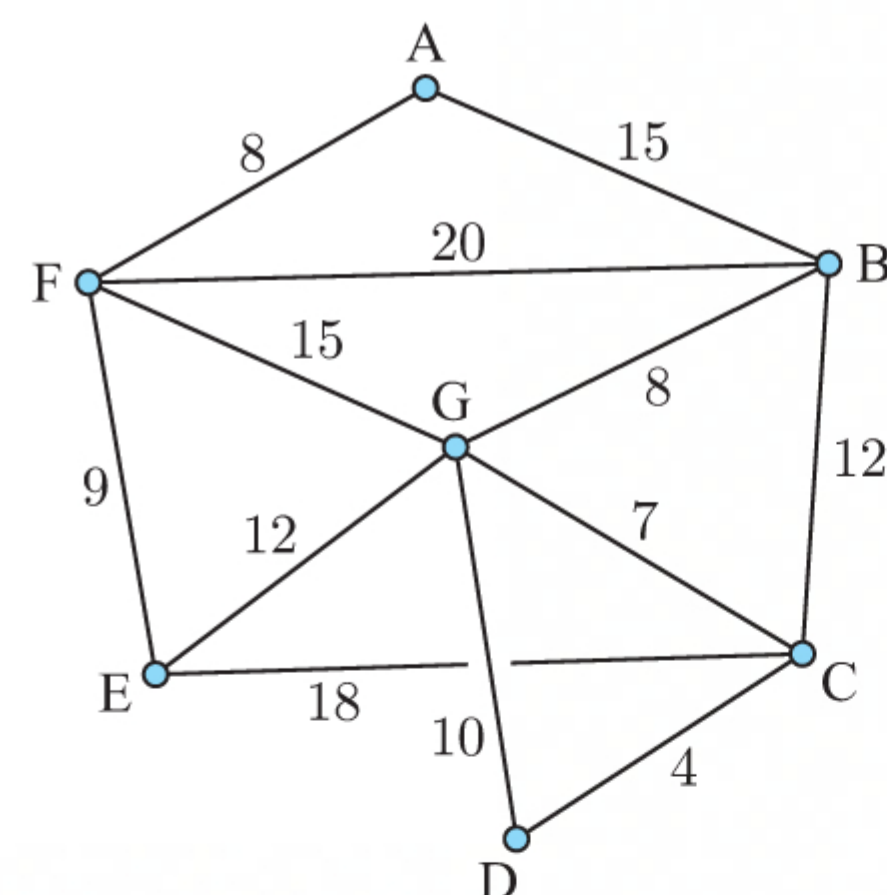
- Find the initial position of each ship.
- Find the speed of each ship.
- Find the acute angle between the paths of the ships.
- Show that the two ships will pass through the same location, but not at the same time.



- 7** The graph shows the main roads and their length, in kilometres, connecting 7 key locations in a city.

A major flood has damaged all the main roads. The local council wishes to reconnect all locations as quickly as possible by repairing the minimum length of road.

- Use Prim's algorithm to determine which roads should be prioritised, and the minimum length of road to be repaired.
- If road FG must be repaired first, find the roads which should be prioritised, and the minimum length of road to be repaired.



- 8** OABCD is a regular pentagon with side length 1. Let  $z_1 \equiv \overrightarrow{OA}$ ,  $z_2 \equiv \overrightarrow{OB}$ ,  $z_3 \equiv \overrightarrow{OC}$ , and  $z_4 \equiv \overrightarrow{OD}$ .

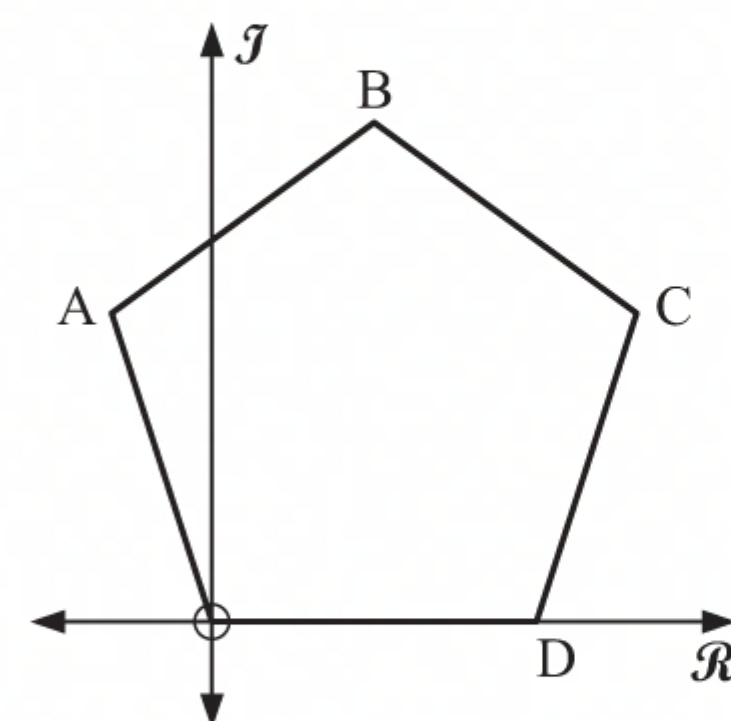
- a** Write in polar form:

**i**  $z_1$

**ii**  $z_2 - z_1$

**iii**  $z_3$

- b** Let  $w = \frac{z_3^2}{k(\sqrt{3} - i)}$ , where  $k \in \mathbb{R}^+$ . Find  $k$  such that successive powers of  $w$  lie on a circle.



- 9** In previous years, the number of injuries per game in junior football has followed a Poisson distribution with an average of 0.8 injuries per game.

To test whether rule changes this year have improved the injury rate, a sample of 30 games were observed. 20 injuries occurred in these games.

- State the null and alternative hypotheses under consideration.
- Describe the null distribution.
- Calculate the  $p$ -value, and hence determine at the 10% level whether the injury rate has improved.

- 10** The population  $P$  of a colony of marsupials on an island satisfies the differential equation  $\frac{dP}{dt} = 0.01P\left(1 - \frac{P}{1000}\right)$ , where  $t$  is the time in years. The slope field for this differential equation is shown alongside.

- a** Suppose the population is 400 after 2 years.

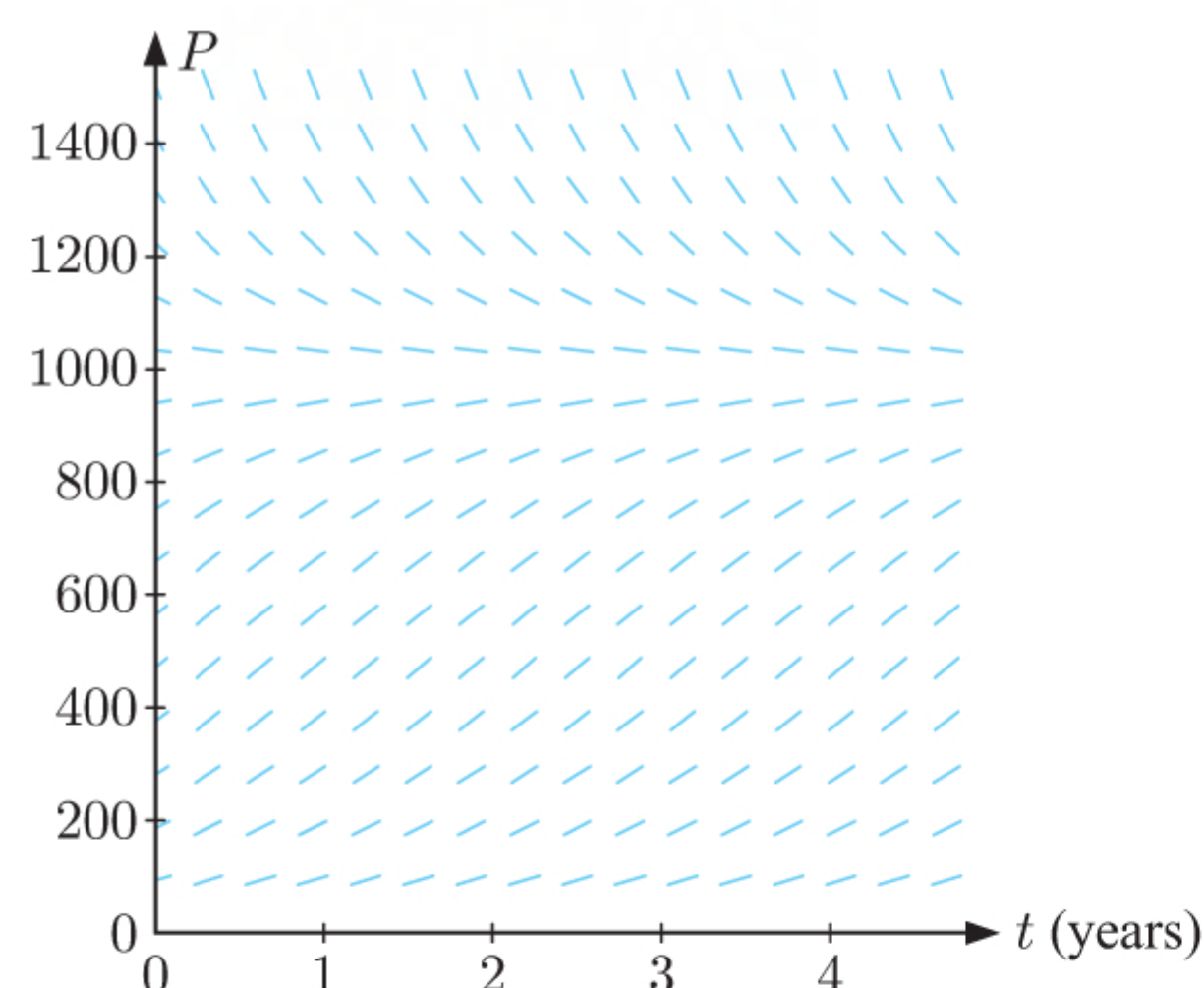
Is the population increasing or decreasing at this time? Justify your answer.

- b** Sketch the solution curve if the initial population was:

**i** 200

**ii** 1400.

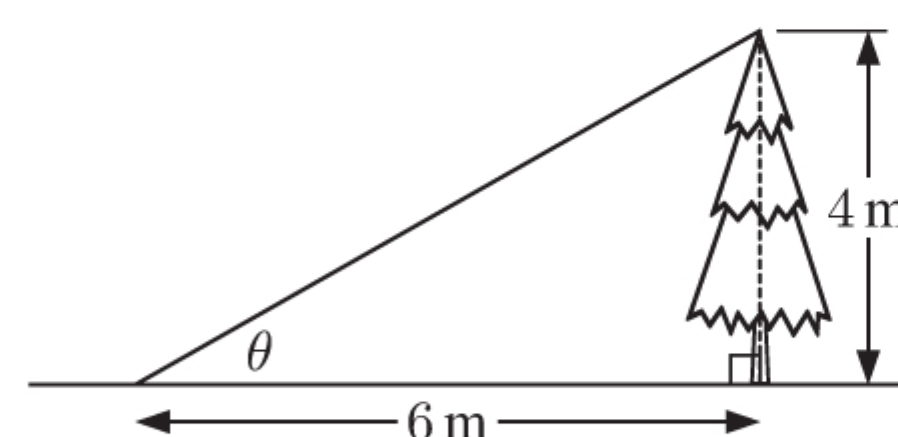
- c** Describe how the population changes over time for each initial population in **b**.



## MIXED QUESTIONS SET 14

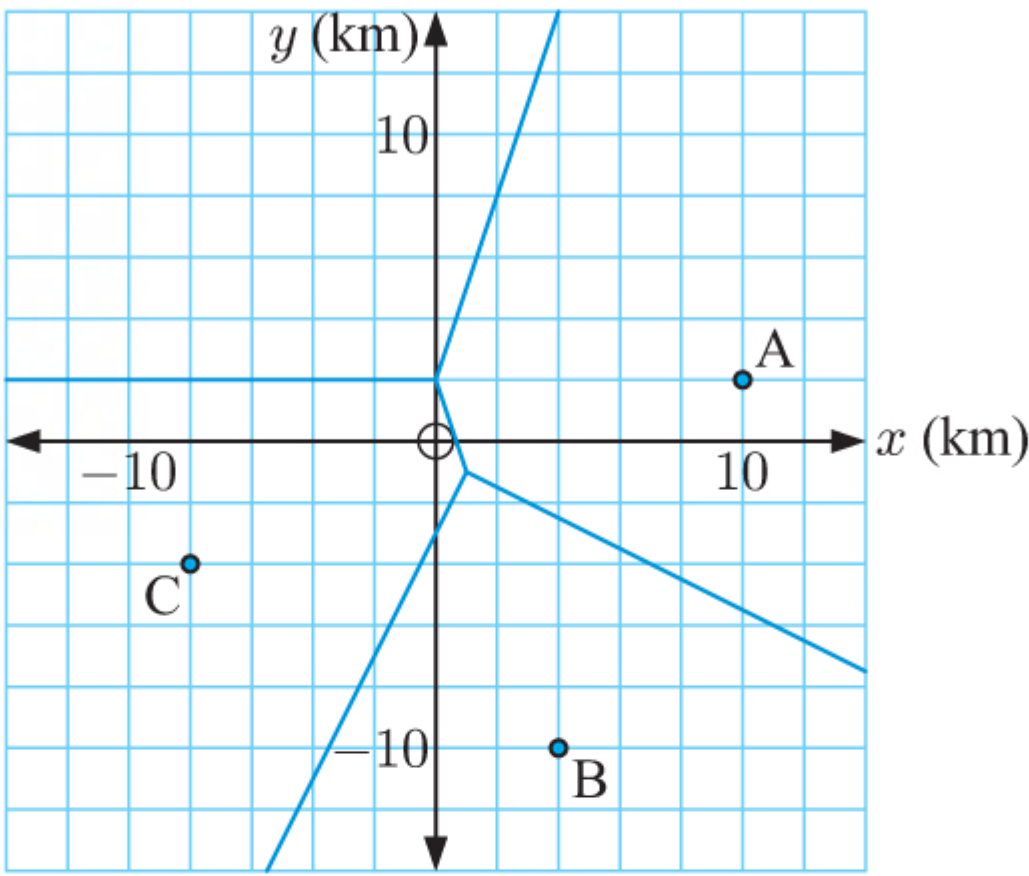
- 1** The measurements alongside have been rounded to the nearest metre.

- Use these measurements to estimate the angle of elevation  $\theta$ .
- Find the boundary values of  $\theta$ .
- Hence find the maximum percentage error in the estimate in **a**.





- 2 This Voronoi diagram shows the locations of hospitals in a particular city.
- a Copy the diagram and shade the cell with the missing hospital.
  - b Find the coordinates of the missing hospital X.
  - c Find the hospital closest to:
    - i  $(-1, 2)$
    - ii  $(4, -1)$
  - d An ambulance is transferring a patient from hospital A to hospital X. A stop is made at the patient's house, which is closest to hospitals A and X. The total distance travelled is  $12\sqrt{5}$  km. Assuming the ambulance travelled in a straight line for each leg of the journey, determine the location of the patient's house.



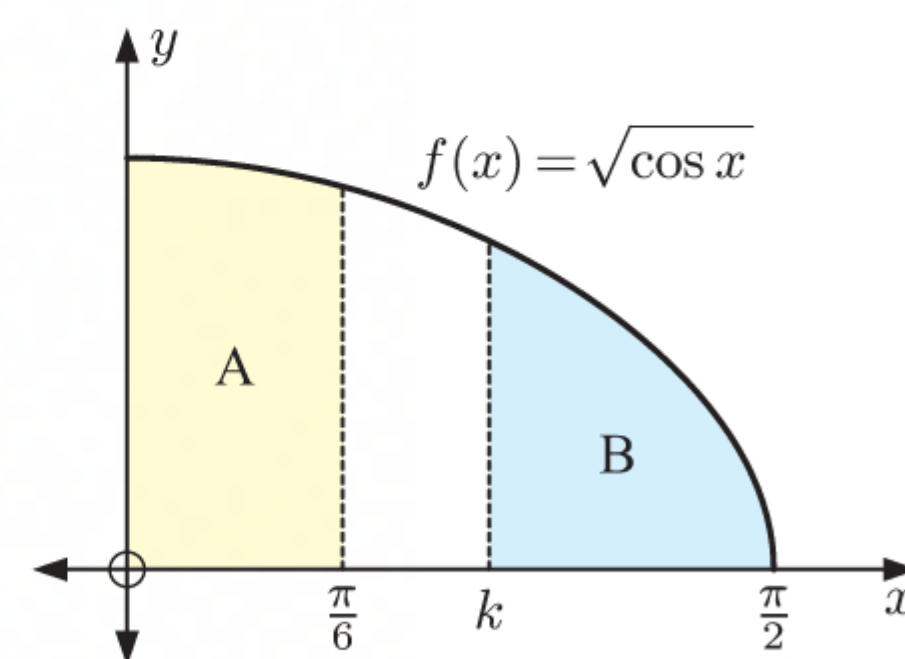
- 3 A bag contains 6 red balls and 4 white balls. A game is played in which the player draws 3 balls from the bag without replacement. The player wins if 3 red balls are drawn.
- a Find the probability of the player winning a single game.
  - b Let  $X$  be the number of wins when the game is played 60 times.
    - i Find the mean  $\mu$  and standard deviation  $\sigma$  of  $X$ .
    - ii Find  $P(X = \mu)$ .
    - iii Find  $P(\mu - \sigma \leq X \leq \mu + \sigma)$ .
- 4 Consider the function  $f(x) = \frac{\sqrt{x}}{x-3}$ .
- a State the domain of  $f(x)$ .
  - b Find  $f'(x)$ .
  - c For what value(s) of  $x$  is  $f'(x)$ :
    - i zero
    - ii undefined?
- 5 The maximum velocity  $V \text{ m s}^{-1}$  of a body varies with the height,  $h$  metres from which it is dropped, as shown in the following table.

Height ( $h \text{ m}$ )	5	25	45	70	75
Maximum velocity ( $V \text{ m s}^{-1}$ )	17.9	40	53.6	66.9	69.3

- a Draw scatter diagrams of:
    - i  $V$  against  $h$
    - ii  $V$  against  $\ln h$
    - iii  $\ln V$  against  $h$
    - iv  $\ln V$  against  $\ln h$
  - b Which model is most appropriate for the data? Explain your answer.
  - c Find the model connecting  $h$  and  $V$ .
  - d Estimate the maximum velocity of the body if it is dropped from a height of 50 m.
  - e Explain why the model may not be appropriate for heights greater than 90 m.
- 6 A shuttle bus company claims that its buses travel from the airport to the city in 45 minutes. To test this claim, 15 of the company's bus trips are sampled at random. The mean travel time was  $\bar{x} = 47.8$  minutes, and the standard deviation of the sample was  $s_n = 3.5$  minutes.
- a Find  $s_{n-1}$  for the sample.
  - b Find a 90% confidence interval for the population mean time.
  - c Does the company's claim appear to be justified? Explain your answer.
- 7 Suppose  $f(x) = 25 - x^2$  and  $g(x) = \frac{2}{\sqrt{x}}$ .
- a Find  $(g \circ f)(x)$ , and state its domain.
  - b Solve  $(g \circ f)(x) = 1$ .
  - c Find the asymptotes of  $y = (g \circ f)(x)$ .
- 8 The distances travelled by a ski jumper are normally distributed with mean 170 m and standard deviation 15 m. Find the probability that, over three consecutive jumps, the ski jumper will travel a combined distance of at least 500 m.
- 9 Nassa's drone takes off from  $(9, 20, 0)$  and moves with velocity vector  $0.5\mathbf{i} - 0.2\mathbf{j} + 0.6\mathbf{k}$ . At the same time, Gemma's drone takes off from  $(15, 8, 0)$  and moves at speed  $0.9 \text{ m s}^{-1}$  with velocity vector  $0.1\mathbf{i} + 0.8\mathbf{j} + a\mathbf{k}$ , where  $a > 0$ .
- a Find the value of  $a$ .
  - b Find the acute angle between the paths of the drones.
  - c How far apart are the drones when they take off?
  - d
    - i At what time are the drones closest to each other?
    - ii Find the distance between the drones at this time. Round your answer to the nearest cm.

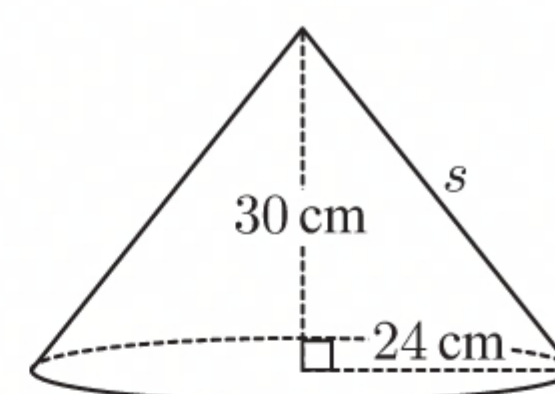


- 10** Consider the function  $f(x) = \sqrt{\cos x}$  alongside.
- Find the volume of the solid formed when region  $A$  is revolved  $2\pi$  about the  $x$ -axis.
  - The solid formed when region  $B$  is revolved  $2\pi$  about the  $x$ -axis has half the volume of the solid in **a**. Find  $k$ .



## MIXED QUESTIONS SET 15

- The first four terms of a geometric sequence are 0.125, 0.25, 0.5, and 1.
  - Write down the common ratio  $r$ .
  - Find the 20th term  $u_{20}$ .
  - Find the sum of the first 10 terms.
- A solid right-circular cone has base radius 24 cm and vertical height 30 cm.
  - Show that the slant height  $s$  is 38.4 cm, correct to 3 significant figures.
  - Determine the total surface area of the cone. Give your answer in the form  $a \times 10^k$  where  $1 \leq a < 10$  and  $k \in \mathbb{Z}$ .



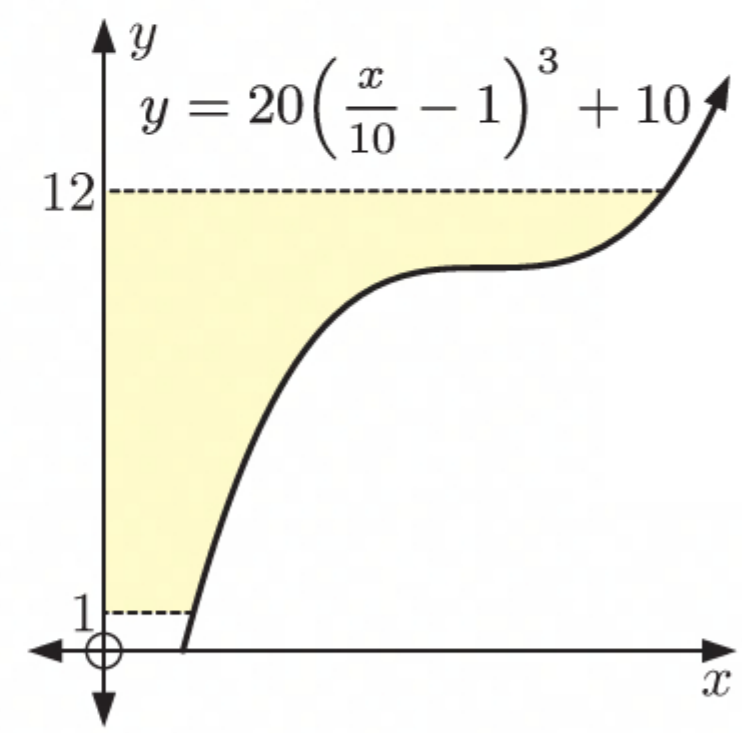
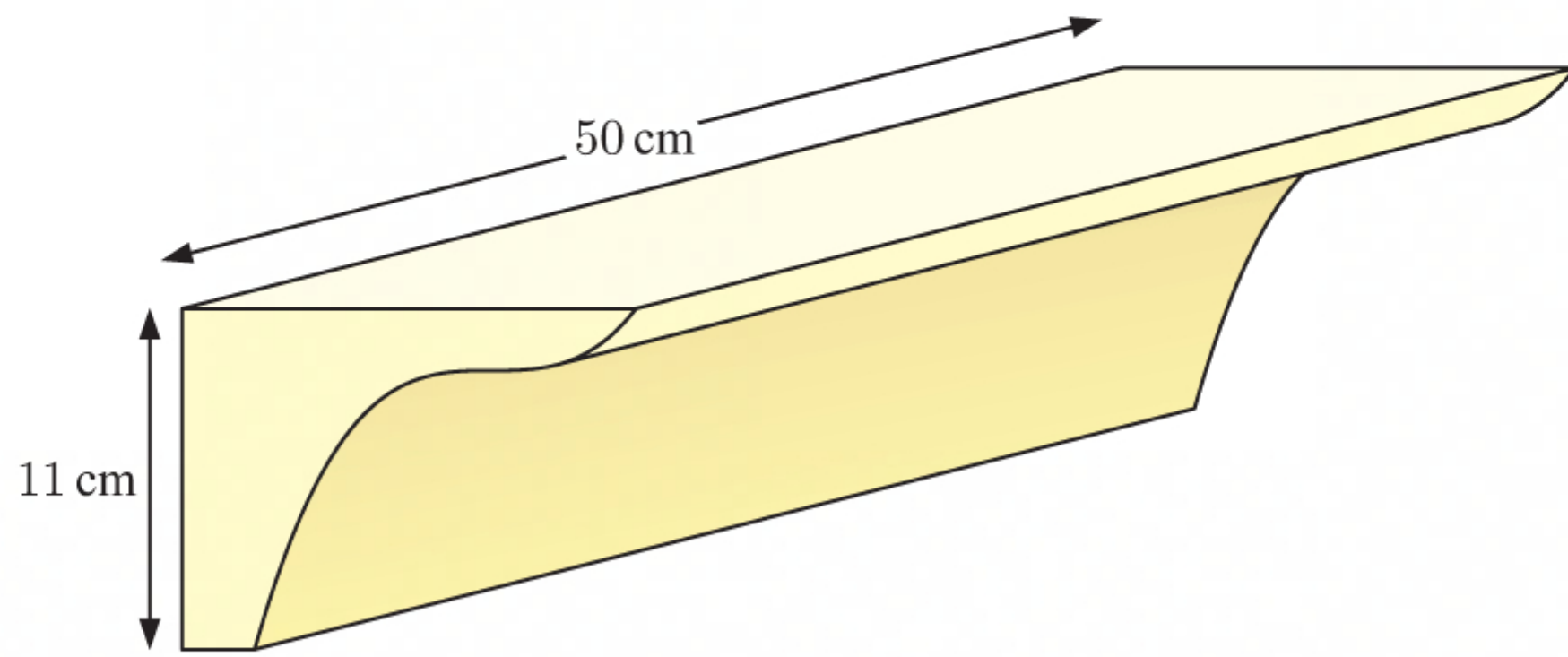
- A winemaker wants to examine the effect of weed spray in his vineyard. He randomly selects 50 sample spots, each of area  $1 \text{ m}^2$ , and counts the number of weeds in each spot. The results are shown in the table alongside.
  - Determine the value of  $p$ .
  - Estimate the mean number of weeds per spot.
  - What percentage of sample spots had fewer than 10 weeds?

Number of weeds	Frequency
0 - 4	9
5 - 9	15
10 - 14	10
15 - 19	$p$
20 - 24	5
25 - 29	2

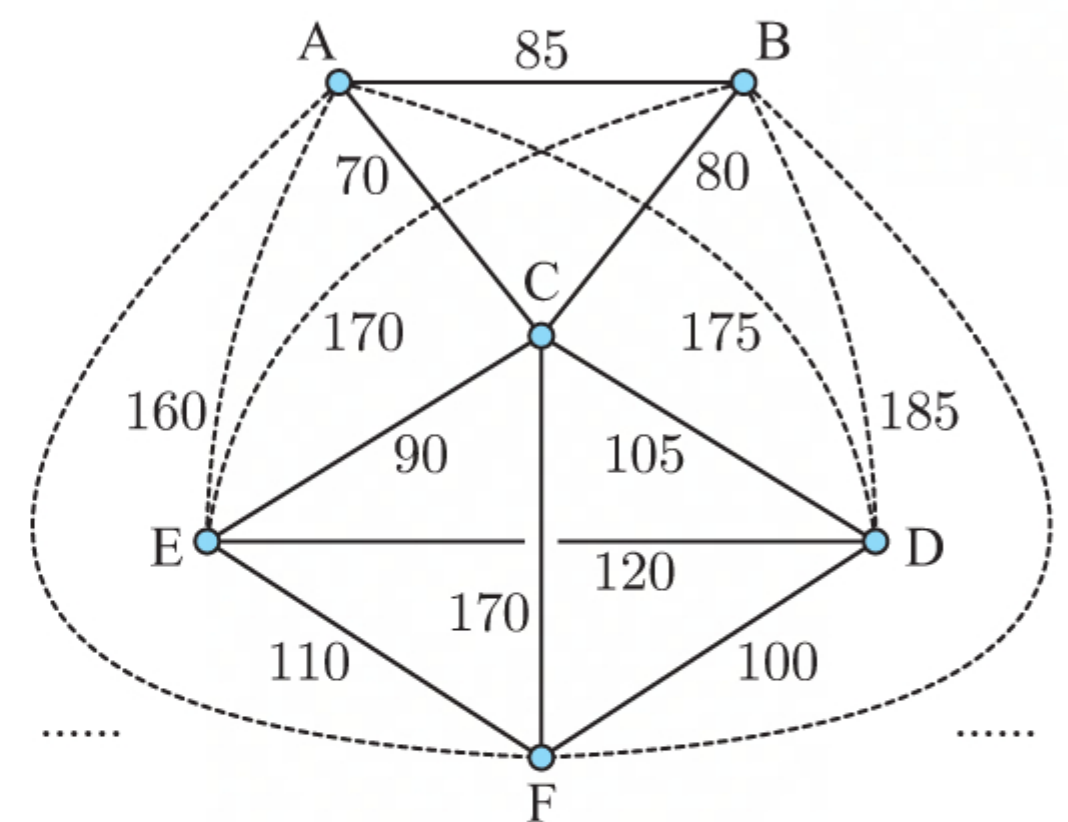
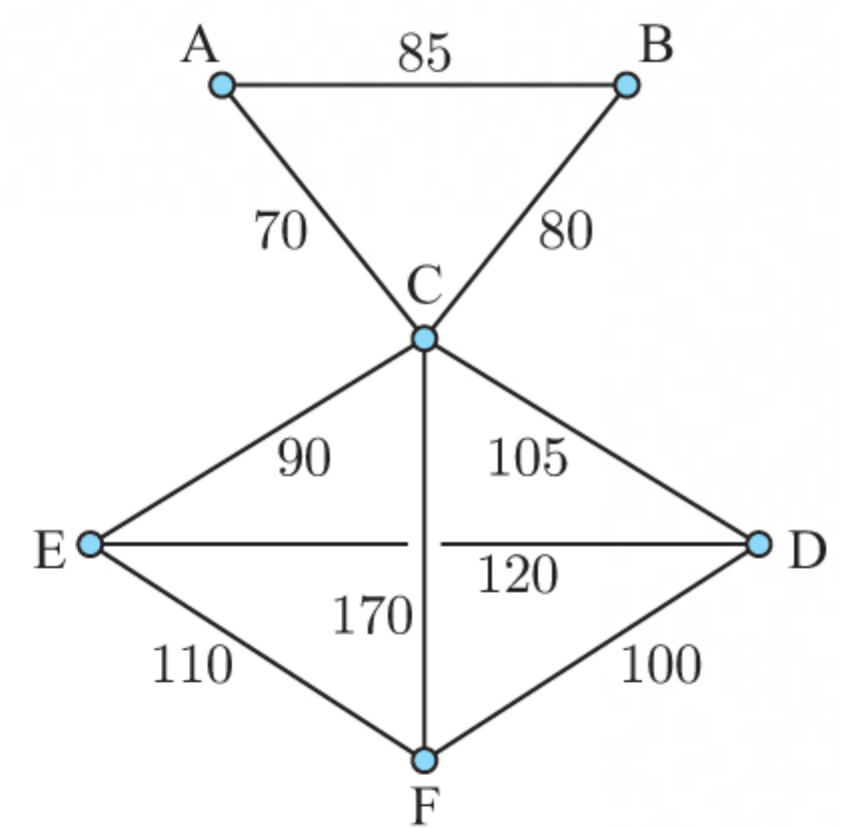
- Tariq takes out a loan for £7000. The loan is to be repaid over 3 years at 8.5% p.a. interest compounded monthly.
  - Find the monthly repayments.
  - How much interest must Tariq pay in total?
- The speeds  $X$  of Jerry's baseball pitches are normally distributed with mean  $140 \text{ km h}^{-1}$  and standard deviation  $5 \text{ km h}^{-1}$ . The speeds  $Y$  of Eric's baseball pitches are normally distributed with mean  $136 \text{ km h}^{-1}$  and standard deviation  $4 \text{ km h}^{-1}$ . Assume that  $X$  and  $Y$  are independent.
  - Find the distribution of  $X - Y$ .
  - Jerry and Eric each pitch a baseball. Find the probability that:
    - Jerry's pitch is faster than  $143 \text{ km h}^{-1}$
    - Eric's pitch is faster than Jerry's pitch.
- In a chemical reaction, the concentration of a substance after  $t$  seconds is given by the logistic model  $C = \frac{L}{1 + 9e^{-kt}}$  units. The concentration was initially 5 units, and after 6 seconds, the concentration was 20 units.
  - Find  $L$  and explain what it means.
  - Find  $k$ .
  - Find the concentration of the substance after 10 seconds.
  - At what time was the concentration 40 units?
- Find the transformation matrix  $\mathbf{A}$  for a vertical stretch with scale factor  $\frac{3}{5}$  followed by a clockwise rotation through  $\frac{\pi}{4}$  about  $O$ .
  - Find the image of  $(2, -1)$  under this transformation.
  - The point  $(1, k)$  is mapped to  $(2\sqrt{2}, \sqrt{2})$  under this transformation. Find the value of  $k$ .



- 8 The uniform cross-section of this shelf is formed by the region between the  $y$ -axis and the curve  $y = 20\left(\frac{x}{10} - 1\right)^3 + 10$ , from  $y = 1$  to  $y = 12$ .



- a Rearrange  $y = 20\left(\frac{x}{10} - 1\right)^3 + 10$  so that  $x$  is the subject.
- b Find the area of the cross-section.
- c Hence find the volume of the shelf.
- 9 This graph shows the direct train lines between towns, and the distances in km of the lines.
- a Write down the adjacency matrix for the graph.
- b Hence calculate the number of ways to travel from A to D in exactly 4 trips.
- c Is the graph Eulerian? Explain your answer.
- d A safety inspector would like to travel along each train line, starting and finishing at A. Find the shortest distance she must travel, and write down a possible route she could take.
- e Is the graph Hamiltonian? Explain your answer.
- f Copy and complete this graph, which shows the least distance to travel between each pair of towns.
- g A tourist would like to visit each of the towns, starting and finishing at B.
- i Use the nearest neighbour algorithm to find an upper bound for the total distance of the trip.
- ii By deleting vertex B, use the deleted vertex algorithm to find a lower bound for the total distance of the trip.



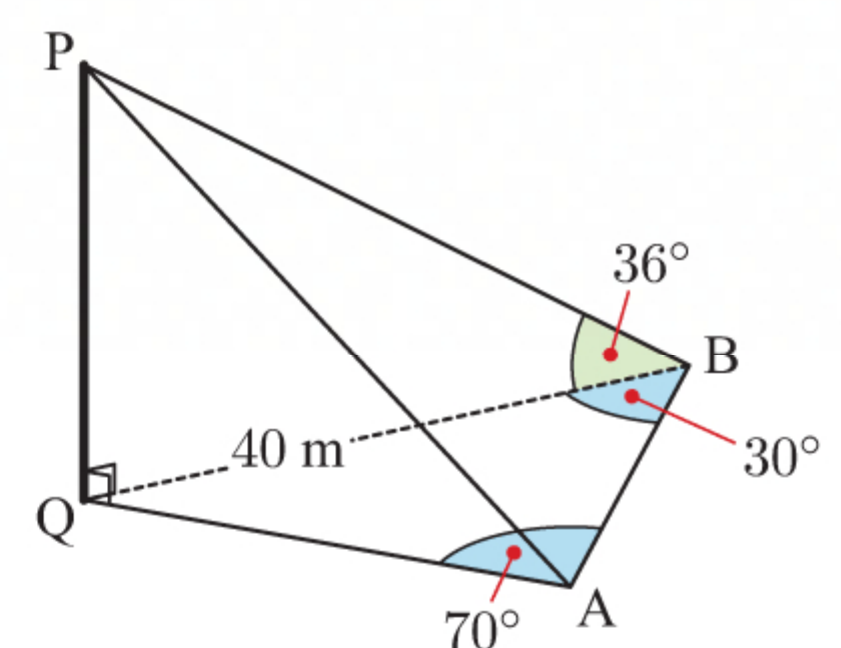
- 10 A vessel contains 100 litres of liquid industrial waste. A faulty tap at the base of the vessel allows liquid to escape at a rate proportional to the square root of the quantity of liquid remaining in the vessel.
- a Show that the volume  $V$  remaining after  $t$  hours is given by  $V = \left(\frac{20 - kt}{2}\right)^2$  litres, where  $k$  is a positive constant.
- b If, after 4 hours, 19 litres have escaped, how long will it take for the vessel to empty?

## MIXED QUESTIONS SET 16

- 1 The diagram shows a vertical pole  $[PQ]$ , which is supported by two wires fixed to the horizontal ground at A and B.  $\widehat{PBQ} = 36^\circ$ ,  $\widehat{BAQ} = 70^\circ$ ,  $\widehat{ABQ} = 30^\circ$ , and the distance BQ is 40 m.

Find:

- a the height of the pole
- b the distance between A and B.





- 2** Robin is an amateur archer. Using her regular bow, it takes on average 1.523 seconds for her arrow to hit a target 50 m away. Her coach suggests trying a longer bow which should increase the speed of her arrows, and therefore decrease their flight time.

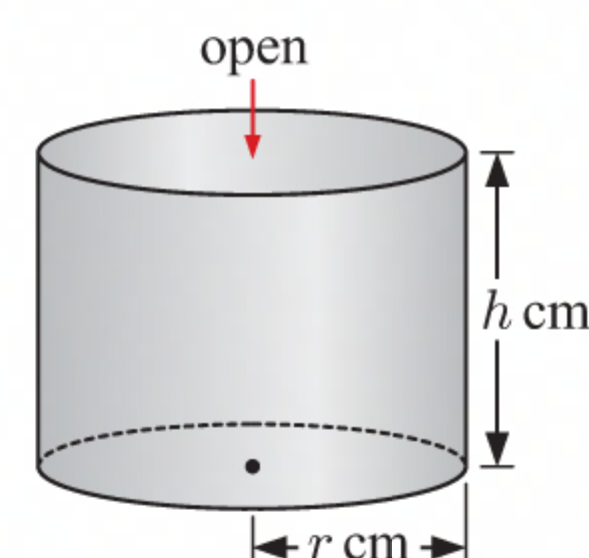
She trials the longer bow with 10 shots, and the flight time in seconds for each arrow is recorded:

1.501   1.516   1.543   1.492   1.498   1.487   1.522   1.511   1.486   1.538

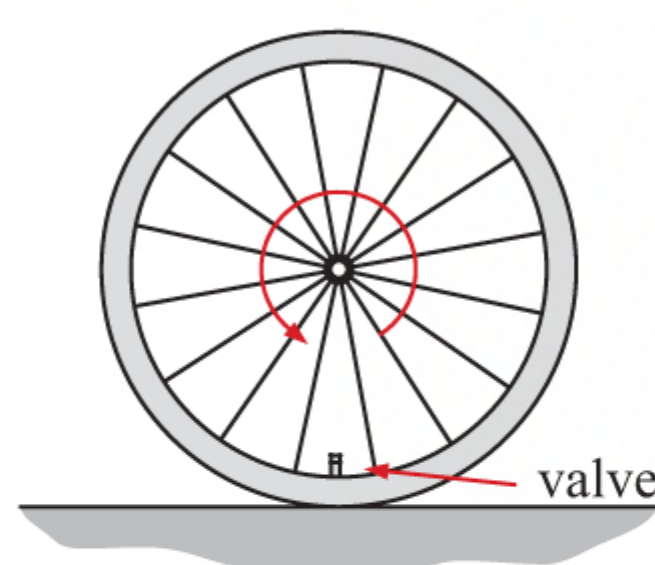
Robin is interested in whether the longer bow has indeed decreased the flight time of her arrows.

- State the null and alternative hypotheses.
  - Calculate the:
    - test statistic
    - $p$ -value.
  - At the 1% significance level, determine whether the longer bow has decreased the flight time of Robin's arrows.
- 3** An open cylindrical bin is to be made from PVC plastic and is to have capacity 500 litres.

- Show that  $h = \frac{500\,000}{\pi r^2}$ .
- Hence write a formula for the surface area  $A$  in terms of  $r$  only.
- Use calculus to find the dimensions of the bin which minimises the amount of PVC plastic used.



- 4** A bicycle wheel sits on the road so its valve is at the bottom. The tyre has inner radius 35 cm and outer radius 40 cm.



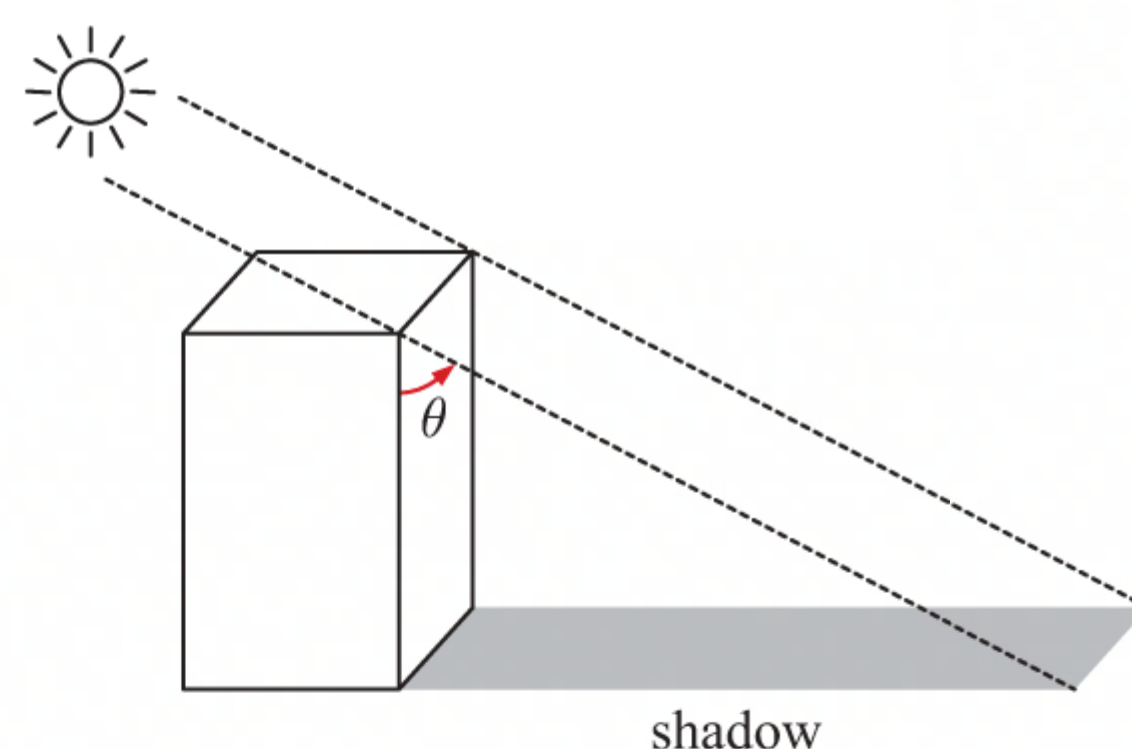
The wheel begins to rotate at a constant speed of 4 revolutions per second.

- Find the height of the valve above the road after:
  - 0 seconds
  - $\frac{1}{12}$  second.
- The height of the valve above the road after  $t$  seconds can be modelled by the function  $H(t) = a \sin(b(t - c)) + d$  cm.  
Find:
  - $a$
  - $d$
  - $b$
  - $c$
- How long does it take the valve to rise to 60 cm above the road?

- 5** The angle  $\theta$  the sun makes with the building and the area  $A$  of the shadow cast is recorded in a table at various times during the day.

$\theta$	$3^\circ$	$22^\circ$	$37^\circ$	$52^\circ$	$74^\circ$
$A$ (m <sup>2</sup> )	10.48	80.81	150.71	255.99	697.48

- Robert suggests that  $A \propto \theta$ .  
  - Assuming this relationship is true, use the first data point to find the model connecting  $A$  and  $\theta$ .
  - Hence show that Robert's suggestion is incorrect.
- David thinks that  $A \propto \tan \theta$ .



- Copy and complete the table.

$\theta$	$3^\circ$	$22^\circ$	$37^\circ$	$52^\circ$	$74^\circ$
$\tan \theta$					
$A$ (m <sup>2</sup> )	10.48	80.81	150.71	255.99	697.48

- Hence use technology to obtain a power model connecting  $A$  and  $\tan \theta$ .
  - Does the power model support David's claim?
  - Use the power model to estimate  $A$  when  $\theta = 43^\circ$ .
- 6** Timothy wants to estimate the number of people caught speeding in his suburb. He asks 100 people the question "Have you ever driven over the speed limit?"
- Explain why this question is likely to produce a measurement error.
  - How could the question be worded to accurately measure the number of people who have been caught speeding?



- 7** For the first 6 seconds of its motion, a particle moving in a straight line has velocity given by  $v = t^3 - 9t^2 + 24t$  m s<sup>-1</sup>, where  $t$  is the time in seconds.
- Find the acceleration function for the particle.
  - Find the greatest velocity of the particle in the first 6 seconds.
  - At what times in the first 6 seconds is the speed of the particle decreasing?

- 8** The matrix  $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$  has eigenvalues 1, 2 with corresponding eigenvectors  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

- Write down matrices  $\mathbf{P}$  and  $\mathbf{D}$  such that  $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ .
- Show that  $\mathbf{A}^n = \begin{pmatrix} 1 & 1 - 2^n \\ 0 & 2^n \end{pmatrix}$ .
- Given that  $\det(\mathbf{A}^k) = 512$  for some  $k \in \mathbb{Z}^+$ , find  $\mathbf{A}^k$ .

- 9** A mountain railway runs straight up a mountainside with the aid of a cable. The train begins at point A with position vector  $\mathbf{a}$ , and ends at point B with position vector  $\mathbf{b}$ .

After  $t$  minutes, the train is at point P with position vector  $\mathbf{p} = \left(1 - \frac{t}{12}\right)\mathbf{a} + \frac{t}{12}\mathbf{b}$ .

- Locate the train at time  $t = 0$ .
- How long does it take for the train to reach B?
- Suppose  $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$  where the units are kilometres.
  - Find the distance between A and B.
  - Find the average speed of the train, giving your answer exactly.

- 10** The population  $P$  of a colony of frogs grows according to the differential equation

$$\frac{d^2P}{dt^2} = \frac{dP}{dt} - \frac{1}{40}P \frac{dP}{dt}$$

where  $t$  is the time in months.

- Write the equation as a coupled system of first order differential equations.
- Initially,  $P = 20$  and  $\frac{dP}{dt} = 15$ .
  - Use Euler's method, with step size 0.5, to predict the population of frogs after 1 month.
  - By repeated application of Euler's method, approximate the limiting population of frogs.
  - The analytic solution to the differential equation is  $P = \frac{80}{1 + 3e^{-t}}$ . Use this model to find the correct limiting population of frogs, and comment on the accuracy of the Euler approximation.

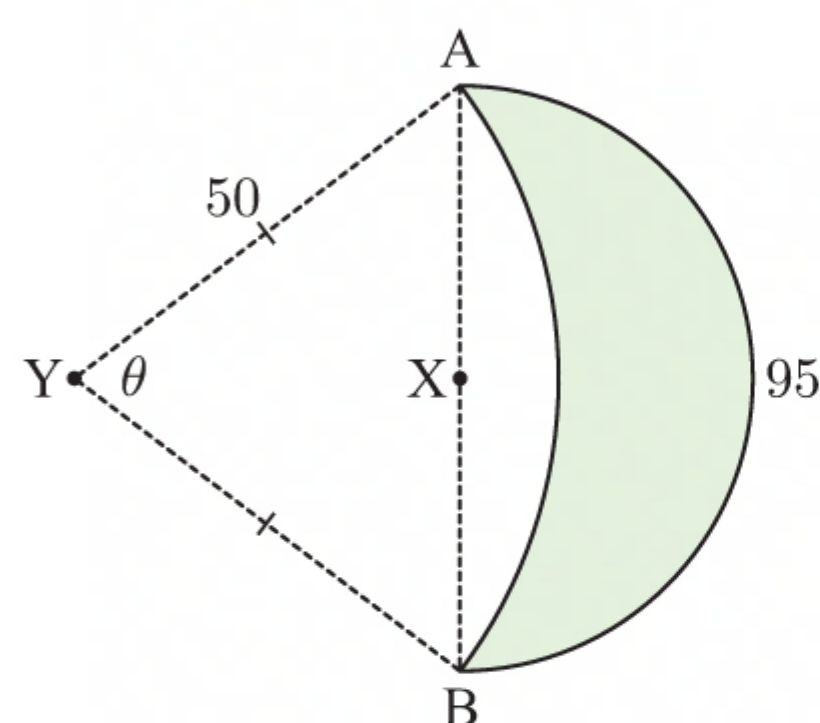
## MIXED QUESTIONS SET 17

- 1** Hayley and Patrick were training for a road cycling race. During the first week they both cycled 60 km. Hayley cycled an additional 20 km each subsequent week, whereas Patrick increased his distance by 20% each subsequent week.
- How far did each of them cycle in the 5th week of training?
  - Who was the first to cycle 210 km in one week?
  - Who cycled a greater total distance in the first 12 weeks? Explain your answer.

- 2** X and Y are the centres of the two arcs AB shown.

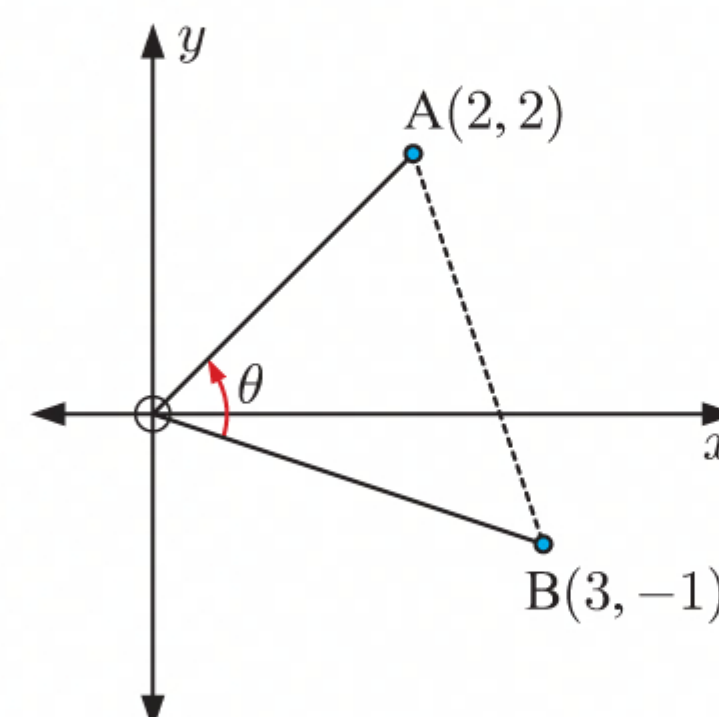
Find:

- the length AX
- the angle  $\theta$
- the shaded area.





- 3 a** Use the trapezoidal rule to estimate the area between  $f(x) = xe^x$  and the  $x$ -axis from  $x = 0$  to  $x = 1$  using:
- 2 subintervals
  - 6 subintervals.
- b** Given that the exact area is  $1 \text{ unit}^2$ , discuss the accuracy of your estimates in **a**.
- 4** The masses of sea lions on a particular island are normally distributed with mean 700 kg and standard deviation 80 kg.
- Given that 65% of the sea lions weigh less than  $a$  kg, find  $a$ .
  - Find the probability that a randomly selected sea lion weighs more than 600 kg.
  - Let  $Y$  be the number of sea lions in a group of 20 who weigh less than 600 kg.
    - Find the mean and standard deviation of  $Y$ .
    - Find  $P(Y > 3)$ .
- 5** The following data shows Craig's weekly grocery bills, in dollars, for the last 5 months.
- 181, 155, 163, 200, 149, 185, 160, 159, 164, 171,  
173, 212, 303, 191, 169, 161, 207, 140, 132, 165
- Find the median, lower quartile, and upper quartile of the data set.
  - Find the interquartile range of the data set.
  - The bill of \$303 occurred when Craig bought groceries for a large Christmas lunch. Show that this value is an outlier.
  - Draw a box plot of the data set.
- 6** Consider the curve  $y = xe^{2x}$ .
- Find the exact value of  $k \in \mathbb{R}$  such that  $y = k$  is a horizontal tangent to the curve.
  - For which values of  $k \in \mathbb{R}$  does the line  $y = k$  meet the curve at:
    - exactly one point
    - two distinct points
    - no points?
  - Now consider the family of curves  $y = xe^{ax}$ ,  $a \in \mathbb{R}$ ,  $a > 0$ .
    - Show that  $y = x$  is a tangent to all such curves and find the point of contact.
    - Find the equation of the normal to  $y = xe^{ax}$ ,  $a \in \mathbb{R}$ ,  $a > 0$ , when  $x = 0$ , and find the acute angle this normal makes with the  $x$ -axis.
- 7** Let  $p$  be the probability of rolling a six with a six-sided die.
- Consider the hypotheses  $H_0: p = \frac{1}{6}$  and  $H_1: p > \frac{1}{6}$ .
- The hypotheses will be tested by rolling the die 10 times, with significance level 5%.
- Find:
    - the critical region  $\mathcal{C}$
    - the acceptance region  $\mathcal{A}$
    - the critical value  $c$ .
  - The die was rolled 10 times, and 3 sixes appeared. Is this sufficient evidence to reject  $H_0$ ?
- 8** Consider the diagram alongside.
- Show that  $\cos \theta = \frac{1}{\sqrt{5}}$ .
  - Hence find:
    - $\sin \theta$
    - the area of triangle OAB.
  - The transformation  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  where  $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$  is applied to triangle OAB.
    - Find the eigenvalues and corresponding eigenvectors of  $\mathbf{A}$ .
    - Explain why the points on [OB] do not move under this transformation.
    - Copy and complete:  
 "Under this transformation, the points on [OA] are enlarged with scale factor ....."
    - Hence find the area of the image of triangle OAB under this transformation.
    - Verify your answer to **iv** using  $|\det \mathbf{A}|$ .



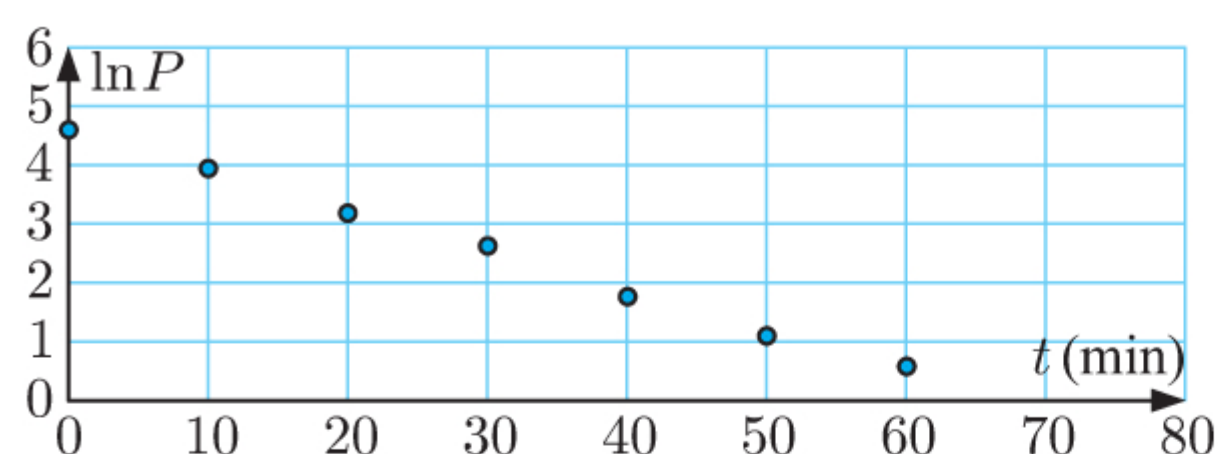


- 9 During scuba diving, nitrogen inhaled by the diver dissolves into the bloodstream and is then transported into body tissue. When the diver returns to the surface of the water, the nitrogen within the tissue is gradually expelled.

A scientist studied the variation in the percentage  $P$  of nitrogen that remains within a particular type of body tissue (type A)  $t$  minutes after a diver returns to the surface.

$t$ (minutes)	0	10	20	30	40	50	60
$P$ (%)	100	52.3	24.1	14	5.9	3.0	1.8
$\ln P$							

- a Copy and complete the table. Give your values correct to three decimal places.
- b A graph of  $\ln P$  versus  $t$  is given below.



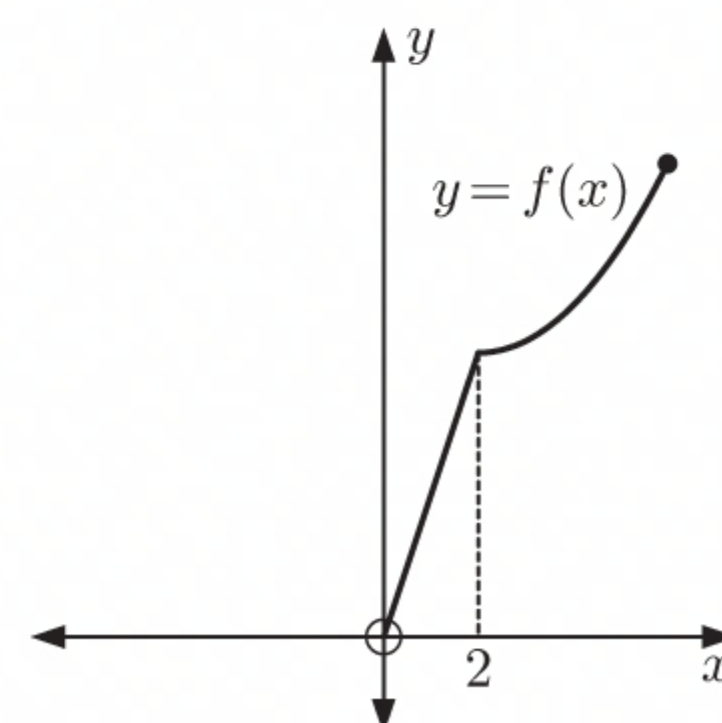
The relationship between  $\ln P$  and  $t$  is approximately linear. What does this suggest about the relationship between  $P$  and  $t$ ?

- c The method of least squares is used to fit a line of best fit to the graph in b. The following values for gradient and vertical intercept were obtained:

$$\begin{aligned}\text{gradient} &= -0.0685 \\ \text{vertical intercept} &= 4.604\end{aligned}$$

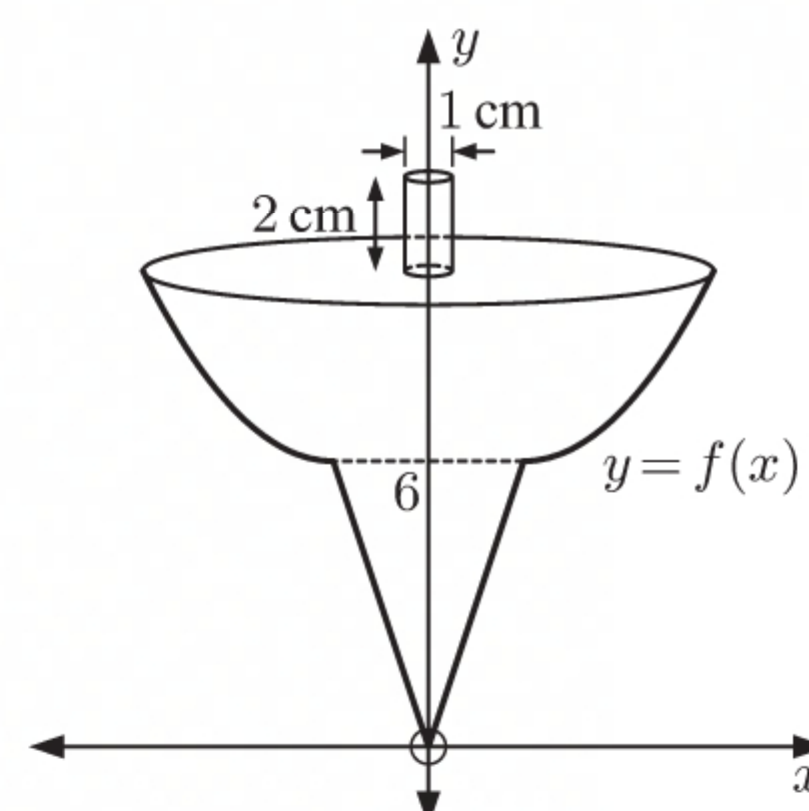
Use these values to show that  $P \approx 99.88e^{-0.0685t}$ .

- d i Predict the level of nitrogen that will remain in type A tissue 8 hours after a diver reaches the surface of the water.
- ii Comment on the accuracy of the prediction in i.
- 10 Consider the piecewise function  $f(x) = \begin{cases} kx, & 0 \leq x \leq 2 \\ \frac{1}{4}(x-2)^2 + 6, & 2 < x \leq 6 \end{cases}$
- a Find the value of  $k$ .
- b Find the range of the function.



- c A spinning top is to be made by rotating  $y = f(x)$   $360^\circ$  about the  $y$ -axis, and adding a cylindrical handle with the dimensions shown.

Find the total volume of the spinning top.



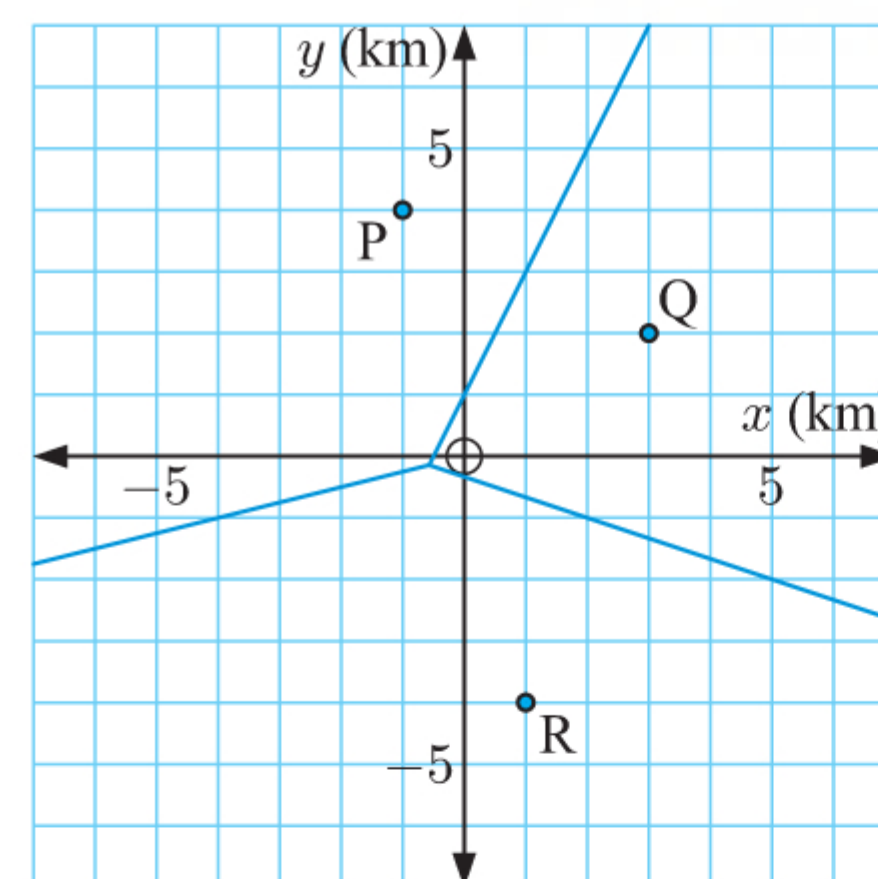
## MIXED QUESTIONS SET 18

- 1 The weight of a radioactive substance after  $t$  years is given by  $W(t) = 5 \times (0.965)^t$  grams,  $t \geq 0$ .
- a Find the percentage decrease in weight of the substance each year.
- b Find the weight of the substance after 300 years. Write your answer in the form  $a \times 10^k$  where  $1 \leq a < 10$ ,  $k \in \mathbb{Z}$ .
- c How long will it take for the weight to fall below 1 g?



- 2 The lead concentration of soil was measured at sites P, Q, and R on a farm. The results are shown below, in parts per million.

Site	Lead concentration (ppm)
P	28
Q	47
R	62



- a Use nearest neighbour interpolation to estimate the lead concentration at:
- (2, 4)
  - (-2, -3)
  - (-4, -1)
- b An additional measurement revealed a lead concentration of 55 ppm at  $S(-1, -6)$ .
- Redraw the Voronoi diagrams with site S added.
  - Where necessary, update your estimates in a.
- 3 Raman transfers €500 000 of savings into an annuity fund which earns 4.6% p.a. interest compounded monthly. She wants to withdraw €3000 per month.
- How long will her money last?
  - Find the balance of the fund after 4 years.
  - After 4 years, Raman decides that she only needs her money to last a further 15 years. How much can she withdraw each month for the remaining 15 years?
- 4 ABC is an equilateral triangle with sides 10 cm long. P is a point within the triangle. P is 7 cm from A and 4 cm from B.
- Draw a diagram clearly showing all of the information provided.
  - Calculate:
    - $\widehat{BAP}$
    - $\widehat{CAP}$
  - Hence find the length of [CP].

- 5 Students in an art course were asked to select one specialisation: painting, sketching, or sculpting. The number of students who chose each activity is shown alongside.

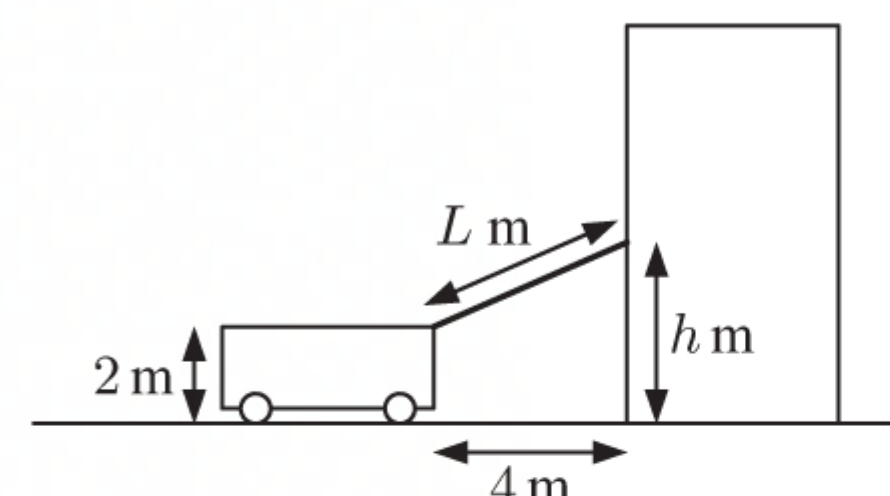
	Painting	Sketching	Sculpting
Male	30	35	15
Female	20	15	25

A  $\chi^2$  test for independence is to be used to determine whether gender is independent of the specialisation chosen. The test is performed at the 1% significance level, with corresponding critical  $\chi^2$  value 9.210.

- Assuming gender is independent of the specialisation chosen, find the expected number of male sculptors.
  - Calculate the value of the  $\chi^2$  test statistic.
  - Hence determine whether gender is independent of the specialisation chosen. Explain your reasoning.
- 6 One end of a fire truck's ladder is 4 m from the edge of a building, and 2 m above ground level.
- As the ladder is extended at  $0.1 \text{ m s}^{-1}$ , the other end of the ladder moves up the edge of the building.
- Show that  $L = \sqrt{h^2 - 4h + 20}$ .
  - Find the rate at which  $h$  is changing at the instant when:
    - $h = 5$
    - $L = 6$
- 7 This table shows the number of games played and runs scored for some players in a cricket team.

Number of games ( $x$ )	10	4	7	15	3	11	9	15	12	10
Number of runs ( $y$ )	107	82	150	212	40	165	185	241	101	183

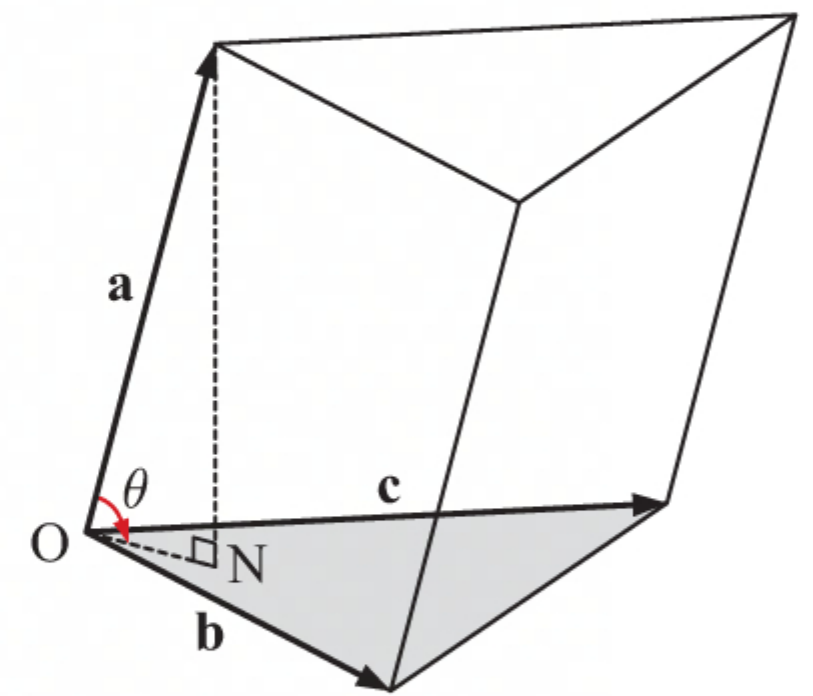
- Find the correlation coefficient  $r$ , and interpret your answer.
- Find the equation of the least squares regression line.
- State and interpret the gradient of the regression line.
- Estimate the number of runs scored by a player who played 6 games.





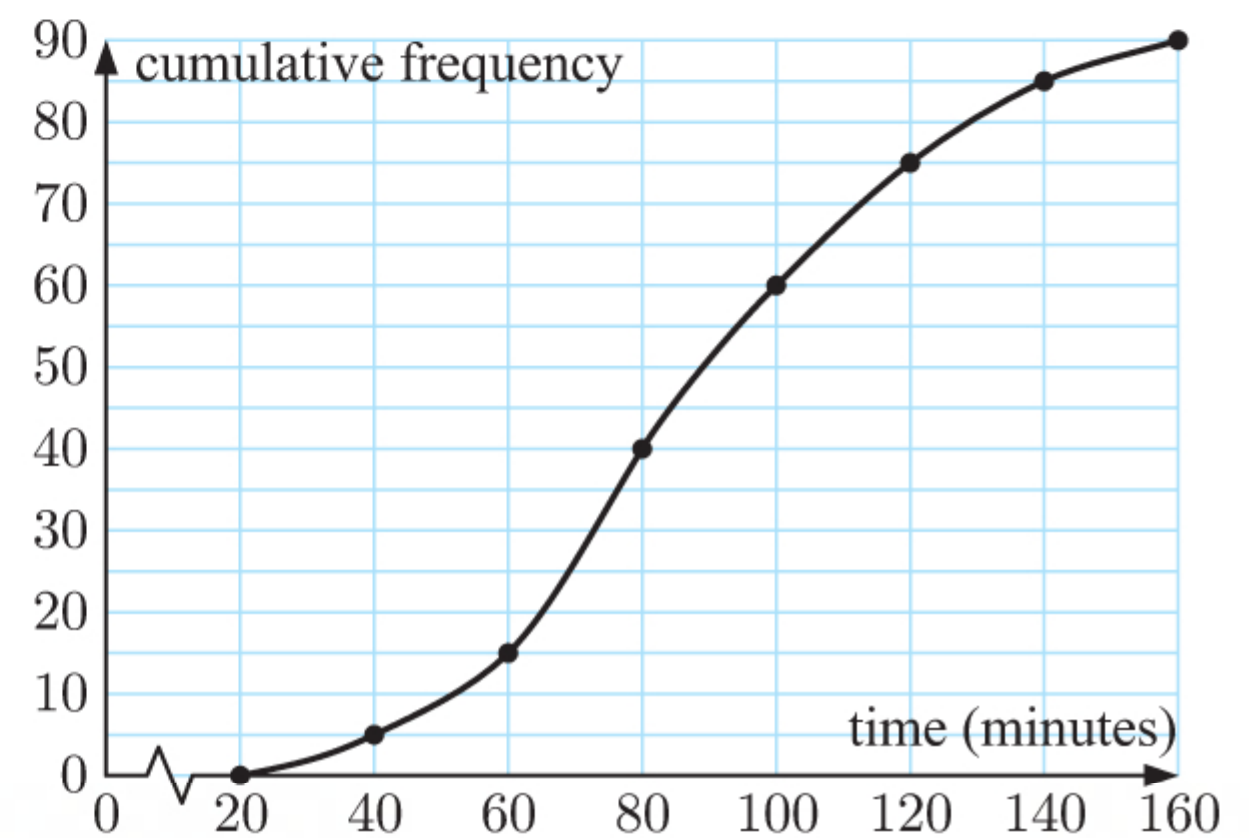
- 8 Consider the 3-dimensional shape alongside with defining vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ .

Let  $\theta$  be the angle between  $\mathbf{a}$  and the base plane.



- a Find:
- the area of the base plane
  - the perpendicular height of the shape.
- b Hence show that the volume of the shape is  $\frac{1}{2} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \text{ units}^3$ .
- c  $A(2, 0, 1)$ ,  $B(0, 2, 0)$ , and  $C(3, 1, 2)$  are vertices of this shape, adjacent to another vertex  $O(0, 0, 0)$ . Find the volume of this shape.

- 9 The lengths, in minutes, of games in a chess tournament are displayed in this cumulative frequency graph.



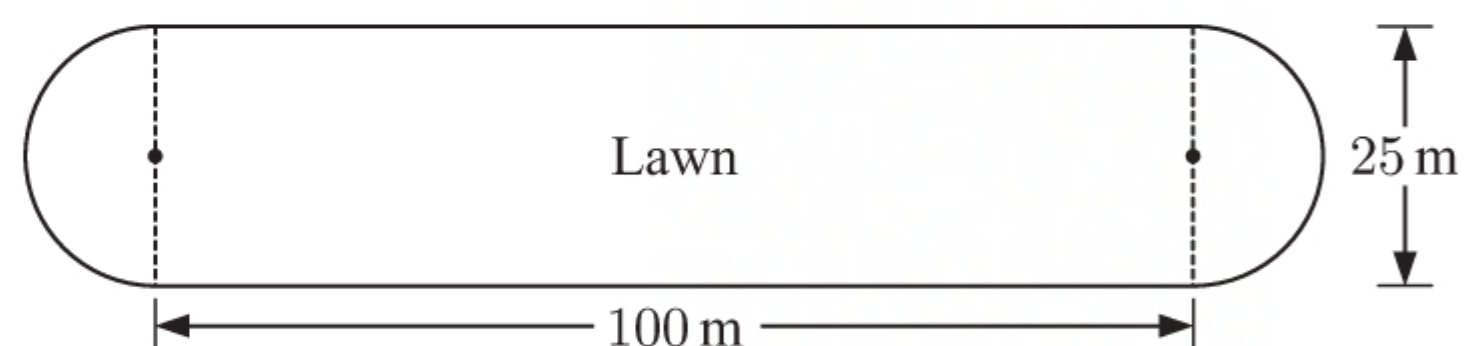
- a How many games were played during the tournament?
- b Find the median game length.
- c Estimate the interquartile range for the data.
- d 10% of the games took less than  $k$  minutes. Estimate the value of  $k$ .
- e Draw a frequency histogram to represent the data.
- 10 Suppose the rate of change of population of a bacterial culture is proportional to the population.
- Show that the population  $P$  at time  $t$  is given by an equation of the form  $P = Ae^{kt}$ .
  - Given that the initial population is 1 million and the population doubles every 3 hours, find  $A$  and  $k$ .
  - How long does it take for the population to reach 10 million bacteria?
  - Suppose the bacteria consume a nutrient at a rate proportional to the population, and that the initial population consumes nutrient at 0.05 grams per hour. Let  $N$  be the mass of the nutrient consumed after  $t$  hours.
    - At what rate will the nutrient be consumed after  $t$  hours?
    - How many grams of nutrient will be consumed in the time that it takes the population to grow to 10 million bacteria?

## MIXED QUESTIONS SET 19

- 1 Gordon needs to fertilise the large lawn shown in the diagram.

The ends of this lawn are semi-circular, and the middle is rectangular.

Gordon calculates the area of the lawn using correct formulae, but approximates the value of  $\pi$  to one significant figure, so  $\pi \approx 3$ .

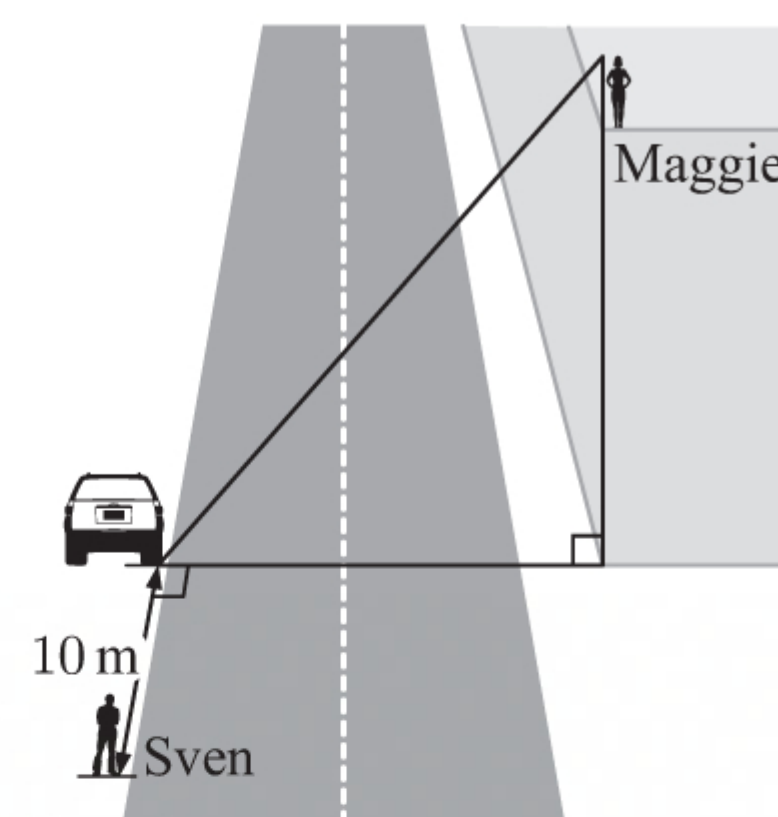


- Show that the area of the lawn is  $(156.25\pi + 2500) \text{ m}^2$ .
  - Write down Gordon's approximation for the area of the lawn. Give your answer to the nearest whole number.
  - Calculate the percentage error in Gordon's approximation. Round your answer to 2 significant figures.
- 2 The volume of water in a tank is given by  $V(t) = 10t^2 - \frac{1}{3}t^3$  litres, where  $t$  is the time in minutes and  $0 \leq t \leq 30$ .
- Find  $V(5)$  and explain what this means.
  - Find  $V'(t)$ . Do not forget to include units.
  - Find  $t$  when  $V'(t) = 0$ .
  - Find  $V'(5)$  and  $V'(25)$ .
  - Determine the time(s) at which the volume is increasing by 75 litres per minute.



- 3** Maggie is 155 cm tall and is standing on top of a building 50 m tall. A car is parked on the far kerb of the road, directly opposite Maggie. To see the car, Maggie looks down at an angle  $67^\circ$  below horizontal.

- a** How far is the car from the base of the building?
- b** Maggie's friend Sven is walking on the same side of the road that the car is parked. He is currently 10 m from the car.
  - i** Find the distance between Maggie and Sven.
  - ii** At what angle must Sven look up to see Maggie?

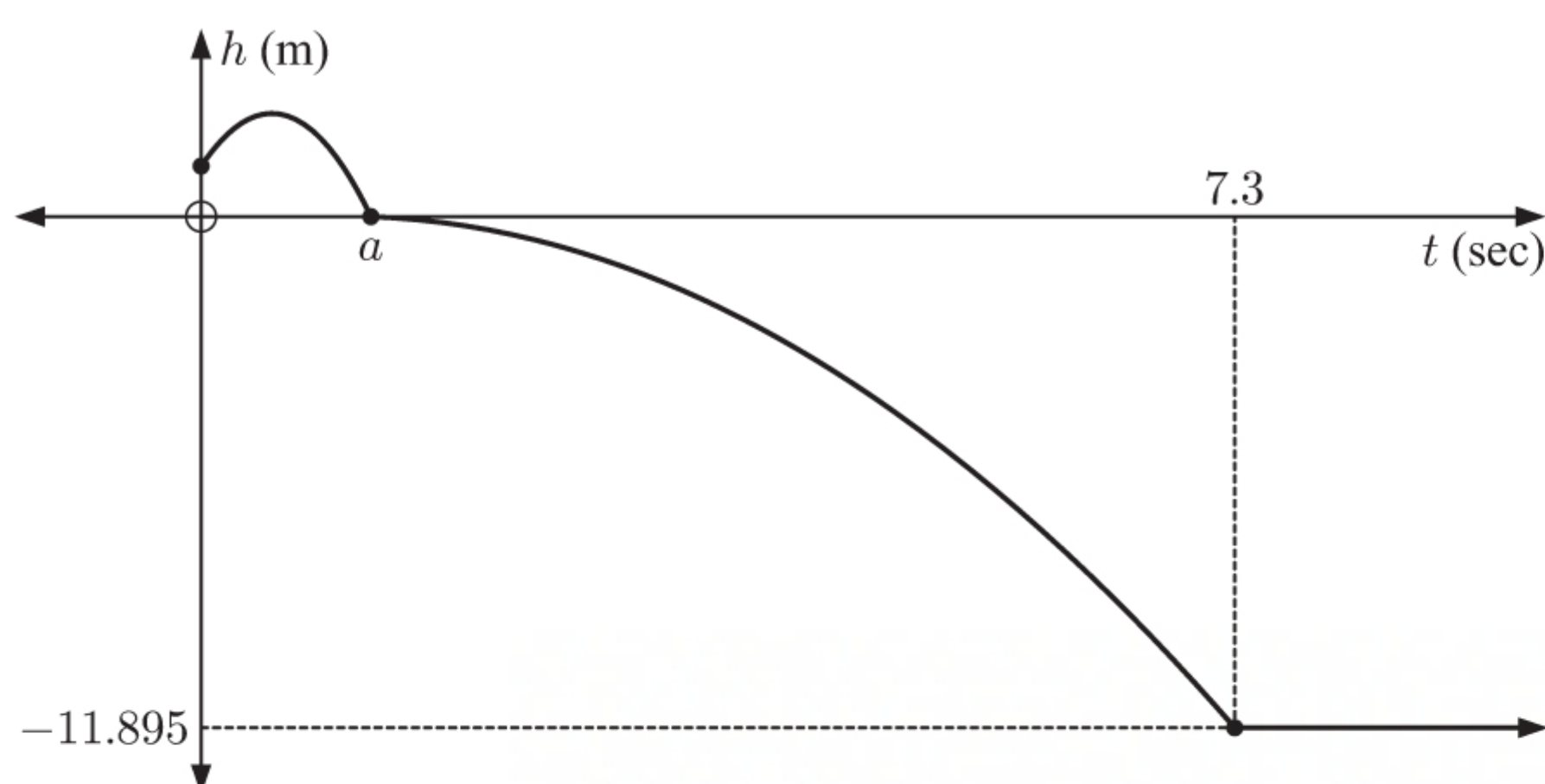


- 4** A scientist measures the pressure of a fixed mass of gas, as the volume it occupies is increased.

Volume ( $V \text{ m}^3$ )	0.5	0.8	1.2	1.5	2.4
Pressure ( $P \text{ kPa}$ )	7.2	4.5	3	2.4	1.5

- a** Draw a scatter diagram of the data. Explain why an inverse variation model appears to be appropriate.
  - b** Given  $V$  and  $P$  are inversely proportional, determine the model connecting  $V$  and  $P$ .
  - c** Find the pressure of the gas when it occupies  $3 \text{ m}^3$ .
- 5** Kapil invested 2000 rupees in a bank account on January 1st 2012. The account pays 8.25% per annum compounded annually.
- a** Find the total value of Kapil's investment on January 1st 2019.
  - b** Would it have been a better option for Kapil to invest his money in an account paying 8% per annum interest compounded monthly? Justify your answer.
- 6** A rock is thrown into a lake. The height of the rock in metres above the surface of the lake after  $t$  seconds is given by:

$$h(t) = \begin{cases} -4.9t^2 + 4.9t + 1.176, & 0 \leq t < a \\ -0.3t^2 + 0.6t - 0.288, & a \leq t < 7.3 \\ -11.895, & t \geq 7.3 \end{cases}$$



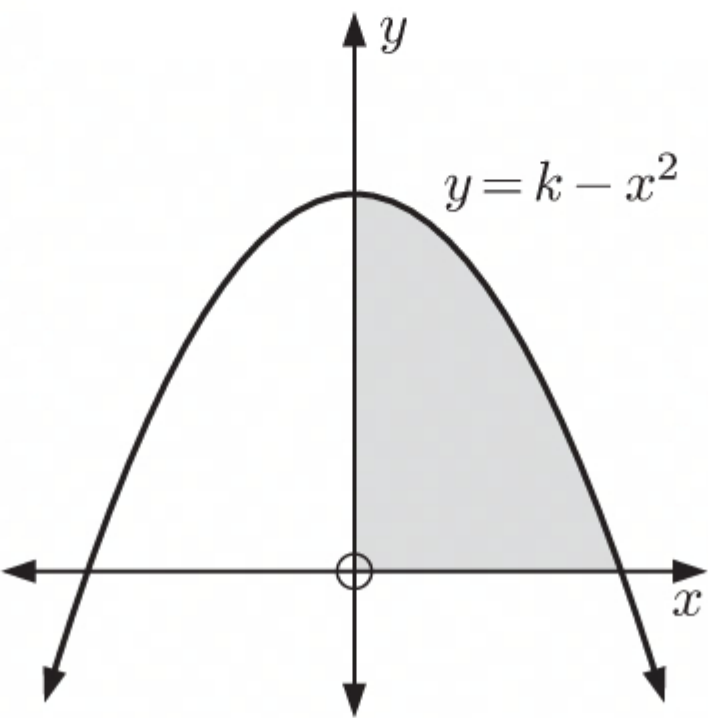
- a** State:
  - i** the time at which the rock reaches the bottom of the lake
  - ii** the depth of the lake.
- b** Find  $a$ , and interpret your answer.
- c** Find the height of the rock above or below the surface of the lake after:
  - i** 1 second
  - ii** 5 seconds
  - iii** 9 seconds.
- d** Find the time(s) at which the rock is falling at  $1 \text{ m s}^{-1}$ .



7 Consider the graph  $f(x) = k - x^2$  alongside, where  $k > 0$ .

The volume when the shaded area is rotated about the  $x$ -axis is equal to the volume when it is rotated about the  $y$ -axis.

Find the value of  $k$ .



8 A town has 5000 adults. Each year, 10% of the employed adults become unemployed, and 30% of the unemployed adults become employed.

- a Write down a transition matrix **T** representing the relationship between employed and unemployed adults.
- b Find the eigenvalues and corresponding eigenvectors of **T**.
- c Write down a matrix **P** which diagonalises **T**.
- d The town initially had 4000 employed and 1000 unemployed adults.
  - i Find an expression for the number of employed adults after  $n$  years,  $n \in \mathbb{N}$ .
  - ii How many employed adults will the town have in the long term?

9 Jessica has left crayfish nets in the ocean at locations Q, R, S, T, and U. From port P, she must travel by boat to check each net before returning to P. She would like to travel the shortest possible distance when performing this task.

The distances in metres between each pair of locations is shown below.

	P	Q	R	S	T	U
P	—	620	900	750	600	500
Q	620	—	500	800	280	730
R	900	500	—	350	300	600
S	750	800	350	—	310	300
T	600	280	300	310	—	340
U	500	730	600	300	340	—

- a Which net is closest to P?
  - b Use the nearest neighbour algorithm to find an upper bound for the distance travelled.
  - c Jessica’s friend suggests the route  $P \rightarrow Q \rightarrow T \rightarrow R \rightarrow U \rightarrow S \rightarrow P$ .  
Explain why this cannot be the shortest route.
  - d By deleting Q, use the deleted vertex algorithm to find a lower bound for the distance travelled.
- 10 A team of researchers decide to investigate whether or not some couples are more likely to have children of one gender than the other. The researchers intend to base their study on families with four children and record the number of children in each family who are girls. Let  $X$  denote the number of girls.
- a Assume that each newborn child has probability 0.5 of being a girl.
    - i On the basis of this assumption, what is the probability distribution for  $X$ ?
    - ii Find the probability that a randomly chosen family with four children will have:  
**(1)** no girls                                      **(2)** at least two girls                                      **(3)** exactly four girls.
    - iii Hence find the probability that a randomly chosen family with four children will have either all girls *or* all boys.
  - b The researchers’ null hypothesis is that each newborn child has probability 0.5 of being a girl, independent of the gender of the other children in the family.  
Let  $p$  be the proportion of families with four children that have either all girls or all boys.
    - i Explain why the researchers’ null hypothesis can be written as  $H_0: p = 0.125$ .
    - ii What would a value of  $p$  greater than 0.125 indicate in this context?
    - iii The researchers randomly select 120 families with four children, and find that 21 have either all girls or all boys. Conduct an appropriate test at the 10% significance level. What conclusion should the researchers draw?



## MIXED QUESTIONS SET 20

- 1 The population  $P$  of a species after  $n$  months follows the rule  $P = 1000 + ae^{kn}$ .

The initial population was 2000, and after 1 year the population was 4000.

- a Find:      i  $a$       ii  $k$ .
- b How long will it take for the population to reach 15 000?

- 2 Two four-sided dice are rolled simultaneously. The faces of each die are labelled 1, 2, 3, and 4.

Let  $S$  represent the sum of the results from the dice.

- a The grid alongside shows the values of  $S$  for each possible set of results. Copy the grid and fill in the missing values.

- b Find:

i  $P(S = 5)$       ii  $P(S > 5)$       iii  $P(S = 5 \mid S > 3)$

- c Suppose points are awarded according to the value of  $S$ .

Value of $S$	$S = 2$	$3 \leq S \leq 5$	$S > 5$
Number of points won or lost	32	16	-8

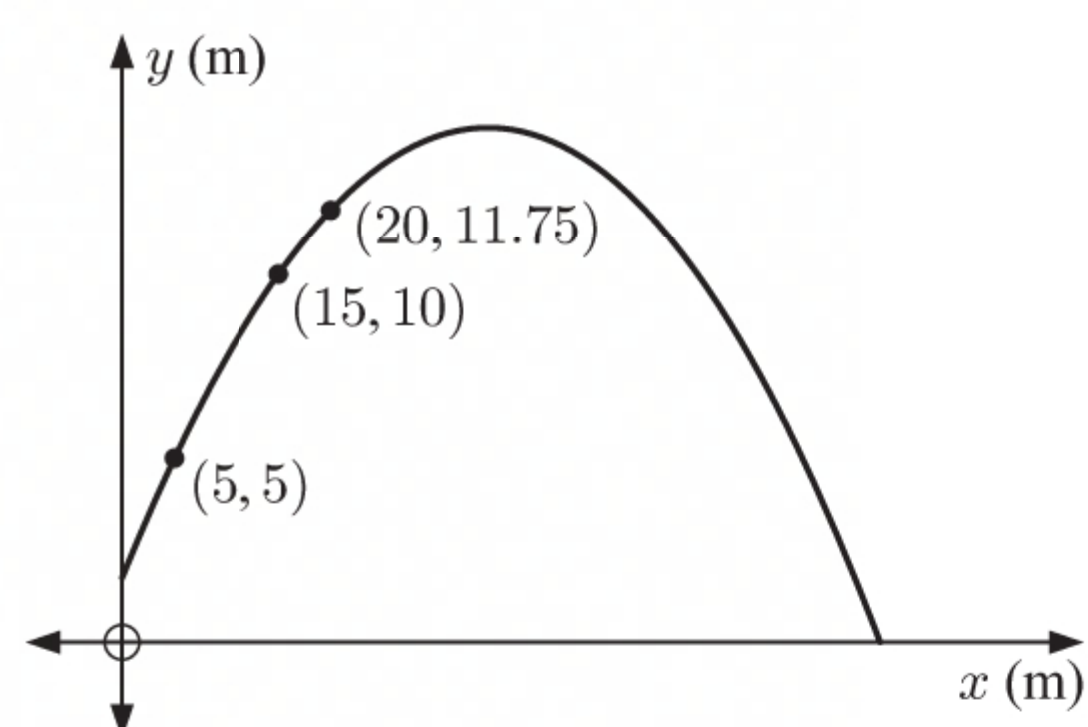
- i Determine the expected number of points for each roll of the dice.

- ii What should the number of points for  $S = 2$  be changed to so that the expectation for the game is zero?

- 3 This graph shows the path travelled by a discus thrown during an athletics event.

$x$  and  $y$  are connected by the quadratic model  $y = ax^2 + bx + c$ .

- a Write down three equations involving  $a$ ,  $b$ , and  $c$ .
- b Use technology to find  $a$ ,  $b$ , and  $c$ .
- c Find the maximum height reached by the discus.
- d How far did the discus travel horizontally before it hit the ground?



- 4 Consider this Voronoi diagram for the towns A, B, C, and D.

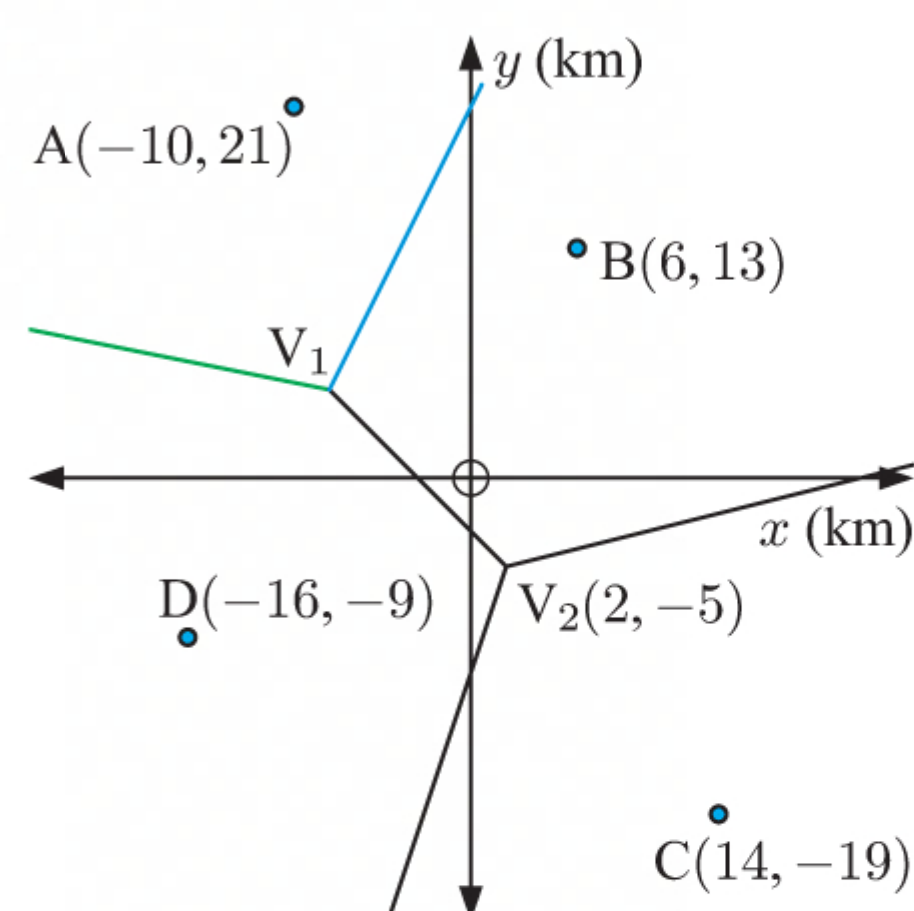
- a Find the equation of:

- i the blue edge      ii the green edge.

- b Hence find the coordinates of  $V_1$ .

- c Ivan wants to open a weekend retreat, located so that it is as far as possible from the nearest town.

- i Find the optimal position for the retreat.
- ii Find the towns which are closest to the retreat in this case.



- 5 The function  $f(x) = x + \frac{k}{x^2}$  is graphed alongside.

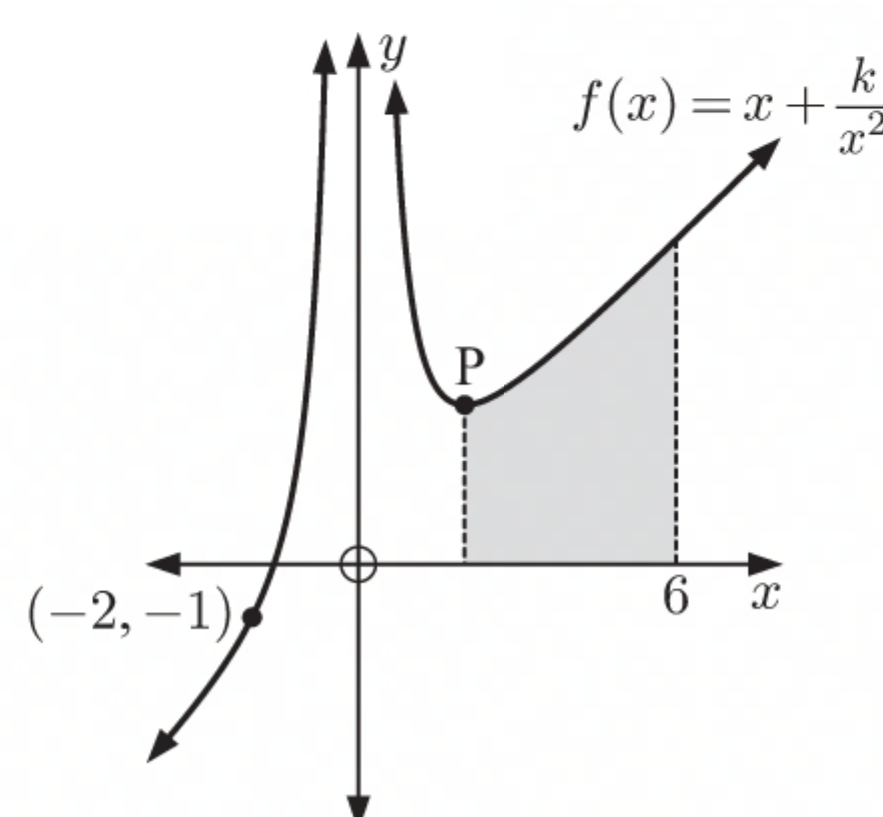
- a Find  $k$ .

- b Find  $f'(x)$ .

- c Hence find the coordinates of P.

- d Find  $\int f(x) dx$ .

- e Hence find the shaded area.

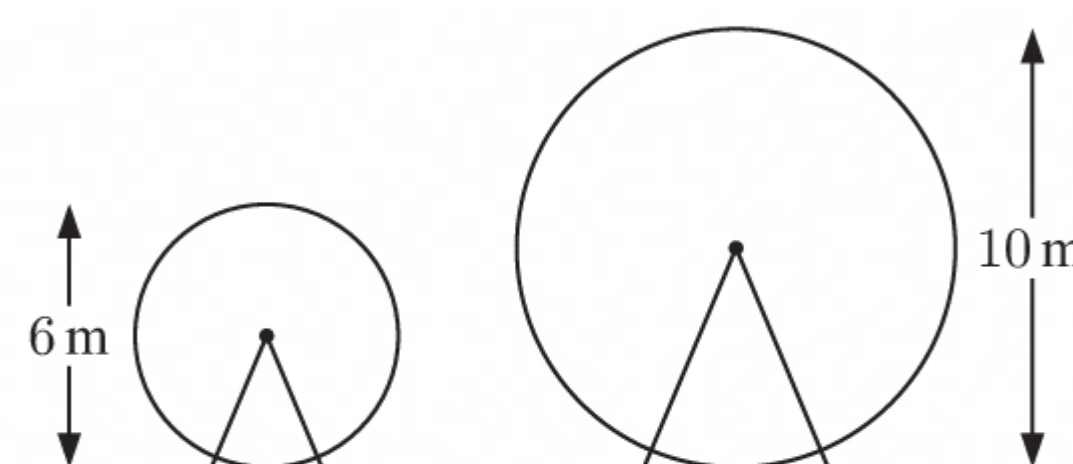




- 6** At a fairground there is a miniature Ferris wheel for children, and a standard Ferris wheel.

The miniature Ferris wheel is 6 m high, and the standard Ferris wheel is 10 m high. Each wheel takes 40 seconds to complete a revolution.

Alysa gets on the miniature Ferris wheel, and 10 seconds later her mother Martha gets on the standard Ferris wheel.



Let  $H_1(t)$  be Alysa's height above ground level, and  $H_2(t)$  be Martha's height above ground level,  $t$  seconds after Alysa gets on the Ferris wheel.

- a** Show that:

**i**  $H_1(t) = 3 - 3 \cos(\frac{\pi}{20}t), \quad t \geq 0$

**ii**  $H_2(t) = 5 - 5 \cos(\frac{\pi}{20}t - \frac{\pi}{2}), \quad t \geq 10.$

- b** How much higher is Alysa than Martha when Martha first gets on the Ferris wheel?

- c** Write  $H_1(t) - H_2(t)$  in the form  $A \cos(\frac{\pi}{20}t + B) + C$ , where  $A > 0$ ,  $-\pi < B \leq \pi$ .

- d** Find the maximum amount by which:

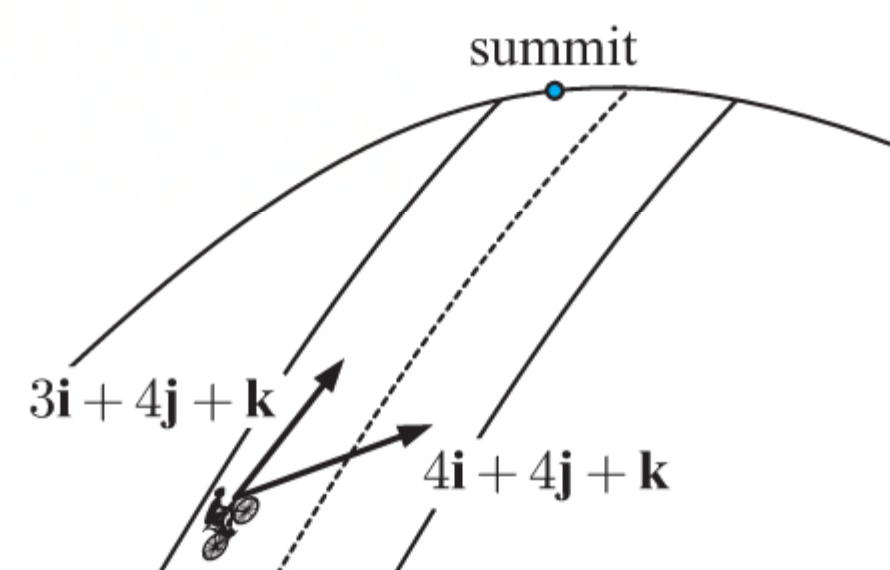
**i** Alysa is higher than Martha

**ii** Martha is higher than Alysa.

- e** Find the first time at which Alysa and Martha are the same height above ground level.

- 7** Emma is cycling up a hill. The summit is in the direction  $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$  relative to Emma.

As the hill is quite steep, Emma cycles from one side of the road to the other in the direction  $4\mathbf{i} + 4\mathbf{j} + \mathbf{k}$  at  $3 \text{ m s}^{-1}$ .



- a** Calculate Emma's velocity vector.

- b** How fast is Emma travelling in the direction of the summit?

- c** The width of the road is 4.5 m. How long will it take for Emma to cycle from one side of the road to the other?

- 8 a** Show that if  $\tan \theta = 4$  then  $\cos^2 \theta = \frac{1}{17}$ .

- b** The normal to  $f(x) = a\sqrt{\tan x} + b$  at the point where  $x = \frac{\pi}{4}$  has equation  $x + \sqrt{\pi}y = \pi$ .

- i** Find  $a$  and  $b$ .

- ii** Find the gradient of the normal to  $f(x)$  at the point where  $x = \tan^{-1}(4)$ .

- 9** This table shows the distances of drives hit by a golfer. It is known that the standard deviation  $\sigma$  of her drive lengths is 30 m.

Distance ( $d$ m)	Frequency
$180 \leq d < 200$	8
$200 \leq d < 220$	10
$220 \leq d < 240$	15
$240 \leq d < 260$	21
$260 \leq d < 280$	32
$280 \leq d < 300$	14

- a** Construct a histogram for the data.

- b** Does the data appear to be normally distributed? Explain your answer.

- c** Use the data to estimate the mean  $\mu$  of the golfer's drive lengths.

- d** Assuming the golfer's drive lengths are normally distributed, construct a table of expected frequencies. Where necessary, combine rows such that the expected value in each column is greater than 5.

- e** Conduct a  $\chi^2$  test with 5% significance level to determine whether the data is normally distributed with standard deviation 30 m.

- 10** A group of scientists would like to document the wildlife on a remote island. The proportion  $p$  of wildlife observed by the scientists after  $t$  weeks is modelled by  $2 \frac{d^2p}{dt^2} + \frac{dp}{dt} = 0$ .

- a** Use the substitution  $y = \frac{dp}{dt}$  to write this as a coupled system of first order differential equations.

- b** The equation for  $\frac{dy}{dt}$  is separable and independent of  $p$ . Solve this equation for  $y(t)$ .

- c** Hence find a general solution for  $p(t)$ .

- d** Initially  $p = 0$ , and after 2 weeks,  $p = 0.45$ .

- i** Find the particular solution for  $p(t)$ .

- ii** What proportion of the wildlife will *never* be observed by the scientists?



# Trial examination 1

## PAPER 1

## CALCULATOR, 120 MINUTES

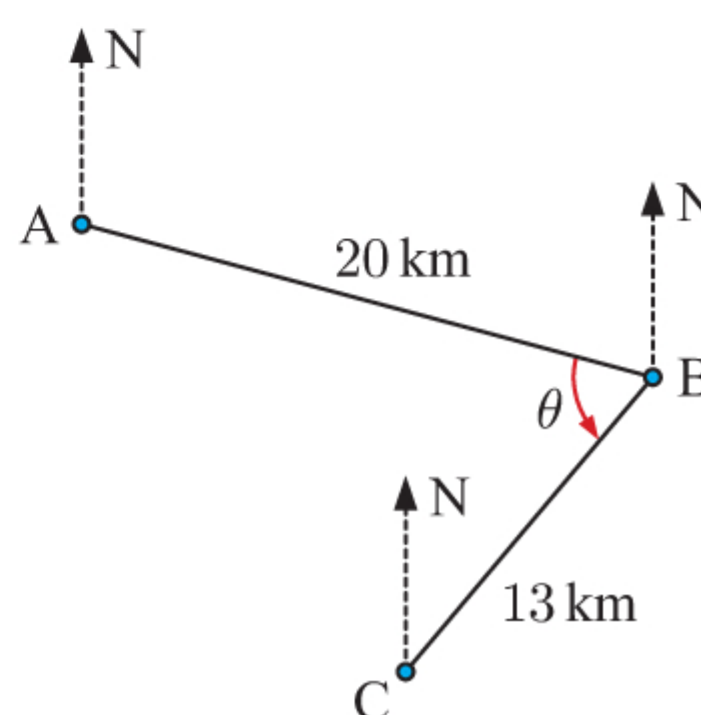
### 1 [Maximum mark: 5]

Erin surveys the 100 students in her school. She finds that 80 of them come to school by bus, while 30 of them bring a packed lunch. There are 14 students who do not come by bus and do not bring a packed lunch.

- Find the number of students who come to school by bus and bring a packed lunch. [1]
- Given that a student does not bring a packed lunch, find the probability they come to school by bus. [2]
- Determine whether coming to school by bus is independent of bringing a packed lunch for the students at this school. [2]

### 2 [Maximum mark: 7]

A ship sails from port A for 20 km on a bearing of  $105^\circ$  until it arrives at port B. It then sails on a bearing of  $220^\circ$  for 13 km to reach the port C.



- Show that the angle on the diagram,  $\theta = 65^\circ$ . [1]
- Find the distance the ship must sail from port C to return to port A. [3]
- Find the bearing the ship must sail on from port C to return to port A. [3]

### 3 [Maximum mark: 5]

A pot of soup is boiled and left to cool down to room temperature. The temperature  $T$  in  $^\circ\text{C}$  can be modelled by  $T = 22 + 78(0.85)^t$ ,  $t > 0$  at time  $t$  minutes after boiling.

- State the temperature of the room suggested by this model. [1]
- Find the temperature of the soup after 5 minutes. [2]
- Find the time taken for the temperature of the soup to drop below  $30^\circ\text{C}$ . [2]

### 4 [Maximum mark: 8]

Ann and Greg work for the same company which pays them at the end of each month. They sign the following contracts for 24 months, starting in January 2020 and finishing in December 2021.

Ann: Paid \$1000 in January 2020, increasing by \$50 in each subsequent month.

Greg: Paid \$600 in January 2020, increasing by 10% in each subsequent month.

- Find Greg's monthly salary for February 2021. [2]
- Find the total amount received by Ann in the first year of her contract. [2]
- Find the month and year in which the total amount received from Greg's contract first exceeds the total amount received from Ann's contract. [4]



**5 [Maximum mark: 6]**

Pumpkins produced by a farm have masses which can be modelled by a normal distribution with mean of 4.7 kg and a standard deviation of 0.9 kg.

The farmer does not sell any pumpkins which have a mass of less than 3 kg.

The heaviest 10% of pumpkins are sold as “Super Pumpkins” by the farmer.

- a** Find the probability that a randomly chosen pumpkin from the farm is below 3 kg. [2]
- b** Find the lowest mass for a pumpkin to be sold as a “Super Pumpkin”. [2]
- c** Two pumpkins are selected at random. Find the probability that one is too light to sell and the other is a “Super Pumpkin”. [2]

**6 [Maximum mark: 6]**

Anya wants to take out a loan of \$5000 to buy a car. A finance company offers the following repayment scheme.

The loan will be repaid in monthly payments, over 3 years with an interest rate of 4.8% p.a., compounded monthly.

- a** Find the amount Anya must pay each month. [2]
- b** Calculate the amount still remaining to be repaid after one year. [2]

After one year the finance company changes its interest rate on the loan to 5.3% p.a., compounded monthly.

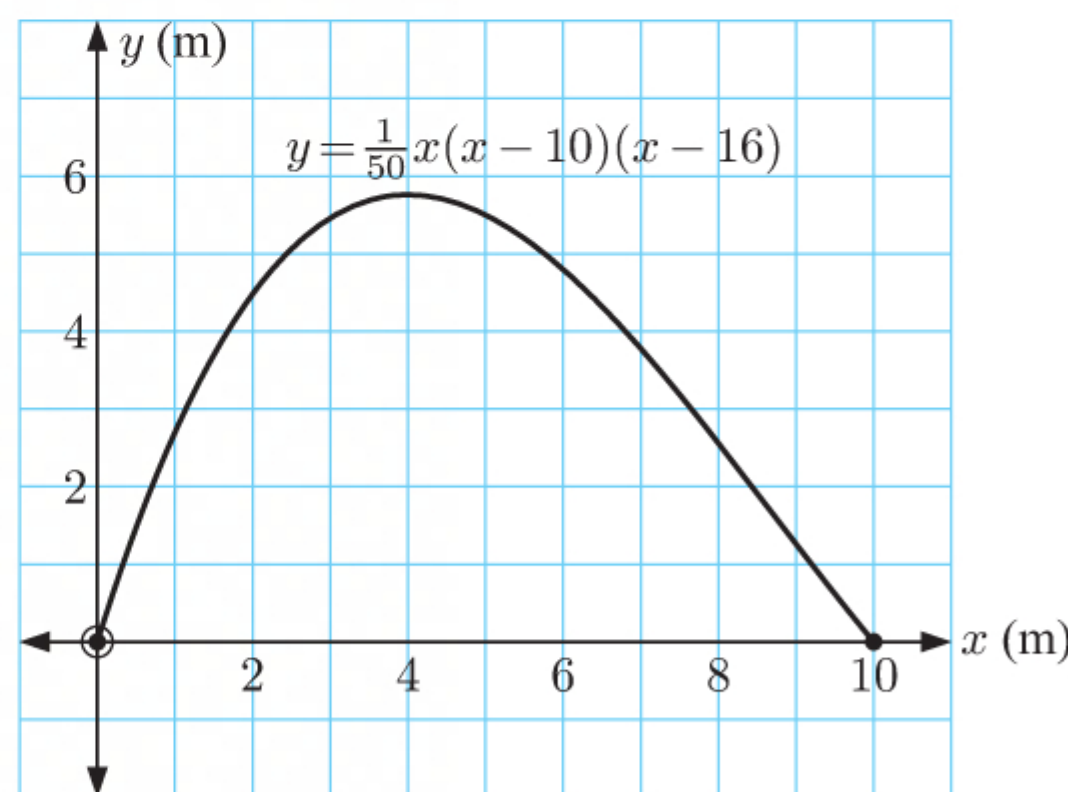
- c** Find Anya’s new monthly payment for the remaining two years of the loan. [2]

**7 [Maximum mark: 7]**

A railway tunnel is 100 m long. Its constant cross-section, shown below, can be modelled by the curve

$$y = \frac{1}{50}x(x - 10)(x - 16), \quad 0 \leq x \leq 10$$

where  $x$  and  $y$  are horizontal and vertical distances measured in metres.



Engineer A uses integration to find the exact volume of the tunnel.

- a** Find the volume obtained by Engineer A. [3]

Engineer B says the trapezoidal rule with 5 strips will produce a good enough estimate for the volume of the tunnel.

- b** Find the volume obtained by Engineer B. [3]
- c** Find the percentage error in Engineer B’s answer for the volume of the tunnel in comparison to the one obtained by Engineer A. [1]

**8 [Maximum mark: 7]**

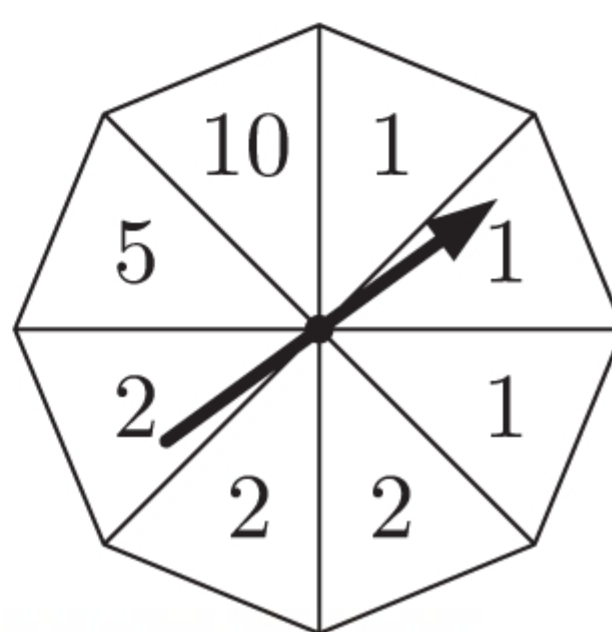
A particle moves in a straight line so that its position (in mm) relative to the origin O at time  $t$  (in seconds) is given by  $s(t) = t^2 \sin 2t$ ,  $0 \leq t \leq 4$ .

- a** Find an expression for the velocity of the particle at time  $t$ . [3]
- b** Find the times in the interval  $0 \leq t \leq 4$  when the particle is stationary. [2]
- c** Find the total distance travelled by the particle in the 4 seconds of motion. [2]



**9 [Maximum mark: 5]**

Let the discrete random variable  $X$  be the score obtained when the fair spinner shown below is spun once.



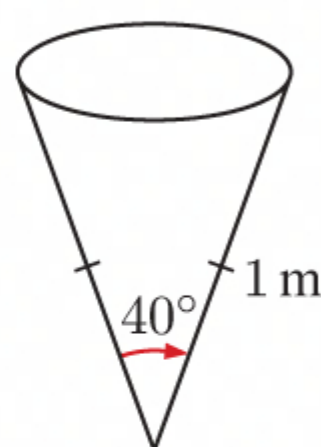
- a** Find  $E(X)$ . [2]

A maths teacher allocates her son's weekly pocket money by letting him spin the spinner once. She then calculates the amount he receives,  $P$  in dollars, using the following formula,  $P = 2X + 5$ .

- b** Find  $E(P)$ . [1]  
**c** Given that  $\text{Var}(X) = 8.5$  find  $\text{Var}(P)$ . [2]

**10 [Maximum mark: 8]**

The diagram shows a large inverted cone which collects water in a factory.



- a** Find the maximum amount of water the cone holds, to the nearest mL. [3]

Water flows into the cone at a rate of 500 mL/s.

- b** Find the rate at which the water level is rising when the cone is filled to half its maximum volume. [5]

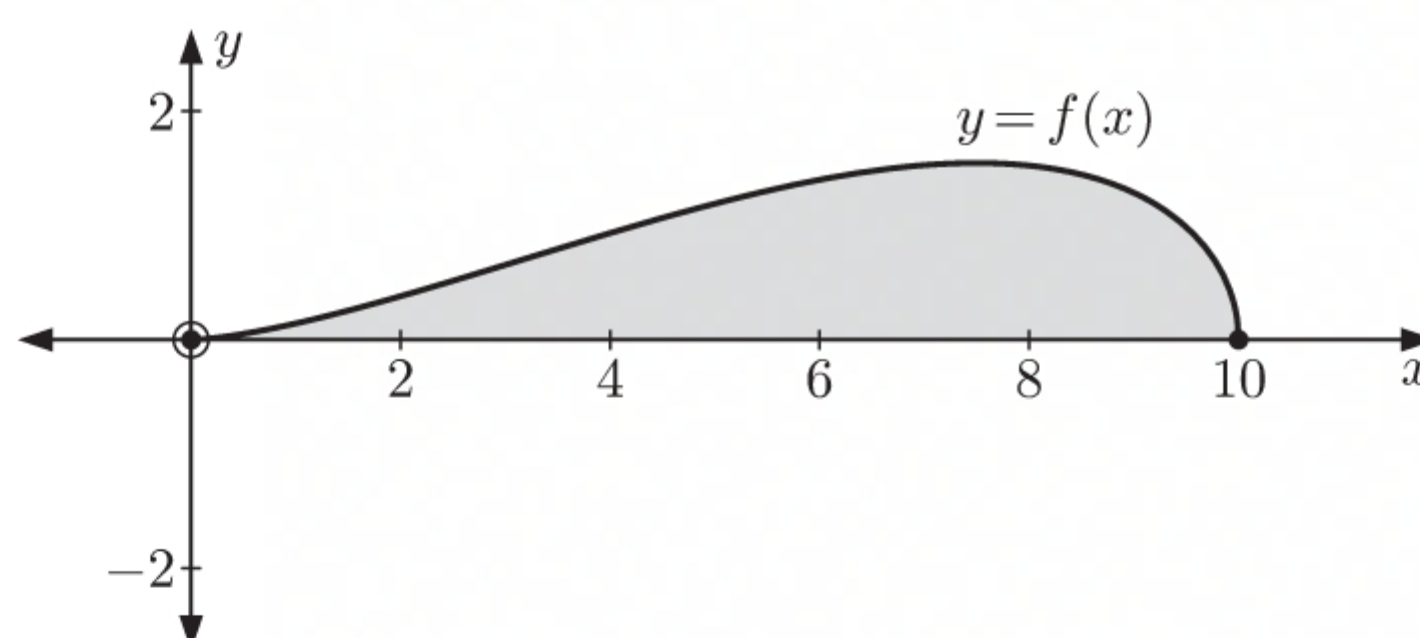
**11 [Maximum mark: 7]**

During baseball practice, a batter strikes the ball at a point 1 m above the ground. The ball is assumed to travel in a vertical plane with velocity of  $\begin{pmatrix} 14 \\ 35 - 20t \end{pmatrix}$  m/s at time  $t$  seconds after it leaves the bat.

- a** Find the initial speed of the ball as it leaves the bat in m/s. [2]  
**b** Find the maximum height obtained by the ball. [3]  
**c** The ball is caught by a fielder at a point 2.5 m above the ground. Find the horizontal distance from the batsman to the fielder. [2]

**12 [Maximum mark: 6]**

The area enclosed between the curve  $f(x) = \frac{x}{20}\sqrt{x(10-x)}$  and the  $x$ -axis is shown shaded in the diagram below. The units on the axes are mm.



A jeweller creates a metallic pendant to be hung on the end of a necklace. The pendant can be modelled by rotating the shaded area  $360^\circ$  about the  $x$ -axis.

- a** Find the volume of the metallic pendant. [3]

When hung vertically from the necklace the pendant has a length of 10 mm.

- b** Find the circumference of the pendant at its widest point. [3]



**13 [Maximum mark: 6]**

A rectangular pond has a width of 50 m and a length of 400 m. The area of the pond covered by an algae is denoted by  $A$  (in  $\text{m}^2$ ) and is measured at time  $t$  (in weeks) after a biologist begins to observe the growth.

The rate at which  $A$  is changing can be modelled as being proportional to  $\sqrt{A}$ . Initially the algae covers an area of  $900 \text{ m}^2$  and three weeks later this has increased to  $1296 \text{ m}^2$ .

How many days after the initial observation will it take for the algae to cover more than 10% of the pond's surface?

**14 [Maximum mark: 8]**

A sound engineer is testing a speaker he is installing on the outside of a building. He measures the sound intensity ( $I \text{ W/m}^2$ ) at different distances ( $x \text{ m}$ ) away from the speaker. For safety reasons he does not go within 10 m of the speaker.

Distance ( $x \text{ m}$ )	10	20	40	100
Sound intensity ( $I \text{ W/m}^2$ )	118	46.1	25.9	21.3

The engineer suspects a function of the form  $I = \frac{10\,000}{x^2} + 20$ ,  $x \geq 10$  will best model the data.

- Calculate the sum of the square residuals for this model and comment on the goodness of fit. [3]
- Find the limit of the sound intensity the engineer records as he moves a large distance away from the speaker. [1]
- Find an expression for  $x$  in terms of  $I$ . [2]
- Hence or otherwise, find the range of sound intensities predicted by the model. [2]

**15 [Maximum mark: 6]**

The matrix  $\mathbf{M}$  is defined as

$$\mathbf{M} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$$

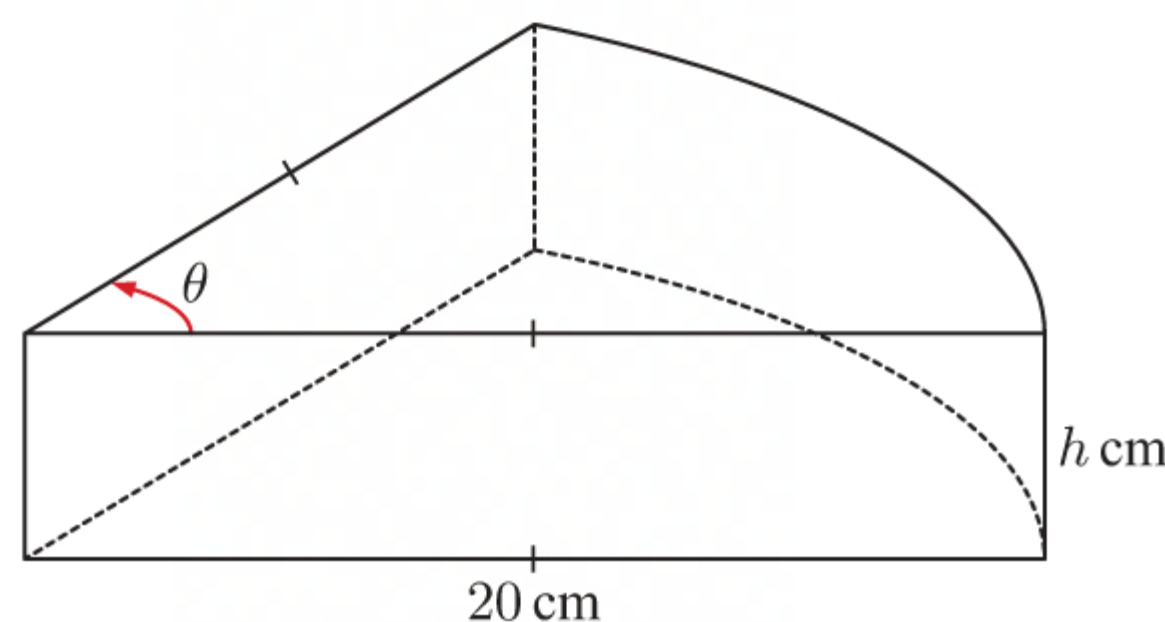
The object point  $(x, y)$  is transformed to its image point  $(x', y')$  using the following matrix equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Describe the transformation fully. [2]
- Find the exact coordinates of the image of the point  $(4, 6)$  after this transformation. [2]
- Find the least positive integer  $n$  such that  $\mathbf{M}^{-1} = \mathbf{M}^n$ . Justify your answer geometrically. [2]

**16 [Maximum mark: 7]**

A cheese company is creating a box for a special edition piece of cheese. The box is a prism with a cross-section in the shape of a sector. The radius of the sector must be 20 cm and the volume of the box must be  $1400 \text{ cm}^3$ . The angle of the sector ( $\theta$  in **radians**) and the height ( $h$  in cm) of the box are to be decided.



- Show that  $h = \frac{7}{\theta}$ . [1]
- Show that the total surface area of the box  $A = 400\theta + \frac{280}{\theta} + 140$ . [2]
- Find an expression for  $\frac{dA}{d\theta}$ . [2]
- Hence, or otherwise, find the minimum surface area possible for the box and the value of  $\theta$  that will be required to produce it. [2]



17 [Maximum mark: 6]

The amount of sugar in each box of a company’s healthy breakfast cereal is normally distributed. The company claims there is “less than 10 g of sugar on average” in each 300 g box. To test this claim an inspector randomly samples twelve boxes of the cereal and records the sugar content in each box. The values she obtains are shown below.

10.3 g	11.1 g	9.8 g	11.3 g	9.2 g	10.1 g	10.8 g	10.6 g	10.4 g	9.9 g	10.1 g	10.5 g
--------	--------	-------	--------	-------	--------	--------	--------	--------	-------	--------	--------

Let  $\mu$  and  $\sigma^2$  be the mean and variance of the amount of sugar in the population of 300 g boxes.

- a Find unbiased estimators for  $\mu$  and  $\sigma^2$ . [2]
- b Carry out an appropriate test, at the 5% significance level, to determine the validity of the company’s claim. State clearly the  $p$ -value and justify the conclusion you make. [4]

PAPER 2

CALCULATOR, 120 MINUTES

1 [Maximum mark: 14]

Reuben owns a small boat and so wishes to investigate the depth of water in his local harbour. On June 1st, during low tide at 07:00 the depth was 1.73 m and during the subsequent high tide at 13:15 the depth was 4.18 m. Letting  $w(t)$  be the depth of water (metres) at time  $t$  (hours after midnight), Reuben finds a model for the depth of water on June 1st in the form

$$w(t) = a \sin(b(t - c)) + d, \quad 0 \leq t \leq 24.$$

- a Show that  $d = 2.955$ . [1]
- b Given that  $a > 0$  find the value of  $a$ . [1]
- c Find the value of  $b$  in the form  $\frac{p\pi}{q}$  where  $p$  and  $q$  are integers. [2]
- d Given that  $0 < c < 20$  find the exact value of  $c$ . [2]
- e Describe, in the correct order, the transformations which would transform the curve  $y = \sin t$  to the curve  $y = w(t)$ . [5]
- f Reuben plans to take his boat out at high tide, in the afternoon of June 5th. Find the time this will occur, rounded to the nearest 5 minutes. [3]

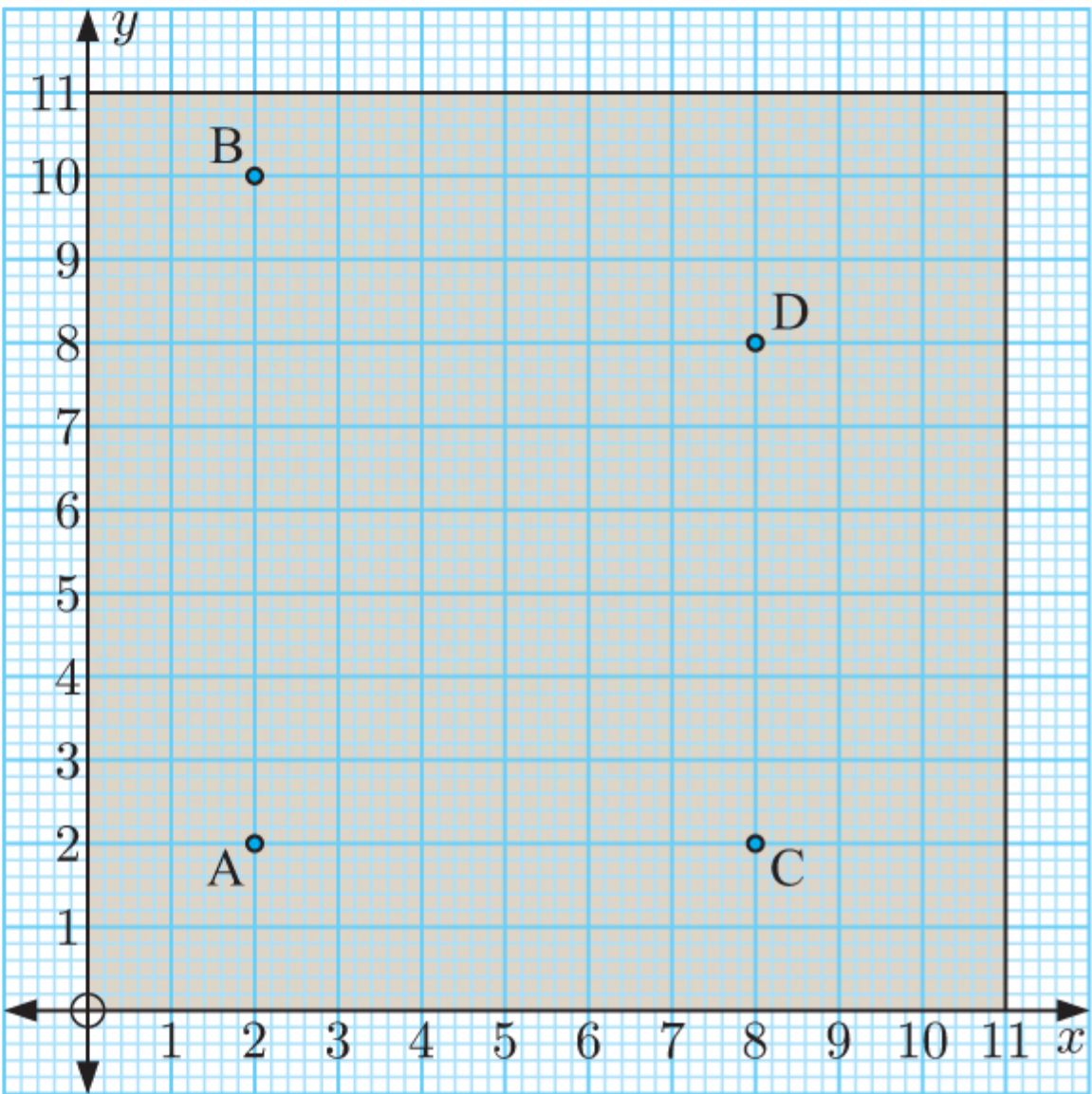
2 [Maximum mark: 13]

A farmer’s field is modelled by the square shown below with each unit representing 10 metres. Feeding stations for the cattle are positioned at points A(2, 2), B(2, 10), C(8, 2), and D(8, 8) within the field.

- a Find the midpoint of line segment BD. [1]
- b Calculate the gradient of the line segment BD. [1]
- c Show that the equation of the perpendicular bisector of the line segment BD is  $y - 3x + 6 = 0$ . [3]

It is assumed that cattle will feed at the nearest station within the field, so the farmer wishes to construct a Voronoi diagram.

- d Given that the perpendicular bisector of the line segment AD is  $y = 10 - x$ , construct the Voronoi diagram for the sites A, B, C, and D, showing all edges clearly on the diagram below. [5]





The farmer now wishes to place a fifth feeding station in the field. The new feeding station should be at least 10 m away from the boundary wall, and as far away as possible from any of the present feeding stations A, B, C, or D.

- e** Determine the optimum location of the new feeding station. [3]

**3 [Maximum mark: 17]**

A hot air balloon's position is given by the coordinates  $(x, y, z)$ , where  $x$  and  $y$  are the balloon's position east and north of a viewing platform, and  $z$  is the height of the balloon above sea level. All displacements are given in units of 100 metres. The time  $t$  is measured as minutes after 08:00.

At 08:00 the hot air balloon takes off from the point  $(-4, 0, 0)$  and moves with constant velocity until at 8:03 it is at position  $(2, 3, 3)$ .

- a** Show that the vector equation for the position  $\mathbf{r}_b$  of the balloon at time  $t$  is given by [2]

$$\mathbf{r}_b = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

- b** Find the speed of the balloon in metres per minute. [1]

At 08:05 a helicopter sets off from a nearby hill to film the hot air balloon. Using the same coordinate system, its take-off point is given by  $(0, 20, 1)$ . It moves with a constant velocity of  $\mathbf{v}_h = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$ .

- c** Explain why the position of the helicopter at time  $t$  is given by [1]

$$\mathbf{r}_h = \begin{pmatrix} 0 \\ 20 \\ 1 \end{pmatrix} + (t - 5) \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

- d** Find the height of the helicopter when it is due east of the hot air balloon. [3]

- e** Find the distance between the helicopter and the hot air balloon at 08:10. [3]

- f** If the distance between the hot air balloon and the helicopter at time  $t$  is denoted by  $d$  then show that [4]

$$d^2 = 21t^2 - 362t + 1562, \quad t \geq 5$$

Regulations state that a helicopter is not allowed to come within 200 m of a hot air balloon.

- g** For how many seconds are the helicopter and the hot air balloon contravening this regulation? [3]

**4 [Maximum mark: 14]**

Rose's height as she grows from a child into an adult is to be modelled by a function  $h(t)$ , where  $h$  is her height (cm) at age  $t$  (years).

From the age of 2 years to 13 years her height can be modelled by a constant rate of increase of 6.9 cm/year. Rose was measured to be 84.5 cm tall when she was 2 years old.

- a** How tall will Rose be at age 10? [1]

- b** Find the function  $h(t)$  for  $2 \leq t \leq 13$ . [2]

From the age of 13 years to 20 years, the rate at which her height changes is modelled by the differential equation

$$h'(t) = \frac{32}{5(t-12)}, \quad 13 < t \leq 20$$

- c** Show that  $h(t) = 6.4 \ln(t - 12) + 160.4$  for  $13 < t \leq 20$ . [5]

The height of a particular chimpanzee in cm at age  $t$  years old can be modelled by the following logistic equation

$$g(t) = \frac{165}{1 + 4e^{-0.78t}}, \quad t \geq 0$$

- d** How tall was the chimpanzee when it was born? [1]

- e** Find the maximum height that the chimpanzee will grow to. [2]

- f** The chimpanzee was born on Rose's 5th birthday. Using the above models, find the ages Rose will be when she and the chimpanzee are the same height. [3]



**5 [Maximum mark: 14]**

The time taken (in minutes) to travel along the train lines between the seven stations in a town are shown in the table below. Blank cells indicate no direct train line between those stations.

	A	B	C	D	E	F	G
A		7				6	
B	7		8			4	6
C		8		6			7
D			6		4		
E				4		9	4
F	6	4			9		5
G		6	7		4	5	

- a** Show the train lines between the stations and associated times on a weighted graph. [2]

Train driver Elsie does a safety check early each morning by driving the train down each line exactly once.

- b** Explain why Elsie cannot start and finish this safety check at the same station. [1]

- c** If Elsie starts the safety check at station C, where will she finish? Explain your answer. [2]

Elsie lives close by to station A and so wishes to change her route to start and finish the safety check there each morning. She realises she will now have to travel down some of the lines more than once.

- d** Determine the shortest possible time for the safety check starting and finishing at station A and a possible route to achieve this. [4]

The network is developed to add two new train lines. The new line from A to G takes 4 minutes in either direction. The new line from G to D takes 3 minutes in either direction.

- e** Determine the percentage increase in the time taken for the safety check. [5]

**6 [Maximum mark: 20]**

Consider the matrix  $\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$ .

- a** Calculate  $\mathbf{M} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$  and hence write down one of the eigenvalues of  $\mathbf{M}$ . [2]

- b** Show that 3 is an eigenvalue of  $\mathbf{M}$  and find the corresponding eigenvector. [2]

Two species of mammal,  $X$  and  $Y$ , are to be introduced onto a small island. Letting  $x$  be the population of Species  $X$  and  $y$  be the population of Species  $Y$ , a conservationist wishes to model the populations of both species  $t$  months after they are released onto the island.

The conservationist thinks the growth of the populations may be modelled by the following system of differential equations.

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{100}(2x + y) \\ \frac{dy}{dt} &= \frac{1}{100}(4x - y) \end{aligned}$$

- c** Given that initially  $x = 100$  and  $y = 50$ , show that  $x = 90e^{0.03t} + 10e^{-0.02t}$  and  $y = 90e^{0.03t} - 40e^{-0.02t}$ . [3]

- d** Using this model, find the number of both species 30 months after they are introduced. [2]

- e** Describe what happens to the populations of both species, using this model, in the long term. [2]

After considering this model the conservationist decides it is unrealistic. She realises that Species  $Y$  is in fact a predator which will feed on Species  $X$  as its prey. She therefore proposes the following system of differential equations as a model.

$$\begin{aligned} \frac{dx}{dt} &= \frac{x(100 - y)}{1000} \\ \frac{dy}{dt} &= \frac{y(2x - 300)}{1000} \end{aligned}$$

- f** Taking the same initial populations of  $x = 100$  and  $y = 50$ , use Euler's method with a step length of 1 month to predict the populations of both species to the nearest integer after 4 months. [4]



The conservationist realises that this model will produce periodic behaviour in the populations of the mammals. She wishes to determine the maximum and minimum number of predators that will occur over time.

- g** Find the value of  $x$  for which the maximum and minimum values of  $y$  will occur. [2]

The equation

$$2x + y + 100 \ln\left(\frac{50\,000\,000}{x^3y}\right) = 250$$

gives an exact relationship between  $x$  and  $y$  for this system of differential equations.

- h** Determine, to the nearest integer, the maximum and minimum sizes of the predator population. [3]

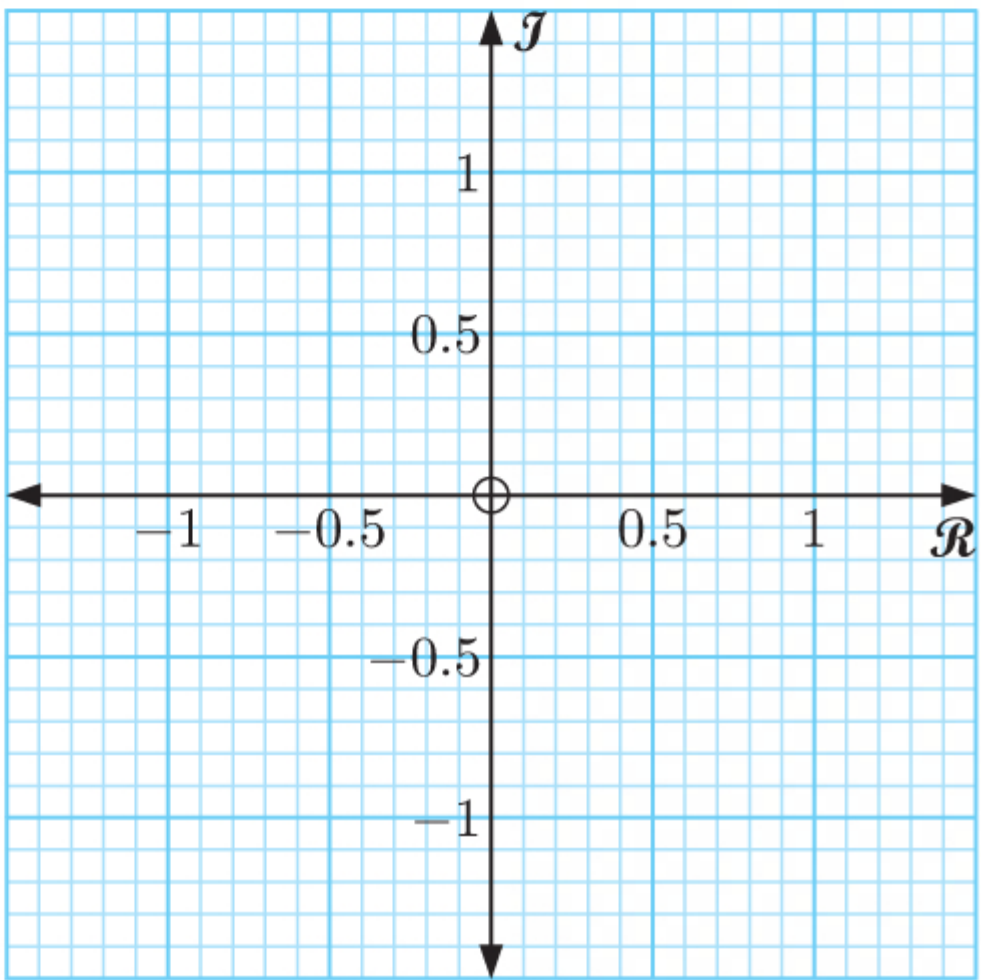
**7 [Maximum mark: 18]**

Let  $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ .

- a** Copy and complete the table giving all answers in exact form. [5]

	$w$	$w^2$	$w^3$
Cartesian form $a + bi$	$\frac{1}{2} + \frac{\sqrt{3}}{2}i$		
Euler's form $e^{i\theta}$ , $0 < \theta < 2\pi$			

- b** Hence or otherwise, draw  $w, w^2, w^3, w^4, w^5$ , and  $w^6$  on the following Argand diagram. [3]



- c** State the period of the function  $f(t) = e^{\frac{\pi}{3}ti}$ . [1]

- d** Show, giving values correct to 4 significant figures, that  $50f\left(\frac{3}{\pi}\right) \approx 27.02 + 42.07i$ . [2]

A circuit has two voltage sources, P and Q. The voltage from P is given by  $p(t) = 20 \cos\left(\frac{\pi}{3}t\right)$  volts, and the voltage from Q is given by  $q(t) = 50 \cos\left(\frac{\pi}{3}t + 1\right)$  volts, where  $t$  is the time in milliseconds.

- e** Show that  $p(t) = \Re[20f(t)]$ . [2]

- f** Hence or otherwise, find the total voltage in the circuit after  $t$  milliseconds in the form  $T(t) = A \cos\left(\frac{\pi}{3}t + B\right)$ . [5]

PAPER 3

CALCULATOR, 60 MINUTES

**1 [Maximum mark: 28]**

An online media company is considering the different packages it offers to its customers.

Under a two package system it would offer customers the choice between Standard and Deluxe. Customers would only change between the packages at the end of a month.

Research suggests that 30% of customers on the Standard package would change to Deluxe every month, while 20% of customers on the Deluxe package would change to Standard every month.

- a** Write down a transition state diagram and a transition matrix  $\mathbf{T}$  to represent the movement between the packages at the end of a particular month. [2]

- b** Find the eigenvalues and eigenvectors of the matrix  $\mathbf{T}$ . [4]

The company has 24 000 customers who would all begin on the Standard package.

- c** By using the relevant eigenvector of the matrix  $\mathbf{T}$  find the number of Standard customers the company can expect to have in the long term. [2]



- d

Write down matrices **P** and **D** such that  $\mathbf{T} = \mathbf{PDP}^{-1}$ .

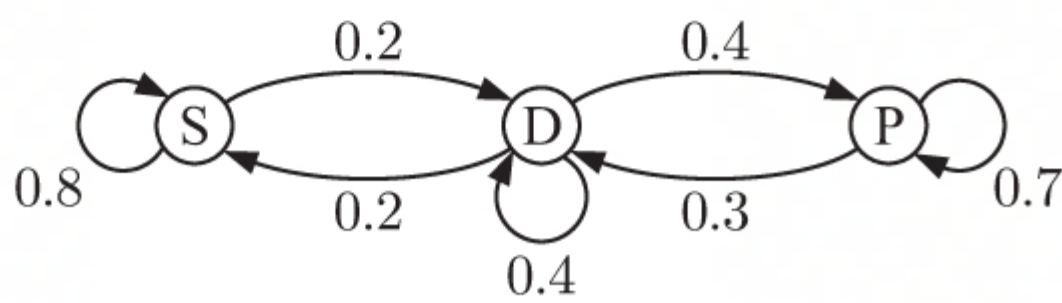
[2]
- e

Hence, show that the number of Standard customers under the two package system after  $n$  months is given by  $S_2(n) = 9600 + 14\,400(0.5)^n$ .

[5]

Under a three package system the company would also offer a Platinum package. This would only be available to customers who are already on the Deluxe package and changes would still only be allowed at the end of a month.

Research suggests that monthly changes between the packages would now be represented by the following transition state diagram.



- f

Write down the new transition matrix **M**.

[1]
- The company would again start with all 24 000 customers on the Standard package.
- g

Use the matrix **M** to find the number of Standard customers after 5 months.

[2]
- h

By solving a relevant system of equations, show that the number of Standard package customers in the steady state under this three package system is 7200.

[3]
- A team member suggests a model of the form  $S_3(n) = p + qr^n$  for the number of Standard customers after  $n$  months under the three package system.
- i

Using the number of Standard customers initially, after 5 months, and in the steady state, find suitable values of  $p$ ,  $q$ , and  $r$ .

[3]
- j

On the same set of axes sketch the graphs of  $y = S_2(n)$  and  $y = S_3(n)$ .

[2]
- k

Hence, find the first month in which the number of Standard customers under the three package system will be less than the number of Standard customers under the two package system.

[2]

2 [Maximum mark: 27]

Mr Stitch runs a clothing shop which orders scarves of length 1 metre from a local manufacturer. The scarves are all the same style but come in five different colours. As a quality check he takes a stratified sample of 100 scarves and records the number of flaws found in each individual scarf.

<i>Number of flaws</i>	0	1	2	3	4	5
<i>Frequency</i>	12	37	21	19	8	3

- a

Describe a possible way in which Mr Stitch may have selected the stratified sample.

[2]
- The local manufacturer claims that the number of flaws  $X$  in a scarf can be modelled by a Poisson distribution  $X \sim \text{Po}(1.7)$ . Mr Stitch tests this claim with a  $\chi^2$  goodness of fit test carried out at the 5% significance level.
- b

Write down the null hypothesis for this test.

[1]
- c

Complete the table for the expected frequencies in the sample of 100 scarves, giving your answers correct to 2 decimal places.

[2]

<i>Number of flaws</i>	0	1	2	3	$\geq 4$
<i>Expected frequency</i>		31.06		14.96	

- d

Write down the number of degrees of freedom.

[1]
- e

Find the  $p$ -value for the test.

[2]
- f

State the conclusion of the test. Give a reason for your answer.

[2]

A customer enters the shop and buys 5 of the scarves.

- g

Find the probability that at least 3 of the scarves have no flaws.

[3]

The scarves from the local manufacturer are delivered in boxes of 50. Mr Stitch calculates the mean number of flaws per scarf  $\bar{X}$  for the scarves from a single box.

- h

Explain why it can be assumed that  $\bar{X}$  follows a normal distribution.

[2]



An overseas manufacturer of the same scarves also delivers them in boxes of 50. The overseas manufacturer claims that their scarves have less flaws than those from the local manufacturer. Mr Stitch takes 8 boxes from each supplier and records the mean number of flaws per scarf in a box.

Local manufacturer				Overseas manufacturer			
1.54	1.78	2.00	1.52	1.72	1.62	1.28	1.44
1.66	1.88	1.74	1.50	1.54	1.48	1.82	1.38

- i

Use an appropriate test to determine whether there is evidence, at the 5% significance level, that the scarves from the overseas manufacturer have fewer flaws than those from the local manufacturer. State any assumptions you have made in performing the test.
- [5]

Mr Stitch decides to order from both local and overseas manufacturers. After a year he has identical looking boxes from both manufacturers scattered randomly in his warehouse. To determine whether a box is from the local or overseas manufacturer, he performs a hypothesis test where:

$H_0$ : the box is from the local manufacturer

$H_1$ : the box is from the overseas manufacturer

He opens a box and records the mean number of flaws per scarf  $\bar{f}$ . If  $\bar{f} > 1.55$  he accepts the null hypothesis, if not then he accepts the alternative.

- j

Given that for the local manufacturer  $\bar{X} \sim N(1.7, (0.025)^2)$  find the probability of a Type I error.
- [2]
- k

Given that for the overseas manufacturer  $\bar{Y} \sim N(1.5, (0.025)^2)$  find the probability of a Type II error.
- [2]

Mr Stitch knows that 70% of the boxes in his warehouse are from the local manufacturer and 30% are from the overseas manufacturer. A box is randomly chosen and Mr Stitch’s hypothesis test concludes that the box is from the local manufacturer.

- l

Find the probability that the box is actually from the overseas manufacturer.
- [3]



# Trial examination 2

## PAPER 1

## CALCULATOR, 120 MINUTES

### 1 [Maximum mark: 5]

The number of words in the first 20 sentences of Chapter 1 of *The Hunger Games* by Suzanne Collins are as follows:

12, 17, 12, 4, 7, 7, 9, 18, 13, 18, 7, 5, 11, 14, 14, 3, 5, 25, 9, 10

- a Calculate the IQR for the data. [2]
- b What is the largest data value that is *not* an outlier? [3]

### 2 [Maximum mark: 9]

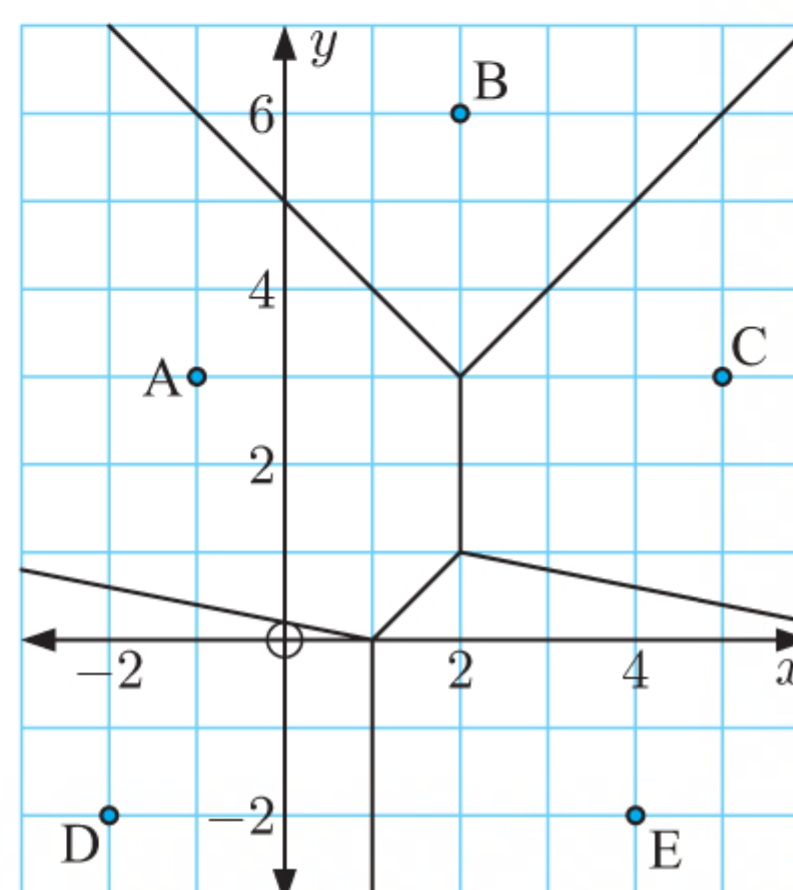
- a i Evaluate  $\log_{0.5} 7$ . [1]
- ii Find the value of  $x$  for the equation  $\log_9 x = 2$ . [1]
- b Solve  $9^x \times \left(\frac{1}{27}\right)^{1-x} = 1$ . [3]
- c Solve for  $a$  and  $b$ : [4]

$$1.65^a \times 2.03^b = 9.8$$

$$1.42^a \times 2.59^b = 13.6$$

### 3 [Maximum mark: 4]

The Voronoi diagram with points A(−1, 3), B(2, 6), C(5, 3), D(−2, −2), and E(4, −2) below represents the locations of recycling centres in a city.



- a Charlotte lives at (4, 1). Which recycling centre should she visit? [1]
- b A new recycling centre is to be built as far away as possible from the existing recycling centres. Assuming it must be located inside the region defined by the existing recycling centres, where should the new recycling centre be placed? [3]

### 4 [Maximum mark: 5]

A savings scheme is offering an interest rate of 2.5% per annum. Astria wants to save £10 000. She works out that she can save £500 a year, which she will deposit on the 1st of January each year.

- a How much will she have saved after 5 years? [3]
- b How many years will it take for Astria to save the full amount? Give your answer as an integer. [2]

### 5 [Maximum mark: 7]

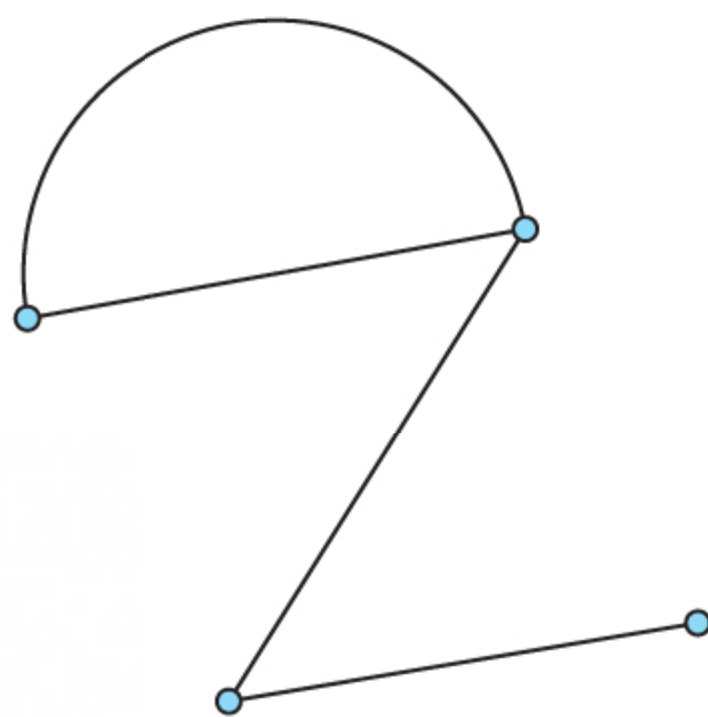
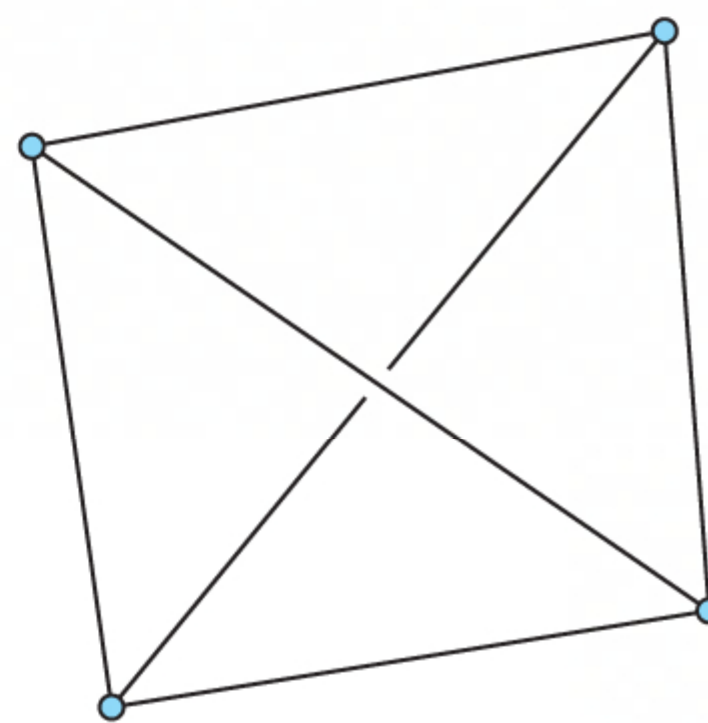
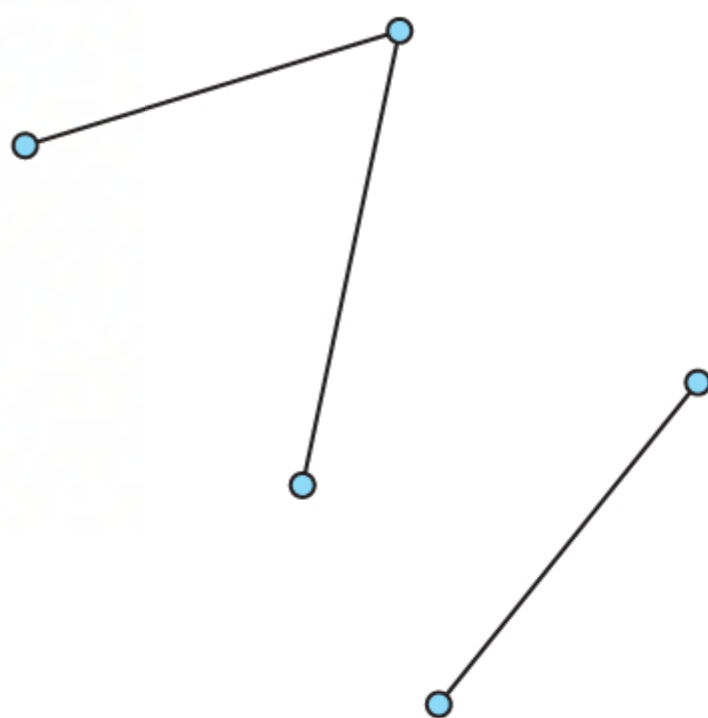
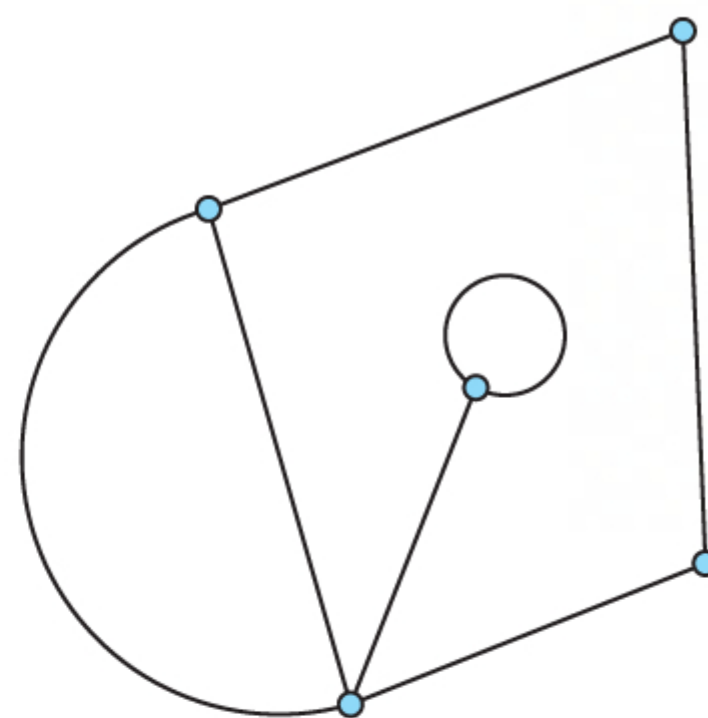
Sarah will cycle 2000 miles over a number of days for charity. She cycles 12 miles on day 1, and increases this distance by 10% each day.

- a How many days will it take for her to complete this challenge? [4]
- b What is the greatest number of miles that she will complete in a single day? [3]



**6 [Maximum mark: 5]**

Below are different types of graphs:

**A****B****C****D**

**a** Which of the graphs are:

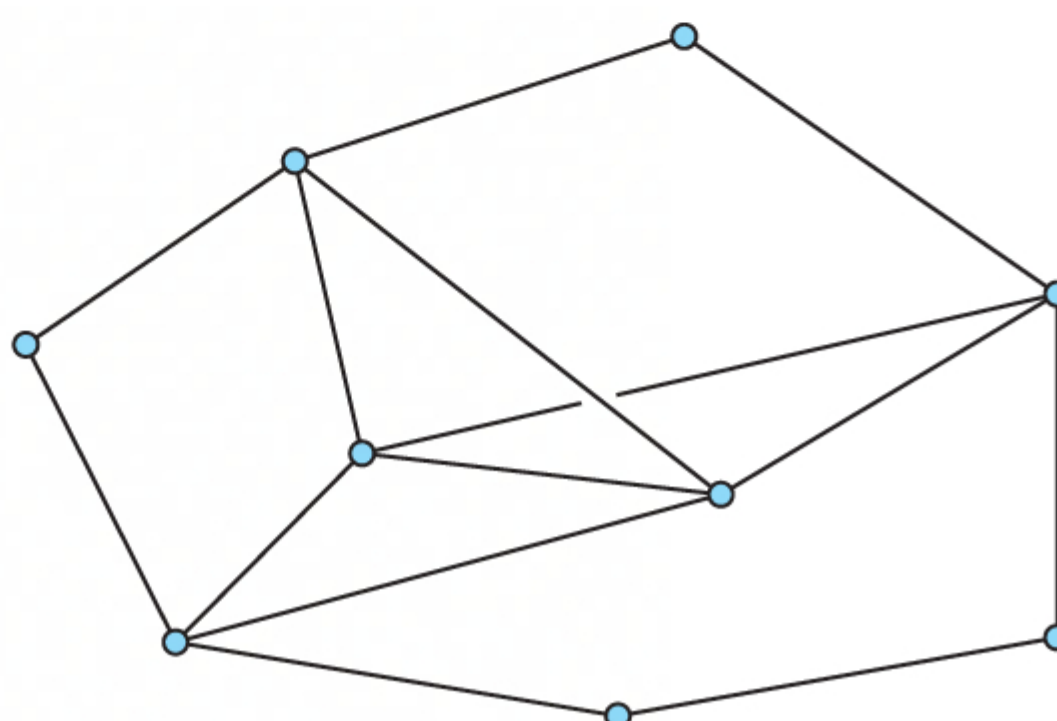
- i** simple
- ii** connected
- iii** complete?

[1]

[1]

[1]

An electrician was studying the network of wires connecting electricity to different houses in a neighbourhood. Here is a simplified image of the network:



His daughter, who was studying graph theory in school, said “That looks like an Eulerian graph”.

**b** Is the electrician’s daughter correct? Give a reason for your answer.

[2]

**7 [Maximum mark: 6]**

The function  $f(x)$  is defined as  $f(x) = \frac{3x+5}{x-3}$ .

**a** State the domain of the function.

[1]

**b** Find  $f^{-1}(-4)$ .

[3]

Another function  $h(x)$  is defined as  $h(x) = 2^{x+1}$ .

**c** Solve  $f(x) = h(x)$ .

[2]



**8 [Maximum mark: 6]**

A theory claims that, when sweet peas with red flowers and sweet peas with white flowers are crossed, the next generation of sweet peas have red, white, and pink flowers in the proportions  $\frac{1}{6}$ ,  $\frac{1}{6}$ , and  $\frac{2}{3}$  respectively.

The outcomes in an actual experiment are as follows: 24 with red flowers, 34 with white flowers, and 62 with pink flowers.

A  $\chi^2$  goodness of fit test at a 1% significance level was conducted on the data.

- a** State the null and alternative hypotheses. [1]
- b** State the number of degrees of freedom. [1]
- c** Find the  $p$ -value. [2]
- d** What is the conclusion of the test? Give reasons for your answer. [2]

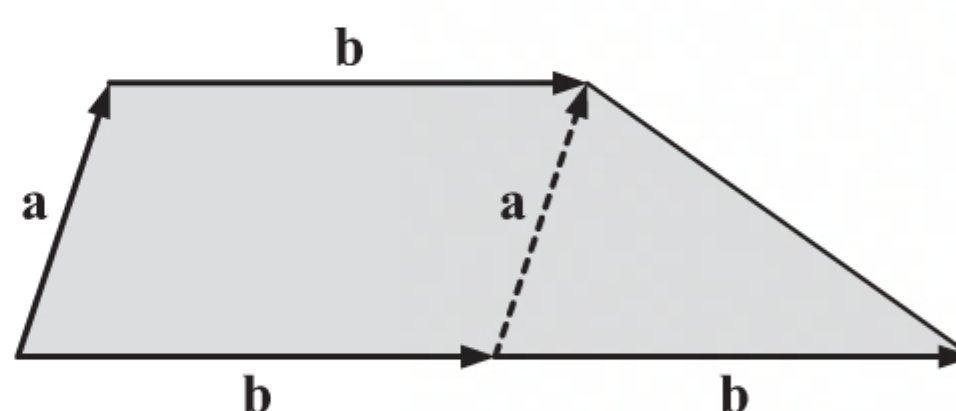
**9 [Maximum mark: 5]**

A telecommunications company offers a phone and broadband plan for new customers. It costs £27 a month for the first 12 months with a £35 set up fee. It then costs £49 a month thereafter.

- a** Set up a piecewise function for the cost,  $C$ , of the phone and broadband plan after  $m$  months. [3]
- b** Hence find how much you would have paid after 3.5 years. [2]

**10 [Maximum mark: 6]**

In the trapezium below,  $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$  and  $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$ .



Calculate the area of the trapezium.

**11 [Maximum mark: 4]**

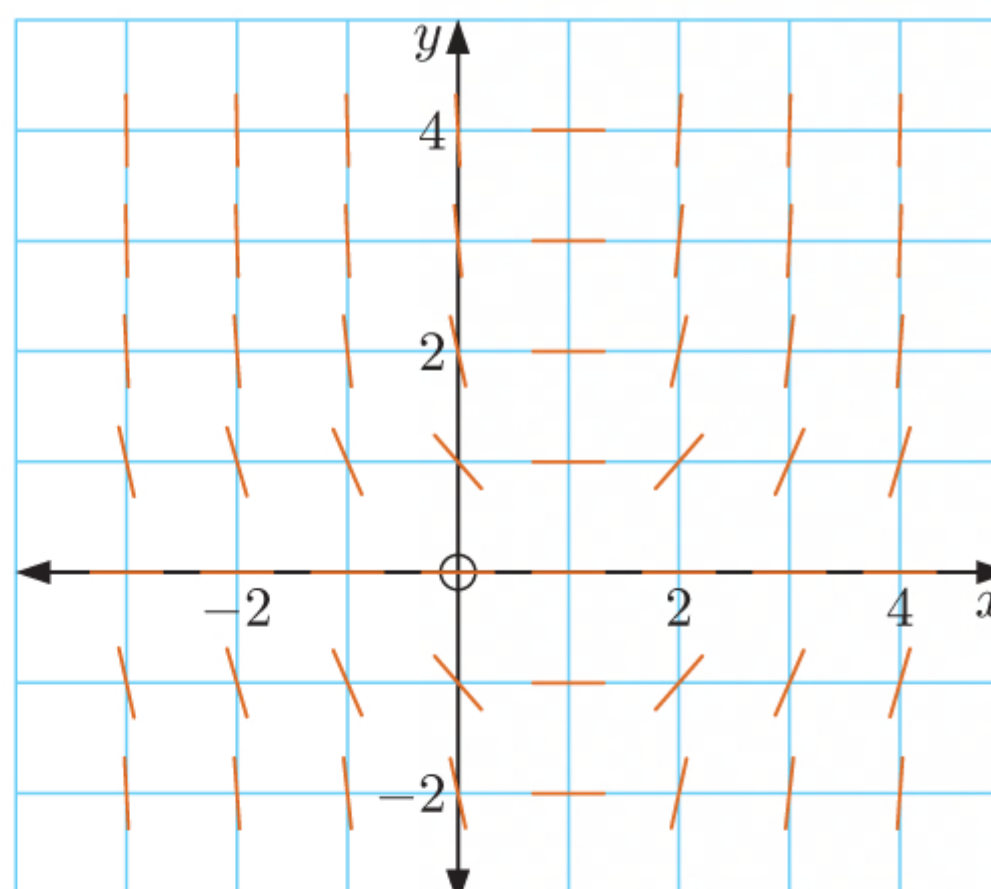
A renowned butcher is blindfolded and asked to taste and arrange eight cuts of meat in order of price. The correct order is A, B, C, D, E, F, G, H while the order chosen by the butcher was A, (B, D), C, G, (E, F, H). The brackets indicate cuts of meat which the butcher assigned the same price.

Determine the value of  $r_s$  as a measure of the correlation between the butcher's opinion and the correct order.

**12 [Maximum mark: 7]**

Consider the following differential equation and its slope field.

$$\frac{dy}{dx} = y^2(x - 1)$$

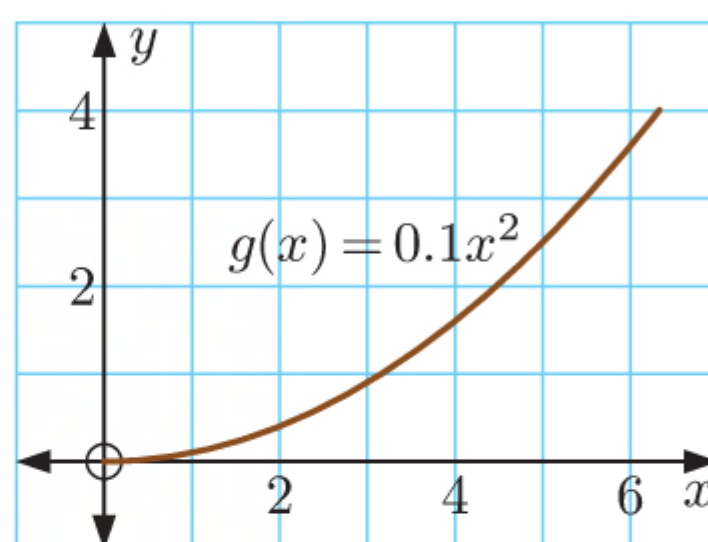


- a** Sketch the solution curve which passes through the point  $(1, 2)$ . [2]
- b** Find the equation of the solution curve drawn in part **a**. [5]



**13 [Maximum mark: 7]**

Emma was on holiday in Spain and bought a lovely bowl at a market. The shape of the bowl can be modelled by rotating the function  $g(x) = 0.1x^2$ , for  $0 \leq x \leq 6.33$ , about the  $y$ -axis through  $360^\circ$ .



- a** Find  $g(6.33)$ . [1]
- b** Find  $g^{-1}(x)$ . [2]
- c** Calculate the volume of the bowl. [4]

**14 [Maximum mark: 7]**

A lecturer is marking an end of year test. The number of mistakes they make whilst marking is 2.1 mistakes per paper.

- a** State an appropriate probability distribution for this context. [1]
- b** Find the probability that, in a randomly selected paper, there are at least 2 mistakes. [3]

There are 10 students in this exam.

- c** Find the probability that there were 15 mistakes made in total. [3]

**15 [Maximum mark: 6]**

A rectangle with vertices  $A(1, 1)$ ,  $B(5, 1)$ ,  $C(5, 3)$ , and  $D(1, 3)$  is transformed by the matrix  $\begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix}$  onto image  $A'B'C'D'$ .

- a** Find the coordinates of  $A'$  and  $D'$ . [3]
- b** Find the area of the image  $A'B'C'D'$ . [3]

**16 [Maximum mark: 8]**

A fruit shop is selling boxes of fruit to a super store. The weights of apple boxes are normally distributed with mean 25.7 kg and standard deviation 3.1 kg. The weights of banana boxes are normally distributed with mean 17.3 kg and standard deviation 1.5 kg. The weights of orange boxes are normally distributed with mean 22.1 kg and standard deviation 2.2 kg.

- a** Find the probability that double the weight of a box of apples is greater than the sum of the weights of a box of bananas and a box of oranges. [5]

10 boxes of apples are selected at random.

- b** Find the probability that the mean weight is greater than 26 kg. [3]

**17 [Maximum mark: 8]**

An aeroplane is at  $A(5, 7, 2)$  and moves at a constant velocity. After 20 seconds it is at point  $B(4, 7.5, 3)$ .

- a** Assuming coordinates are in km, find the speed of the plane in km/min. [4]
- b** At what angle is the plane flying to the horizontal? [4]

**18 [Maximum mark: 5]**

The velocity of an object can be modelled by the following differential equation

$$\frac{dx}{dt} = xt + 30$$

Use Euler's method with step size 0.1 to estimate  $x(1)$  given  $x(0) = 0$ .



PAPER 2

CALCULATOR, 120 MINUTES

1 [Maximum mark: 14]

Clare has forgotten her graphics calculator and instead just has a normal calculator in her bag. She therefore has to estimate the area under the curve  $y = \sqrt[4]{20x^2 + 1}$  between  $x = 2$  and  $x = 6$  using the trapezoidal rule.

- a Copy and complete the table below, giving your answers to 4 decimal places. [2]

$x$	2	2.5	3	3.5	4	4.5	5	5.5	6
$y = \sqrt[4]{20x^2 + 1}$	3		3.6679		4.2328		4.7311		5.1818

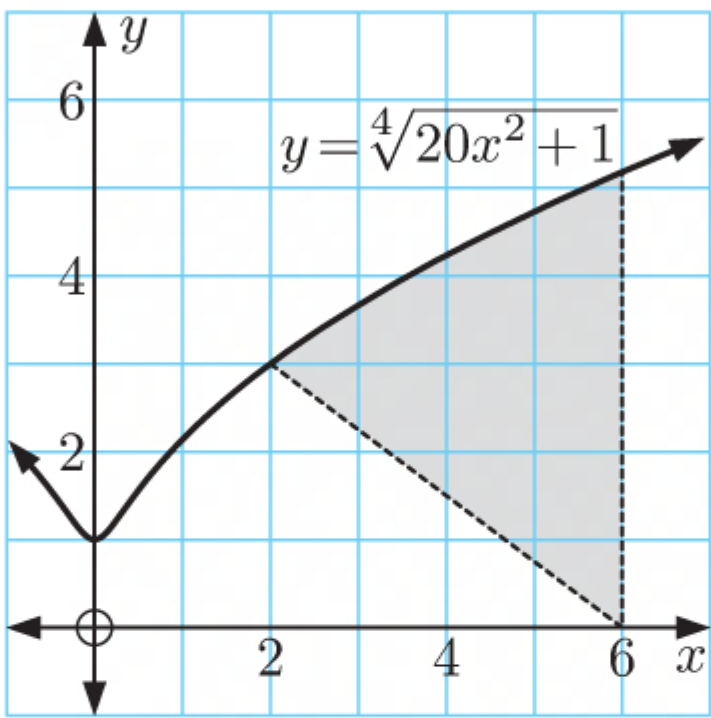
- b Using the table in part a and the trapezoidal rule, estimate  $\int_2^6 \sqrt[4]{20x^2 + 1} \, dx$  to 3 decimal places. [4]

Clare’s friend Roy lets Clare borrow his graphics calculator when he was not using it.

- c Calculate the value of  $\int_2^6 \sqrt[4]{20x^2 + 1} \, dx$ , giving your answer to 3 decimal places. [2]

- d Calculate the percentage error in Clare’s estimate using the value in part c. [3]

- e [3]



Clare has designed a pool in the shape of the shaded region above, where the units are metres.

Use your answer to part c to find the area of the pool, to 3 decimal places.

2 [Maximum mark: 15]

A survey was done in a school in which every 5th student was asked how much screen time, on average, they had each evening. The results are shown below:

	Year 9	Year 10	Year 11	Year 12
Less than 1 hour	4	6	8	11
Between 1 and 3 hours	6	8	6	14
More than 3 hours	9	7	7	9

- a Name the sampling method used in this survey. [1]

- b Given that a student chosen at random is in Year 12, find the probability that they had more than 3 hours of screen time. [2]

- c One of the teachers in the school wanted to test whether the amount of screen time was dependent on the Year group a student was in. He performs a  $\chi^2$  test at a 10% significance level.

- i Write down the null hypothesis for this test. [1]

- ii Write down the number of degrees of freedom. [1]

- iii Find the  $p$ -value for this test. [2]

- iv State the conclusion of the test, giving a reason for your answer. [2]

- d Last year, only 20% of Year 12 students had less than one hour of screen time each evening. Another teacher suspects that the proportion has increased this year. She performs a one-tailed test at a 10% significance level.

- i Write down the null and alternative hypotheses for this test. [1]

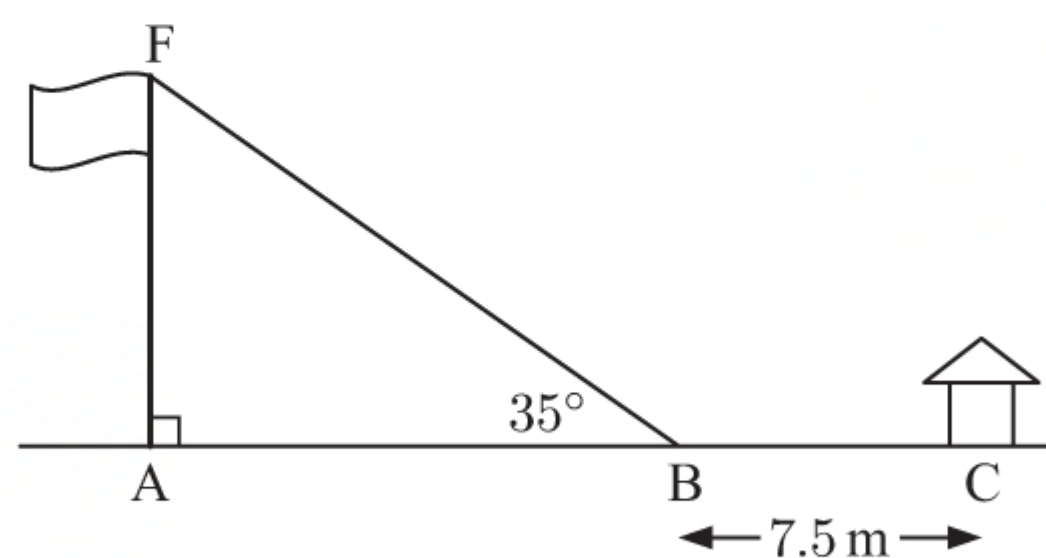
- ii Find the  $p$ -value for this test. [3]

- iii State the conclusion of the test, giving a reason for your answer. [2]



**3 [Maximum mark: 12]**

A vertical flagpole is positioned at A and is supported by a wire which runs from the top of the flagpole to the ground at B. The wire makes an angle of  $35^\circ$  with the ground as shown in the diagram below:



- a** Find the angle of depression from the top of the flagpole to B. [1]

There is a storm forecast for the upcoming week. The site manager wants to add another wire for extra support from the top of the flagpole to his cabin at C. The angle that the wire will make with the ground is  $29^\circ$ .

- b** Calculate the length of the wire that is required to go from the top of the flagpole to C. [5]

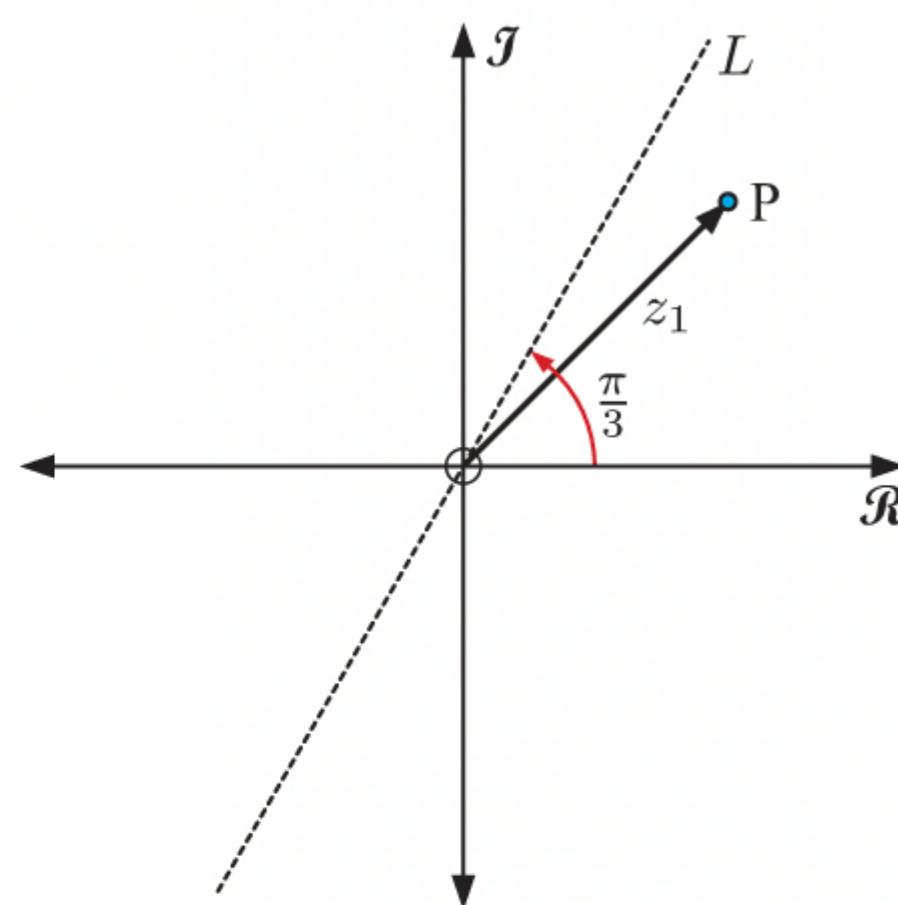
- c** Calculate the height of the flagpole. [3]

Every morning the site manager must walk from his cabin at C to the base of the flagpole to raise the flag, then return to his cabin. Once the working day is finished, he must walk there again to bring the flag down before returning to his cabin to lock up.

- d** Calculate the **total distance** the site manager walks each day to raise and bring down the flag. [3]

**4 [Maximum mark: 20]**

A real quadratic has a root  $z_1 = 3 + 3i$ .



- a** State the other root,  $z_2$ . [1]

- b** Draw  $z_2$  on the Argand diagram. [1]

- c** Find a quadratic with roots  $z_1$  and  $z_2$ . [4]

- d** Write  $z_1$  in polar form. [2]

- e** Suppose  $z_1 = \overrightarrow{OP}$ ,  $z_2 = \overrightarrow{OQ}$ , and  $z_3 = \overrightarrow{OR}$  where R is obtained by reflecting P in the line L.

- i** Find the matrix for this reflection. [3]

- ii** Hence find the exact coordinates of R. Write each coordinate in the form  $a + b\sqrt{3}$ , where  $a, b \in \mathbb{Q}$ . [2]

- iii** Explain geometrically why  $z_3$  has modulus  $3\sqrt{2}$  and argument  $\frac{5\pi}{12}$ . [4]

- iv** Hence show that  $\cos \frac{5\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$  and  $\sin \frac{5\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$ . [3]

**5 [Maximum mark: 19]**

A farmer supplies eggs to his local shop. Last year the weights of eggs,  $X$ , were normally distributed with a mean of 51.6 g and a standard deviation of 7.5 g.

- a** Find the percentage of eggs that weighed less than 40 g. [2]

Medium eggs are classified as having a weight between 49 grams and  $k$  grams, where  $k > 49$ .

- b** Given that medium eggs made up 35.69% of eggs, find the value of  $k$ . [4]



40 eggs were selected at random.

- c** Find the probability that exactly  $\frac{2}{5}$  of the eggs were medium. [3]

The shop owner will purchase the eggs if at least 10 of them are medium.

- d** What is the probability that the shop owner purchases the eggs? [2]

The farmer claims that the eggs he produced this year are heavier than last year. Below is a random sample of the weights of 30 eggs:

33 45 54 70 36 37 55 50 81 32 40 37 58 57 53  
52 51 40 65 75 78 68 63 53 54 52 55 56 53 40

- e** Calculate the mean and standard deviation for this sample. [3]

The farmer performs a  $t$ -test at a 10% significance level to test his claim.

- f i** State the null and alternative hypotheses for this test. [1]

- ii** Find the  $p$ -value and  $t$ -statistic. [2]

- g** Conclude, giving reasons for your answer, whether the farmer was correct in his claim. [2]

**6 [Maximum mark: 13]**

A stone is dropped from various heights, and the time of flight from that height was recorded in the table below:

Distance ( $D$ m)	1.1	6.95	24.32	65.1	103.2
Time ( $T$ s)	0.4	1.5	2.1	3.6	4.4

- a** Copy and complete the table below giving your answers to 4 significant figures: [2]

$\ln D$					
$\ln T$					

The relationship between  $\ln D$  and  $\ln T$  can be modelled by the regression equation  $\ln T = a \ln D + b$ .

- b** Find the values of  $a$  and  $b$ . [2]

The relationship between the distance,  $D$ , and the time,  $T$ , can be modelled by the exponential model  $T = kD^n$  where  $n \neq 0$ ,  $n \neq 1$ , and  $k \neq 0$ .

- c** Find the values of  $k$  and  $n$ . [5]

- d** Hence predict the time taken for the object to fall 150 m. [2]

- e** Comment on the reliability of your prediction in part **d**. [2]

**7 [Maximum mark: 17]**

Alanna has moved into her new house and is working on maintaining a nice tidy flower bed in the garden. She purchases two plants in the garden centre, marigolds and dianthus. The heights  $x$  and  $y$ , of the marigolds and dianthus respectively, can be modelled by the following set of equations, where  $t$  is in years.

$$\begin{aligned}\frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 4x + 3y\end{aligned}$$

- a** Write these equations in the matrix form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ . [1]

Initially, the height of the marigolds was 15 cm, and the height of the dianthus was 20 cm.

- b** Find  $\dot{\mathbf{x}}$  when  $t = 0$ . [2]

- c** Calculate the eigenvalues and corresponding eigenvectors of matrix  $\mathbf{A}$ . [6]

- d** Find the particular solution for this set of differential equations. [5]

Alanna wants to make sure that the flowers in the flower bed do not grow above 1 m, as it will make the garden look untidy.

- e** Find the height of the plants after 6 months. Does Alanna need to cut back any of the flowers she has planted? [3]

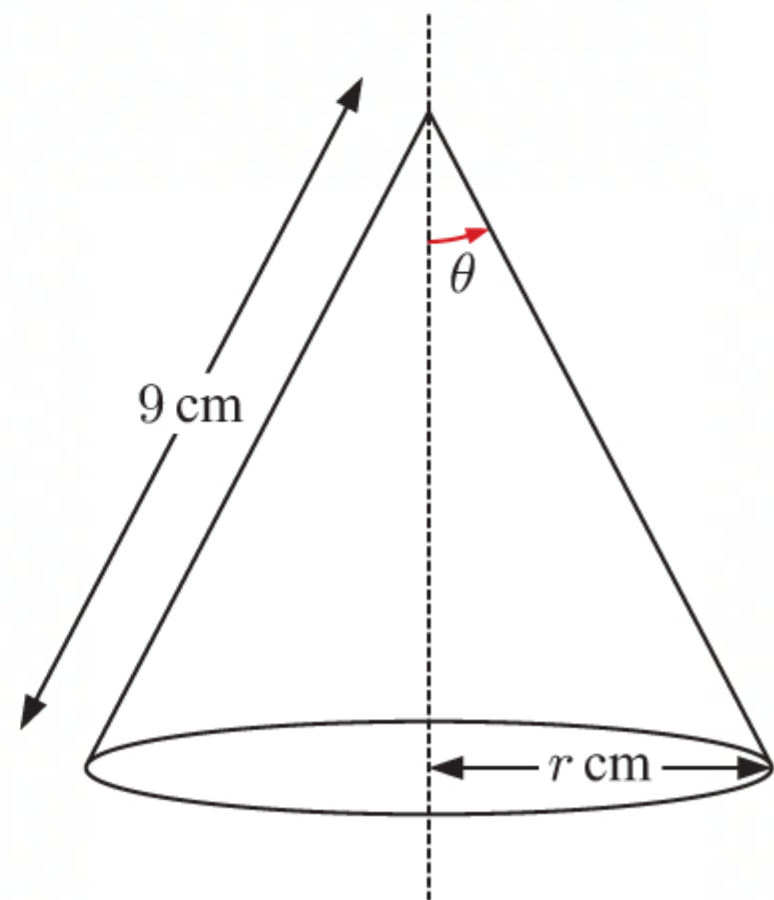


PAPER 3

CALCULATOR, 60 MINUTES

1 [Maximum mark: 29]

A company P is producing conical paper cups for a school. Below is a model of the paper cup.



- a Write down an expression for the volume,  $V$ , of the cup in terms of  $\theta$ . [4]
- b Find  $\frac{dV}{d\theta}$ . [3]
- c Find, in radians, the value of  $\theta$  which maximises the volume of the cup. [2]
- d Convert this value of  $\theta$  into degrees. [1]
- e Find the maximum volume that the cup can hold. [2]

The water dispenser in the school holds 11.7 litres of water. It is known that  $1\text{ cm}^3 = 1\text{ mL}$ . On average people fill the paper cups so they are 90% full.

- f Given that the paper cups produced can hold the maximum volume and assuming that the staff do not reuse the cups, how many cups will the water dispenser fill? [4]

A rival company Q also produces conical paper cups. Staff members are asked to choose which of the cups they prefer. 30% of people who choose P one year continue to choose P the next year, and 45% of people who choose Q one year continue to choose Q the next year.

- g Show this information in a transition diagram. [3]
- h Show this information in a transition matrix  $\mathbf{M}$ . [2]

This year 100 staff members voted for company P and 50 voted for company Q.

- i State the initial state matrix. [1]
- j Calculate the number of staff members who will vote for company Q in two year's time. [4]
- k In the long term, how many staff members will vote for company P? [3]

2 [Maximum mark: 26]

Ryan and Poppy were visiting the zoo. Below is a weighted adjacency table which tells you approximately how many minutes it takes to walk between different sections of the zoo.

	A	B	C	D	E	F	G	H
A		10		6	5			
B	10		7		8	9		
C		7				5		
D	6				4		9	
E	5	8		4		11	11	14
F		9	5		11			4
G				9	11			7
H					14	4	7	

- A: Flamingos
- B: Reptile house
- C: Elephants and giraffes
- D: Penguins and seals
- E: Chimpanzees
- F: Bird house
- G: Lions and tigers
- H: Meerkats and lemurs

- a Draw the weighted graph which represents this weighted adjacency table. [3]

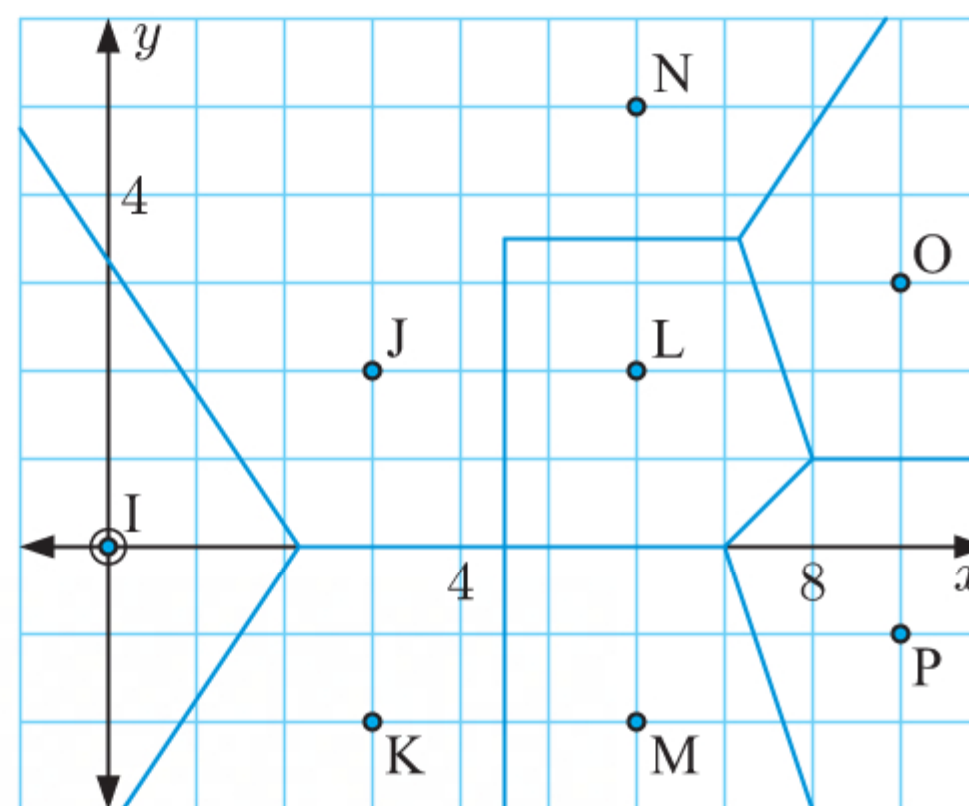


- b i** State the degree of vertex E. [1]
- ii** What does this mean in the context of the question? [1]
- c** Starting at vertex A, apply Prim's algorithm to the weighted adjacency table to find the weight of the minimum spanning tree. [4]

The entrance and exit to the zoo are at vertex A. To make sure they do not miss any of the animals, Ryan and Poppy will start at A, walk every path in the zoo, and return to A. Ryan wants to know the shortest amount of time they can do this in.

- d** Name the algorithm that will help Ryan find the shortest time. [1]
- e** Perform this algorithm and find the shortest time. [7]
- f** Ryan and Poppy start their walk at 3:30 pm. They have a dinner reservation at 6 pm, and the restaurant is a 15 minute drive from the zoo. Will they make their reservation? [2]

Recycling bins have been placed throughout the zoo. The sites on the incomplete Voronoi diagram below show the locations of the recycling bins.



- g** What can we say about the point  $(7, 0)$  in the context of this Voronoi diagram? [2]
- h** Find the equation of the perpendicular bisector of  $[JN]$ . Give your answer in the form  $ax + by = c$  where  $a, b$ , and  $c \in \mathbb{Z}$ . [4]
- i** The reptile house is situated at the point  $(2, 3)$ . Which recycling bin should the public use if they are at the reptile house? [1]



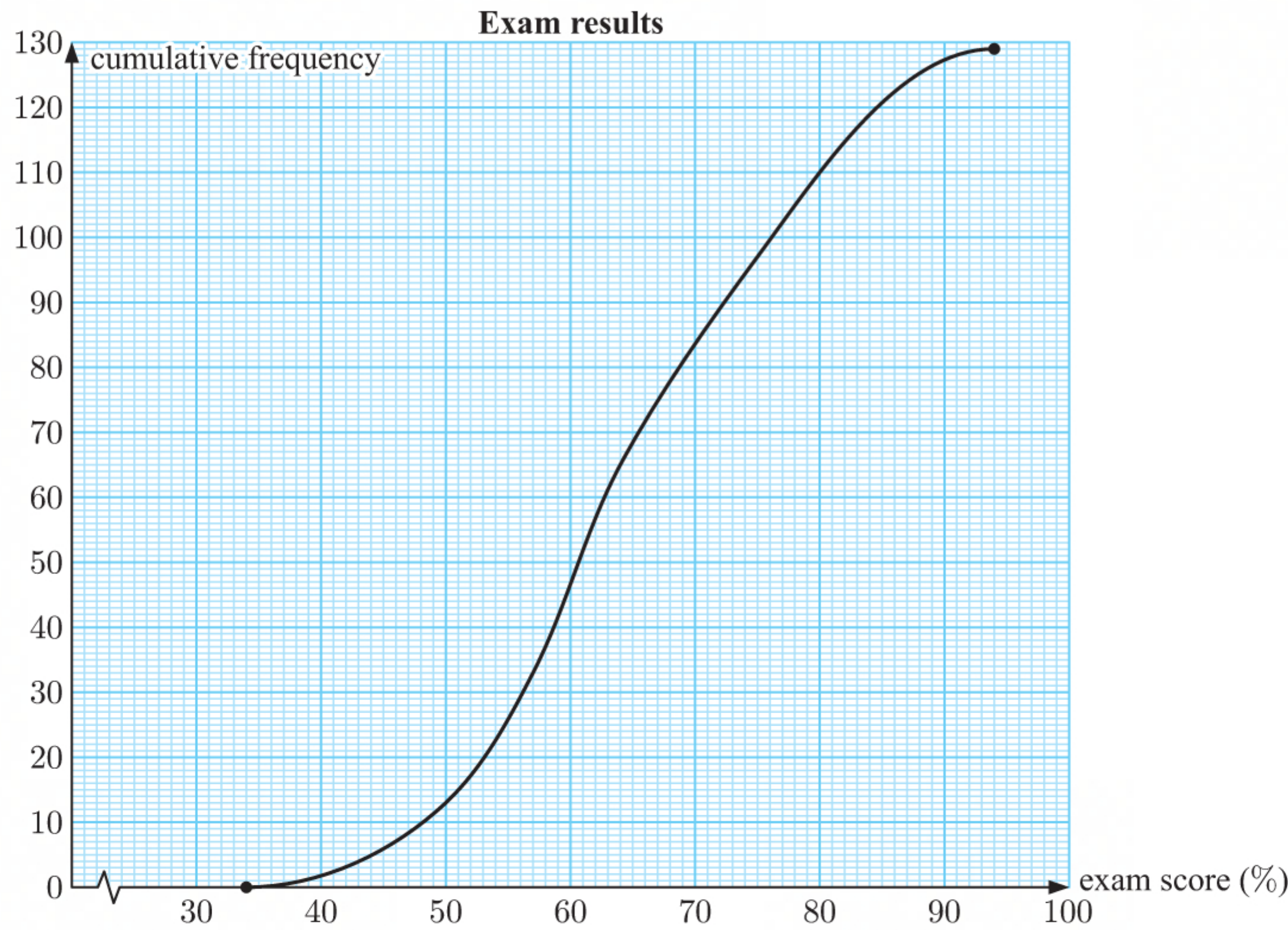
# Trial examination 3

PAPER 1

CALCULATOR, 120 MINUTES

1 [Maximum mark: 6]

The cumulative frequency graph below displays the performances of 129 students in an examination.



- a Find the percentage of students who obtained at least 50% for their examination.

b Construct a box and whisker diagram to summarise the results.

c Find the interquartile range for the results and explain what it means.
- [1]

[3]

[2]

2 [Maximum mark: 6]

The rent Georgio pays for his apartment increases by 2.5% each year. In 2020, Georgio pays a total of €12 000 in rent.

- a Explain why Georgio’s rent follows a geometric sequence, and find its common ratio.

b How much rent will Georgio pay for his apartment in 2025?

c Find the total amount Georgio will pay in rent from 2020 to 2025 inclusive.
- [2]

[2]

[2]

3 [Maximum mark: 6]

The manager of a shopping mall wants to survey customers to ask how many times per week, on average, they visit the mall. She knows from a previous survey that the age of her customers follows the distribution alongside.

Age	Percentage
under 30	19.7%
31 - 50	30.4%
51 - 70	28.3%
over 70	21.6%

The manager uses a stratified sample of 250 customers to obtain these results:

Visits per week	1	2	3	4	5	6	7
Customers	55	$x$	56	42	$y$	17	6

The mean number of visits per week for these customers was found to be 3.08 .

- a Find the number of customers sampled from the age range 51 - 70.

b Write a pair of equations involving  $x$  and  $y$ .

c Find  $x$  and  $y$  by solving your pair of equations simultaneously.
- [2]

[2]

[2]



4 [Maximum mark: 4]

The weights of Pedro’s potatoes are normally distributed with known standard deviation  $\sigma = 30$  grams.

Pedro wants to estimate the population mean  $\mu$  using a 95% confidence interval. He collected a sample of 50 potatoes and found that their mean weight was 152 grams.

- a

Which distribution should Pedro use to construct the confidence interval?

[2]
- b

Hence calculate a 95% confidence interval for  $\mu$ .

[2]

5 [Maximum mark: 6]

In Florence, Italy, a street vendor sells replicas of the statue of David by Michelangelo. The replicas for sale have the following dimensions:

Size	extra small	small	medium	large	extra large
Height (cm)	8.0	10.0	12.5	16.0	25.0
Volume (cm <sup>3</sup> )	8.2	16.0	31.3	65.5	250

- a

Find the variation model which best fits the data, including the value of  $r$ .

[2]
- b

Do you think the replicas are mathematically *similar* to one another? Explain your answer.

[2]
- c

The real statue of David is approximately 5.17 m high, with volume 2.098 m<sup>3</sup>.  
Are the replicas to scale? Explain your answer.

[2]

6 [Maximum mark: 5]

The volume of a sphere with radius  $r$  is given by  $V(r) = \frac{4}{3}\pi r^3$ ,  $r > 0$ .

- a

If the radius of a sphere is doubled, what happens to its volume?

[1]
- b

Show that  $(V^{-1} \circ V)(r) = r$ .

[3]
- c

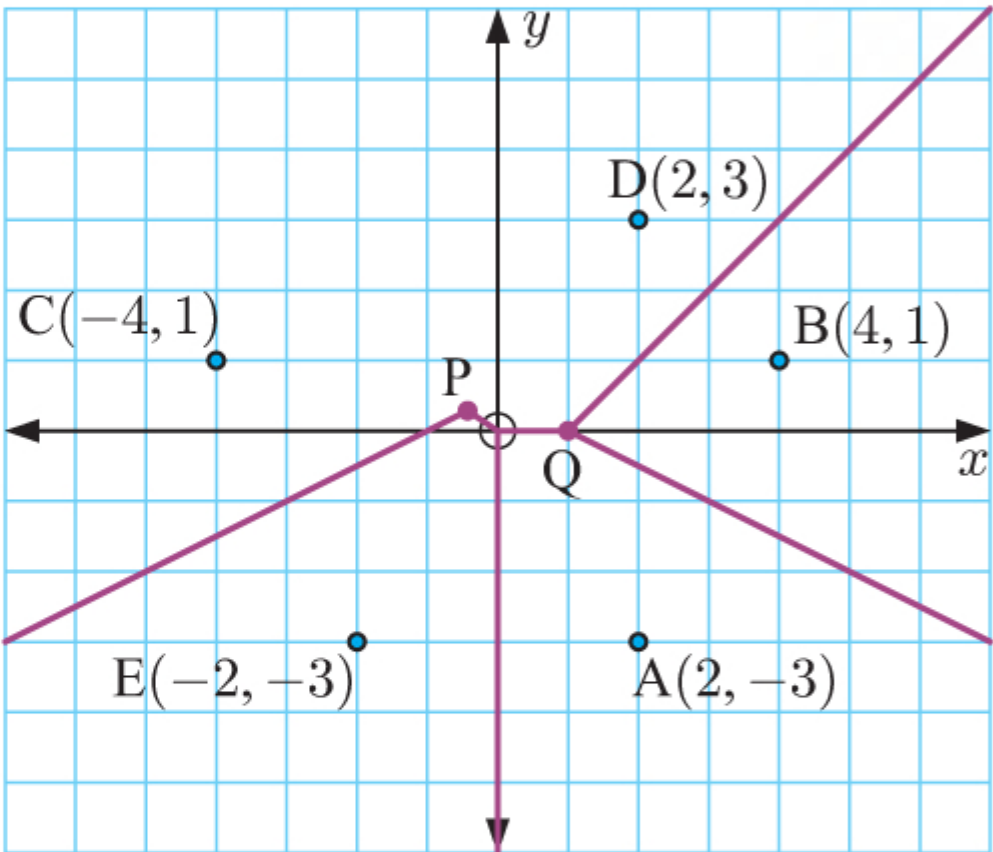
Hence or otherwise, explain the significance of  $V^{-1}(V)$ .

[1]

7 [Maximum mark: 8]

Hamsika wants to open a new Indian restaurant not more than 4 km from the centre of her city. It should be strategically placed to avoid competition from existing Indian restaurants.

The map below shows the centre of the city (the origin) and the existing Indian restaurants A to E. The grid units are in km.



- a

All but one edge of a Voronoi diagram has been drawn on the grid.  
Find the equation of the missing edge, giving your answer in the form  $ax + by + d = 0$ ,  $a, b, d \in \mathbb{Z}$ .

[3]
- b

The equation of OP is  $y = -\frac{2}{3}x$ .  
Find exactly the coordinates of P.

[2]
- c

Suggest the most appropriate location for Hamsika’s restaurant. Justify your answer.

[3]



8 [Maximum mark: 6]

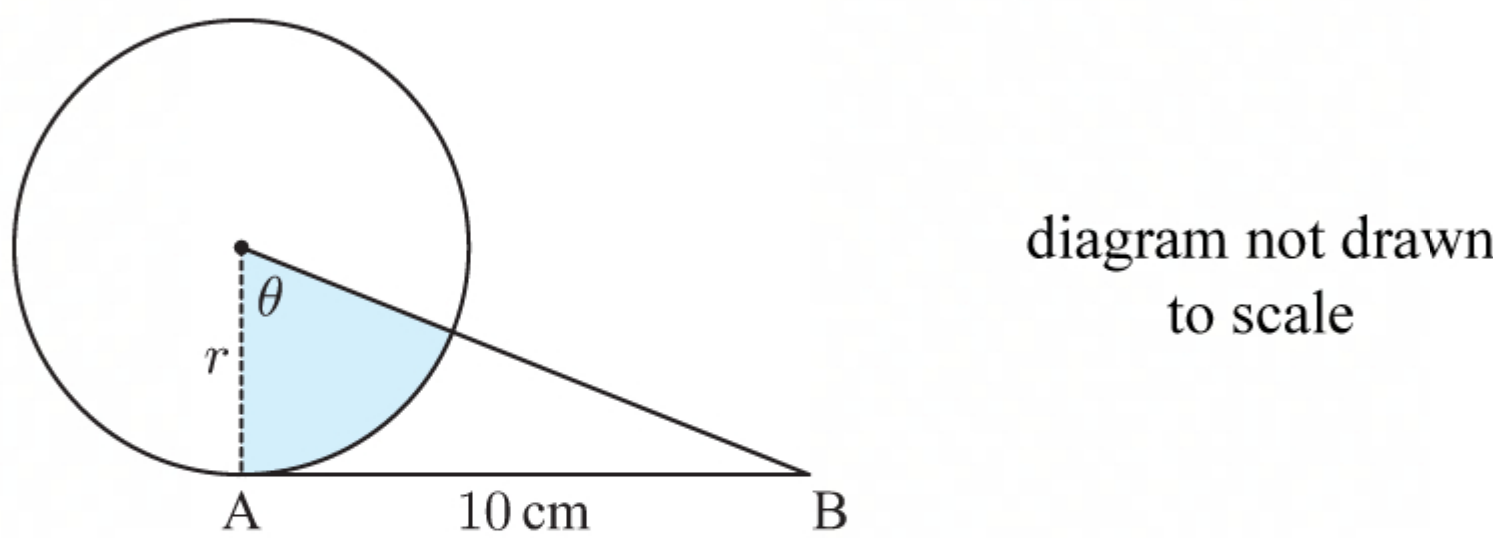
The weights of a sample of newborn calves are recorded in the table below:

Male weight (kg)	38.2	41.7	31.9	46.8	37.5	43.0	
Female weight (kg)	36.8	37.2	42.6	39.1	45.7	42.2	34.3

A hypothesis test at a 5% level of significance is conducted to see whether the mean weight  $\mu_m$  of male calves, is the same as the mean weight  $\mu_f$  of female calves.

- a State the null and alternative hypotheses. [2]
- b Assuming the weights of both male and female calves are normally distributed with the same standard deviation, calculate the  $p$ -value for this test. [2]
- c State, with a reason, whether the null hypothesis should be accepted. [2]

9 [Maximum mark: 7]



AB is a tangent to the circular flywheel above. The shaded area is 20 cm<sup>2</sup>. Find the radius of the flywheel  $r$ , and the angle at its centre  $\theta$  in radians.

10 [Maximum mark: 5]

When people go shopping in the fish market, the number of types of fish each person buys has the following distribution:

Types of fish	0	1	2	3	4	5	$\geq 6$
Probability	$k$	0.46	0.15	0.06	0.03	0.01	0

- a Find the proportion of people who leave without buying any fish. [1]
- b Calculate the expected number of fish types a person will buy. [2]
- c Three people leaving the fish market are selected at random. Find the probability that at least one has bought more than one type of fish. [2]

11 [Maximum mark: 9]

Orienteers Morris and Eleanor leave point P at the same time. Morris runs 3.1 km on the bearing 148°, then a further 2.2 km on the bearing 214°, to point F. Eleanor runs directly to point F.

- a Find the distance Eleanor runs. [3]
- b Find the bearing on which Eleanor runs. [3]
- c Morris averages 8.4 km h<sup>-1</sup> whereas Eleanor averages 6.9 km h<sup>-1</sup>. Who arrives first and by how much? [3]

12 [Maximum mark: 8]

Boxes of chocolate frogs are each sold with a collectible card featuring a famous professor. It is claimed that the cards are distributed as follows:

Professor	D	M	T	F	S
Percentage	60%	20%	10%	7%	3%

Eager student H buys a crate containing 150 boxes. The following cards are found:

Professor	D	M	T	F	S
Observed frequency	85	37	12	12	4

To investigate whether his sample is consistent with the claim, student H conducts a  $\chi^2$  goodness of fit test at a 5% significance level.

- a Write down the null hypothesis. [1]



- b** Copy and complete the following table of expected frequencies: [2]

Professor	D	M	T	F	S
Expected frequency					

- c** Write down the number of degrees of freedom. [1]  
**d** Find the  $p$ -value for the test. [2]  
**e** State the conclusion of the test, giving a reason for your answer. [2]

**13 [Maximum mark: 4]**

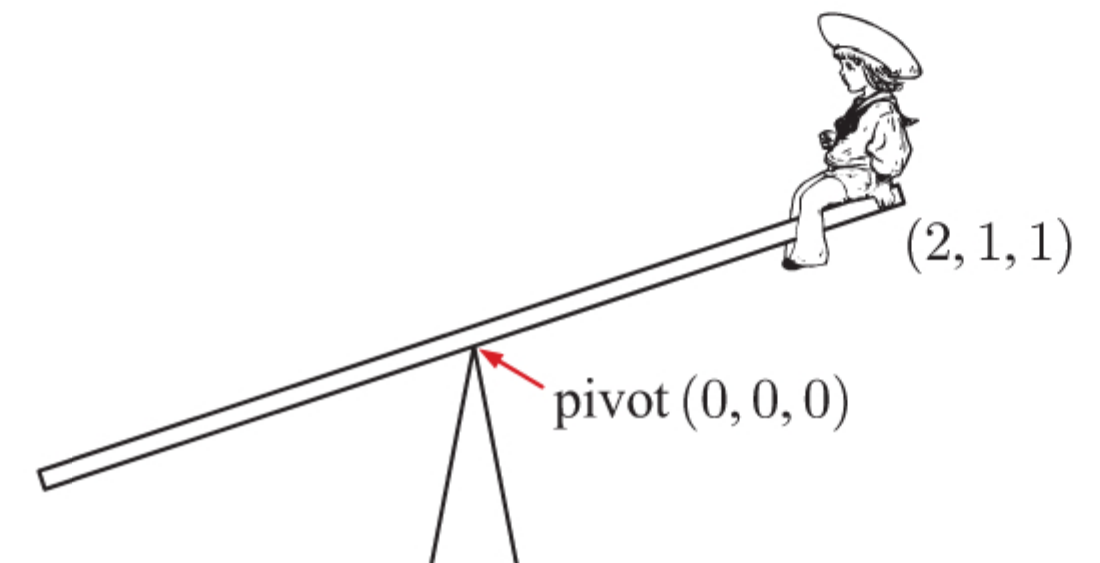
When a linear force  $\mathbf{F}$  is applied at position  $\mathbf{r}$  relative to the pivot of a lever, the resulting *torque* or “turning force” is  $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$  Nm.

Suppose the pivot of a seesaw is the origin.

A child is sitting on one end at  $(2, 1, 1)$ .

Her weight due to gravity is 250 N downwards, so  $\mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ -250 \end{pmatrix}$ .

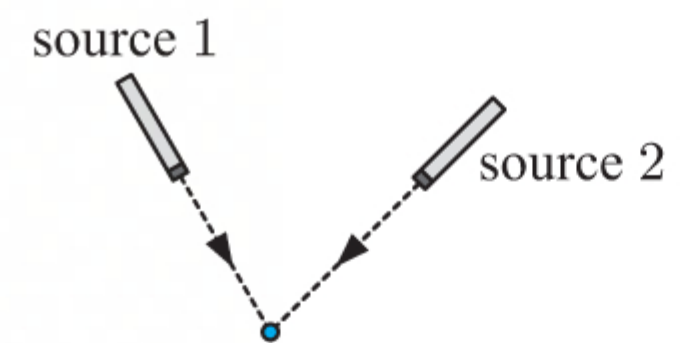
Find the torque on the seesaw due to the child’s weight.



**14 [Maximum mark: 6]**

Two electromagnetic pulses intersect at a point. They have the same frequency, but not the same phase. The strength of the pulse after  $t$  microseconds is given by

$$S(t) = S_1(t) + S_2(t) \\ = \sqrt{3} \cos 2t + \cos(2t + \frac{\pi}{2})$$

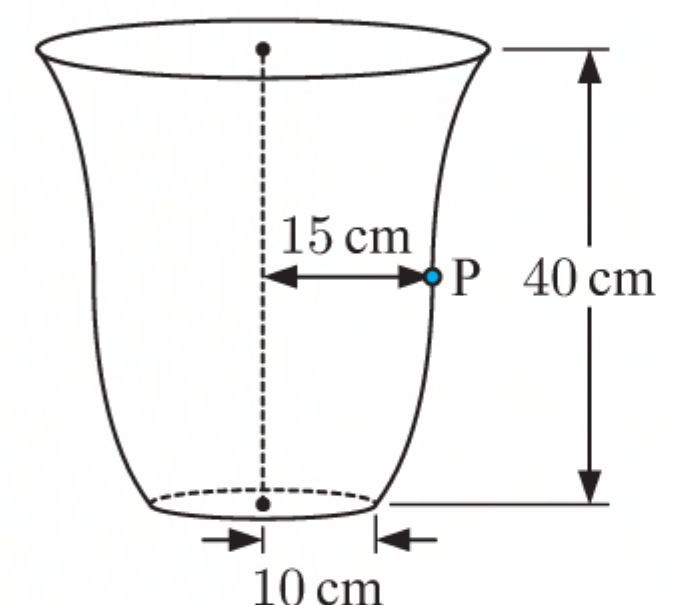


- a** By considering  $\Re(\sqrt{3}e^{2ti} + e^{(2t+\frac{\pi}{2})i})$ , write  $S(t)$  in the form  $A \cos(2t + B)$ . [4]  
**b** Hence find the maximum sum of the pulses. [2]

**15 [Maximum mark: 7]**

Cedric is designing a garden pot with the dimensions shown.

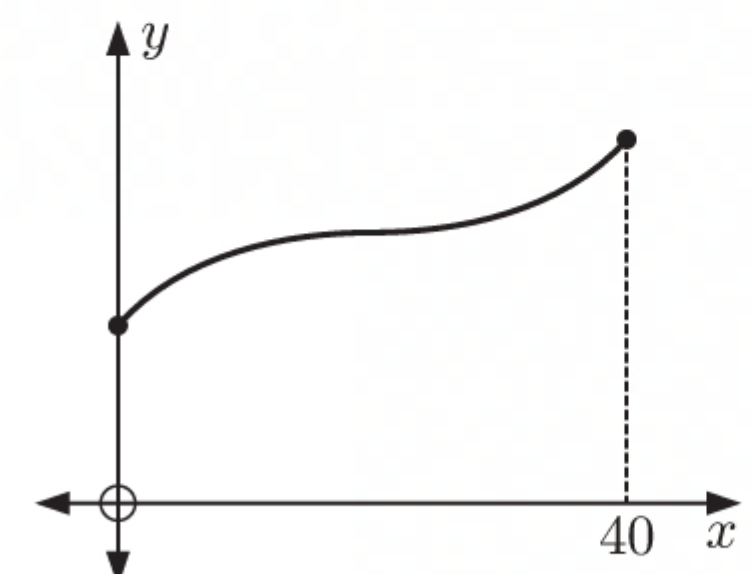
At the point P halfway up the side of the pot, the surface is vertical.



The curved surface of the pot is obtained by revolving

$$y = a(x - b)^3 + c, \quad 0 \leq x \leq 40$$

through  $2\pi$  about the  $x$ -axis.



- a** Find the constants  $a$ ,  $b$ , and  $c$ . [4]  
**b** Write an integral expression for the capacity of the pot. [1]  
**c** Evaluate the integral using technology, and write the capacity of the pot in litres. [2]

**16 [Maximum mark: 7]**

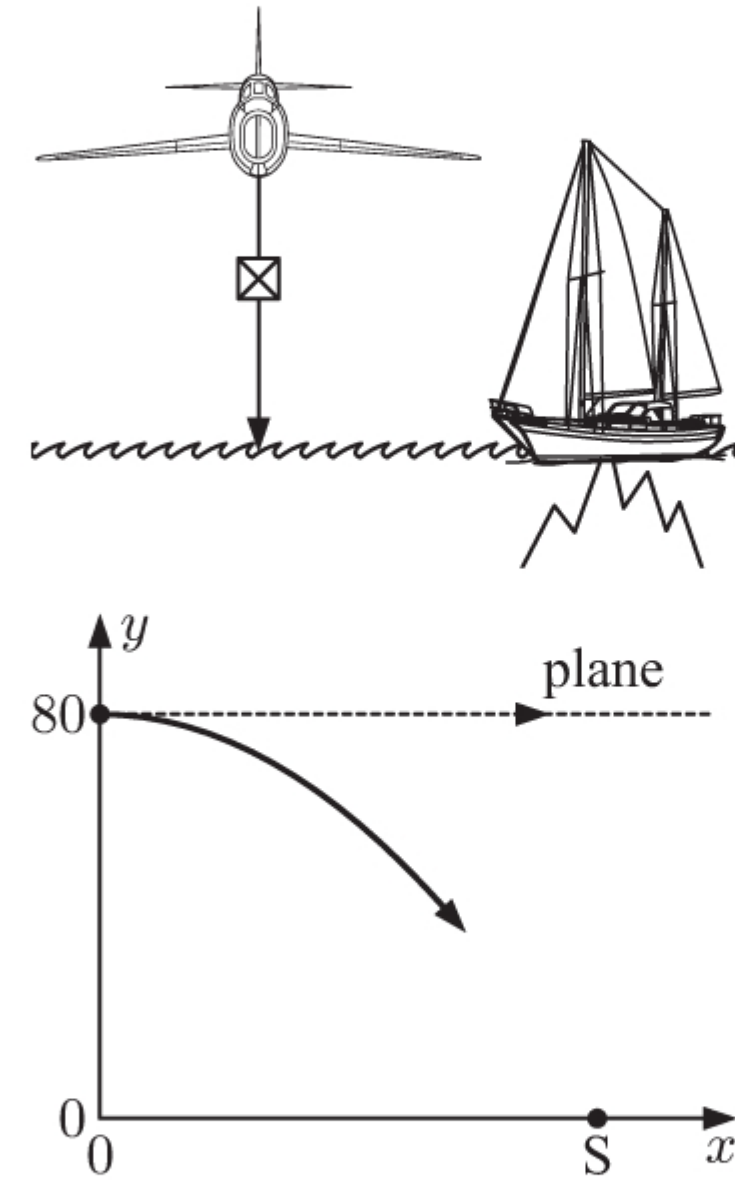
A courier service offers a fixed delivery fee of €10 for parcels up to 4 kg. For heavier parcels, customers pay an extra €1 per additional kg of weight.

- a** Let  $P(x)$  be the price of delivering an  $x$  kg parcel.  
**i** Sketch  $P(x)$ . [2]  
**ii** Write a piecewise formula for  $P(x)$ . [2]  
**b** Let  $C(x)$  be the cost per kilogram of delivering an  $x$  kg parcel. Sketch the graph of  $C(x)$ . [3]



**17 [Maximum mark: 10]**

A ship is stuck on a reef and is breaking apart. The pilot of a small plane will fly in a line alongside the ship, so she can drop emergency supplies as near to the sailors as possible but without endangering them.



The pilot flies at  $40 \text{ m s}^{-1}$  at the constant altitude 80 m above the water. The supplies are released at  $t = 0$ , at which time the plane is at  $(0, 80)$ . The velocity vector of the supplies is  $\begin{pmatrix} 40 \\ -gt \end{pmatrix} \text{ m s}^{-1}$ , where  $g \approx 9.8 \text{ m s}^{-2}$  is the acceleration due to gravity.

- Find the time taken for the supplies to reach the water. [4]
- Find the speed and angle at which the supplies will hit the water. [4]
- Horizontally, how far should the pilot be from the ship when she releases the supplies? [2]

**PAPER 2****CALCULATOR, 120 MINUTES****1 [Maximum mark: 11]**

At age 30, Corina borrows some money to purchase a new boat. She is offered a 5 year loan at an interest rate of 11.8% p.a., compounded quarterly. Her quarterly loan repayments are \$869.77.

- How much money did Corina borrow? [3]
- Find the interest paid on the loan. [2]

At age 60, Corina retires with \$865 400 in a savings fund. She finds an annuity fund for her money which returns 5.2% p.a. compounded monthly.

- Corina wants the money to last until she is 90 years old. How much can she afford to withdraw each month? [2]
- At age 66, Corina realises that with rising medical and utility costs, she will need to withdraw more money each month. She decides to withdraw \$5400.
  - How much money is in the account at the time Corina makes this decision? [2]
  - What age will Corina be when her annuity runs out? [2]

**2 [Maximum mark: 15]**

A chain of department stores surveys all its staff to better understand their external care responsibilities. They found that 46.8% of their staff were responsible for children under the age of 18, and 11.5% were carers for their parents or other adults. 43.4% of their staff had no external care responsibilities.

- Construct a Venn diagram to display this information, letting  $C$  be the event that a staff member is responsible for a child under the age of 18, and  $P$  be the event that a staff member is a carer for a parent or other adult. [4]
- Find the probability that a randomly selected staff member is responsible for a child under the age of 18 *and* is a carer for a parent or other adult. [1]
- A randomly chosen staff member is not responsible for a child under the age of 18. Find the probability that they do not have a parent or other adult in their care. [2]



- d** A sample of 500 staff is randomly selected to answer a more extensive questionnaire. Before they begin, the Human Resources manager wants to be confident that the sample will be a good representation of the population by conducting a  $\chi^2$  goodness of fit test with a 5% level of significance.

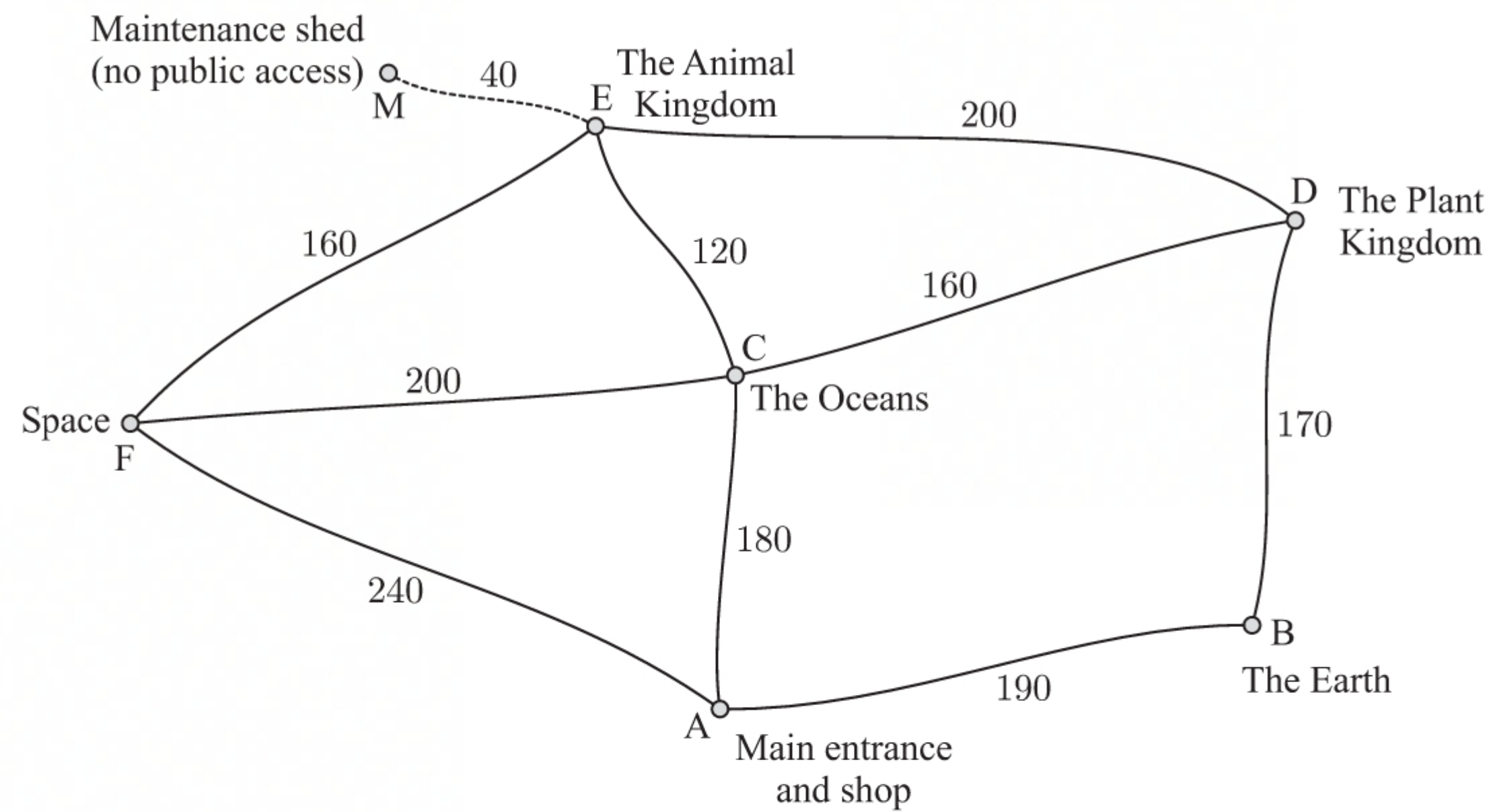
The table alongside summarises the people selected in the sample:

Group	Frequency
$C \cap P$	9
$C \cap P'$	236
$C' \cap P$	
$C' \cap P'$	205
Total	500

- i** How many people in the sample are not responsible for a child under the age of 18 but *are* carers for a parent or other adult? [1]
- ii** Let  $p_1 = P(C \cap P)$ ,  $p_2 = P(C \cap P')$ ,  $p_3 = P(C' \cap P)$ ,  $p_4 = P(C' \cap P')$ . [1]
- The null hypothesis for this test is
- $H_0: p_1 = 0.017, p_2 = 0.451, p_3 = 0.098, p_4 = 0.434$ .
- Write down the alternative hypothesis.
- iii** Construct a table of expected frequencies for each group using the proportions in the whole population. [2]
- iv** Calculate the test statistic  $\chi^2_{\text{calc}}$  for the test. [2]
- v** Given that the critical value for this test is  $\chi^2_{\text{crit}} \approx 7.81$ , write down the conclusion of the test. [2]

**3 [Maximum mark: 17]**

A new theme park is based on the science of the natural world. Its map is given below, including the distances between attractions in metres.



- a** Suppose visitors to the park are equally likely to choose any available road from whichever attraction they are at.
- i** Construct a transition matrix **T** for the probabilities. [3]
- ii** If a visitor starts at the main entrance A, where are they most likely to be after travelling three roads? Explain your answer. [2]
- iii** Give a reason why the supposition is not necessarily appropriate when answering **ii**. [1]
- b** Suppose instead that visitors are equally likely to choose any road from A, then any *other* available road next. Find the expected distance travelled along the first two roads. [4]
- c** A new shuttle is to provide less mobile visitors with access to each of the attractions B to F, starting and finishing at A. [2]
- Find a minimum weight cycle which the shuttle could follow, and find the weight of this cycle.
- d** Every night, maintenance staff drive a street cleaner around the park. They begin and end at their Maintenance shed, and must drive down every road.
- i** Explain why some roads will need to be driven down more than once. [1]



- ii

Find the shortest distance the street cleaner needs to travel.

[2]
- iii

Provide an example route of minimum distance for the street cleaner.

[2]

4 [Maximum mark: 16]

The mass of cherries in a harvest is normally distributed with mean 5.38 g and standard deviation 0.62 g.  
Let  $X$  be the mass of a randomly selected cherry.

- a

Find the value of  $k$  such that  $P(X < k) = 0.4$ . Explain what this value means.

[3]
- b

Christina takes a sample of 50 cherries.

i

Find the probability that a randomly selected cherry has mass greater than 6 g.

[1]

ii

How many cherries would Christina expect to have mass greater than 6 g?

[2]

iii

Find the probability that at least 4 cherries will have mass greater than 6 g.

[2]
- c

Pedro takes a sample of 10 cherries and analyses their sugar content. He records the following results:

Mass (g)	5.17	5.84	6.01	5.74	4.88	5.41	5.62	4.78	4.89	5.20
Sugar content (g)	0.66	0.83	0.92	0.75	0.59	0.71	0.75	0.60	0.58	0.65

- i

Sketch the scatter diagram for the data.

[2]
- ii

Explain why Spearman’s rank correlation coefficient might be appropriate for this data.

[1]
- iii

Find the ranks for each variable.

[2]
- iv

Calculate Spearman’s rank correlation coefficient  $r_s$ .

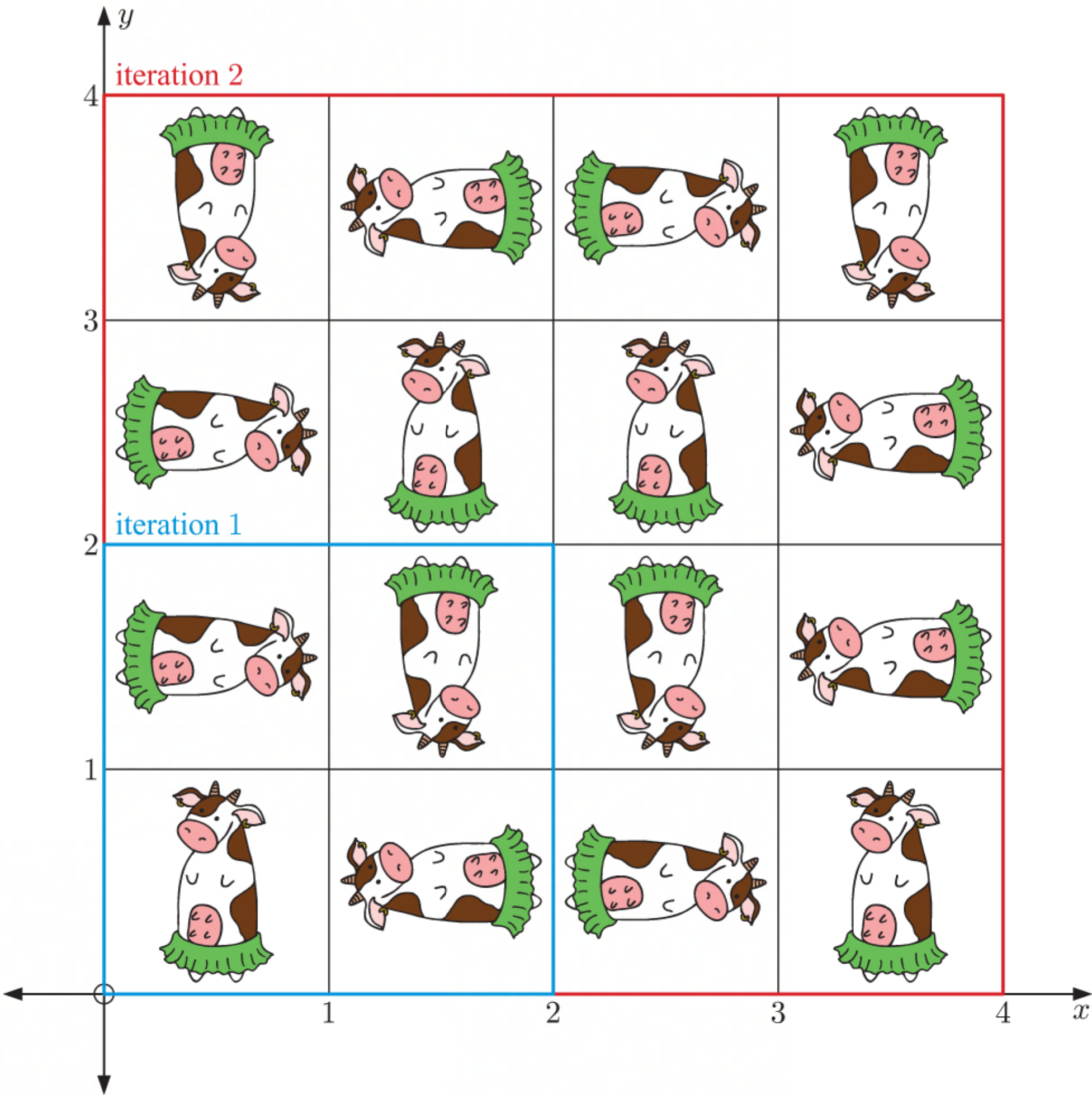
[2]
- v

Hence describe the correlation between the variables.

[1]

5 [Maximum mark: 13]

Carol is knitting a square blanket for her son Stephen.  
The blanket will be made up of squares containing Stephen’s favourite character, Betty the Cow.  
Carol uses an iterative procedure to construct the blanket. The first two iterations are illustrated below.  
The origin is the bottom left corner of the blanket, as shown.



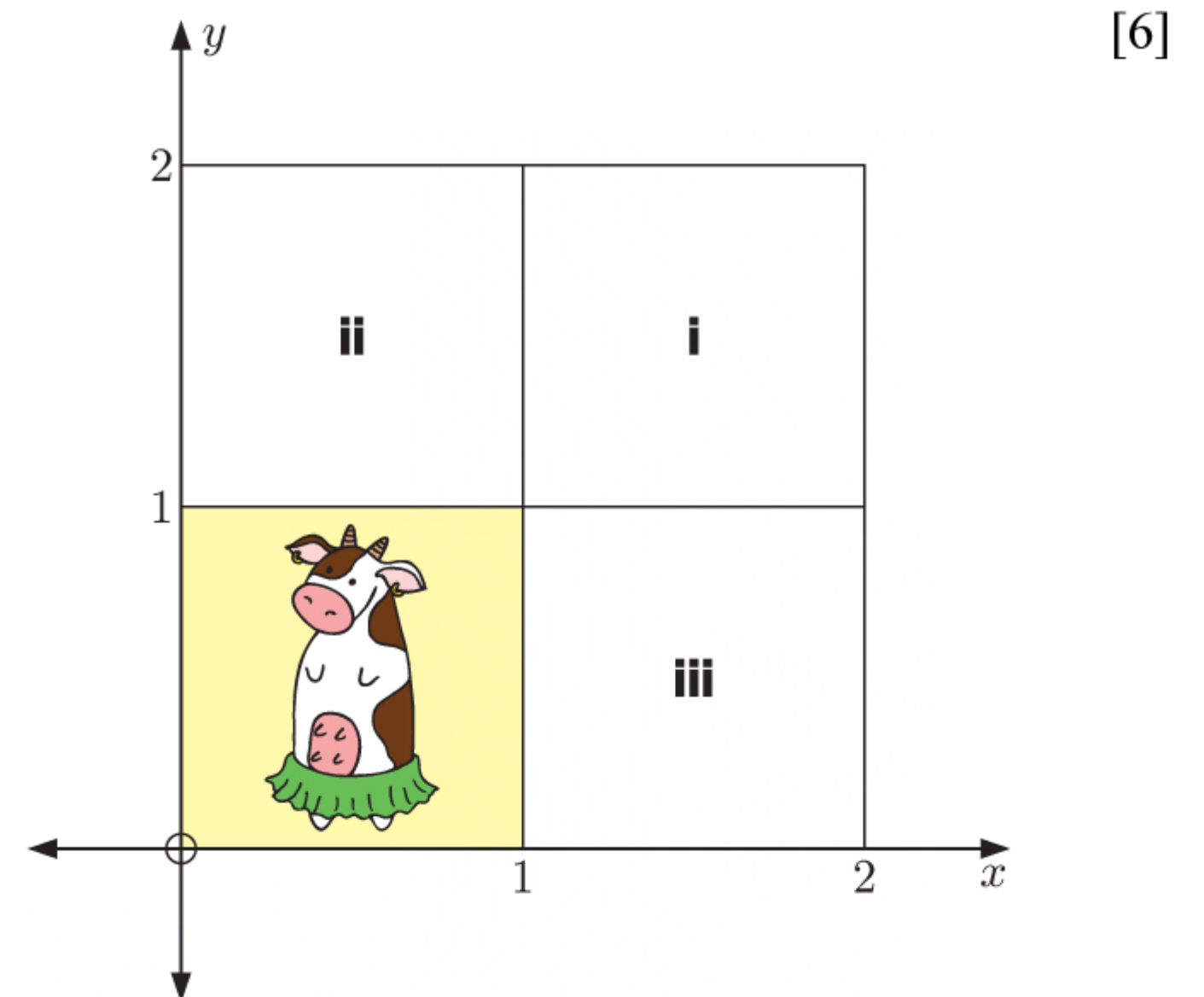
- a

How many pictures of Betty will there be after  $n$  iterations?

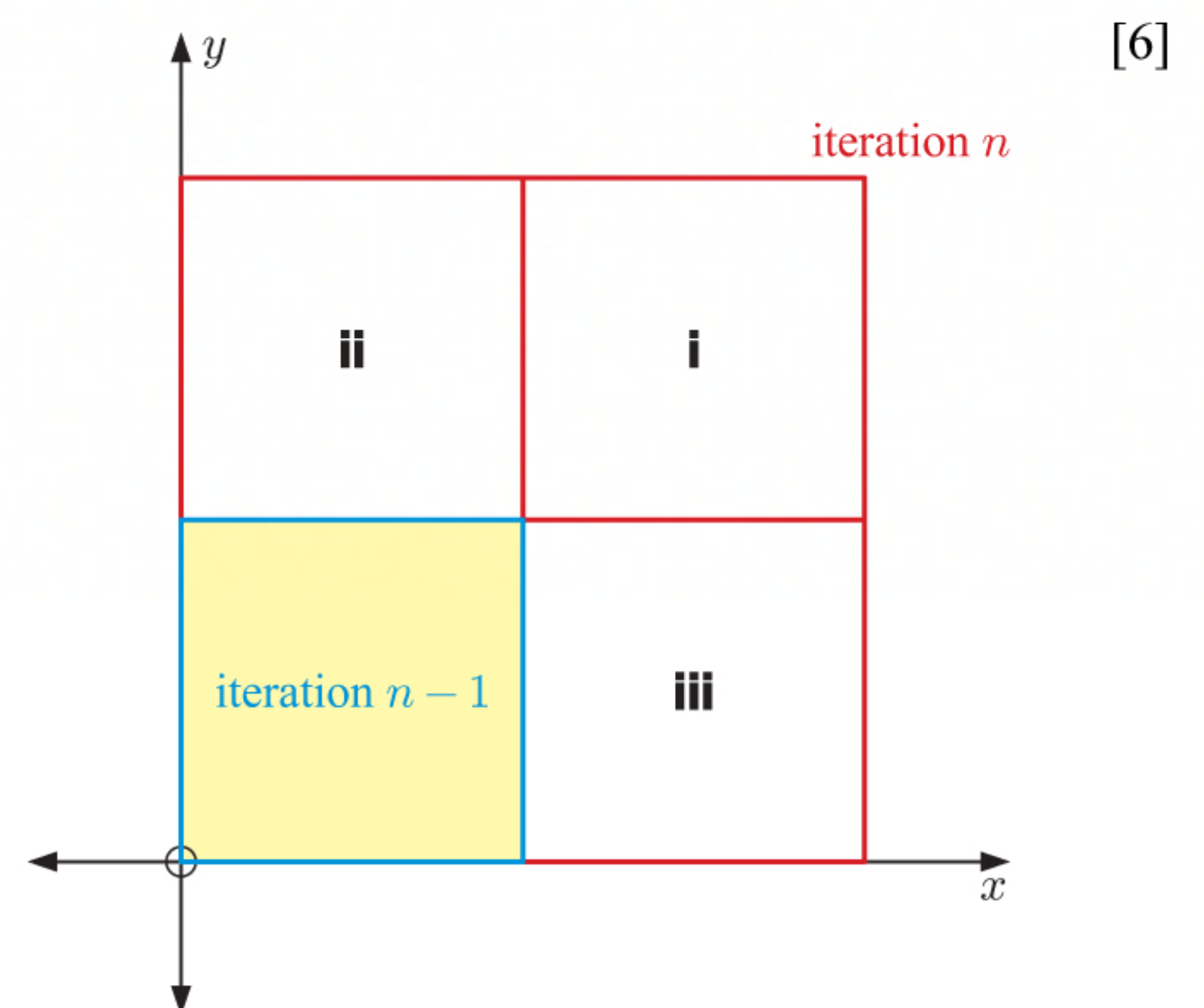
[1]



- b** The first iteration begins with the Betty in the bottom left corner of the blanket.
- Describe a sequence of transformations which maps the shaded Betty onto the Betty in squares **i**, **ii**, and **iii**.
- Write the transformation equation, in the form  $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ , for each sequence of transformations.



- c** Hence or otherwise, write down the transformation equation which maps iteration  $n - 1$  onto squares **i**, **ii**, and **iii** of iteration  $n$ .



**6 [Maximum mark: 20]**

An RLC circuit contains a resistor, an inductor, and a capacitor.

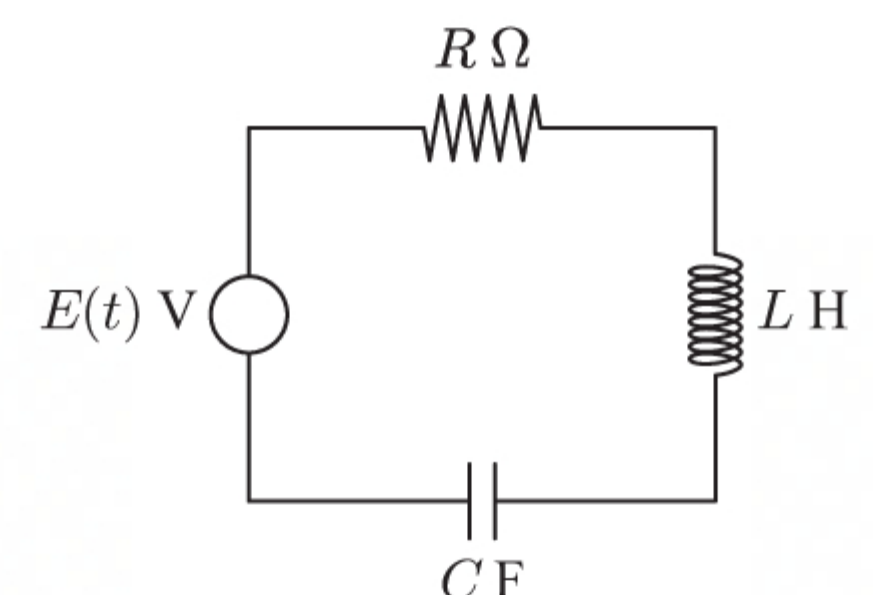
Suppose the resistance is  $R$  ohms ( $\Omega$ ), the inductance is  $L$  henrys (H), and the capacitance is  $C$  farads (F).

Let  $E(t)$  be the electrical potential driving the circuit, in volts (V).

At time  $t$ , the charge  $q$  coulombs on the capacitor is determined by the differential equation

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = E(t).$$

In a particular RLC circuit,  $L = 1$  H,  $R = 10 \Omega$ , and  $C = \frac{1}{41}$  F.



- a** The current in the circuit is  $I = \frac{dq}{dt}$  amps. [2]
- Write the differential equation as a system of coupled linear differential equations involving  $I$  and  $q$ .
- b** Suppose the power supply is shut off, so  $E(t) = 0$  V. At  $t = 0$ , the current in the system is  $I = 1$  Ampere and there is no charge.
- i** Write the system in the matrix form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  where  $\mathbf{x} = \begin{pmatrix} q \\ I \end{pmatrix}$ . [1]
- ii** Find  $\dot{\mathbf{x}}$  when  $t = 0$ . [2]
- iii** Find the eigenvalues of  $\mathbf{A}$ . [3]
- iv** Hence *sketch* the phase portrait of the system. [2]
- v** Describe the behaviour of the system in the long term. [2]
- c** Suppose the power supply is constant when  $E(t) = 300$  V, and there is no initial charge or current in the system.
- i** Find the steady state values of  $I$  and  $q$ . [2]



ii Euler's method will be applied with  $h = 0.05$  to generate values of  $t_k, q_k, I_k$  for  $0 \leq t \leq 1$ .

(1) Write down the equations for the  $k$ th iteration of Euler's method for this system. [2]

(2) Write down your estimates for  $q(1)$  and  $I(1)$ . [2]

(3) Sketch the trajectory of the data points  $\{(q_k, I_k)\}$  on a phase portrait. [2]

## 7 [Maximum mark: 18]

A national park was declared in the Andes to try to protect an endangered group of condor. When first established, there were 38 females in the region. After 5 years, this number had increased to 55 females.

A biologist tries to model the population using an exponential model of the form  $P = A \times b^t$  where  $t$  is the time in years after the park is established.

a Find the values of  $A$  and  $b$ . [3]

b Use the biologist's model to estimate the number of females after 10 years. [2]

c After 10 years, the population was actually found to include 68 females. Calculate the percentage error in the estimate from the biologist's model. [2]

d Explain why a logistic model is more appropriate for modelling the population. [1]

e Consider a logistic model of the form  $P = \frac{L}{1 + ae^{-kt}}$ .

i Show that  $\ln a = \ln\left(\frac{L-P}{P}\right) + kt$ . [2]

ii Use the known data values to explain why  $\ln\left(\frac{L-38}{38}\right) + \ln\left(\frac{L-68}{68}\right) + 10k = 2\ln\left(\frac{L-55}{55}\right) + 10k$ . [3]

iii Write a quadratic equation in terms of  $L$ . [3]

iv Hence predict the long-term number of female condor in the region. [2]

## PAPER 3

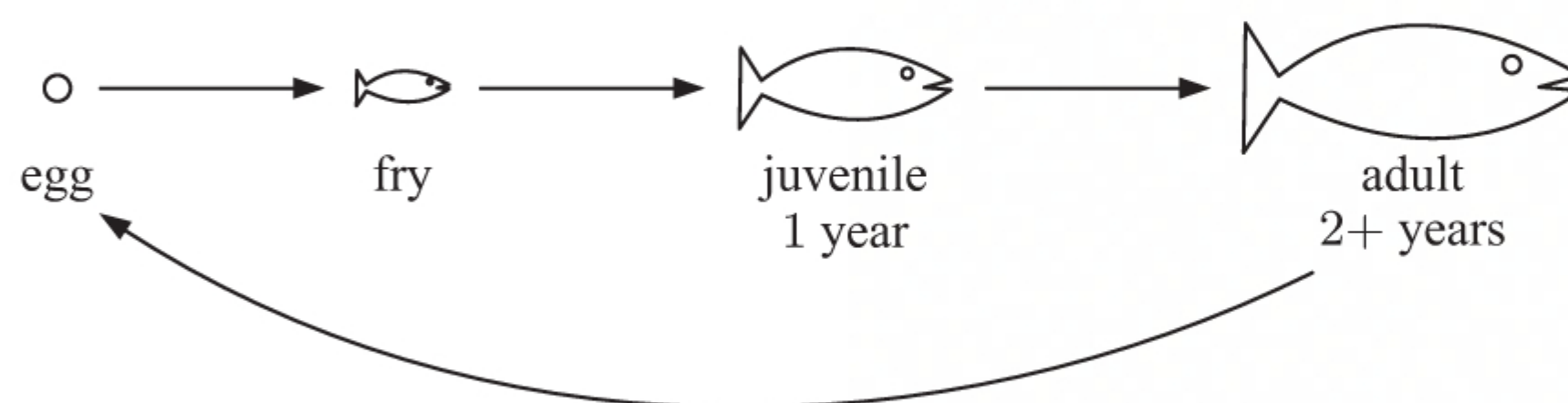
## CALCULATOR, 60 MINUTES

### 1 [Maximum mark: 31]

The salmon family are some of the most commercially fished species on the planet, with more than a billion fish caught each year.

Individual populations of salmon can be modelled because salmon always return to spawn (reproduce) in the same river that they were spawned in.

When a female salmon spawns, she will lay on average 2000 eggs, half of which will contain a female *fry*.



A proportion  $\alpha$  of the fry survive to become juvenile salmon. In an average year,  $\alpha \approx 0.02$ . Each year, a proportion  $\beta$  of the juvenile salmon survive to adulthood, and the same proportion  $\beta$  of the adult salmon survive to the point of reproduction. This means they avoid being fished, eaten by predators, or dying from other causes. Under present fishing conditions,  $\beta \approx 0.03$ .

A population of female salmon can be described by the state matrix  $\mathbf{s}_n = \begin{pmatrix} J_n \\ A_n \end{pmatrix}$  where  $J_n$  and  $A_n$  are the numbers of juvenile and adult female salmon respectively, after  $n$  years.

The population changes each year according to  $\mathbf{s}_n = \mathbf{L}\mathbf{s}_{n-1}$  where  $\mathbf{L} = \begin{pmatrix} 0 & 1000\alpha \\ \beta & \beta \end{pmatrix}$ .

a Explain the non-zero elements of  $\mathbf{L}$ . [2]

b Show that  $\mathbf{s}_n = \mathbf{L}^n \mathbf{s}_0$ . [2]

c At the start of a research project in a particular river, there are 14 000 juvenile female salmon and 600 adult female salmon.

i Write down the initial state matrix  $\mathbf{s}_0$  and the transition matrix  $\mathbf{L}$ . [2]



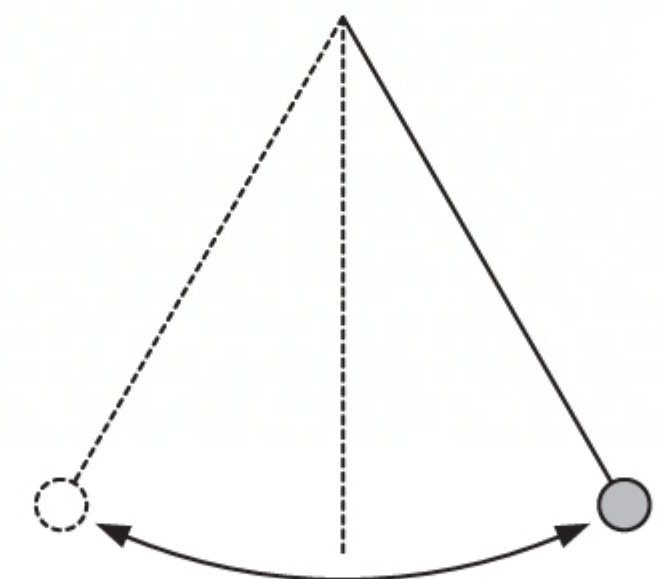
- ii Use matrix multiplication on your calculator to complete this table: [2]

$n$	1	2	3	4	5	6	7	8
$A_n$								

- iii Sketch the scatter diagram of  $\ln A_n$  against  $n$ . [2]
- iv Explain why an exponential model is appropriate. [1]
- v Use regression to find the exponential model connecting  $A_n$  and  $n$ . [2]
- d The eigenvalues of  $\mathbf{L}$  are  $\lambda_1 \approx 0.7897$  and  $\lambda_2 \approx -0.7597$  with corresponding eigenvectors  $\mathbf{v}_1 \approx \begin{pmatrix} 25.32 \\ 1 \end{pmatrix}$  and  $\mathbf{v}_2 \approx \begin{pmatrix} -26.32 \\ 1 \end{pmatrix}$  respectively.
- i Write  $\mathbf{s}_0$  in the form  $a\mathbf{v}_1 + b\mathbf{v}_2$ . [2]
- ii Show that  $\mathbf{s}_n \approx a\lambda_1^n \left( \mathbf{v}_1 + \frac{b}{a} \left( \frac{\lambda_2}{\lambda_1} \right)^n \mathbf{v}_2 \right)$ . [3]
- iii Describe what happens as  $n \rightarrow \infty$ , and how this relates to your exponential model in c v. [3]
- e In order for fishing to be sustainable in the long-term, the global population of salmon must remain stable.
- i State the value of the positive eigenvalue necessary for the population to be stable in the long-term. [1]
- ii Hence find the critical value which  $\beta$  must be increased to, in order that salmon fishing is sustainable. [3]
- f Due to changes in fishing practices,  $\beta$  is increased to 0.1.
- i Calculate the eigenvalues of  $\mathbf{L}$  in this case. [2]
- ii Find the eigenvector corresponding to the *positive* eigenvalue. [3]
- iii What does this eigenvector tell us about the population in the long-term? [1]

**2 [Maximum mark: 24]**

This diagram shows a simple pendulum which, when released from rest, travels through a small angle along its arc of motion. Its velocity along its arc of motion is given by  $v(t) = -32e^{-\frac{t}{24}} \sin 2t \text{ cm s}^{-1}$ .



- a Find the initial acceleration of the pendulum. [3]
- b Show that the displacement of the pendulum at time  $t$  is given by  $s(t) = e^{-\frac{t}{24}}(a \sin 2t + b \cos 2t)$  where  $a \approx \frac{1}{3}$  and  $b \approx 16$ . [7]
- c Find the first two times when the pendulum changes direction. [2]
- d Hence estimate, correct to 4 significant figures, the distance the pendulum travels in:
- i its initial swing to the other extreme [3]
- ii the swing back towards its initial position. [2]
- e i Estimate the percentage loss of distance with each swing of the pendulum. [2]
- ii Hence estimate the total distance the pendulum will travel before it comes to rest. [3]
- f Write an exact expression for the total distance the pendulum will travel. [2]



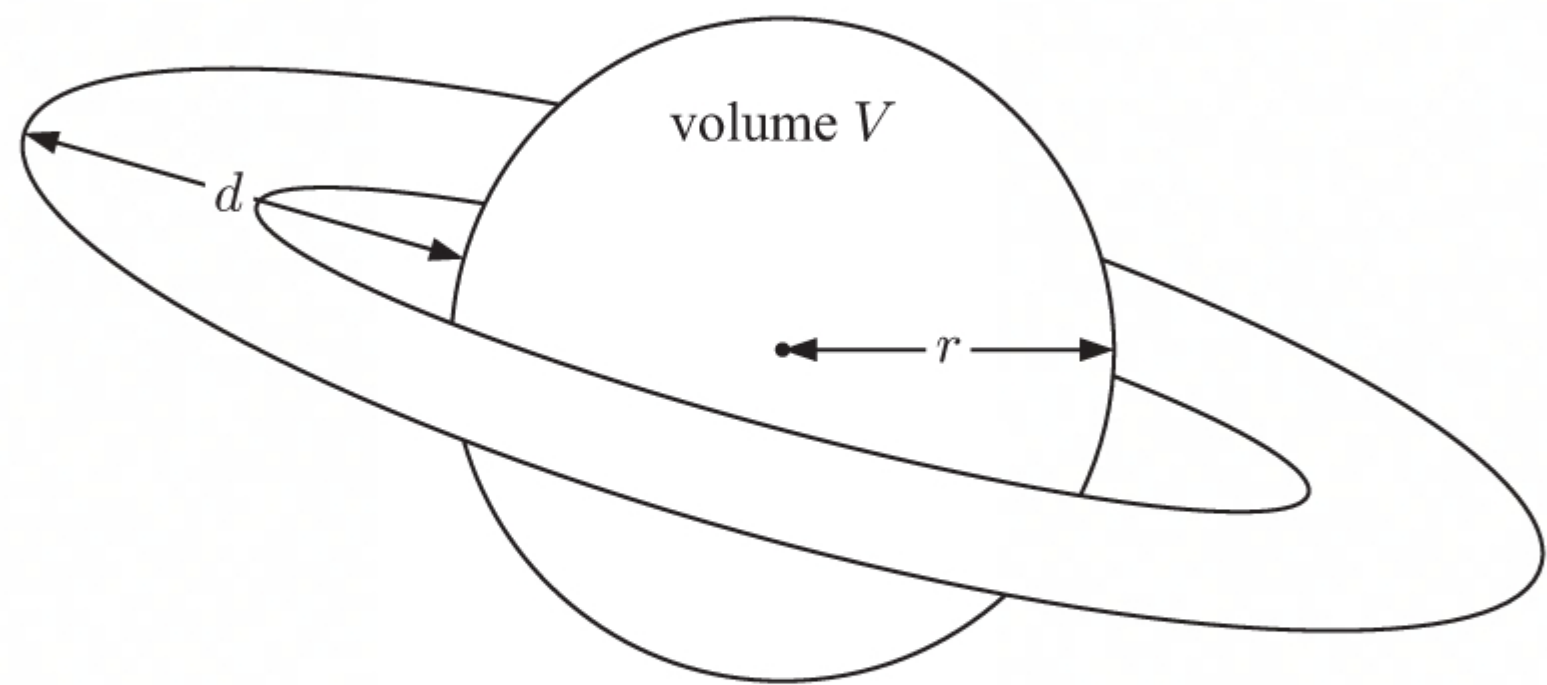
# Trial examination 4

PAPER 1

CALCULATOR, 120 MINUTES

1 [Maximum mark: 5]

The data below is given by NASA for the planet Saturn.



Equatorial radius	$r \approx 5.8232 \times 10^4 \text{ km}$
Volume	$V \approx 8.2713 \times 10^{14} \text{ km}^3$
Ring system	up to $d \approx 2.82 \times 10^5 \text{ km}$ from planet

- a Use the equatorial radius given to estimate the volume of Saturn. [2]
- b Estimate the circumference of the orbit of an ice-covered rock at the far edge of Saturn’s rings. [3]

2 [Maximum mark: 5]

In a board game, a player is required to roll a pair of ordinary dice.

- a A pair of dice is rolled. Find the probability that the sum of the dice is 5. [2]
- b A pair of dice is rolled 10 times. Find the probability that their sum will be 5 at least twice. [3]

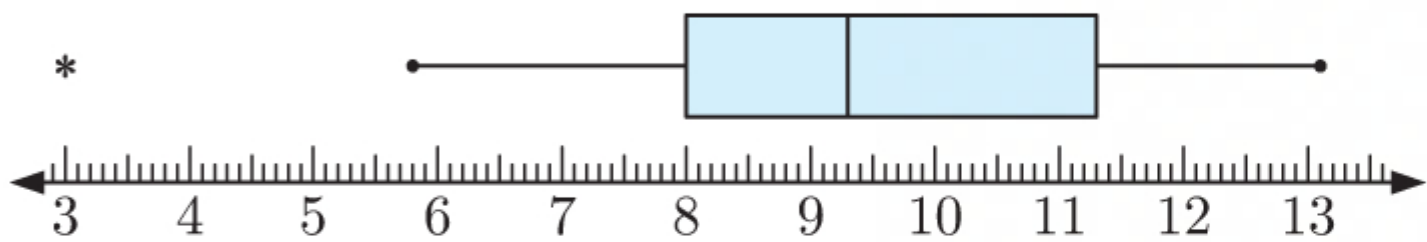
3 [Maximum mark: 9]

A nursery wants to compare the growth of seedlings under different lighting conditions. Two samples A and B are compared after 30 days.

- a The heights (in cm) of the sample A seedlings are:  
6.8 7.8 8.2 8.5 8.7 9.6 9.9 10.0 10.1 10.3 11.3 11.5 12.1 13.2 14.2  

i Find the five-number summary for this data. [3]

ii Assuming there are no outliers, construct a box plot to display the data. [3]
- b The box plot for the sample B seedlings is:



- i Find the interquartile range for this sample. [1]
- ii Compare the two samples to decide which growing conditions are more favourable. [2]



**4 [Maximum mark: 5]**

A group of adults were surveyed regarding their opinions of their government's handling of the COVID-19 pandemic. The results are shown in the table:

<i>Age group</i>	<i>Opinion</i>		
	poor	fair	excellent
18 - 25	14	12	6
26 - 39	19	20	11
40 - 59	8	15	13
60+	12	17	9

Genevieve conducts a  $\chi^2$  test for independence at a 10% level of significance.

- State the null hypothesis. [1]
- Calculate the  $p$ -value for this test. [2]
- State, giving a reason, whether the null hypothesis should be accepted. [2]

**5 [Maximum mark: 6]**

A geologist is studying a radioactive sample he has recently collected. The initial weight of the sample is 18.61 g. When each atom emits its  $\alpha$ -particle, the mass of the atom is reduced by 2.2%.

The mass of the sample is given by  $M(t) = A + B\left(\frac{1}{2}\right)^{\frac{t}{400}}$ ,  $t \geq 0$  days.

- Explain why  $A$  is the final mass of the sample after every atom in the sample has emitted its  $\alpha$ -particle, and find its value. [2]
- Find the value of  $B$ . [2]
- Estimate the mass of the sample after 2 years. [2]

**6 [Maximum mark: 8]**

Sam and Markus are on holiday exploring the coast at Wollongong. Sam has climbed the 25 m high lighthouse, which stands at the top of a cliff. Markus is in a boat some distance offshore. He measures the angle of elevation to the base of the lighthouse is  $6.4^\circ$ , and the angle of elevation to Sam is  $10.3^\circ$ .

- Find the distance from Markus to the base of the cliff below the lighthouse. [3]
- Find the height of the cliff. [1]
- Markus starts paddling towards the lighthouse at  $2 \text{ m s}^{-1}$ . Find the rate at which his distance from Sam is changing when Markus is 100 m from the base of the cliff. [4]

**7 [Maximum mark: 4]**

The size or magnitude of an earthquake is described using a logarithmic scale called the Richter scale.

If an earthquake releases  $E$  joules of energy, then the magnitude  $M$  of the earthquake is given by

$$M = \frac{2}{3} \log_{10} E - 3.6.$$

- Find the magnitude of an earthquake which releases  $7.8 \times 10^{13} \text{ J}$  of energy. [2]
- How much energy is released by an earthquake of magnitude 2.6? [2]

**8 [Maximum mark: 6]**

This table shows the number of tickets issued to people caught using their phone while driving, and the number of car accidents in a particular city each month:

<i>Month</i>	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
<i>Number of tickets (<math>x</math>)</i>	13	16	10	11	20	23	27	28	10	30	30	19
<i>Number of accidents (<math>y</math>)</i>	6	4	7	5	4	1	11	8	3	11	11	7

- Calculate Pearson's correlation for this data and hence describe the relationship between the variables. [2]
- Calculate  $r^2$  and interpret its value. [2]



- c

A researcher wants to test the following hypotheses at a 5% level of significance:

[2]

$$H_0: \rho = 0$$
$$H_1: \rho \neq 0$$

where  $\rho$  is the population correlation coefficient.

Calculate the  $p$ -value and hence state the conclusion of the test.

9 [Maximum mark: 6]

Line  $L_1$  has vector equation  $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, s \in \mathbb{R}.$

Line  $L_2$  has parametric equations  $x = 1 + 2t, y = -1, z = a + t, t \in \mathbb{R}$  where  $a$  is a constant.

The two lines intersect. Find:

- a

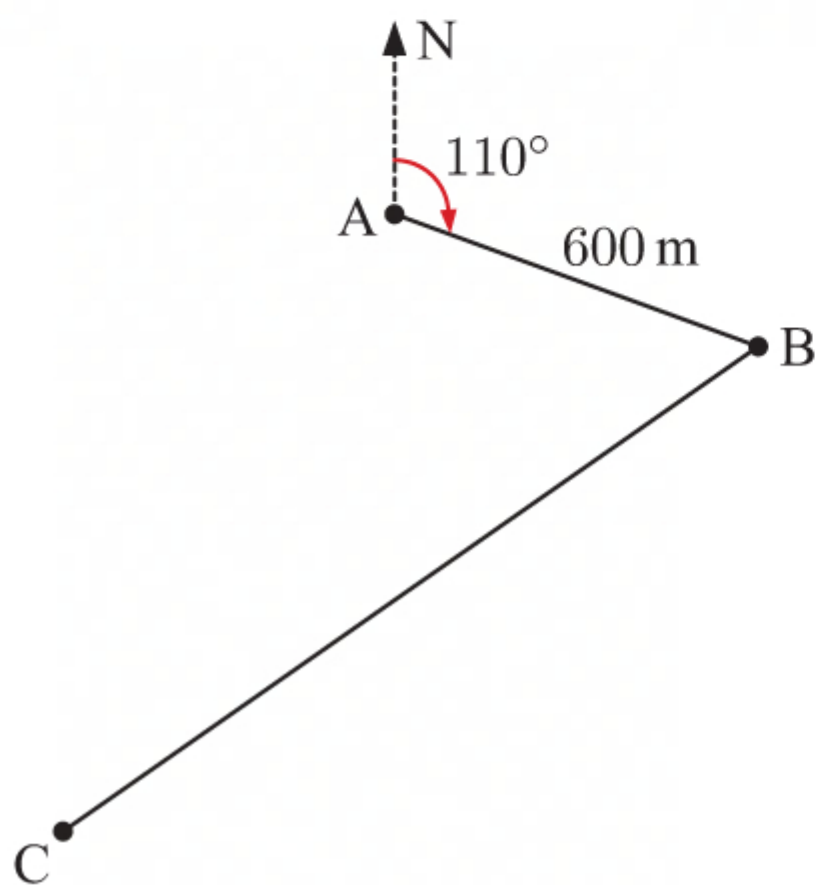
the value of  $a$

[3]
- b

the angle between the lines.

[3]

10 [Maximum mark: 6]



Alan runs at  $3 \text{ m s}^{-1}$  from A to B. At exactly the same time, Belinda starts cycling at  $8 \text{ m s}^{-1}$  on the bearing  $230^\circ$  from B to C.

- a

Find  $\widehat{ABC}$ .

[2]
- b

Find the distance between Alan and Belinda after 2 minutes.

[4]

11 [Maximum mark: 5]

The number of vehicles crossing an intersection follows a Poisson distribution with rate 31 vehicles per hour.

Let  $X$  be the number of cars crossing the intersection in  $2\frac{1}{2}$  hours.

- a

Write down the distribution of  $X$ .

[1]
- b

State the mean and variance of  $X$ .

[2]
- c

Calculate:

i

$P(X \leq 70)$

[1]

ii

$P(X > 70).$

[1]

12 [Maximum mark: 10]

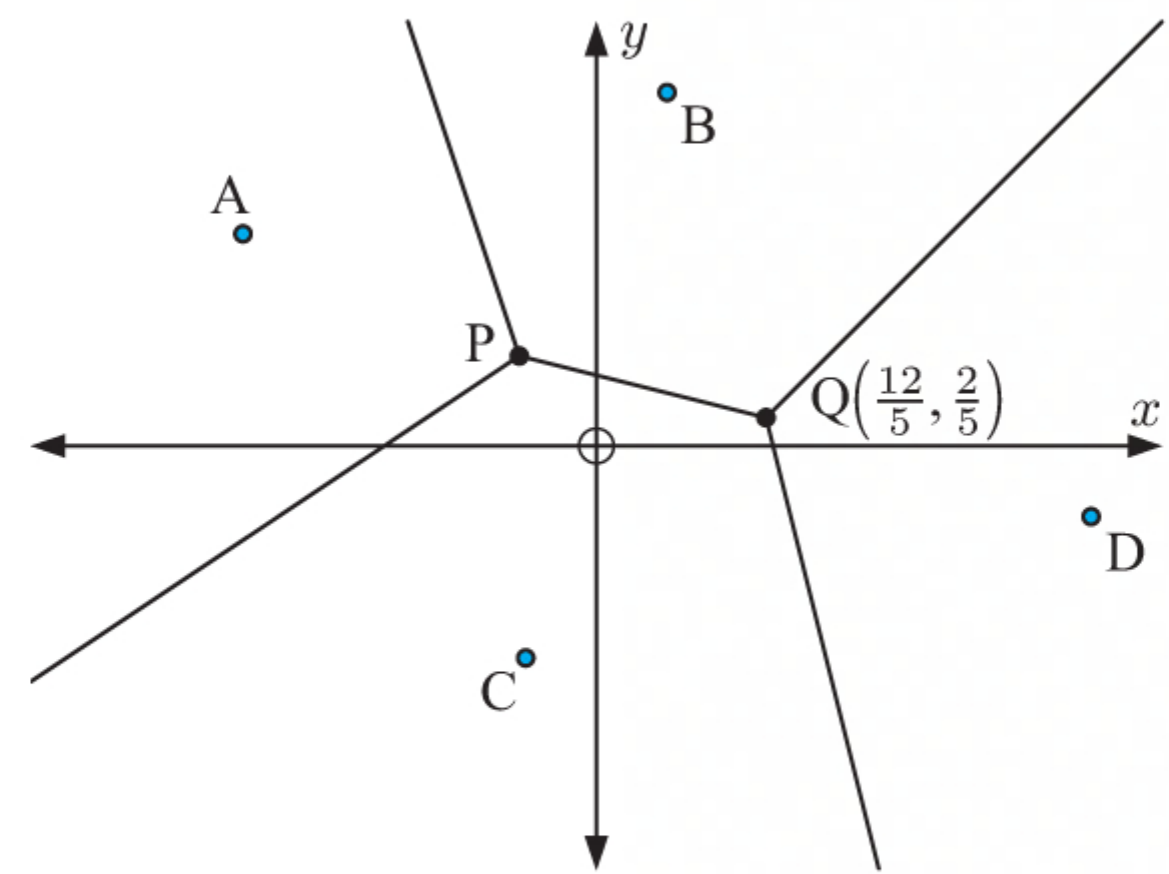
An alpine ski region has four weather stations whose coordinates and altitudes are given in the table below. The grid units are kilometres.

Weather station	Coordinates	Altitude (m)
A	$(-5, 3)$	2635
B	$(1, 5)$	$z$
C	$(-1, k)$	2307
D	$(7, -1)$	2683



The Voronoi diagram alongside has been constructed to help people understand which weather station will be closest to them.

The points P and Q are the intersections of the Voronoi edges.



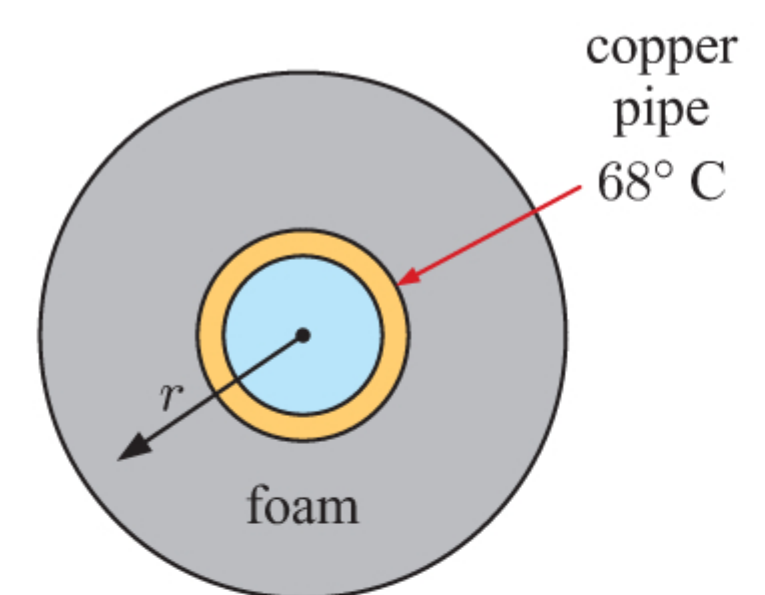
- Find the equation of the perpendicular bisector of line segment AB. [3]
- The equation of the perpendicular bisector of line segment AC is  $y = \frac{2}{3}x + 2$ .
  - Find the value of  $k$ . [2]
  - Find the coordinates of point P. [3]
- Using nearest neighbour interpolation, the altitude of Q is estimated to be 2602. Find the value of  $z$ . [2]

**13 [Maximum mark: 6]**

Hot water leaves a heater in a copper pipe of radius 1 cm. The pipe is the same temperature as the water,  $68^\circ\text{C}$ , so the pipe must be insulated with foam. The safety standards state that the outer surface temperature of the foam should be no more than  $40^\circ\text{C}$ .

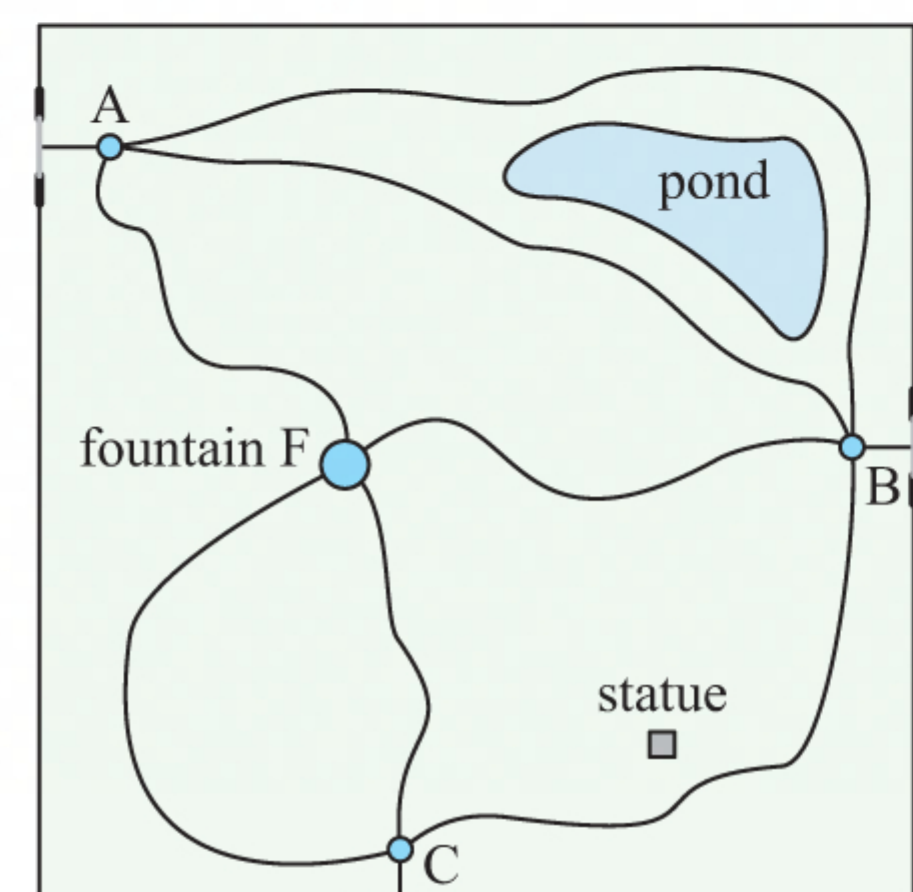
The temperature  $T(r)$  at distance  $r$  cm from the centre of the pipe is given by Fourier's law  $\frac{dT}{dr} = -\frac{28}{r}$ ,  $r \geq 1$ .

Calculate the minimum necessary thickness of the foam.



**14 [Maximum mark: 7]**

The map alongside shows the paths in a city park.



- Construct an adjacency matrix  $\mathbf{A}$  to show the paths between A, B, C, and the fountain F. [2]
- Calculate  $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$  and explain its meaning. [2]
- A runner starts at gate A and runs along 3 randomly selected paths. Find the probability that the runner is now at the fountain. [3]

**15 [Maximum mark: 7]**

A branch drops from a tree and lands in a lake. The branch drifts towards the edge of the lake with velocity  $v = \frac{(5-s)^2}{5} \text{ cm s}^{-1}$ , where  $s$  cm is the displacement of the branch from where it entered the lake.

- Find the velocity of the branch when it first entered the lake. [2]
- Find the acceleration of the branch in terms of  $s$ . [3]
- Find the displacement of the branch when its acceleration is  $-0.64 \text{ cm s}^{-2}$ . [2]



16 [Maximum mark: 9]

A written driving test consists of 15 multiple choice questions with 4 choices each. 10 correct answers are required for an applicant to obtain their learner’s permit.

Maya thinks that her friend, Ricky, will randomly guess answers on the test. She decides to conduct a hypothesis test to test her claim. She will concede that he did not guess the answers if he passes the test.

Let  $p$  be the probability that Ricky answers a given question correctly.

- a The null hypothesis for Maya’s test is  $H_0: p = \frac{1}{4}$ . [1]  
State the alternative hypothesis.
- b Find the significance level of Maya’s test. [3]
- c Ricky answered 11 questions correctly. [2]  
Calculate the  $p$ -value for Maya’s test.
- d Suppose Ricky knew the correct answer to 7 out of the 15 questions. For each of the remaining questions, he was able to eliminate two incorrect choices. He then guessed these answers from the remaining choices. [3]  
Find the probability that Maya was going to make a Type II error in her test.

17 [Maximum mark: 6]

A model train with mass 0.38 kg moves with parametric equations  $x(t) = \frac{1}{2} \cos \frac{t}{10}$ ,  $y(t) = -\frac{1}{2} \sin \frac{t}{10}$  where the distance units are metres, and  $t \geq 0$  seconds.

- a Show that the train moves in a circle. [2]
- b The centripetal force which causes an object with mass  $m$  kg and speed  $v$  m s<sup>-1</sup> to travel in a circle with radius  $r$  m, has magnitude  $F_C = \frac{mv^2}{r}$  N. [4]  
Find the magnitude of the centripetal force acting on the train.

PAPER 2

CALCULATOR, 120 MINUTES

1 [Maximum mark: 15]

The Eiffel Tower in Paris is 324 m high, and has base 125 m × 125 m.



- a A tourist shop sells scale models of the tower which have base 8 cm × 8 cm, and which are encased in a glass pyramid with dimensions 2% bigger than the model. [2]
  - i Find the height of the model, to the nearest mm. [2]
  - ii Find the volume of the glass pyramid which encases the model. [3]
- b The shop owner is keen to maximise the profit he makes from selling the models. [4]  
He knows that the more he orders, the cheaper the models will be for him to buy. However, he will eventually buy more than he can sell, at which point he would make less profit.

His orders from the previous years have generated the profits shown in the table:

The shop owner decides to fit a cubic model of the form

$$P(x) = ax^3 + bx^2 + cx + d$$

for the profit  $P$  when he buys  $x$  thousand models.

Quantity	Profit (€)
4000	32 000
9000	85 500
12 000	115 200

- i Explain why  $d = 0$ . [1]
- ii Construct three linear equations for  $a$ ,  $b$ , and  $c$ . Hence find  $P(x)$ . [4]



- iii Find  $P'(x)$ . [1]
- iv Solve  $P'(x) = 0$ . [2]
- v Hence find, to the nearest hundred, the optimum number of models the owner should buy, and the profit he will make in this case. [2]

## 2 [Maximum mark: 14]

Portia has €10 000 she wants to invest. She is given two options:

A: 5% per annum simple interest paid quarterly.

B: 4.4% per annum interest compounded quarterly.

- a Identify which investment would result in an arithmetic sequence and which would result in a geometric sequence. [2]
- b Write a formula for the value of the simple interest investment after  $n$  quarters. [2]
- c Write a formula for the value of the compound interest investment after  $n$  quarters. [2]
- d Find how long Portia would need to invest her money, for the compound interest investment to be the better option. [3]
- e Portia chooses the compound interest investment, and invests her money for 15 years. At this time, she withdraws the total and places it in an annuity fund which returns 2.8% p.a. compounded monthly.
  - i Find the starting balance for the annuity fund. [2]
  - ii How much can Portia withdraw each month, for the fund to last 5 years? [3]

## 3 [Maximum mark: 15]

James buys bales of wool from different producers, then sells the wool to clothing manufacturers.

James needs to be confident that the bales he sells have a mean pressed weight of more than 160 kg. He therefore tests the weight of every 20th bale that is sent to him.

- a State the type of sampling James is using. [1]
- b During one month, the weights in kg of the bales James tests are:

166.5	158.2	170.7	153.4	165.2	168.6	161.3
155.9	162.3	164.7	159.6	167.1	158.3	162.7

Let  $\mu_J$  be the population mean of James' bales.

- i Find the sample mean and sample standard deviation. [2]
- ii What type of hypothesis test should be used to determine whether James can confidently sell the bales he is receiving? Explain your answer. [2]
- iii The null hypothesis for this test is  $H_0: \mu_J = 160$ . State the alternative hypothesis. [1]
- iv Conduct the appropriate hypothesis test at a 5% level of significance to determine whether James can confidently sell the bales he is receiving. [3]
- c In the same month, James' business partner Susan samples 12 bales she has received. She finds their mean pressed weight is 164.5 kg. James wants to test whether the pressed weights of the bales he is receiving are different from those Susan is receiving.

Let  $\mu_S$  be the population mean of Susan's bales.

- i What type of hypothesis test should James perform? [1]
- ii State one assumption James will need to make in his test. [1]
- iii The null hypothesis for this test is  $H_0: \mu_J = \mu_S$ . State the alternative hypothesis. [1]
- iv Conduct the appropriate hypothesis test at a 5% level of significance to determine whether the mean pressed weights of the bales are different. [3]



**4 [Maximum mark: 17]**

When a skydiver jumps out of a plane, their speed after  $t$  seconds is given by the function  $v(t)$ .

After 1 second, the skydiver is falling at  $9.8 \text{ m s}^{-1}$ .

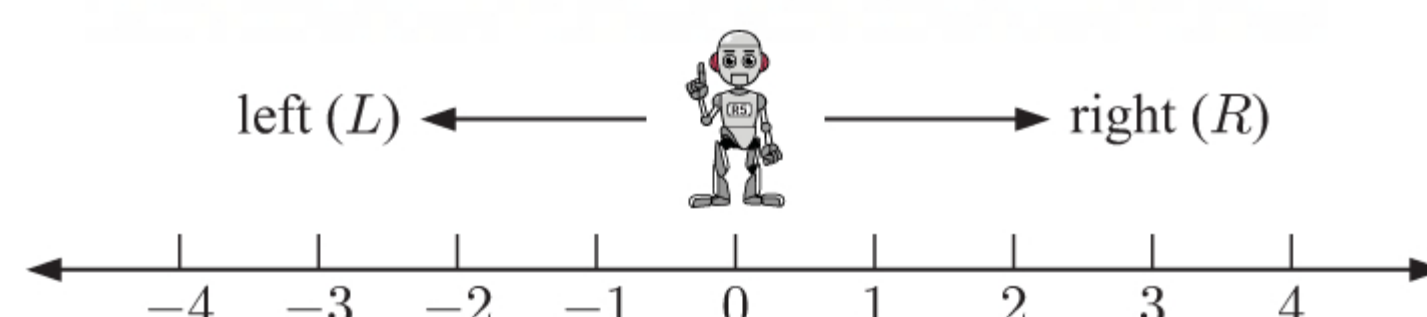
- a** The distance the skydiver has fallen after  $\tau$  seconds is the area between  $v(t)$  and the  $t$ -axis on the interval  $0 \leq t \leq \tau$ . [1]

Write this distance as an integral.

- b** If we ignore air resistance,  $v$  increases in proportion to  $t$ . [3]
- i** Find the proportionality constant, and hence write  $v$  as a function of  $t$ . [3]
- ii** Find, by integration, the distance the skydiver falls in the first 2 seconds. [3]
- c** As the speed of the skydiver increases, air resistance quickly becomes important, and we need to include it in our model. A more accurate model for higher speeds has the form  $V(t) = 53(1 - e^{-kt})$  where  $k \in \mathbb{R}$ . [2]
- i** Find  $V(0)$  and explain why this is consistent with the previous model. [2]
- ii** Find the value of  $k$ , to 3 decimal places. [3]
- iii** Sketch the graph of  $V(t)$ . Discuss what happens to the speed of the skydiver over time. [3]
- iv** Calculate the distance the skydiver falls in the first 5 seconds, giving your answer to the nearest metre. [3]

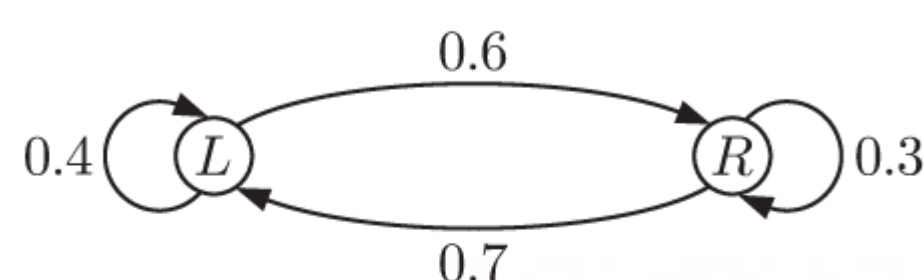
**5 [Maximum mark: 16]**

Ryu5 is a new model Japanese dancing robot. It begins its dance at the origin, and its movement is tracked on a number line.



When Ryu5 first moves, it moves left or right with equal probability.

Following this move, the probabilities for each subsequent move depend on the move which has just taken place, as shown in the transition diagram below.



Suppose Ryu5's movement is modelled using a Markov chain.

- a** List the possible states for the Markov chain. [1]
- b** Write down the: [1]
- i** transition matrix  $\mathbf{T}$  [1]
- ii** initial state matrix  $\mathbf{s}_0$ . [3]
- c** Calculate  $\mathbf{s}_5$  and interpret your answer. [3]
- d** Show that  $\mathbf{T}$  has eigenvalues  $\lambda_1 = 1$  and  $\lambda_2 = -0.3$ . [3]
- e** Describe the relationship between the steady state of the Markov chain and the eigenvectors of  $\mathbf{T}$ . [1]
- f** Hence find the steady state matrix  $\mathbf{s}$  for the system. [5]
- g** In the long term, will Ryu5 be to the left or right of the origin? [1]



**6 [Maximum mark: 16]**

*Allometry* is the study of the relationship between the body size of an organism and its physical attributes and abilities.

In *The Simple Science of Flight*, Hendrik Tennekes investigated the relationship between the *weight* and *cruising velocity* of a number of species of birds.

The following table shows the weights and cruising velocities for 10 species of migratory birds.

Species	Weight ( $W$ kg)	Cruising velocity ( $v$ m s <sup>-1</sup> )
Common tern	0.115	7.8
Black-headed gull	0.23	9
Black skimmer	0.30	9.4
Kittiwake	0.39	10.1
Royal tern	0.47	10.7
Herring gull	0.94	11.7
Great skua	1.35	12.9
Great black-billed gull	1.92	13.6
Black-browed albatross	3.8	16.7
Wandering albatross	8.7	19.2

**a** Draw scatter diagrams of:

**i**  $\ln v$  against  $W$

[2]

**ii**  $v$  against  $\ln W$

[2]

**iii**  $\ln v$  against  $\ln W$ .

[2]

**b** Explain why a power model is most appropriate for the data.

[2]

**c** Find a suitable linear model connecting  $\ln v$  and  $\ln W$ .

[1]

**d** Hence or otherwise, find the power model connecting  $v$  and  $W$ .

[2]

**e** The sooty albatross has mean weight 2.8 kg.

[3]

Use your power model to predict the mean cruising velocity of the sooty albatross. Comment on the reliability of your estimate.

**f** The *wing loading*  $S$  of a bird, in  $\text{N m}^{-2}$ , is related to its *cruising velocity* by the power model  $v \propto \sqrt{S}$ .

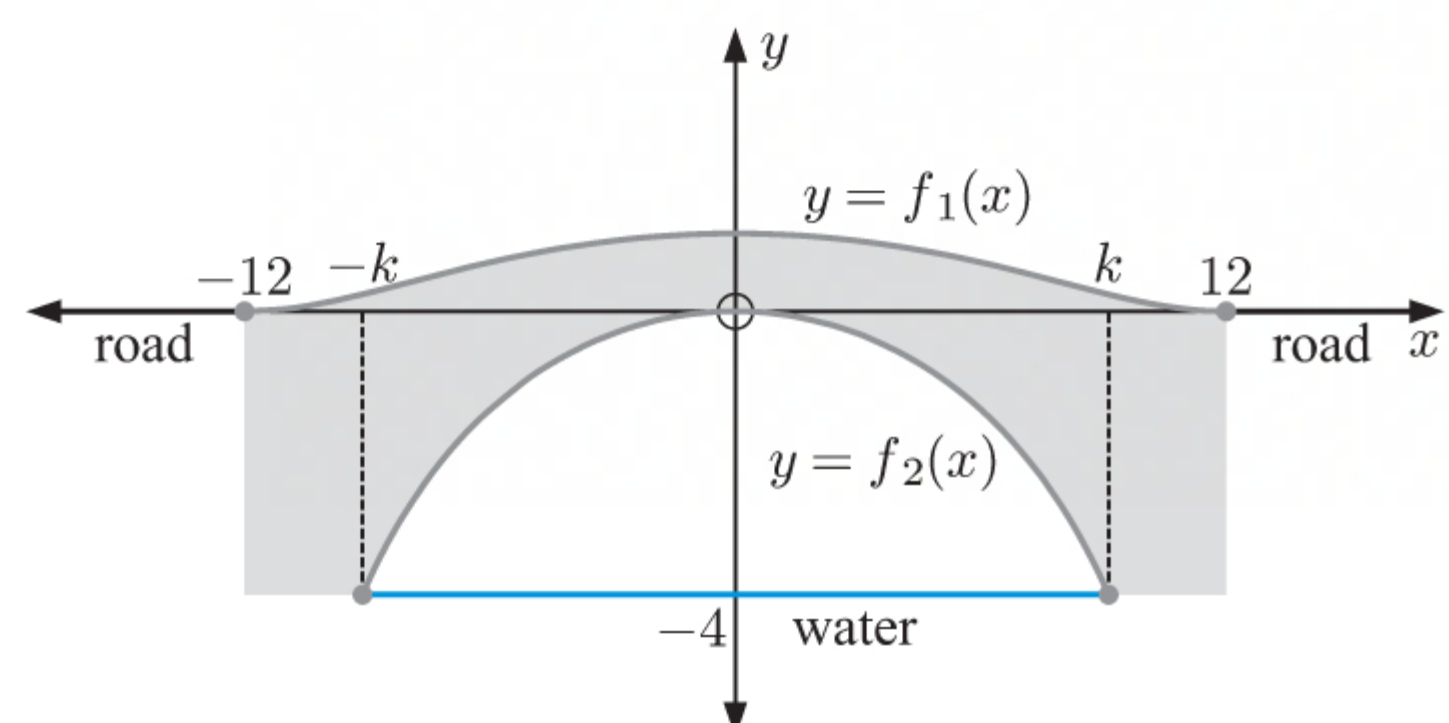
[2]

Find the value of  $n$  such that the relationship between the *wing loading* and *weight* of a bird is  $S \propto W^n$ .

**7 [Maximum mark: 17]**

An arched bridge over a river is shown in the diagram.

$x$  and  $y$  are both in metres.



The defining functions are  $f_1(x) = \ln\left(\cos \frac{\pi x}{12} + 2\right)$ ,  $-12 \leq x \leq 12$

and  $f_2(x) = 2 \ln\left(\cos \frac{\pi x}{12} + 1\right) + a$ ,  $-k \leq x \leq k$ .

**a** Find the value of  $a$ .

[2]

**b** Hence find the value of  $k$ .

[3]

**c** Find exactly the maximum gradient of the road, and when this occurs.

[8]

**d** Find the shaded cross-sectional area of the bridge.

[4]



## PAPER 3

## CALCULATOR, 60 MINUTES

## 1 [Maximum mark: 25]

20 years ago, a large national park was formed to enable native fauna to rejuvenate.

- a** Initially, at time 0 years, there were 500 white tailed deer. With limited predators, its population  $x(t)$  grew logistically according to the differential equation  $\frac{dx}{dt} = 0.15x\left(1 - \frac{x}{80\,000}\right)$ .

- i** Find the values of  $a$ ,  $b$ , and  $k$  such that  $x(t) = \frac{a}{1 + be^{-kt}}$  is a solution to the differential equation with given initial condition. [6]
- ii** Find the population of white tailed deer in the national park now, at time 20 years. [2]
- iii** Find the limiting population which the park could sustain if the present conditions were maintained. [1]

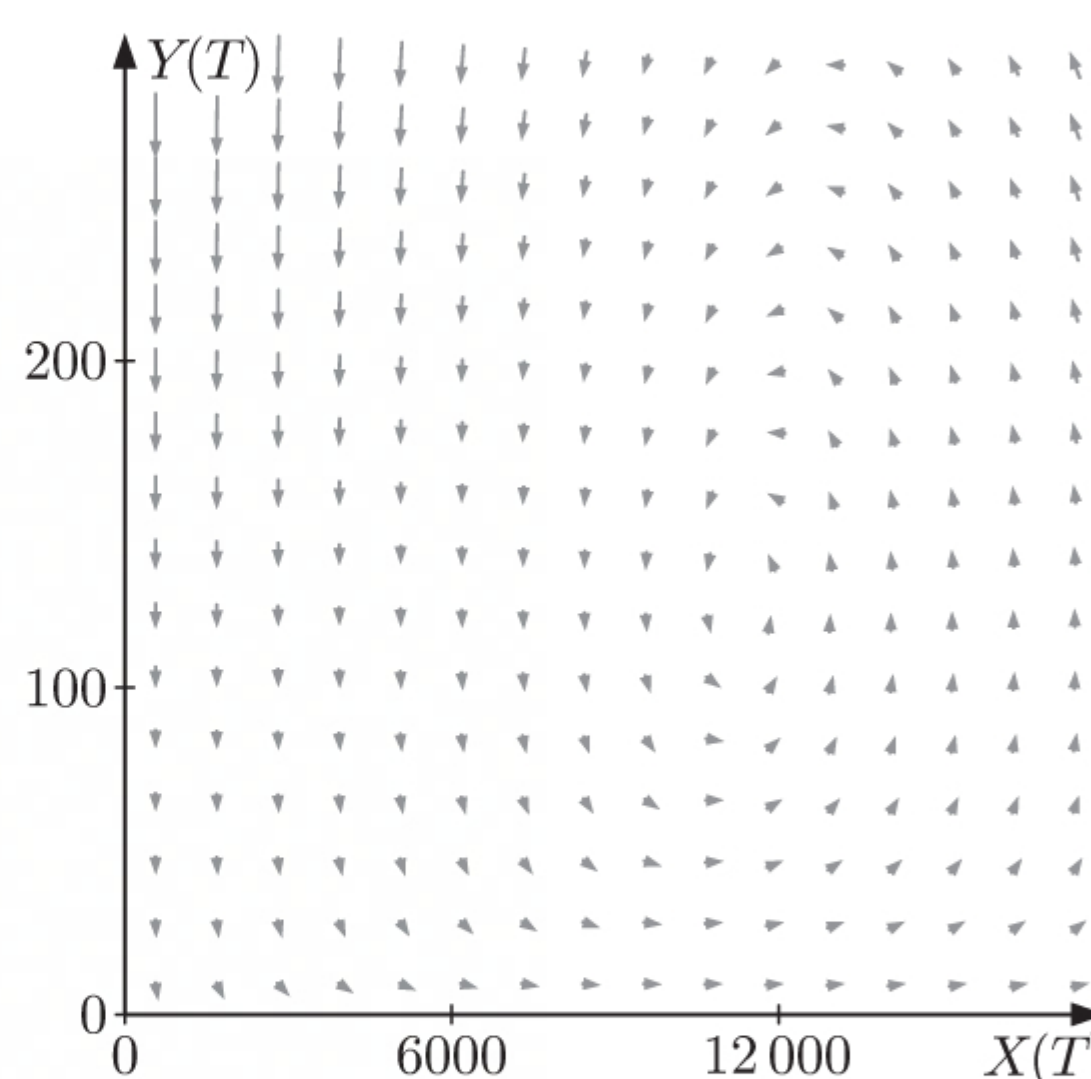
- b** The park rangers believe that the population of deer is now sufficient that a pack of 20 wolves can be introduced to the park.

Suppose that  $T$  years from now, the population of deer is  $X(T)$  and the population of wolves is  $Y(T)$ .

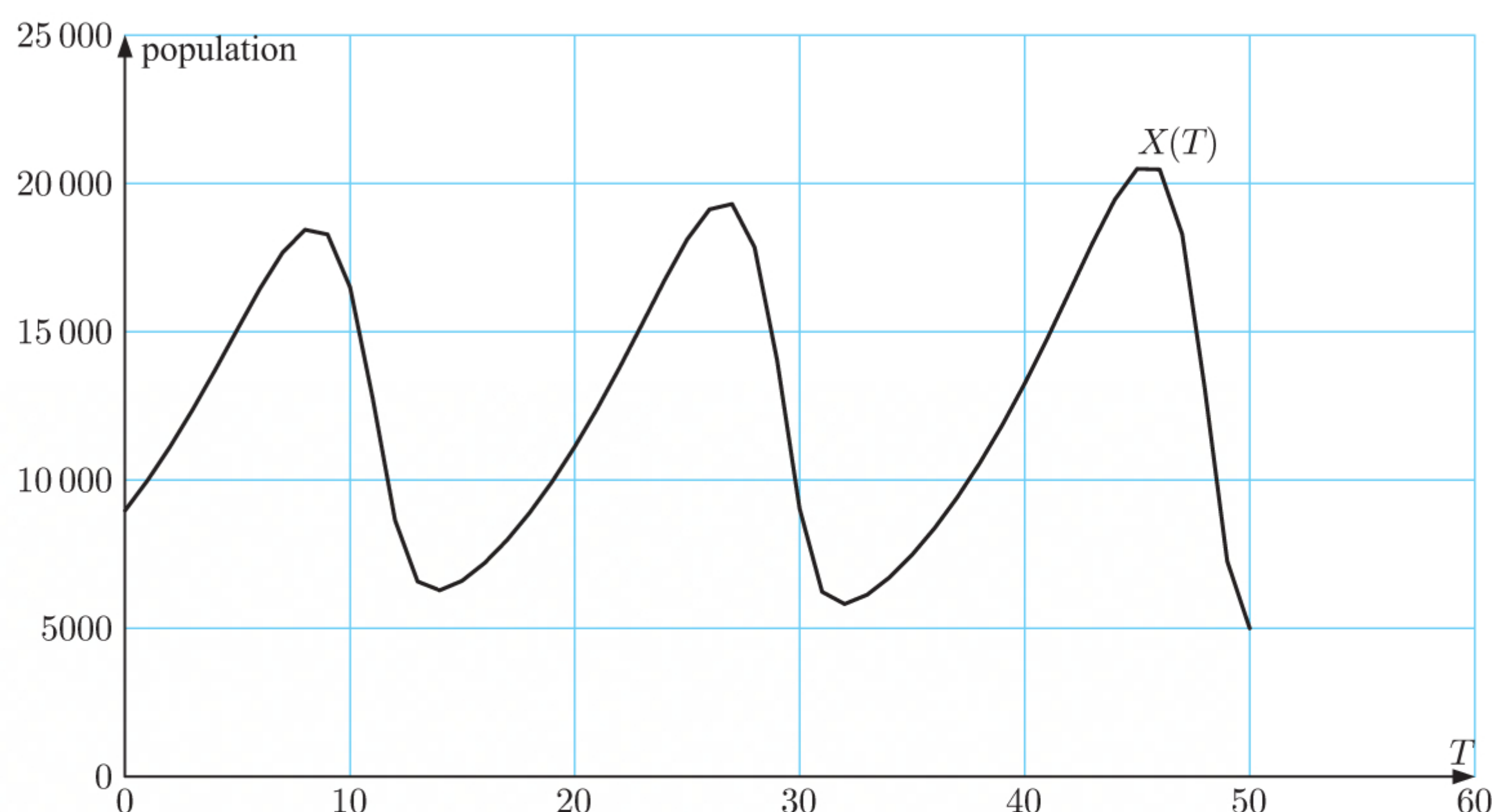
The behaviours of  $X$  and  $Y$  are described by the Kolmogorov predator-prey model:

$$\begin{cases} \frac{dX}{dT} = 0.15X\left(1 - \frac{X}{80\,000} - \frac{Y}{150}\right) \\ \frac{dY}{dT} = Y\left(\frac{X}{10\,000} - 1 - \frac{Y}{1000}\right) \end{cases}$$

The phase portrait for these equations is shown alongside.



- i** Evaluate  $\frac{dX}{dT}$  and  $\frac{dY}{dT}$  when  $T = 0$ . Describe what these values tell us about the deer and wolf populations in the first year after the wolves are introduced. [3]
- ii** The origin is an equilibrium point. [5]  
Find the coordinates of the equilibrium point with  $X > 0$  and  $Y > 0$ , and describe its nature.
- iii** Apply Euler's method with step size 0.1 to estimate:
  - (1)** the maximum population which will be reached by the deer, and the year when this will occur [2]
  - (2)** the maximum population which will be reached by the wolves, and the year when this will occur. [2]
- iv** Sketch the trajectory of the populations on a phase portrait, showing the features you have found. [2]
- v** Ranger Roy applied Euler's method with step size 1 to predict the population  $X(T)$  over time. He generates the following graph: [2]



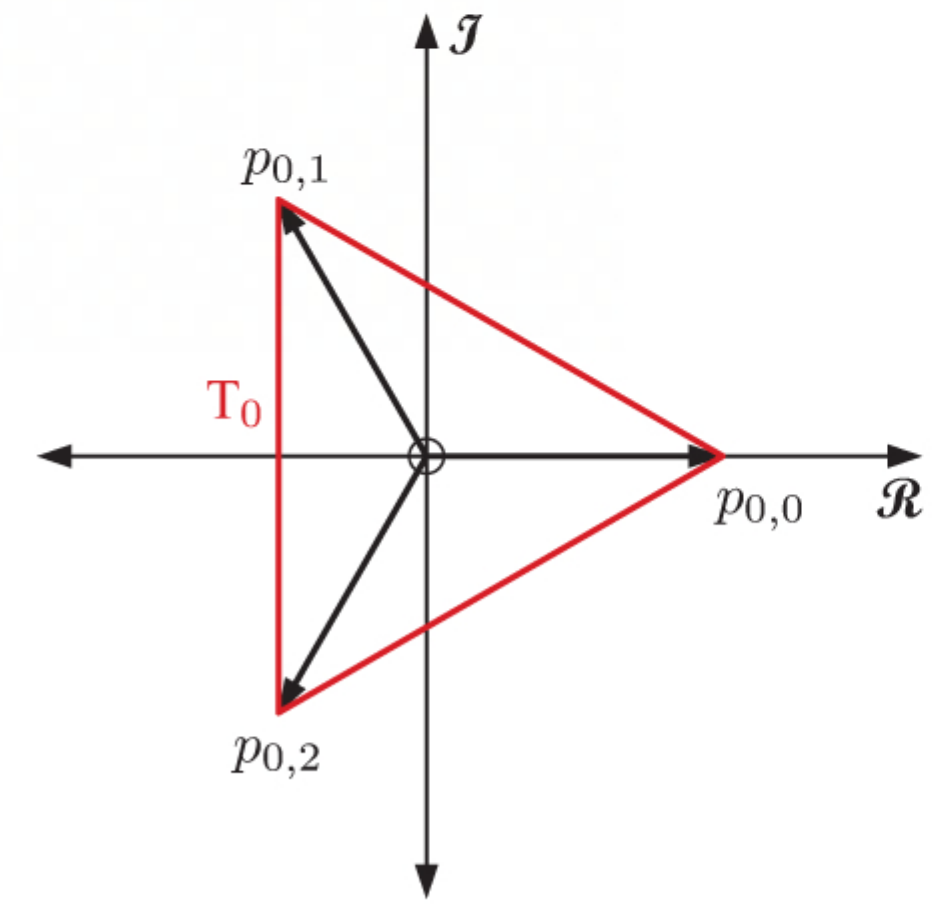
Explain what Ranger Roy has done wrong, and describe the error it has caused.



**2 [Maximum mark: 30]**

The diagram shows the triangle  $T_0$  which is defined in the complex plane by the numbers  $p_{0,n} = e^{i\frac{2n\pi}{3}}$ ,  $n = 0, 1, 2$ .

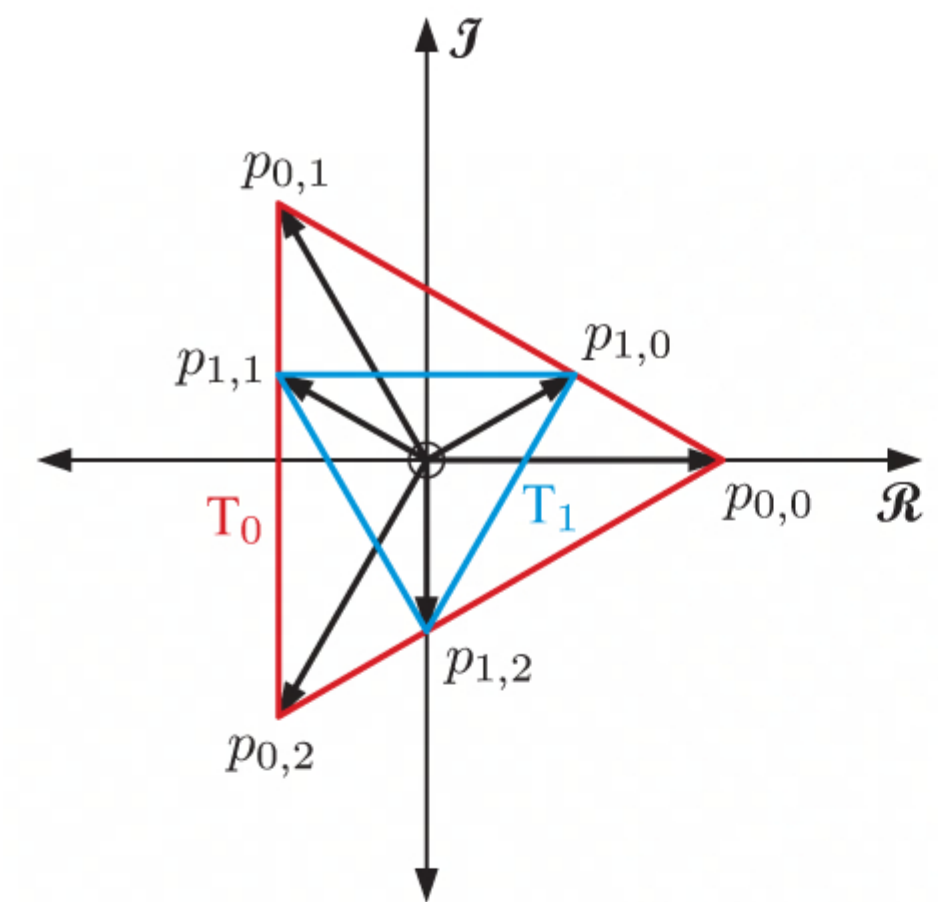
So,  $p_{0,0} = e^{i0} = 1$ .



**a** Write  $p_{0,1}$  and  $p_{0,2}$  in Cartesian form. [2]

**b** Suppose  $T_0$  is rotated anticlockwise through  $\frac{\pi}{6}$  about 0, then reduced to produce triangle  $T_1$  whose vertices lie on the sides of  $T_0$ .

The image of the vertex  $p_{0,n}$  on  $T_0$  is the vertex  $p_{1,n}$  on  $T_1$ .



**i** Explain why multiplying  $p_{0,n}$  by  $e^{i\frac{\pi}{6}}$  results in an anticlockwise rotation through  $\frac{\pi}{6}$  about O. [2]

**ii** State the real part of  $p_{1,1}$ . [1]

**iii** Hence or otherwise, find the scale factor  $k$  of the reduction such that  $p_{1,1} = p_{0,1} \times ke^{i\frac{\pi}{6}}$ . [2]

**c** Suppose the transformation from  $T_0$  to  $T_1$  is repeated to produce a sequence of triangles  $T_m$ ,  $m \in \mathbb{Z}^+$  with vertices given by

$$p_{m,n} = p_{m-1,n} \times ke^{i\frac{\pi}{6}}, \quad m \in \mathbb{Z}^+, \quad n = 0, 1, 2.$$

**i** Sketch  $T_0$ ,  $T_1$ ,  $T_2$ , and  $T_3$  on the same Argand diagram. [2]

**ii** Show that  $p_{m,n} = 3^{-\frac{m}{2}} e^{i\pi(\frac{2n}{3} + \frac{m}{6})}$ ,  $m \in \mathbb{Z}^+$ ,  $n = 0, 1, 2$ . [2]

**iii** Hence find  $p_{10,0}$  exactly. [2]

**d** The transformations that produce the sequence of triangles  $T_m$  can also be generated by matrices.

Let  $\mathbf{p}_{m,n}$  be the vector representing  $p_{m,n}$  in the complex plane.

**i** Find the transformation matrix  $\mathbf{A}$  which corresponds to an anticlockwise rotation through  $\frac{\pi}{6}$  about O, followed by a reduction with scale factor  $k$ . [3]

**ii** Find  $p_{4,2}$  using matrices. [2]

**iii** Find  $\det \mathbf{A}$ . [2]

**iv** Find the ratio of the area of  $T_m$  to the area of  $T_0$ . [1]

**e** Now suppose the procedure for generating  $T_m$  is changed slightly.

At each iteration, the rotation is now through  $\frac{\pi}{6}$  anticlockwise with probability  $q$ , and through  $\frac{\pi}{6}$  clockwise with probability  $1 - q$ .

**i** Let  $X$  be the number of iterations of anticlockwise rotations in  $m$  iterations. [1]

Write down the distribution of  $X$ .

**ii** Let  $Y$  be the argument of  $p_{m,0}$ .

**(1)** Write  $Y$  in terms of  $X$ . [2]

**(2)** Write an expression for  $E(Y)$ . [2]

**iii** If  $q = \frac{2}{5}$ , find the probability that  $p_{10,0}$  lies above the real axis. [4]



# Paper 3 practice

The questions in this section are intended to give students practice at answering the style of questions that will appear in Paper 3 examinations. The questions do not have a markscheme attached to them, and may not necessarily be of a similar length to questions that will appear in the Paper 3 examination.

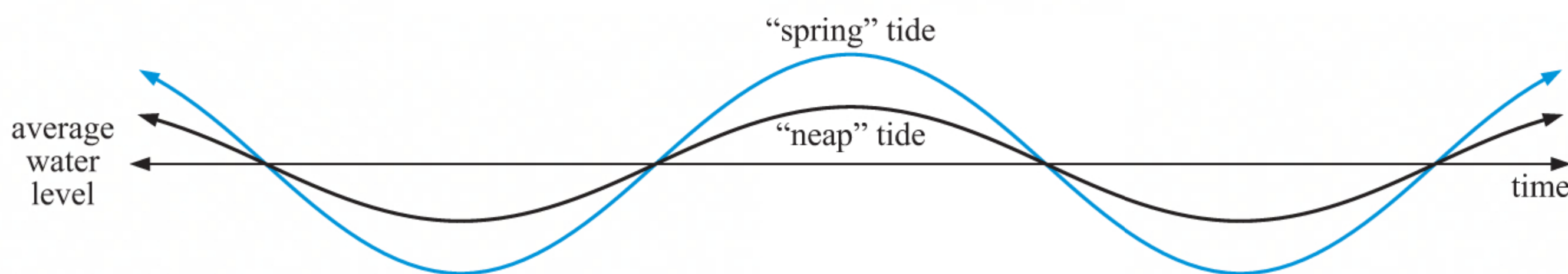
- 1** Herlina lives in a fishing village in Indonesia. She wants to model the tides in the harbour to help the fishermen avoid the coral reefs.

- a** Herlina's first model is a cosine model of the form  $h(t) = a \cos(bt)^\circ + d$  metres, where  $t$  is the time in hours after the first high tide, and  $a$ ,  $b$ , and  $d$  are constants.

From talking to the fishermen, she knows that an average high tide is 3.3 m, an average low tide is 0.9 m, and there are 12 hours and 25 minutes between high tides.

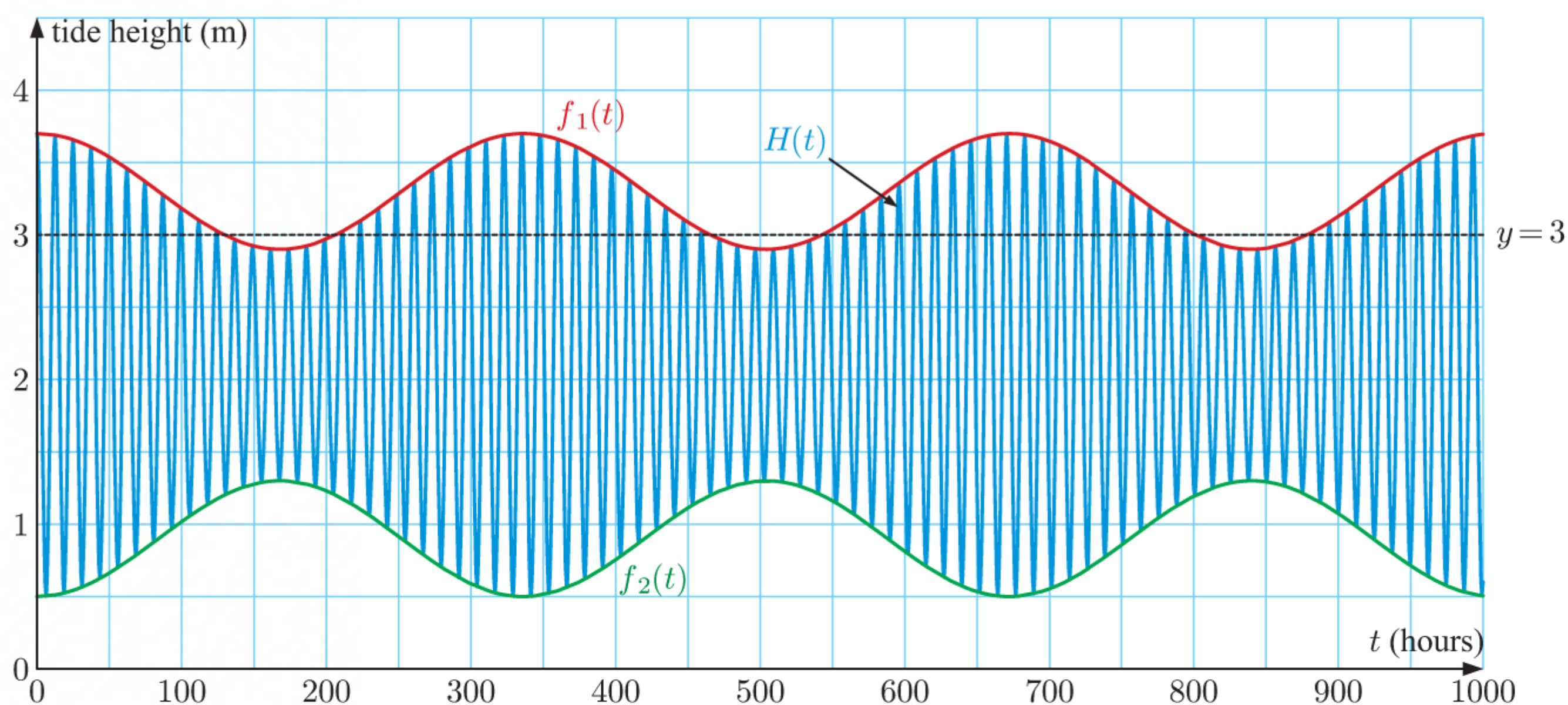
- Find the values of  $a$ ,  $b$ , and  $d$ .
  - The fishing boats can only pass over the reef when the tide is at least 3.0 m. Use the model to find the average number of hours per day for which the fishing boats can pass over the reef.
- b** When Herlina presents her findings to the fishermen, they tell her that there are several days per month when they cannot pass over the reef at all. She therefore realises that she needs to improve her model.

When Herlina looks at the tide marker by the wharf, she notices a rust stain which shows the highest tide is actually 3.7 m. She realises that this must be the “spring” high tide, which occurs every 14 days at either “new” moon or “full” moon. In between the “spring” high tides are “neap” high tides which are the lowest high tides.



Herlina therefore amends her model so that the amplitude  $a$  is now itself a function of time, and  $t$  corresponds to the time in hours after a “spring” high tide.

- Explain why  $a(t) = 0.4 \cos\left(\frac{15}{14}t\right)^\circ + 1.2$  is an appropriate model for the amplitude of the tide model.
- The graph below show Herlina's new model for the tide, which is the function  $H(t) = a(t) \times \cos(bt)^\circ + d$  for  $0 \leq t \leq 1000$  hours.



- Write down the equations of the curves  $f_1(t)$  and  $f_2(t)$ . Explain what these curves tell us.
- Use the graph to estimate the number of high tides there are in a lunar (28 day) month at which the fishing boats are *not* able to pass over the reef.



- Herlina understands how important fishing is for families in her village. She proposes that a narrow section of the reef is cut back, just wide enough so the fishing boats will be able to pass through one at a time, every day of the month. However, she wants to do the minimum damage possible to the reef.

The fishermen inform her that to safely cross the reef, she should allow 3 hours for the boats to cross at high tide.

What is the minimum height of coral that will need to be removed from the reef, in order to give the fishermen safe passage? Justify your answer.



- 2 a** One of the simplest models for infectious disease is the SI model. It is used to model infections such as the herpes simplex virus which causes cold sores.

Once an individual is infected with herpes simplex, they carry the virus for life, and can pass the virus on to other people.

We let  $S$  be the number of susceptible people

and  $I$  be the number of infected people

in a constant population of  $N = S + I$  people.

If we ignore birth and death, the SI model can be written as the system of coupled differential equations

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta SI}{N}, & S(0) = S_0 \\ \frac{dI}{dt} = \frac{\beta SI}{N}, & I(0) = I_0 \end{cases}$$

where  $\beta$  is a constant describing the rate at which one individual passes the virus onto another, and  $t$  is the time in days.

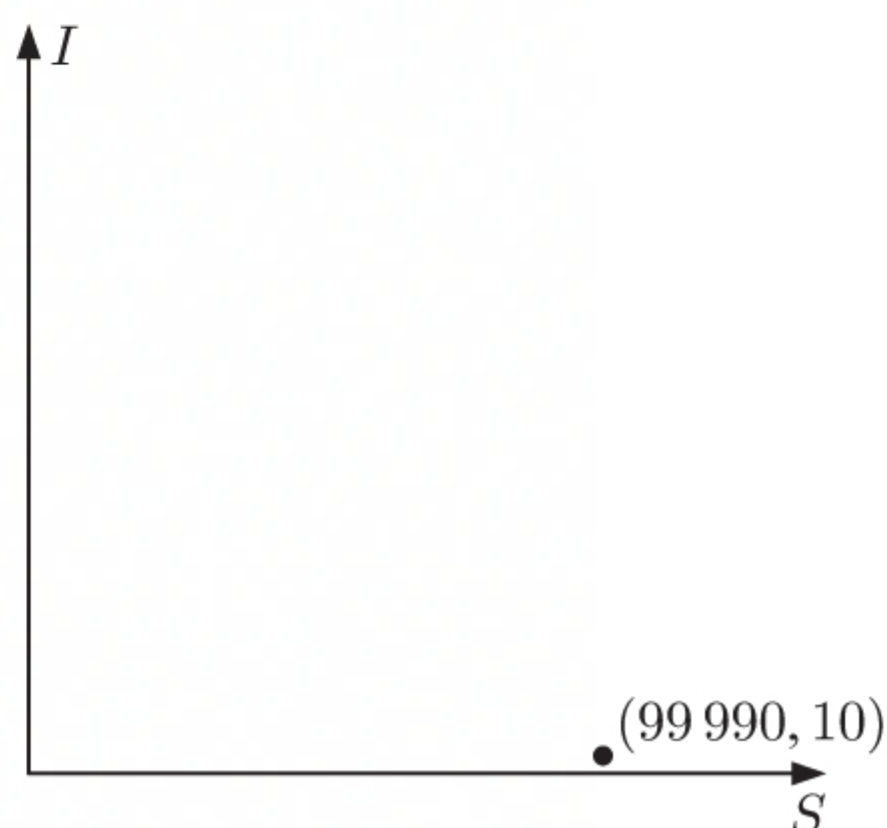
- i** Write  $\frac{dI}{dt}$  in terms of  $I$  only.
  - ii** Show that  $I(t) = \frac{N}{1 + ae^{-\beta t}}$  is a solution to the differential equation for  $\frac{dI}{dt}$  for some constant  $a$ .
  - iii** Hence find  $S(t)$ .
  - iv** Let  $\beta = 0.4$ ,  $S_0 = 99\,990$ , and  $I_0 = 10$ .
    - (1)** Find the value of  $a$ .
    - (2)** Sketch  $I(t)$  and  $S(t)$  on the same set of axes.
    - (3)** Estimate the time required for 50% of the population to be infected.
- b** Viruses which cause common colds do not remain in the individual for life. However, our immune systems are poor at remembering these viruses, so once a person has recovered from an infection, they return to the susceptible group.

This leads us to an SIS model which is governed by the system of coupled differential equations

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta SI}{N} + \gamma I, & S(0) = S_0 \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I, & I(0) = I_0 \end{cases}$$

where  $\gamma$  is the rate of recovery from an infection.

- i** Find the values of  $S$  and  $I$  for which the system is in equilibrium.
- ii** Explain why if  $\gamma > \beta$  then one of the equilibrium points cannot be reached.  
What will happen to the infection in this case, and why?
- iii** Assume that  $\gamma < \beta$  and that  $I > 0$ .
  - (1)** For what values of  $\frac{S}{N}$  is  $\frac{dI}{dt} > 0$ ?
  - (2)** What does this tell us about the stability of the equilibrium points?
- iv** Suppose  $\frac{\gamma}{\beta} = 0.75$  for a particular cold virus, and that  $(S_0, I_0)$  is  $(99\,990, 10)$ .
  - (1)** Describe what happens to the virus in the population over time.
  - (2)** Sketch the solution curve on a set of axes like the one below:





- A new and deadly respiratory virus emerges which means a proportion  $\mu$  of those who become infected will die.

This SIS model with mortality for this system is

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta SI}{S+I} + (1-\mu)\gamma I, & S(0) = S_0 \\ \frac{dI}{dt} = \frac{\beta SI}{S+I} - \gamma I, & I(0) = I_0 \end{cases}$$

where  $\gamma$  is the rate at which there is an *outcome* (recovery or death).

Let  $\beta = 0.4$ ,  $\gamma = 0.3$ ,  $S_0 = 99\,990$ ,  $I_0 = 10$ , and suppose  $\mu = 0.02$  be the probability of death.

- i (1)** Apply Euler's method with step size 8 days, to plot the approximate solution curves for  $S(t)$  and  $I(t)$ .

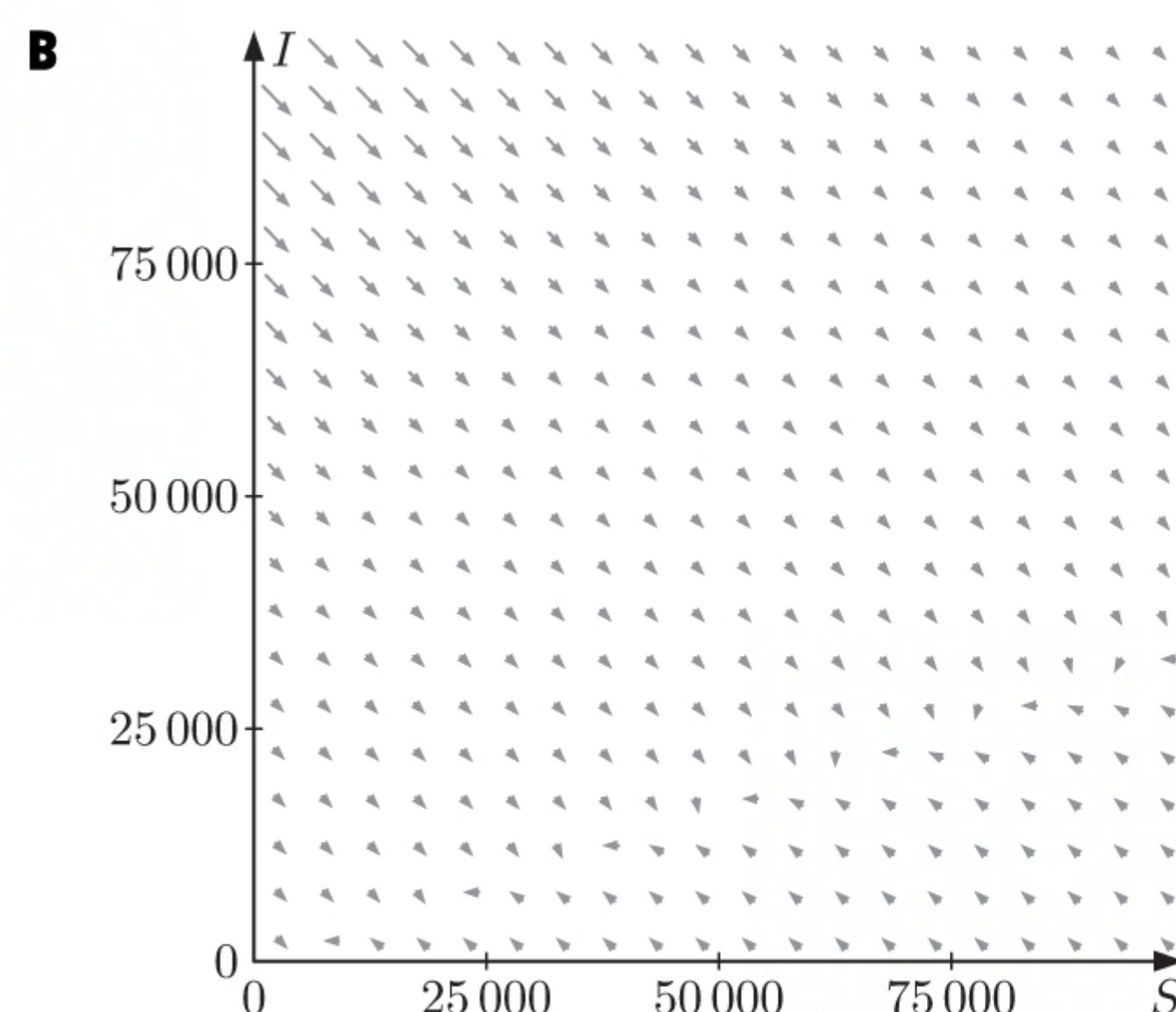
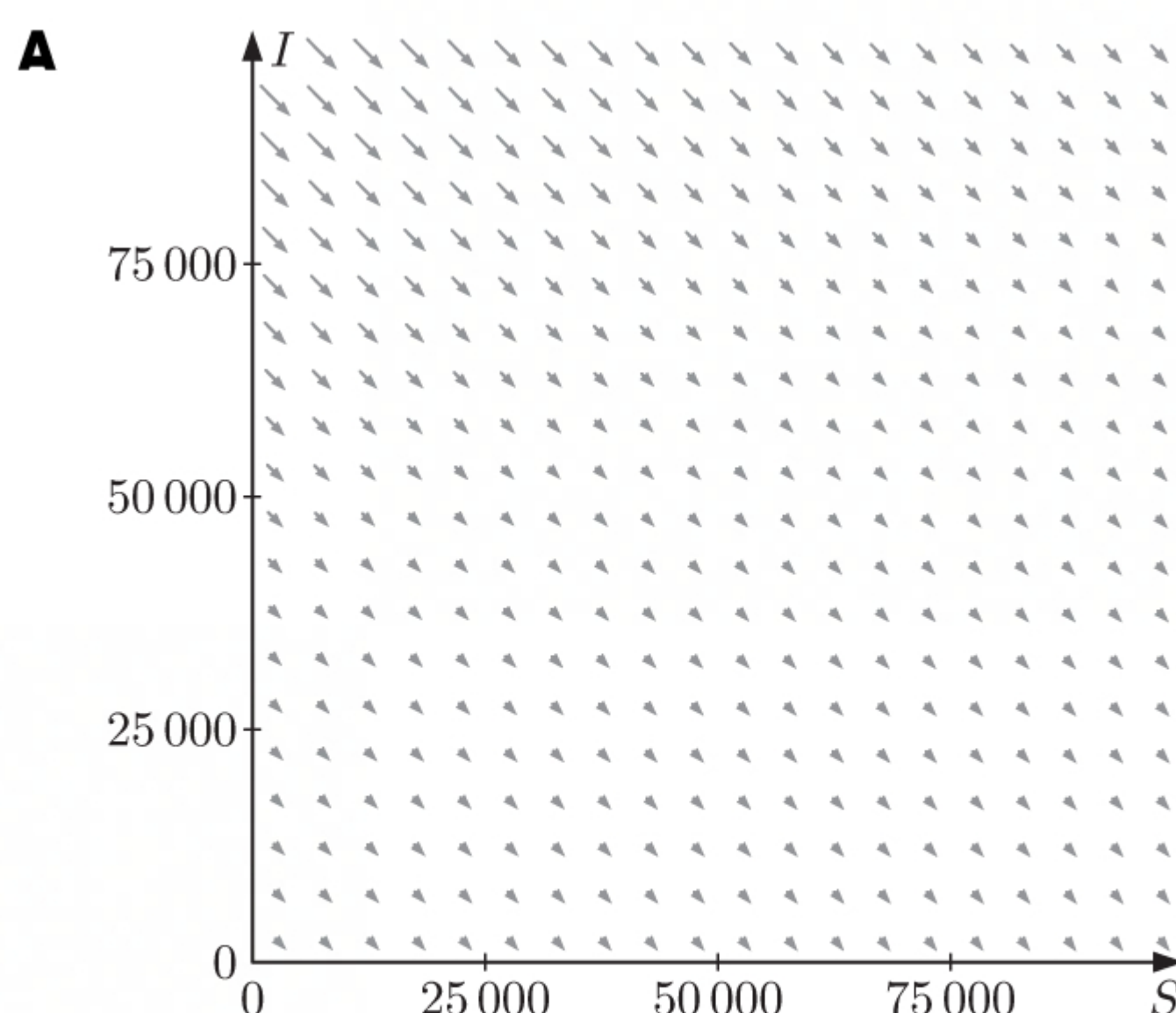
**(2)** Estimate  $S(240)$  and  $I(240)$ . How many people have died so far?

**(3)** Describe what will happen to the population in the long term.

- ii** Suppose  $\mu$  is reduced by improving medical treatment.

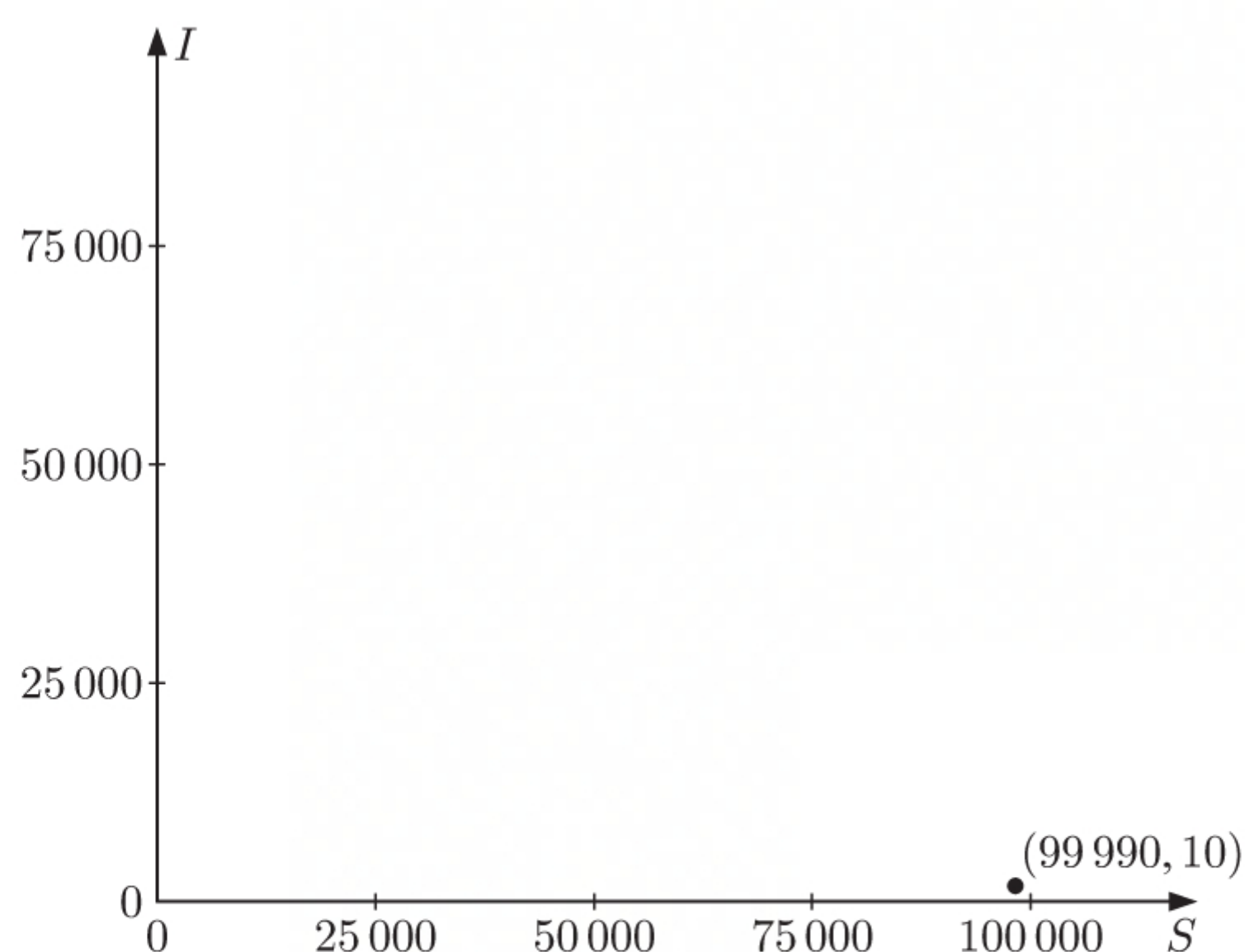
Will reducing  $\mu$  alone be sufficient to avoid the outcome in **i**?

- iii** One of the phase portraits below corresponds to the parameters described above. In the other phase portrait,  $\beta = 0.28$ .



- (1)** Identify which phase portrait corresponds to  $\beta = 0.4$  and which corresponds to  $\beta = 0.28$ . Explain your answer.
- (2)** After 240 days, rigorous measures of social distancing, protective equipment, and quarantine reduce the rate of spread  $\beta$  from 0.4 to 0.28.

Use the phase portraits **A** and **B** to sketch the trajectory of the outbreak on a set of axes like the one below.



- iv** Explain why the cooperation of the population is necessary to avoid the outcome in **i**.



# Worked solutions

## TOPIC 1 SKILL BUILDER QUESTIONS

- 1** The measuring device is accurate to  $\pm \frac{0.1}{2} \text{ km h}^{-1} = \pm 0.05 \text{ km h}^{-1}$ .

$\therefore$  the range of values is  $141.6 \pm 0.05 \text{ km h}^{-1}$ .

The cricket player's actual bowling speed lies between  $141.55 \text{ km h}^{-1}$  and  $141.65 \text{ km h}^{-1}$ .

$\therefore 141.55 \text{ km h}^{-1} < s < 141.65 \text{ km h}^{-1}$

- 2** The width of the block could be from  $16\frac{1}{2} \text{ m}$  to  $17\frac{1}{2} \text{ m}$ .

The length of the block could be from  $21\frac{1}{2} \text{ m}$  to  $22\frac{1}{2} \text{ m}$ .

$\therefore$  the lower boundary of the area is  $16\frac{1}{2} \times 21\frac{1}{2} = 354.75 \text{ m}^2$

and the upper boundary of the area is  $17\frac{1}{2} \times 22\frac{1}{2} = 393.75 \text{ m}^2$ .

$\therefore 354.75 \text{ m}^2 < A < 393.75 \text{ m}^2$

- 3 a** Volume  $\approx 15 \text{ cm} \times 12 \text{ cm} \times 8 \text{ cm}$

$$\approx 1440 \text{ cm}^3$$

- b i** Actual volume  $= 15.3 \text{ cm} \times 11.8 \text{ cm} \times 8.4 \text{ cm}$

$$= 1516.536 \text{ cm}^3$$

- ii** Absolute error  $= |V_A - V_E|$

$$= |1440 - 1516.536| \text{ cm}^3$$

$$= |-76.536| \text{ cm}^3$$

$$= 76.536 \text{ cm}^3$$

$$\text{Percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

$$= \frac{76.536}{1516.536} \times 100\%$$

$$\approx 5.05\%$$

- 4 a** Area  $\approx \pi \times 7^2$

$$\approx 49\pi$$

$$\approx 154 \text{ cm}^2$$

- b** The radius length could be from  $6\frac{1}{2} \text{ cm}$  to  $7\frac{1}{2} \text{ cm}$ .

$\therefore$  the lower boundary of the area is  $\pi \times (6\frac{1}{2})^2 = 42.25\pi \text{ cm}^2 \approx 133 \text{ cm}^2$

and the upper boundary of the area is  $\pi \times (7\frac{1}{2})^2 = 56.25\pi \text{ cm}^2 \approx 177 \text{ cm}^2$ .

- c** If the exact area  $V_E$  was  $42.25\pi \text{ cm}^2$ , the percentage error  $= \frac{|V_A - V_E|}{V_E} \times 100\%$

$$= \frac{|49\pi - 42.25\pi|}{42.25\pi} \times 100\%$$

$$\approx 16.0\%$$

If the exact area  $V_E$  was  $56.25\pi \text{ cm}^2$ , the percentage error  $= \frac{|V_A - V_E|}{V_E} \times 100\%$

$$= \frac{|49\pi - 56.25\pi|}{56.25\pi} \times 100\%$$

$$\approx 12.9\%$$

$\therefore$  the maximum percentage error in the estimate  $\approx 16.0\%$ .

- 5 a**  $64 = 8^2$

$$= (2^3)^2$$

$$= 2^6$$

- b**  $125 \times 5^k = 5^3 \times 5^k$

$$= 5^{3+k}$$

- c**  $\frac{9^m}{81^n} = \frac{9^m}{(9^2)^n}$

$$= \frac{(3^2)^m}{(3^2)^{2n}}$$

$$= \frac{3^{2m}}{3^{4n}}$$

$$= 3^{2m-4n}$$

- 6 a**  $(-3m^3)^4 = (-3)^4 \times (m^3)^4$

$$= 81m^{12}$$

- b**  $\left(\frac{xy^2}{2}\right)^5 = \frac{(xy^2)^5}{2^5}$

$$= \frac{x^5 y^{10}}{32}$$

- c**  $7s^2t \times (4st^3)^3 = 7s^2t \times 4^3 s^3 t^9$

$$= 7 \times 64 s^5 t^{10}$$

$$= 448 s^5 t^{10}$$



$$\begin{aligned} 7 \quad \mathbf{a} \quad 4^0 + 4^{-1} &= 1 + \frac{1}{4} \\ &= \frac{5}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \left(2\frac{3}{4}\right)^{-2} &= \left(\frac{11}{4}\right)^{-2} \\ &= \left(\frac{4}{11}\right)^2 \\ &= \frac{4^2}{11^2} \\ &= \frac{16}{121} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 2^2 + 2^1 + 2^{-1} &= 4 + 2 + \frac{1}{2} \\ &= 6 + \frac{1}{2} \\ &= \frac{13}{2} \end{aligned}$$

$$\begin{aligned} 8 \quad \mathbf{a} \quad (x^2 + x^{-2})^2 &= (x^2)^2 + 2x^2 \times x^{-2} + (x^{-2})^2 \\ &= x^4 + 2 + x^{-4} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (x^4 - x^2)(x^3 + 3) &= x^4 \times x^3 + 3x^4 + (-x^2) \times x^3 + (-x^2) \times 3 \\ &= x^7 + 3x^4 - x^5 - 3x^2 \end{aligned}$$

$$\begin{aligned} 9 \quad \mathbf{a} \quad a^2 b^{-3} &= a^2 \times \frac{1}{b^3} \\ &= \frac{a^2}{b^3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \frac{2m^{-2}n^3}{m^5n^{-5}} &= 2 \times m^{-2-5} \times n^{3-(-5)} \\ &= 2 \times m^{-7} \times n^8 \\ &= 2 \times \frac{1}{m^7} \times n^8 \\ &= \frac{2n^8}{m^7} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \frac{12a^{-3}}{b^{-5}} &= 12 \times a^{-3} \times \frac{1}{b^{-5}} \\ &= 12 \times \frac{1}{a^3} \times b^5 \\ &= \frac{12b^5}{a^3} \end{aligned}$$

$$\begin{aligned} 10 \quad \mathbf{a} \quad 4^{\frac{5}{2}} &= (2^2)^{\frac{5}{2}} \\ &= 2^5 \\ &= 32 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 49^{-\frac{3}{2}} &= (7^2)^{-\frac{3}{2}} \\ &= 7^{-3} \\ &= \frac{1}{7^3} \\ &= \frac{1}{343} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 27^{\frac{5}{3}} &= (3^3)^{\frac{5}{3}} \\ &= 3^5 \\ &= 243 \end{aligned}$$

$$\begin{aligned} 11 \quad \mathbf{a} \quad x^{\frac{1}{2}}(x^{-\frac{1}{2}} + 2x - x^{\frac{1}{2}}) &= x^{\frac{1}{2}} \times x^{-\frac{1}{2}} + x^{\frac{1}{2}} \times 2x - x^{\frac{1}{2}} \times x^{\frac{1}{2}} \\ &= x^0 + 2x^{\frac{3}{2}} - x^1 \\ &= 1 + 2x^{\frac{3}{2}} - x \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 5^x(5^{-x} + 5^{3x}) &= 5^x \times 5^{-x} + 5^x \times 5^{3x} \\ &= 5^0 + 5^{4x} \\ &= 1 + 5^{4x} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 2^{-2x}(2^{2x+3} - 2^{-4x} + 3) &= 2^{-2x} \times 2^{2x+3} - 2^{-2x} \times 2^{-4x} + 3 \times 2^{-2x} \\ &= 2^3 - 2^{-6x} + 3 \times 2^{-2x} \\ &= 8 - 2^{-6x} + 3 \times 2^{-2x} \end{aligned}$$

$$\begin{aligned} 12 \quad \mathbf{a} \quad 42000 &= 4.2 \times 10\,000 \\ &= 4.2 \times 10^4 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 0.0000678 &= 6.78 \times 0.000\,001 \\ &= 6.78 \times 10^{-5} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 526000000 &= 5.26 \times 100\,000\,000 \\ &= 5.26 \times 10^8 \end{aligned}$$

13 Using technology:

$$\mathbf{a} \quad (3.57 \times 10^6) \times (2.38 \times 10^3) = 8.4966 \times 10^9$$

$$\mathbf{c} \quad (0.000\,08)^4 = 4.096 \times 10^{-17}$$

$$\mathbf{b} \quad \frac{4.61 \times 10^{-7}}{3.45 \times 10^8} \approx 1.34 \times 10^{-15}$$

$$\begin{aligned} 14 \quad \mathbf{a} \quad 3x^3 + 7x^2 - 3x &= 2 \\ \therefore 3x^3 + 7x^2 - 3x - 2 &= 0 \end{aligned}$$

Using technology,  $x \approx 0.667$ ,  $-2.62$ , or  $-0.382$

$$\begin{aligned} \mathbf{b} \quad x^4 + 3x^3 + 2 &= 4x^2 - 8x \\ \therefore x^4 + 3x^3 - 4x^2 + 8x + 2 &= 0 \end{aligned}$$

Using technology,  $x \approx -4.33$  or  $-0.222$



**15 a** 
$$\begin{cases} 2x - 3y = 2 \\ 5x + 3y = 5 \end{cases}$$

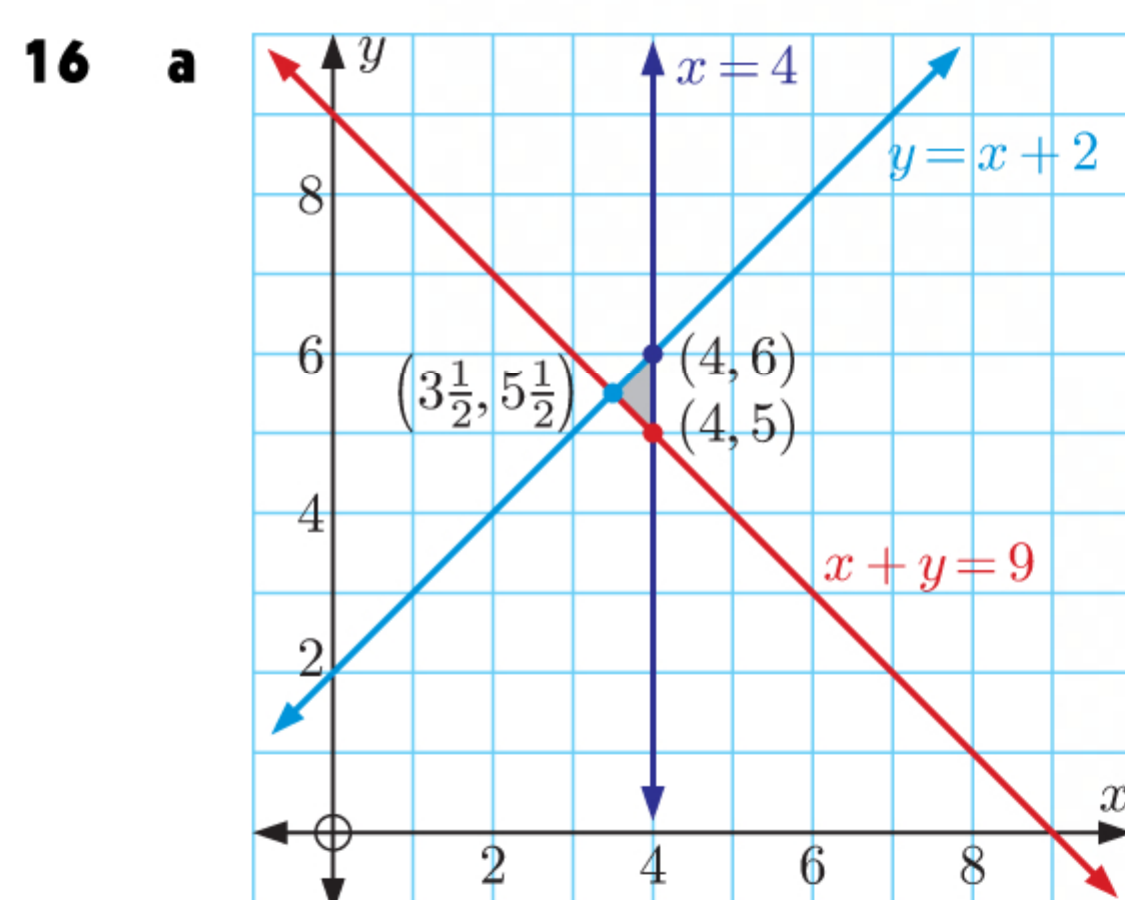
So, the solution is  $x = 1$ ,  $y = 0$ .

**b** 
$$\begin{cases} 3x - 7y = -8 \\ 6x + 11y = 12 \end{cases}$$

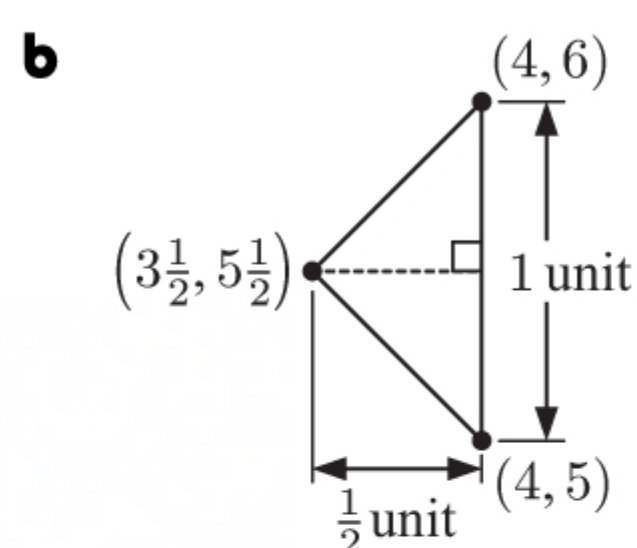
So, the solution is  $x \approx -0.0533$ ,  $y = 1.12$ .

**c** 
$$\begin{cases} 2x + y + 3z = -3 \\ x - y + 2z = 1 \\ 3x - 2y + 5z = 4 \end{cases}$$

So, the solution is  $x = 6$ ,  $y = -3$ ,  $z = -4$ .



The vertices of the triangle are  $(4, 6)$ ,  $(4, 5)$ , and  $(3\frac{1}{2}, 5\frac{1}{2})$ .



$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times 1 \times \frac{1}{2} \\ &= \frac{1}{4} \text{ units}^2 \end{aligned}$$

**17** 8, 13, 18, 23, 28, ....

**a**  $13 - 8 = 5$   
 $18 - 13 = 5$   
 $23 - 18 = 5$   
 $28 - 23 = 5$

The difference between successive terms is constant.

$\therefore$  the sequence is arithmetic with  $u_1 = 8$  and  $d = 5$ .

**b**  $u_n = u_1 + (n - 1)d$   
 $\therefore u_n = 8 + 5(n - 1)$   
 $\therefore u_n = 3 + 5n$

**d i** Let  $u_n = 153$   
 $\therefore 3 + 5n = 153$   
 $\therefore 5n = 150$   
 $\therefore n = 30$

$\therefore 153$  is a member of the sequence, and in fact is the 30th term.

**c**  $u_{42} = 3 + 5(42)$   
 $= 213$

**ii** Let  $u_n = 4067$   
 $\therefore 3 + 5n = 4067$   
 $\therefore 5n = 4064$   
 $\therefore n = 812\frac{4}{5}$

But  $n$  must be an integer, so 4067 is not a member of the sequence.

**18 a** If 3,  $k$ , 11 are consecutive terms of an arithmetic sequence, then

$$\begin{aligned} k - 3 &= 11 - k \quad \{\text{equating differences}\} \\ \therefore 2k &= 14 \\ \therefore k &= 7 \end{aligned}$$



**b** If  $-2$ ,  $k + 4$ ,  $k^2 + 11$  are consecutive terms of an arithmetic sequence, then

$$k + 4 - (-2) = k^2 + 11 - (k + 4) \quad \{\text{equating differences}\}$$

$$\therefore k + 6 = k^2 + 11 - k - 4$$

$$\therefore k^2 - 2k + 1 = 0$$

$$\therefore (k - 1)^2 = 0$$

$$\therefore k = 1$$

**c** If  $k - 5$ ,  $2k$ ,  $2k^2$  are consecutive terms of an arithmetic sequence, then

$$2k - (k - 5) = 2k^2 - 2k \quad \{\text{equating differences}\}$$

$$\therefore k + 5 = 2k^2 - 2k$$

$$\therefore 2k^2 - 3k - 5 = 0$$

$$\therefore (2k - 5)(k + 1) = 0$$

$$\therefore k = \frac{5}{2} \text{ or } -1$$

**19 a** Average mass =  $\frac{\text{total mass} - \text{mass of cage}}{\text{number of hamsters}}$

$$= \frac{1400 - 800}{5}$$

$$= \frac{600}{5}$$

$$= 120 \text{ g}$$

**b**  $u_n = 800 + 120n$

**20 a**  $u_5 = u_1 r^4 = 324 \quad \dots (1)$

and  $u_{10} = u_1 r^9 = 78\,732 \quad \dots (2)$

Now  $\frac{u_1 r^9}{u_1 r^4} = \frac{78\,732}{324} \quad \{(2) \div (1)\}$

$$\therefore r^5 = 243$$

$$\therefore r = \sqrt[5]{243}$$

$$\therefore r = 3$$

Using (1),  $u_1(3)^4 = 324$

$$\therefore 81u_1 = 324$$

$$\therefore u_1 = 4$$

Thus  $u_n = 4 \times 3^{n-1}$

**b**  $u_8 = u_1 r^7 = -10 \quad \dots (1)$

and  $u_{12} = u_1 r^{11} = -160 \quad \dots (2)$

Now  $\frac{u_1 r^{11}}{u_1 r^7} = \frac{-160}{-10} \quad \{(2) \div (1)\}$

$$\therefore r^4 = 16$$

$$\therefore r = \pm \sqrt[4]{16}$$

$$\therefore r = \pm 2$$

If  $r = 2$ , then using (1),  $u_1(2)^7 = -10$

$$\therefore 128u_1 = -10$$

$$\therefore u_1 = \frac{-10}{128} = -\frac{5}{64}$$

Thus  $u_n = -\frac{5}{64} \times 2^{n-1}$

If  $r = -2$ , then using (1),  $u_1(-2)^7 = -10$

$$\therefore -128u_1 = -10$$

$$\therefore u_1 = \frac{10}{128} = \frac{5}{64}$$

Thus  $u_n = \frac{5}{64} \times (-2)^{n-1}$

**21**  $2, 2\sqrt{3}, 6, 6\sqrt{3}$

**a**  $\frac{2\sqrt{3}}{2} = \sqrt{3}, \quad \frac{6}{2\sqrt{3}} = \frac{\cancel{2} \times 3}{\cancel{2}\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}, \quad \frac{6\sqrt{3}}{6} = \sqrt{3}$

Consecutive terms have a common ratio of  $\sqrt{3}$ .

$\therefore$  the sequence is geometric with  $u_1 = 2$  and  $r = \sqrt{3}$ .

**b**  $u_n = u_1 r^{n-1}$

$$= 2(\sqrt{3})^{n-1}$$

**c**  $u_{10} = 2(\sqrt{3})^{10-1}$

$$= 2(\sqrt{3})^9$$

$$= 2\sqrt{3} \times (\sqrt{3})^8$$

$$= 2\sqrt{3} \times ((\sqrt{3})^2)^4$$

$$= 2\sqrt{3} \times 3^4$$

$$= 162\sqrt{3}$$



**d** We need to find  $n$  such that  $u_n = 2(\sqrt{3})^{n-1} > 1000$ .

Using a graphics calculator with  $Y_1 = 2 \times \sqrt{3} \wedge (X - 1)$ , we view a table of values:

X	Y1
11	486
12	841.77
13	1458
14	2525.3

The first term to exceed 1000 is  $u_{13} = 1458$ .

**22** There is a fixed percentage increase each year, so the population forms a geometric sequence with  $u_0 = 217$  and  $r = 1.42$ .

$\therefore$  the population after  $n$  years is  $u_n = 217 \times (1.42)^n$ .

**a i**  $u_5 = 217 \times (1.42)^5$   
 $\approx 1252.86$

The expected population size after 5 years is approximately 1250 birds.

**ii**  $u_{10} = 217 \times (1.42)^{10}$   
 $\approx 7233.41$

The expected population size after 10 years is approximately 7230 birds.

**b** We need to find when  $217 \times (1.42)^n = 30\,000$ .

Using technology,  $n \approx 14.1$ , so it will take approximately 14.1 years for the population to reach 30 000.

Eq: $217 \times 1.42^x = 30000$
$x = 14.05663401$
Lft = 30000
Rgt = 30000
REPEAT

**23 a** If the interest rate per annum is 7.2%, then the interest rate per month  $i = \frac{7.2\%}{12} = 0.6\% = 0.006$ .

$$\begin{aligned} r &= 1 + i \\ &= 1 + 0.006 \\ &= 1.006 \end{aligned}$$

**b** The interest is calculated monthly, so  $n = 3 \times 12 = 36$  time periods.

$$\begin{aligned} u_{36} &= u_0 \times r^{36} \\ &= 500 \times (1.006)^{36} \\ &\approx 620.15 \end{aligned}$$

The value of the account after 3 years is €620.15.

**c**  $\text{real value} \times (1.02)^3 = \text{€}620.15$   
 $\therefore \text{real value} = \frac{\text{€}620.15}{(1.02)^3}$   
 $= \text{€}584.38$

**24** There is 1 time period every 3 months, so  $n = \frac{33}{3} = 11$  time periods.

Each time period the investment increases by  $i = \frac{8\%}{4} = 2\%$ .

$$\begin{aligned} \therefore \text{the amount after 33 months is } u_{11} &= u_0 \times (1 + i)^{11} \\ &= 3500 \times (1.02)^{11} \quad \{2\% = 0.02\} \\ &\approx 4351.81 \end{aligned}$$

The maturing value of the account is £4351.81.

**25** The initial investment  $u_0$  is unknown.

There are  $r = 5 \times 12 = 60$  time periods.

Each time period the investment increases by  $i = \frac{4.8\%}{12} = 0.4\%$ .

$$\begin{aligned} \text{Now } u_{60} &= u_0 \times (1 + i)^{60} \\ \therefore 30\,000 &= u_0 \times (1.004)^{60} \quad \{0.4\% = 0.004\} \\ \therefore u_0 &= \frac{30\,000}{(1.004)^{60}} \approx 23\,610.14 \end{aligned}$$

$\therefore$  I need to invest \$23 610.14 now.

**26 a**  $u_3 = u_0 \times (1 - d)^3$   
 $= 2000 \times (0.7)^3 \quad \{30\% = 0.3\}$   
 $= 686$

**b** Depreciation = £2000 – £686  
 $= \text{£}1314$

So, after 3 years the value of the television is £686.



- 27 a**  $N = 4.5 \times 12 = 54$ ,  $PV = -10\,000$ ,  $PMT = 0$ ,  $FV = 12\,000$ ,  $P/Y = 12$ ,  $C/Y = 12$

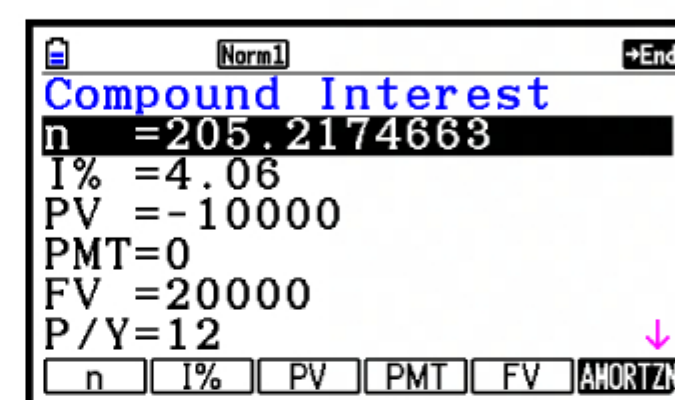
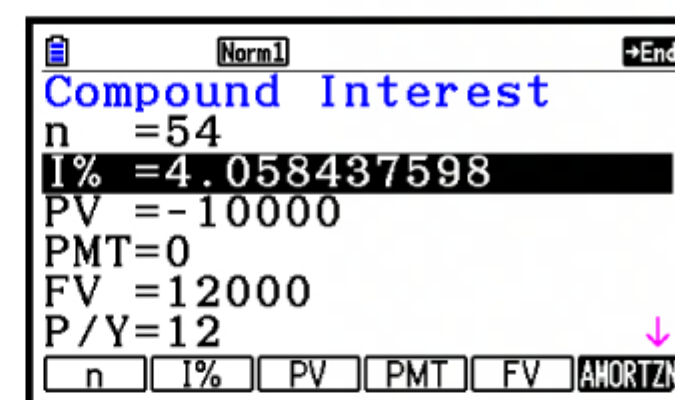
$$\therefore I\% \approx 4.06$$

The account paid about 4.06% interest per annum.

- b**  $I\% = 4.06$ ,  $PV = -10\,000$ ,  $PMT = 0$ ,  $FV = 20\,000$ ,  $P/Y = 12$ ,  $C/Y = 12$

$$\therefore N \approx 205.2$$

It will take 206 months or 17 years and 2 months for Lauren to double her deposit.



- 28 a** The series is arithmetic with  $u_1 = 11$ ,  $d = 4$ , and  $n = 20$ .

$$\text{Now } S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\begin{aligned} \therefore S_{20} &= \frac{20}{2}(2 \times 11 + 19 \times 4) \\ &= 10(22 + 76) = 980 \end{aligned}$$

- b**  $7 + 12.5 + 18 + 23.5 + \dots + 106$

The series is arithmetic with  $u_1 = 7$ ,  $d = 5.5$ , and  $u_n = 106$ .

First we need to find  $n$ .

$$\text{Now } u_n = 106$$

$$\therefore u_1 + (n-1)d = 106$$

$$\therefore 7 + 5.5(n-1) = 106$$

$$\therefore 5.5(n-1) = 99$$

$$\therefore n-1 = 18$$

$$\therefore n = 19$$

$$\text{Using } S_n = \frac{n}{2}(u_1 + u_n)$$

$$\therefore S_{19} = \frac{19}{2}(7 + 106)$$

$$= \frac{19}{2} \times 113$$

$$= 1073.5$$

- c**  $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$  to 100 terms can be expressed as two separate arithmetic series:

$$1 + 3 + 5 + 7 + \dots \text{ where } u_1 = 1, d = 2, n = 50$$

$$\text{and } -2 - 4 - 6 - 8 - \dots \text{ where } u_1 = -2, d = -2, n = 50$$

$$\begin{aligned} \text{Using } S_n &= \frac{n}{2}(2u_1 + (n-1)d), \text{ the sum of the first series} = \frac{50}{2}(2(1) + 49(2)) \\ &= 25(2 + 98) \\ &= 2500 \end{aligned}$$

$$\begin{aligned} \text{and the sum of the second series} &= \frac{50}{2}(2(-2) + 49(-2)) \\ &= 25(-4 - 98) \\ &= -2550 \end{aligned}$$

$$\begin{aligned} \therefore \text{the sum of both series} &= 2500 + (-2550) \\ &= -50 \end{aligned}$$

So,  $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$  to 100 terms is  $-50$ .

- d** The integers from 1 to 200 which are not divisible by 3 are 1, 2, 4, 5, 7, 8, ..., 200.

The sum of these integers can be expressed as two separate arithmetic series  $A$  and  $B$ :

$$S_A = 1 + 4 + 7 + \dots + 196 + 199 \text{ where } u_1 = 1, d = 3, u_n = 199$$

$$\text{and } S_B = 2 + 5 + 8 + \dots + 197 + 200 \text{ where } u_1 = 2, d = 3, u_n = 200$$

$$\text{Now for } S_A, u_n = u_1 + (n-1)d \text{ and for } S_B, u_n = u_1 + (n-1)d$$

$$\therefore 199 = 1 + 3(n-1)$$

$$\therefore 198 = 3(n-1)$$

$$\therefore 66 = n-1$$

$$\therefore n = 67$$

$$\therefore 200 = 2 + 3(n-1)$$

$$\therefore 198 = 3(n-1)$$

$$\therefore 66 = n-1$$

$$\therefore n = 67$$

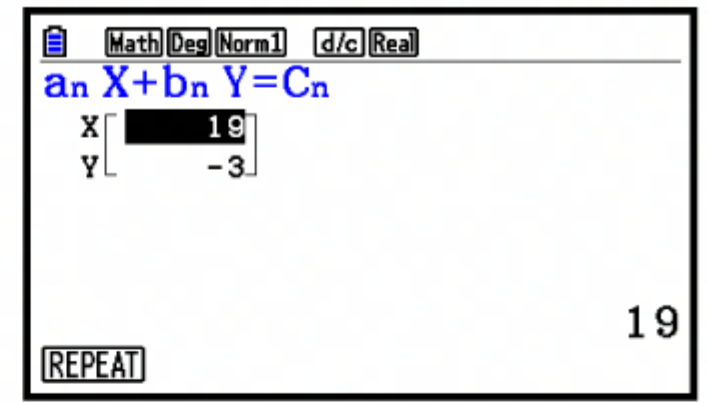
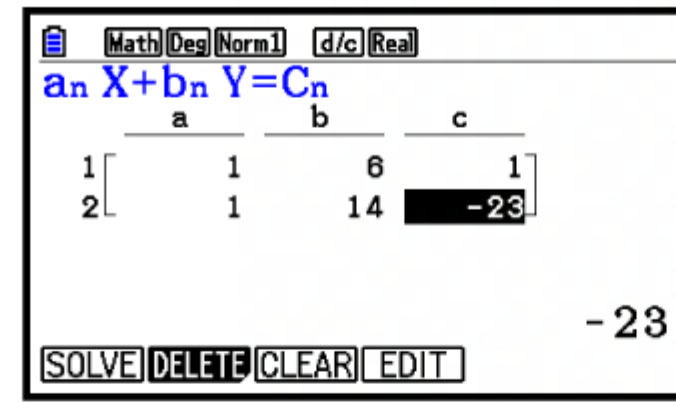
$$\text{Using } S_n = \frac{n}{2}(u_1 + u_n), S_A = \frac{67}{2}(1 + 199) = 6700 \text{ and } S_B = \frac{67}{2}(2 + 200) = 6767$$

$$\begin{aligned} \text{The total sum} &= S_A + S_B \\ &= 6700 + 6767 \\ &= 13\,467 \end{aligned}$$



**29 a**  $u_7 = 1 \quad \therefore u_1 + 6d = 1 \quad \{\text{using } u_n = u_1 + (n-1)d\}$   
 $u_{15} = -23 \quad \therefore u_1 + 14d = -23$

Using technology to solve these equations simultaneously, we find that  $u_1 = 19$  and  $d = -3$ .



**b**  $u_n = u_1 + (n-1)d$   
 $\therefore u_{27} = 19 + 26(-3) \quad \{\text{using a}\}$   
 $= -59$

**c**  $S_n = \frac{n}{2}(u_1 + u_n)$   
 $\therefore S_{27} = \frac{27}{2}(19 + (-59)) \quad \{\text{from b}\}$   
 $= \frac{27}{2} \times (-40)$   
 $= -540$

**30 a**  $S_n = \frac{n}{2}(2u_1 + (n-1)d)$   
 $\therefore -210 = \frac{n}{2}(2 \times 18 - 3(n-1))$   
 $\therefore \frac{n}{2}(36 - 3n + 3) = -210$   
 $\therefore \frac{n}{2}(39 - 3n) = -210$

**b** From **a**,  $\frac{n}{2}(39 - 3n) = -210$   
 $\therefore n(39 - 3n) = -420$   
 $\therefore 39n - 3n^2 = -420$   
 $\therefore 3n^2 - 39n - 420 = 0$   
 $\therefore n^2 - 13n - 140 = 0$   
 $\therefore (n-20)(n+7) = 0$   
 $\therefore n = 20 \quad \{n > 0\}$

**31 a** The sequence is arithmetic with  $u_1 = 7$  and  $d = 3$ .

Now  $S_n = \frac{n}{2}(2u_1 + (n-1)d)$   
 $\therefore S_n = \frac{n}{2}(2(7) + 3(n-1))$   
 $\therefore S_n = \frac{n}{2}(14 + 3n - 3)$   
 $\therefore S_n = \frac{n}{2}(11 + 3n)$

**b**  $S_n = 140$   
 $\therefore \frac{n}{2}(11 + 3n) = 140 \quad \{\text{using a}\}$   
 $\therefore n(11 + 3n) = 280$   
 $\therefore 11n + 3n^2 = 280$   
 $\therefore 3n^2 + 11n - 280 = 0$   
 $\therefore 3n^2 - 24n + 35n - 280 = 0$   
 $\therefore 3n(n-8) + 35(n-8) = 0$   
 $\therefore (n-8)(3n+35) = 0$   
 $\therefore n = 8 \quad \{n > 0\}$

**32 a** The series is geometric with  $u_1 = 10$ ,  $r = \frac{1}{2}$ , and  $n = 8$ .

Now  $S_n = \frac{u_1(r^n - 1)}{r - 1}$   
 $\therefore S_8 = \frac{10((\frac{1}{2})^8 - 1)}{\frac{1}{2} - 1} = 19.921\,875$

**b** The series is geometric with  $u_1 = 2$ ,  $r = 5$ , and  $n = 10$ .

Now  $S_n = \frac{u_1(r^n - 1)}{r - 1}$   
 $\therefore S_{10} = \frac{2(5^{10} - 1)}{5 - 1} = 4\,882\,812$

**c**  $\sum_{k=1}^{20} 3 \times (-2)^{k+2} = 3 \times (-2)^3 + 3 \times (-2)^4 + \dots + 3 \times (-2)^{22}$

The series is geometric with  $u_1 = 3 \times (-2)^3 = -24$ ,  $r = -2$ , and  $n = 20$ .

Now  $S_n = \frac{u_1(r^n - 1)}{r - 1}$   
 $\therefore S_{20} = \frac{-24((-2)^{20} - 1)}{-2 - 1} = 8\,388\,600$

**33 a**  $10 + 14 + 18 + 22 + \dots + 138$  is arithmetic with  $u_1 = 10$ ,  $d = 4$

Now  $u_1 + (n-1)d = 138$   
 $\therefore 10 + 4(n-1) = 138$   
 $\therefore 4(n-1) = 128$   
 $\therefore n-1 = 32$   
 $\therefore n = 33$

So, the sum is  $\frac{n}{2}(u_1 + u_{33}) = \frac{33}{2}(10 + 138)$   
 $= \frac{33}{2}(148)$   
 $= 2442$



- b**  $6 - 12 + 24 - 48 + 96 - \dots + 1536$  is geometric with  $u_1 = 6$ ,  $r = -2$

$$\text{Now } u_1 r^{n-1} = 1536$$

$$\therefore 6 \times (-2)^{n-1} = 1536$$

$$\therefore n = 9 \quad \{\text{using technology}\}$$

$$\begin{aligned} \text{So, the sum is } \frac{u_1(1-r^n)}{1-r} &= \frac{6(1-(-2)^9)}{1-(-2)} \\ &= \frac{6}{3}(1-(-2)^9) \\ &= 2 \times 513 \\ &= 1026 \end{aligned}$$

Math (Des) Norm 1 (d/c) Real  
Eq:  $6 \times (-2)^{x-1} = 1536$   
 $x=9$   
Lft=1536  
Rgt=1536  
[REPEAT]

- 34 a i**  $\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^k = 2\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)^3 + \dots$  is an infinite geometric series with  $u_1 = 2\left(\frac{2}{3}\right) = \frac{4}{3}$  and  $r = \frac{2}{3}$ .

$$\begin{aligned} \text{ii } S &= \frac{u_1}{1-r} \\ \therefore S &= \frac{\frac{4}{3}}{1-\frac{2}{3}} \\ &= \frac{\frac{4}{3}}{\frac{1}{3}} \\ &= 4 \end{aligned}$$

- b i**  $\sum_{k=1}^n (k-4) = -3 - 2 - 1 + 0 + 1 + \dots + (n-4)$  is an arithmetic series with  $u_1 = -3$  and  $d = 1$ .

$$\begin{aligned} \text{ii } S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ \therefore S_n &= \frac{n}{2}(2(-3) + (n-1)) \\ &= \frac{n}{2}(-6 + n - 1) \\ &= \frac{n}{2}(n - 7) \end{aligned}$$

$$\begin{aligned} \text{c } S_n &= S \\ \therefore \frac{n}{2}(n-7) &= 4 \\ \therefore n(n-7) &= 8 \\ \therefore n^2 - 7n - 8 &= 0 \\ \therefore (n-8)(n+1) &= 0 \\ \therefore n &= 8 \quad \{n > 0\} \end{aligned}$$

- 35 a** The series will converge if  $|\text{common ratio}| < 1$

$$\therefore |x-2| < 1$$

$$\therefore -1 < x-2 < 1$$

$$\therefore 1 < x < 3$$

$$\begin{aligned} \text{b } \sum_{k=1}^{\infty} 12(x-2)^{k-1} &= \frac{12}{1-(x-2)} \quad \{u_1 = 12, r = x-2\} \\ &= \frac{12}{3-x} \end{aligned}$$

and  $x = \sqrt{5}$  satisfies  $1 < x < 3$

$$\therefore \text{when } x = \sqrt{5}, \text{ the sum of the series} = \frac{12}{3-\sqrt{5}} \approx 15.7.$$

- 36 a**  $N = 20 \times 12 = 240$ ,  $I\% = 7.2$ ,  $PV = 120\,000$ ,  $FV = 0$ ,  $P/Y = 12$ ,  $C/Y = 12$

$$\therefore PMT \approx -944.82$$

The monthly repayment is \$944.82.

Norm 1 End  
Compound Interest  
n = 240  
I% = 7.2  
PV = 120000  
PMT = -944.8191588  
FV = 0  
P/Y = 12  
[n] [I%] [PV] [PMT] [FV] [AMORTIZ]

- b**  $N = 12$ ,  $I\% = 7.2$ ,  $PV = 120\,000$ ,  $PMT = -944.82$ ,  $P/Y = 12$ ,  $C/Y = 12$

$$\therefore FV \approx 117\,211.33$$

After 1 year, \$117 211.33 is still owing on the loan.

Norm 1 End  
Compound Interest  
n = 12  
I% = 7.2  
PV = 120000  
PMT = -944.82  
FV = -117211.3264  
P/Y = 12  
[n] [I%] [PV] [PMT] [FV] [AMORTIZ]

$$\begin{aligned} \text{c i Amount paid} &= \$944.82 \times 12 \\ &= \$11\,337.84 \end{aligned}$$

$$\text{ii } \$120\,000 - \$117\,211.33 = \$2788.67$$



- iii The load does not decrease by the full amount of the monthly repayment as the payment is used to pay interest as well as to reduce the principal.

d i  $N = 19 \times 12 = 228$ ,  $I\% = 6.95$ ,  $PV = 117\,211.33$ ,  $FV = 0$ ,  
 $P/Y = 12$ ,  $C/Y = 12$

$$\therefore PMT \approx -927.42$$

The new monthly repayment is \$927.42.

ii  $I\% = 6.95$ ,  $PV = 117\,211.33$ ,  $PMT = -944.82$ ,  $FV = 0$ ,  
 $P/Y = 12$ ,  $C/Y = 12$

$$\therefore N \approx 219.5$$

It will take 220 months to pay off the rest of the loan with the original repayments.

$\therefore$  the loan will be paid off 8 months earlier.

37 a  $N = 8 \times 4 = 32$ ,  $I\% = 6.55$ ,  $PMT = -933.62$ ,  $FV = 0$ ,  $P/Y = 4$ ,  
 $C/Y = 4$

$$\therefore PV \approx 23\,110.12$$

Oscar borrowed \$23 110.12.

b Interest =  $\$933.62 \times 8 \times 4 - \$23\,110.12$   
 $= \$6765.72$

c i  $N = 6 \times 4 = 24$ ,  $I\% = 6.55$ ,  $PV = 23\,110.12$ ,  $PMT = -933.62$ ,  
 $P/Y = 4$ ,  $C/Y = 4$

$$\therefore FV \approx -6947.33$$

So, the outstanding balance is \$6947.33.

ii  $I\% = 6.55$ ,  $PV = 6947.33 - 3000$ ,  $PMT = -933.62$ ,  $FV = 0$ ,  
 $P/Y = 4$ ,  $C/Y = 4$

$$\therefore N \approx 4.417$$

It will take another 5 quarters to pay off the rest of the loan.

$$\begin{aligned} \therefore \text{time saved} &= 8 \times 4 - (6 \times 4 + 5) \\ &= 32 - 29 \\ &= 3 \text{ quarters} \end{aligned}$$

38 a real interest rate multiplier  $\times 1.003 = 1.012$

$$\begin{aligned} \therefore \text{real interest rate multiplier} &= \frac{1.012}{1.003} \\ &\approx 1.00897 \end{aligned}$$

$$\therefore \text{real interest rate} \approx 0.897\% \approx 0.9\%$$

b  $N = 5 \times 4 = 20$ ,  $I\% = 1.2$ ,  $PV = -2000$ ,  $PMT = -500$ ,  $P/Y = 4$ ,  
 $C/Y = 4$

$$\therefore FV \approx 12\,413.68$$

After 5 years, Cassie will have €12 413.68 in her savings account.

$$\text{Now real value} \times (1.003)^{5 \times 4} = €12\,413.68$$

$$\begin{aligned} \therefore \text{real value} &= \frac{€12\,413.68}{(1.003)^{20}} \\ &\approx €11\,691.81 \end{aligned}$$

39 a  $N = 4 \times 12 = 48$ ,  $PV = -81\,000$ ,  $PMT = 2000$ ,  $FV = 0$ ,  $P/Y = 12$ ,  
 $C/Y = 12$

$$\therefore I\% \approx 8.60$$

Bill needs to receive 8.60% p.a. compounded monthly.



$$\mathbf{b} \quad N = 48, \quad I\% = 7, \quad PV = -81\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$

$$\therefore PMT \approx 1939.64$$

Bill would receive \$1939.64 per month.

$$\mathbf{40} \quad \mathbf{a} \quad N = 10 \times 12 = 120, \quad I\% = 5.2, \quad PMT = 200 \times 2 = 400, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$

$$\therefore PV \approx 37\,367.19$$

So \$37 367.19 should be invested now to provide such an annuity.

$$\mathbf{b} \quad \text{Total interest} = \$200 \times 2 \times 12 \times 10 - \$37\,367.19$$

$$= \$48\,000 - \$37\,367.19$$

$$= \$10\,632.81$$

$$\begin{aligned} \mathbf{41} \quad \mathbf{a} \quad & \log(10^9 \times 1000^b) \\ &= \log(10^9 \times (10^3)^b) \\ &= \log(10^9 \times 10^{3b}) \\ &= \log(10^{9+3b}) \\ &= 9 + 3b \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log\left(\frac{10^n}{100}\right) \\ &= \log\left(\frac{10^n}{10^2}\right) \\ &= \log(10^{n-2}) \\ &= n - 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log(2^t \times 5^t) \\ &= \log((2 \times 5)^t) \\ &= \log(10^t) \\ &= t \end{aligned}$$

$$\mathbf{42} \quad \mathbf{a} \quad 2 = 10^{\log 2} \\ \approx 10^{0.3010}$$

$$\mathbf{b} \quad 200 = 10^{\log 200} \\ \approx 10^{2.3010}$$

$$\mathbf{c} \quad 0.02 = 10^{\log 0.02} \\ \approx 10^{-1.6990}$$

$$\begin{aligned} \mathbf{43} \quad \mathbf{a} \quad & \ln(e^k \times e^4) \\ &= \ln(e^{k+4}) \\ &= k + 4 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \ln\left(\frac{e}{e^m}\right) \\ &= \ln(e^{1-m}) \\ &= 1 - m \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & e^{2 \ln 6} = (e^{\ln 6})^2 \\ &= 6^2 \\ &= 36 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & e^{-\ln 3} = (e^{\ln 3})^{-1} \\ &= 3^{-1} \\ &= \frac{1}{3} \end{aligned}$$

$$\mathbf{44} \quad \mathbf{a} \quad 47 = e^{\ln 47} \\ \approx e^{3.8501}$$

$$\mathbf{b} \quad 500 = e^{\ln 500} \\ \approx e^{6.2146}$$

$$\mathbf{c} \quad 0.023 = e^{\ln 0.023} \\ \approx e^{-3.7723}$$

$$\begin{aligned} \mathbf{45} \quad \mathbf{a} \quad & \frac{1}{4} \ln 81 + \ln 12 - \ln 4 \\ &= \frac{1}{4} \ln(3^4) + \ln(3 \times 4) - \ln 4 \\ &= \frac{1}{4}(4 \ln 3) + \ln 3 + \cancel{\ln 4} - \cancel{\ln 4} \\ &= \ln 3 + \ln 3 \\ &= 2 \ln 3 \\ &= \ln(3^2) \\ &= \ln 9 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 3 \log 2 - \log 24 \\ &= \log(2^3) - \log 24 \\ &= \log 8 - \log 24 \\ &= \log\left(\frac{8}{24}\right) \\ &= \log\left(\frac{1}{3}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 5 + \log 3 - \frac{1}{2} \log 49 = 5 + \log 3 - \log(49^{\frac{1}{2}}) \\ &= \log(10^5) + \log 3 - \log 7 \\ &= \log(3 \times 10^5) - \log 7 \\ &= \log\left(\frac{300\,000}{7}\right) \end{aligned}$$

$$\mathbf{46} \quad x = \log 5$$

$$\begin{aligned} \mathbf{a} \quad & \log 50 \\ &= \log(5 \times 10) \\ &= \log 5 + \log 10 \\ &= x + 1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log\left(\frac{125}{100}\right) \\ &= \log 125 - \log 100 \\ &= \log(5^3) - \log(10^2) \\ &= 3 \log 5 - 2 \log 10 \\ &= 3x - 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log \sqrt[3]{5} \\ &= \log(5^{\frac{1}{3}}) \\ &= \frac{1}{3} \log 5 \\ &= \frac{1}{3}x \end{aligned}$$



**47**  $A = \log P, \quad B = \log Q, \quad C = \log R$

**a**  $\log(PQ) = \log P + \log Q$   
 $= A + B$

**b**  $\log(P^2Q\sqrt{R})$   
 $= \log(P^2) + \log Q + \log(R^{\frac{1}{2}})$   
 $= 2\log P + \log Q + \frac{1}{2}\log R$   
 $= 2A + B + \frac{1}{2}C$

**c**  $\log\left(\frac{PQ^3}{R^2}\right)$   
 $= \log P + \log(Q^3) - \log(R^2)$   
 $= A + 3\log Q - 2\log R$   
 $= A + 3B - 2C$

**48 a**  $\frac{\log 9}{\log 3} = \frac{\log(3^2)}{\log 3}$   
 $= \frac{2\log 3}{\log 3}$   
 $= 2$

**b**  $\frac{\log 8}{\log 4} = \frac{\log(2^3)}{\log(2^2)}$   
 $= \frac{3\log 2}{2\log 2}$   
 $= \frac{3}{2}$

**c**  $\frac{\log 0.25}{\log 64} = \frac{\log(2^{-2})}{\log(2^6)}$   
 $= \frac{-2\log 2}{6\log 2}$   
 $= -\frac{2}{6}$   
 $= -\frac{1}{3}$

**49 a**  $\ln 20 - \ln 10$   
 $= \ln(2 \times 10) - \ln 10$   
 $= \ln 2 + \cancel{\ln 10} - \cancel{\ln 10}$   
 $= \ln 2$

**b**  $-\ln 13 - 3$   
 $= -(\ln 13 + 3)$   
 $= -(\ln 13 + \ln(e^3))$   
 $= -\ln(13e^3)$   
 $= \ln\left(\frac{1}{13e^3}\right)$

**c**  $\frac{1}{3}\ln 64 + 2\ln 2$   
 $= \ln(64^{\frac{1}{3}}) + \ln(2^2)$   
 $= \ln 4 + \ln 4$   
 $= 2\ln 4$   
 $= \ln(4^2)$   
 $= \ln 16$

**50 a**  $\log x = -2$   
 $\therefore x = 10^{-2}$   
 $\therefore x = 0.01$

**b**  $\log x = \frac{1}{3}$   
 $\therefore x = 10^{\frac{1}{3}}$   
 $\therefore x \approx 2.15$

**c**  $\log x \approx 5.2831$   
 $\therefore x \approx 10^{5.2831}$   
 $\therefore x \approx 192\,000$

**51 a**  $3\log x = \log 24 + \log\left(\frac{1}{3}\right)$   
 $\therefore \log(x^3) = \log\left(\frac{24}{3}\right)$   
 $\therefore \log(x^3) = \log 8$   
 $\therefore x^3 = 8$   
 $\therefore x = 2$

**b**  $\ln x = \ln 12 - \ln(7 - x)$   
 $\therefore \ln x = \ln\left(\frac{12}{7 - x}\right)$   
 $\therefore x = \frac{12}{7 - x}$   
 $\therefore x(7 - x) = 12$   
 $\therefore 7x - x^2 = 12$   
 $\therefore x^2 - 7x + 12 = 0$   
 $\therefore (x - 3)(x - 4) = 0$   
 $\therefore x = 3 \text{ or } 4$

**c**  $\ln(x^2 - 3) - \ln(2x) = 0$   
 $\therefore \ln\left(\frac{x^2 - 3}{2x}\right) = 0$   
 $\therefore \frac{x^2 - 3}{2x} = e^0 = 1$   
 $\therefore x^2 - 3 = 2x$   
 $\therefore x^2 - 2x - 3 = 0$   
 $\therefore (x - 3)(x + 1) = 0$   
 $\therefore x = 3 \text{ or } -1$

But  $x = -1$  does not satisfy the original equation, as  $\ln(-2)$  is undefined.

$\therefore$  the only solution is  $x = 3$ .

**d**  $\log x + \log(x - 2) = 1$   
 $\therefore \log(x(x - 2)) = 1$   
 $\therefore x(x - 2) = 10$   
 $\therefore x^2 - 2x - 10 = 0$   
 $\therefore x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-10)}}{2}$   
 $= \frac{2 \pm \sqrt{44}}{2}$   
 $= \frac{2 \pm 2\sqrt{11}}{2}$   
 $= 1 \pm \sqrt{11}$

But  $x = 1 - \sqrt{11} \approx -2.32$  does not satisfy the original equation, as  $\log(1 - \sqrt{11})$  is undefined.

$\therefore$  the only solution is  $x = 1 + \sqrt{11}$ .



$$52 \quad M = -2.5 \log\left(\frac{F}{F_0}\right)$$

Jupiter has apparent magnitude  $-2.6$ .

$$\text{When } M = -2.6, \quad -2.6 = -2.5 \log\left(\frac{F}{F_0}\right)$$

$$\therefore \log\left(\frac{F}{F_0}\right) = \frac{2.6}{2.5} = 1.04$$

$$\therefore \frac{F}{F_0} = 10^{1.04}$$

$$\therefore F = 10^{1.04} \times F_0$$

$\therefore$  Jupiter has observed flux density  $10^{1.04} \times F_0$ .

$\therefore$  Venus has observed flux density  $5.2 \times 10^{1.04} \times F_0$ .

$$\begin{aligned} \text{When } F = 5.2 \times 10^{1.04} \times F_0, \quad M &= -2.5 \log\left(\frac{5.2 \times 10^{1.04} \times F_0}{F_0}\right) \\ &= -2.5 \log(5.2 \times 10^{1.04}) \\ &\approx -4.39 \end{aligned}$$

$\therefore$  Venus has apparent magnitude  $-4.39$ .

$$53 \quad \mathbf{a} \quad 3x^2 = -27$$

$$\therefore x^2 = -9$$

$$\therefore x = \pm 3i$$

$$\mathbf{b} \quad x^2 + x + 1 = 0$$

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} \quad \{\text{quadratic formula}\}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\mathbf{c} \quad 2x^2 + x + 5 = 0$$

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(2)(5)}}{2(2)} \quad \{\text{quadratic formula}\}$$

$$= \frac{-1 \pm \sqrt{-39}}{4}$$

$$= -\frac{1}{4} \pm \frac{\sqrt{39}}{4}i$$

$$54 \quad z = 2 + i, \quad w = 3 + 5i$$

$$\begin{aligned} \mathbf{a} \quad z - 3w &= (2 + i) - 3(3 + 5i) \\ &= 2 + i - 9 - 15i \\ &= -7 - 14i \end{aligned}$$

$$\therefore \operatorname{Re}(z - 3w) = -7$$

$$\begin{aligned} \mathbf{b} \quad iw^2 &= i(3 + 5i)^2 \\ &= i(9 + 30i + 25i^2) \\ &= i(-16 + 30i) \\ &= -16i + 30i^2 \\ &= -30 - 16i \end{aligned}$$

$$\therefore \operatorname{Im}(iw^2) = -16$$

$$\mathbf{c} \quad \frac{z}{w} = \left(\frac{2+i}{3+5i}\right) \times \left(\frac{3-5i}{3-5i}\right)$$

$$= \frac{6 - 10i + 3i - 5i^2}{9 - 25i^2}$$

$$= \frac{11 - 7i}{34}$$

$$= \frac{11}{34} - \frac{7}{34}i$$

$$\therefore \operatorname{Re}\left(\frac{z}{w}\right) = \frac{11}{34}$$



$$\begin{aligned}
 55 \quad & \frac{z+2}{z-2} = i \\
 \therefore & z+2 = i(z-2) \\
 \therefore & z+2 = iz-2i \\
 \therefore & z-iz = -2-2i \\
 \therefore & z(1-i) = -2-2i \\
 \therefore & z = \left( \frac{-2-2i}{1-i} \right) \times \left( \frac{1+i}{1+i} \right) \\
 & = \frac{-2-2i-2i-2i^2}{1^2-i^2} \\
 & = \frac{-4i}{2} \\
 & = -2i
 \end{aligned}$$

$$\begin{aligned}
 56 \quad \mathbf{a} \quad & (3-2i)(x-yi) = -i \\
 \therefore & 3x-3yi-2xi+2yi^2 = -i \\
 \therefore & (3x-2y) + (-3y-2x)i = -i \\
 & \text{Equating real and imaginary parts,} \\
 3x-2y = 0 & \quad \text{and} \quad -3y-2x = -1 \\
 \therefore 3x = 2y & \quad \text{and} \quad 3y+2x = 1 \\
 \therefore x = \frac{2y}{3} & \quad \text{and} \quad 3y+2x = 1 \\
 \therefore 3y+2\left(\frac{2y}{3}\right) = 1 & \\
 \therefore 9y+4y = 3 & \\
 \therefore 13y = 3 & \\
 \therefore y = \frac{3}{13} & \\
 \text{and } x = \frac{2\left(\frac{3}{13}\right)}{3} = \frac{2}{13} &
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (x+yi)^2 - (x-yi)^2 = x-y+16i \\
 \therefore x^2+2xyi+y^2i^2 - x^2+2xyi-y^2i^2 & = x-y+16i \\
 \therefore 4xyi = x-y+16i & \\
 & \text{Equating real and imaginary parts,} \\
 x-y = 0 & \quad \text{and} \quad 4xy = 16 \\
 \therefore x = y & \quad \text{and} \quad 4xy = 16 \\
 \therefore 4x^2 = 16 & \\
 \therefore x^2 = 4 & \\
 \therefore x = \pm 2 \text{ and } y = \pm 2 &
 \end{aligned}$$

$$\begin{array}{|l}
 \mathbf{57} \quad \mathbf{a} \quad \begin{array}{|l}
 \text{Math Rad Norm1 d/c Real} \\
 (6-5i)^3 \\
 \square \\
 -234-415i \\
 \text{JUMP DELETE ▶ MAT MATH}
 \end{array}
 \end{array}$$

$$(6-5i)^3 = -234-415i$$

$$\begin{array}{|l}
 \mathbf{b} \quad \begin{array}{|l}
 \text{Math Rad Norm1 d/c Real} \\
 \frac{1}{(1-3i)^2} \\
 \square \\
 -0.08+0.06i \\
 \text{JUMP DELETE ▶ MAT MATH}
 \end{array}
 \end{array}$$

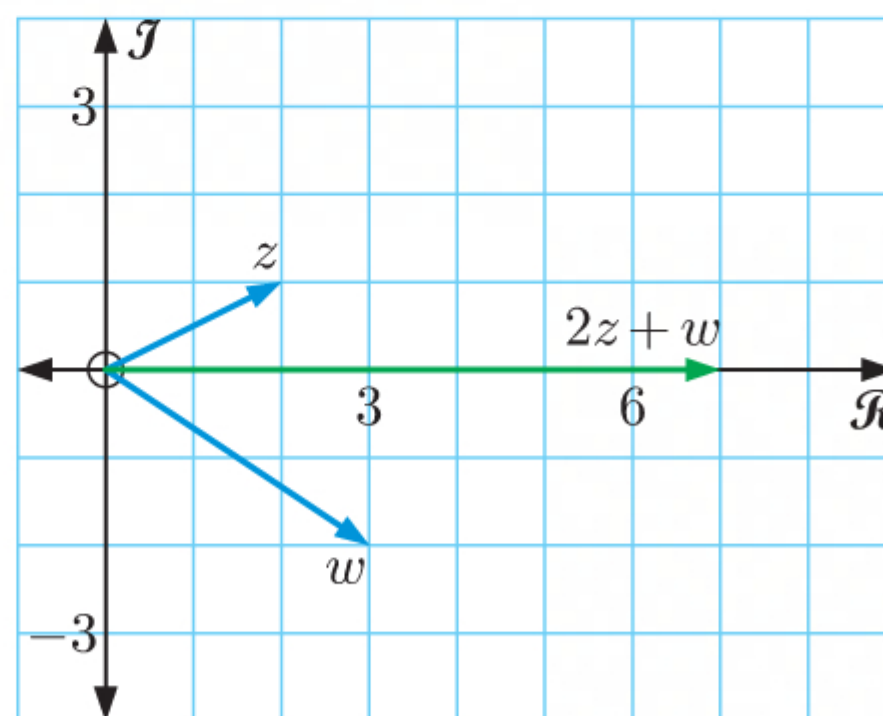
$$\frac{1}{(1-3i)^2} = -0.08+0.06i$$

$$\begin{array}{|l}
 \mathbf{c} \quad \begin{array}{|l}
 \text{Math Rad Norm1 d/c Real} \\
 \frac{(4+3i)^4}{2-i} \\
 \square \\
 -278+29i \\
 \text{JUMP DELETE ▶ MAT MATH}
 \end{array}
 \end{array}$$

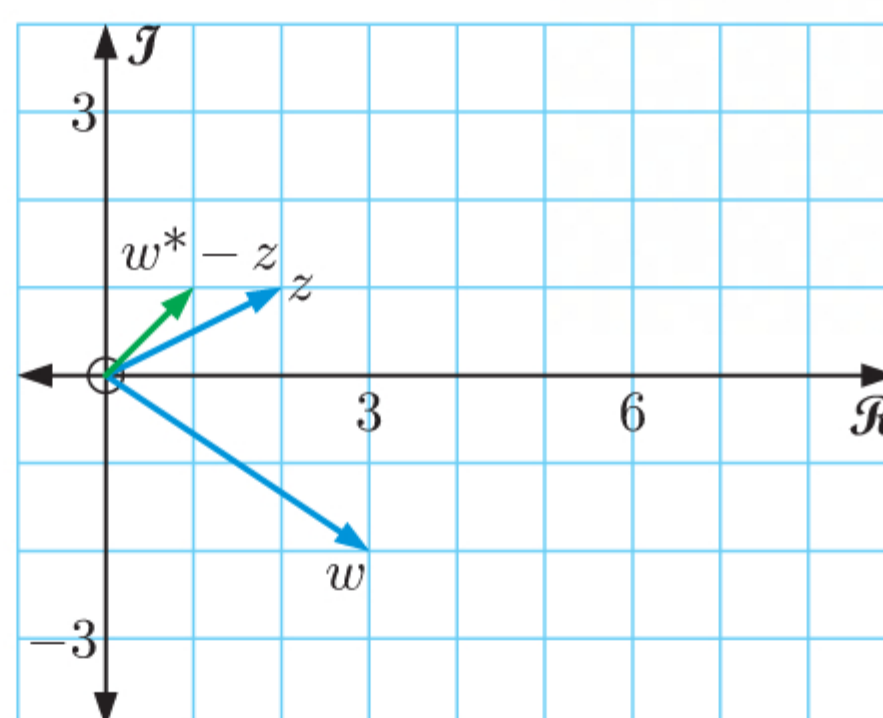
$$\frac{(4+3i)^4}{2-i} = -278+29i$$

$$58 \quad z = 2+i, \quad w = 3-2i$$

$$\begin{aligned}
 \mathbf{a} \quad 2z+w &= 2(2+i) + (3-2i) \\
 &= 4+2i+3-2i \\
 &= 7
 \end{aligned}$$

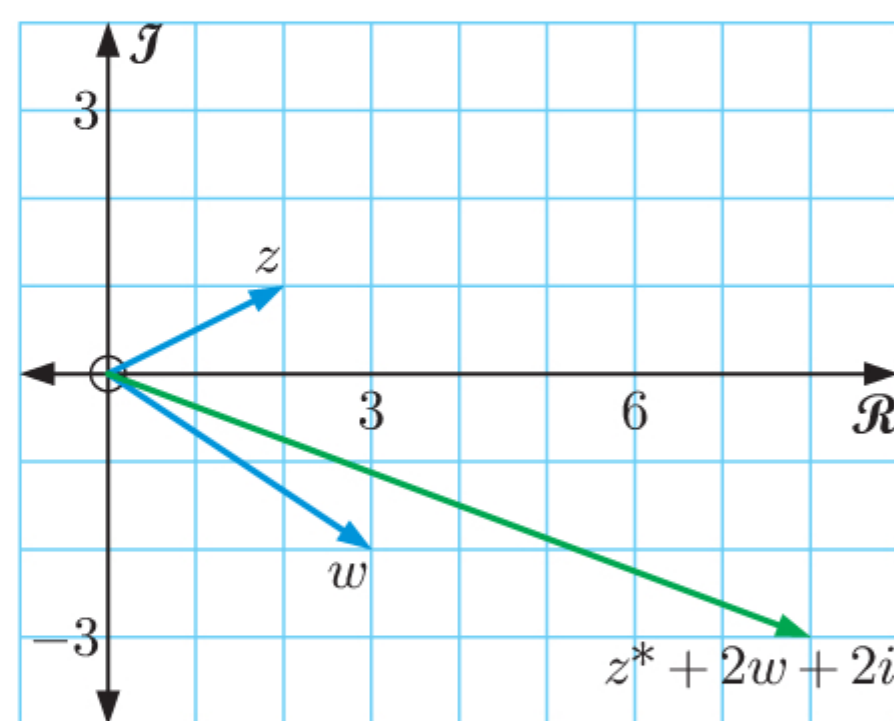


$$\begin{aligned}
 \mathbf{b} \quad w^* - z &= 3+2i - (2+i) \\
 &= 3+2i-2-i \\
 &= 1+i
 \end{aligned}$$





$$\begin{aligned}
 \text{c } z^* + 2w + 2i &= 2 - i + 2(3 - 2i) + 2i \\
 &= 2 - i + 6 - 4i + 2i \\
 &= 8 - 3i
 \end{aligned}$$



$$59 \quad |z| = 3$$

$$\begin{aligned}
 \text{a } |3z| &= 3|z| \\
 &= 3(3) \\
 &= 9
 \end{aligned}$$

$$\begin{aligned}
 \text{b } |(2+i)z| &= |2+i||z| \\
 &= \sqrt{2^2 + 1^2}(3) \\
 &= 3\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \left| \frac{2i}{z^2} \right| &= \frac{|2i|}{|z^2|} \\
 &= \frac{2|i|}{|z|^2} \\
 &= \frac{2(1)}{3^2} \\
 &= \frac{2}{9}
 \end{aligned}$$

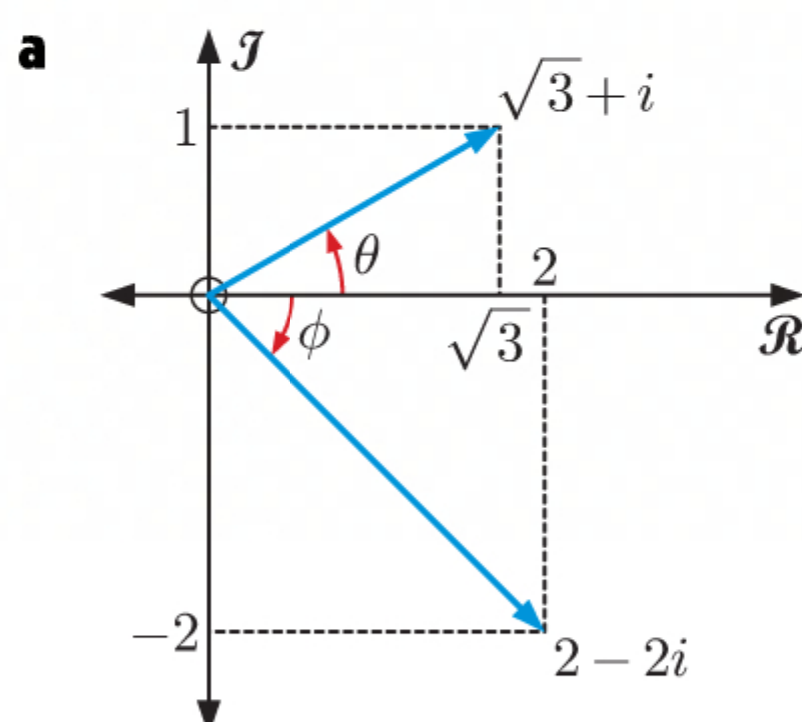
$$\begin{aligned}
 60 \quad \text{a } 2 \operatorname{cis} \frac{\pi}{7} \operatorname{cis} \frac{6\pi}{7} \\
 &= 2 \operatorname{cis} \left( \frac{\pi}{7} + \frac{6\pi}{7} \right) \quad \{ \operatorname{cis} \theta \operatorname{cis} \phi = \operatorname{cis}(\theta + \phi) \} \\
 &= 2 \operatorname{cis} \pi \\
 &= 2(\cos \pi + i \sin \pi) \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \frac{\sqrt{8} \operatorname{cis} \frac{3\pi}{16}}{\sqrt{2} \operatorname{cis} \left( -\frac{5\pi}{16} \right)} \\
 &= \frac{2\sqrt{2}}{\sqrt{2}} \operatorname{cis} \left( \frac{3\pi}{16} - \left( -\frac{5\pi}{16} \right) \right) \quad \left\{ \frac{\operatorname{cis} \theta}{\operatorname{cis} \phi} = \operatorname{cis}(\theta - \phi) \right\} \\
 &= 2 \operatorname{cis} \frac{8\pi}{16} \\
 &= 2 \operatorname{cis} \frac{\pi}{2} \\
 &= 2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \\
 &= 2i
 \end{aligned}$$

$$\begin{aligned}
 \text{b } (\operatorname{cis} \frac{5\pi}{12})^2 \\
 &= \operatorname{cis} \left( 2 \times \frac{5\pi}{12} \right) \quad \{ (\operatorname{cis} \theta)^n = \operatorname{cis} n\theta \} \\
 &= \operatorname{cis} \frac{5\pi}{6} \\
 &= \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \\
 &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \operatorname{cis}(\theta + 15\pi) \\
 &= \operatorname{cis} \theta \operatorname{cis} 15\pi \quad \{ \operatorname{cis}(\theta + \phi) = \operatorname{cis} \theta \operatorname{cis} \phi \} \\
 &= \operatorname{cis} \theta (\operatorname{cis} \pi)^{15} \quad \{ \operatorname{cis} n\theta = (\operatorname{cis} \theta)^n \} \\
 &= (\cos \theta + i \sin \theta)(\cos \pi + i \sin \pi)^{15} \\
 &= (\cos \theta + i \sin \theta)(-1)^{15} \\
 &= -\cos \theta - i \sin \theta
 \end{aligned}$$

$$61 \quad z = \sqrt{3} + i, \quad w = 2 - 2i$$



$$|z| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\text{Now } \tan \theta = \frac{1}{\sqrt{3}}$$

$$\therefore \theta = \frac{\pi}{6}$$

$$\therefore \arg z = \frac{\pi}{6}$$

$$\therefore z = 2 \operatorname{cis} \frac{\pi}{6}$$

$$|w| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\text{Now } \tan \phi = \frac{-2}{2} = -1$$

$$\therefore \phi = -\frac{\pi}{4}$$

$$\therefore \arg w = -\frac{\pi}{4}$$

$$\therefore w = 2\sqrt{2} \operatorname{cis} \left( -\frac{\pi}{4} \right)$$

$$\begin{aligned}
 \text{b } zw &= 2 \operatorname{cis} \frac{\pi}{6} \times 2\sqrt{2} \operatorname{cis} \left( -\frac{\pi}{4} \right) \\
 &= 4\sqrt{2} \operatorname{cis} \left( \frac{\pi}{6} - \frac{\pi}{4} \right) \\
 &= 4\sqrt{2} \operatorname{cis} \left( -\frac{\pi}{12} \right)
 \end{aligned}$$

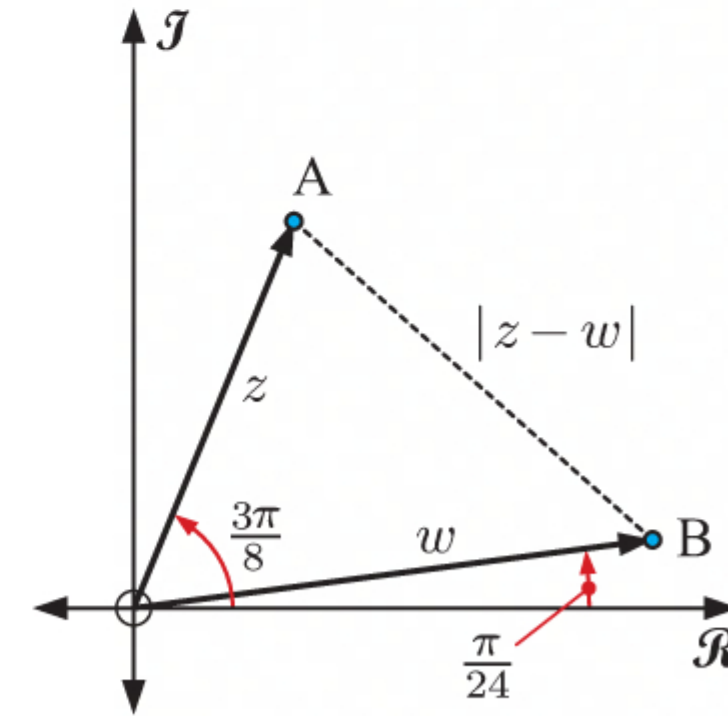
c When  $z$  is multiplied by  $w$ , it is dilated with scale factor  $2\sqrt{2}$ , then rotated clockwise through  $\frac{\pi}{4}$  about the origin.



**62**  $|z| = 4, |w| = 5$

**a i**  $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$   
 $= \frac{3\pi}{8} - \frac{\pi}{24}$   
 $= \frac{8\pi}{24}$   
 $= \frac{\pi}{3}$

**ii**  $|z - w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos\frac{\pi}{3}$   
 $= 16 + 25 - 2 \times 4 \times 5 \times \frac{1}{2}$   
 $= 21$   
 $\therefore |z - w| = \sqrt{21} \quad \{|z - w| \geq 0\}$



**b** Perimeter  $= |z| + |w| + |z - w|$       Area  $= \frac{1}{2}|z||w|\sin\frac{\pi}{3}$   
 $= 4 + 5 + \sqrt{21}$        $= \frac{1}{2} \times 4 \times 5 \times \frac{\sqrt{3}}{2}$   
 $= 9 + \sqrt{21}$  units       $= 5\sqrt{3}$  units<sup>2</sup>

**63 a**  $z_1 - z_2$  represents  $\overrightarrow{BA}$

$z_3 - z_2$  represents  $\overrightarrow{BC}$

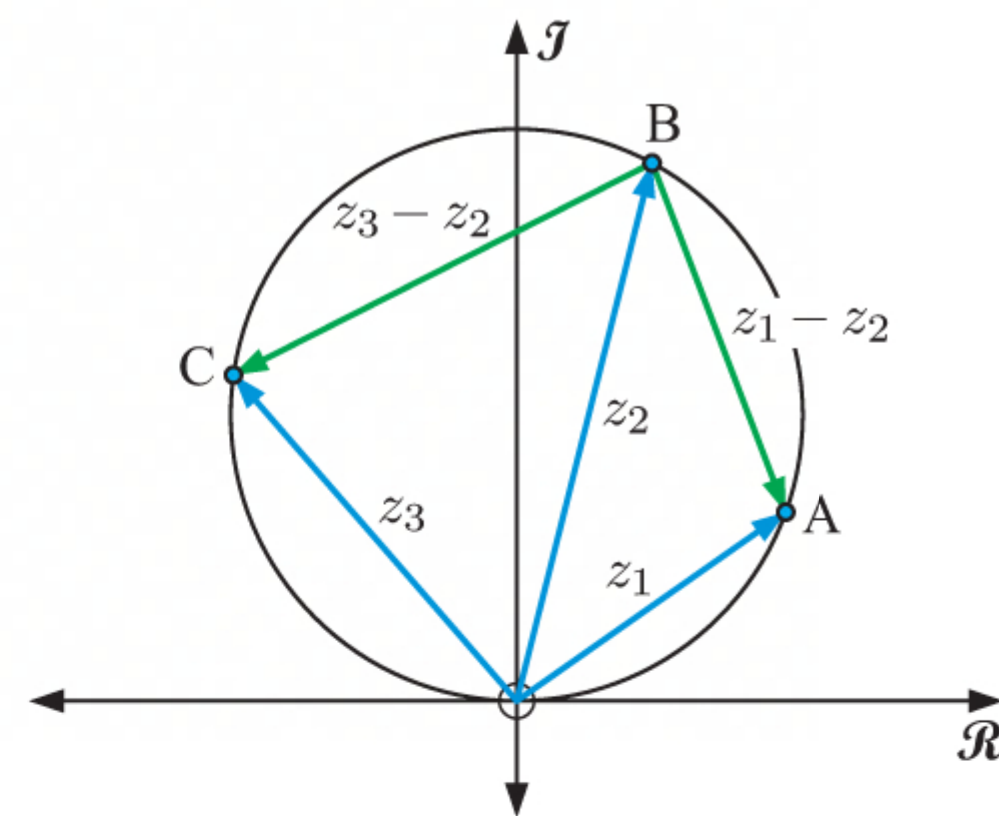
**b** OABC is a cyclic quadrilateral.

$\therefore \widehat{AOC} + \widehat{ABC} = \pi$

Now  $\widehat{AOC} = \arg z_3 - \arg z_1$   
 $= \arg\left(\frac{z_3}{z_1}\right)$

and  $\widehat{ABC} = \arg(z_1 - z_2) - \arg(z_3 - z_2)$  {using **a**}  
 $= \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$

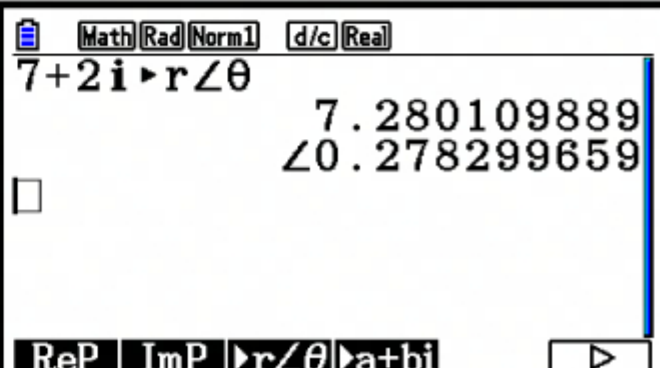
$\therefore \arg\left(\frac{z_3}{z_1}\right) + \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \pi$



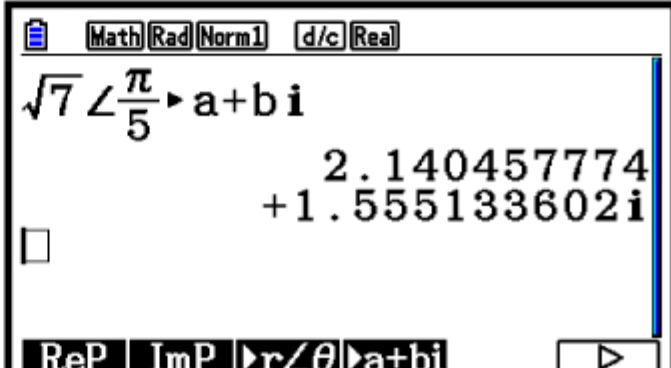
**64 a**  $|\sqrt{3} + i| = 2, \arg(\sqrt{3} + i) = \frac{\pi}{6}$   
 $\therefore \sqrt{3} + i = 2 \operatorname{cis} \frac{\pi}{6}$  (polar form)  
 $= 2e^{i\frac{\pi}{6}}$  (exponential form)

**b**  $2 \operatorname{cis} \frac{5\pi}{6} = 2 \cos \frac{5\pi}{6} + 2i \sin \frac{5\pi}{6}$   
 $= -\sqrt{3} + i$  (Cartesian form)  
 $= 2e^{i\frac{5\pi}{6}}$  (exponential form)

**c**  $5e^{-i\frac{\pi}{4}} = 5 \cos\left(-\frac{\pi}{4}\right) + 5i \sin\left(-\frac{\pi}{4}\right)$   
 $= \frac{5}{\sqrt{2}} - \frac{5}{\sqrt{2}}i$  (Cartesian form)  
 $= 5 \operatorname{cis}\left(-\frac{\pi}{4}\right)$  (polar form)

**65 a** 

$7 + 2i \approx 7.28 \operatorname{cis} 0.278$

**b** 

$\sqrt{7} \operatorname{cis} \frac{\pi}{5} \approx 2.14 + 1.56i$

**66 a**  $|1 + i| = \sqrt{2}, \arg(1 + i) = \frac{\pi}{4}$   
 $\therefore 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$   
 $|\sqrt{3} - i| = 2, \arg(\sqrt{3} - i) = -\frac{\pi}{6}$   
 $\therefore \sqrt{3} - i = 2e^{-i\frac{\pi}{6}}$

**b**  $z = \frac{-1 - i}{\sqrt{3} - i}$   
 $= \frac{-(1 + i)}{\sqrt{3} - i}$   
 $= -\frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{-i\frac{\pi}{6}}}$  {using **a**}  
 $= -\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4} + i\frac{\pi}{6}}$   
 $= -\frac{1}{\sqrt{2}}e^{i\frac{5\pi}{12}}$



$$\begin{aligned} \mathbf{c} \quad z &= -\frac{1}{\sqrt{2}} e^{i\frac{5\pi}{12}} \\ \therefore z^n &= \left(-\frac{1}{\sqrt{2}}\right)^n e^{i\frac{5\pi n}{12}} \\ &= \left(-\frac{1}{\sqrt{2}}\right)^n \left(\cos \frac{5\pi n}{12} + i \sin \frac{5\pi n}{12}\right) \end{aligned}$$

So,  $z^n$  is real when  $\sin \frac{5\pi n}{12} = 0$

$$\therefore \frac{5\pi n}{12} = k\pi, \quad k \in \mathbb{Z}$$

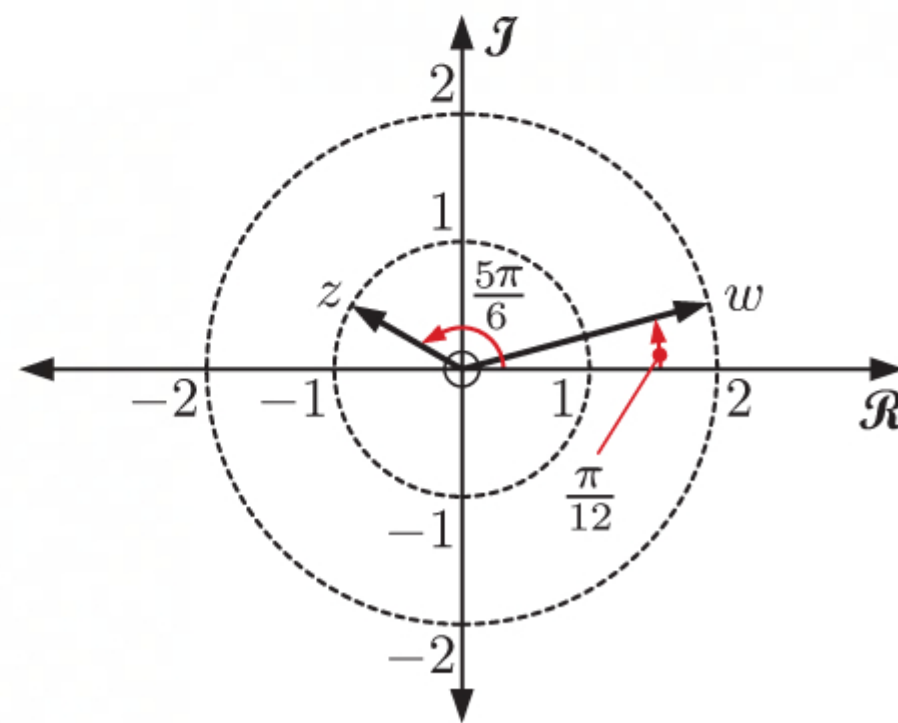
$$\therefore n = \frac{12}{5}k, \quad k \in \mathbb{Z}$$

$\therefore$  the smallest positive integer  $n$  such that  $z^n$  is a real number is  $n = 12$  when  $k = 5$ .

$$\mathbf{67} \quad \mathbf{a} \quad |-1 + i| = \sqrt{2}, \quad \arg(-1 + i) = \frac{3\pi}{4}$$

$$\begin{aligned} \therefore -1 + i &= \sqrt{2} \operatorname{cis} \frac{3\pi}{4} \\ &= \sqrt{2} e^{i\frac{3\pi}{4}} \end{aligned}$$

$$\mathbf{b} \quad z = e^{\frac{5\pi}{6}i}, \quad w = 2e^{\frac{\pi}{12}i}$$



$\mathbf{c} \quad (-1 + i)w^n$  is perpendicular to  $z$  if the difference between their arguments is an odd multiple of  $\frac{\pi}{2}$ .

$$\begin{aligned} (-1 + i)w^n &= \sqrt{2} e^{i\frac{3\pi}{4}} \times (2e^{\frac{\pi}{12}i})^n \quad \{\text{using a}\} \\ &= 2^n \sqrt{2} e^{i\frac{3\pi}{4}} \times e^{\frac{n\pi}{12}i} \\ &= 2^n \sqrt{2} e^{i(\frac{3\pi}{4} + \frac{n\pi}{12})} \end{aligned}$$

Now we require  $\frac{3\pi}{4} + \frac{n\pi}{12} - \frac{5\pi}{6} = (2k + 1)\frac{\pi}{2}$  where  $k \in \mathbb{Z}$ .

$$\therefore \frac{n\pi}{12} - \frac{\pi}{12} = (2k + 1)\frac{\pi}{2}$$

$$\therefore \frac{(n - 1)\pi}{12} = (2k + 1)\frac{\pi}{2}$$

$$\therefore n - 1 = 6(2k + 1)$$

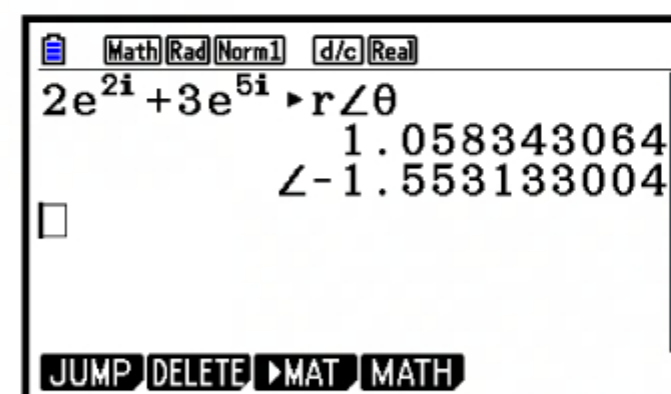
$$\therefore n = 6(2k + 1) + 1$$

We want the *smallest* positive integer, so we try  $k = 0$ .

$$\therefore n = 6(0 + 1) + 1 = 7$$

$\therefore$  the smallest positive integer  $n$  such that  $(-1 + i)w^n$  is perpendicular to  $z$  is  $n = 7$ .

$$\begin{aligned} \mathbf{68} \quad \mathbf{a} \quad &2 \sin(4t + 2) + 3 \sin(4t + 5) \\ &= \operatorname{Im}(2 \operatorname{cis}(4t + 2) + 3 \operatorname{cis}(4t + 5)) \\ &= \operatorname{Im}(2e^{(4t+2)i} + 3e^{(4t+5)i}) \\ &= \operatorname{Im}(e^{4ti}(2e^{2i} + 3e^{5i})) \\ &\approx \operatorname{Im}(e^{4ti}(1.06e^{-1.55i})) \\ &\approx \operatorname{Im}(1.06e^{(4t-1.55)i}) \\ &\approx 1.06 \sin(4t - 1.55) \end{aligned}$$





$$\begin{aligned}
\mathbf{b} \quad & 1.5 \cos(10t - 3) + 4.5 \cos(10t + 7) \\
&= \operatorname{Re}(1.5 \operatorname{cis}(10t - 3) + 4.5 \operatorname{cis}(10t + 7)) \\
&= \operatorname{Re}(1.5e^{(10t-3)i} + 4.5e^{(10t+7)i}) \\
&= \operatorname{Re}(e^{10ti}(1.5e^{-3i} + 4.5e^{7i})) \\
&\approx \operatorname{Re}(e^{10ti}(3.34e^{0.963i})) \\
&\approx \operatorname{Re}(3.34e^{(10t+0.963)i}) \\
&\approx 3.34 \cos(10t + 0.963)
\end{aligned}$$

$$69 \quad \mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 5 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 1 \\ -1 & 3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 4 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

**a i** The order of  $\mathbf{A}$  is  $2 \times 3$ .      **ii** The order of  $\mathbf{B}$  is  $2 \times 2$ .      **iii** The order of  $\mathbf{C}$  is  $2 \times 3$ .

$$\begin{aligned}
\mathbf{b} \quad \mathbf{i} \quad \mathbf{A} + \mathbf{C} &= \begin{pmatrix} 1 & -2 & 3 \\ 5 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1+4 & -2-1 & 3+2 \\ 5+2 & 0+1 & 1+0 \end{pmatrix} \\
&= \begin{pmatrix} 5 & -3 & 5 \\ 7 & 1 & 1 \end{pmatrix}
\end{aligned}$$

**ii**  $\mathbf{B} + \mathbf{C}$  cannot be found as the matrices do not have the same order.

$$\begin{aligned}
\mathbf{iii} \quad 3\mathbf{C} &= 3 \begin{pmatrix} 4 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 3 \times 4 & 3 \times (-1) & 3 \times 2 \\ 3 \times 2 & 3 \times 1 & 3 \times 0 \end{pmatrix} \\
&= \begin{pmatrix} 12 & -3 & 6 \\ 6 & 3 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{iv} \quad \frac{1}{2}\mathbf{A} - \mathbf{C} &= \frac{1}{2} \begin{pmatrix} 1 & -2 & 3 \\ 5 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & -1 & \frac{3}{2} \\ \frac{5}{2} & 0 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 4 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{7}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}
\end{aligned}$$

**v**  $\mathbf{B}$  is  $2 \times 2$  and  $\mathbf{A}$  is  $2 \times 3$   $\therefore \mathbf{BA}$  is  $2 \times 3$ .

$$\begin{aligned}
\mathbf{BA} &= \begin{pmatrix} 4 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 5 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 4 \times 1 + 1 \times 5 & 4 \times (-2) + 1 \times 0 & 4 \times 3 + 1 \times 1 \\ -1 \times 1 + 3 \times 5 & -1 \times (-2) + 3 \times 0 & -1 \times 3 + 3 \times 1 \end{pmatrix} \\
&= \begin{pmatrix} 9 & -8 & 13 \\ 14 & 2 & 0 \end{pmatrix}
\end{aligned}$$

**vi**  $\mathbf{A}$  is  $2 \times 3$  and  $\mathbf{B}$  is  $2 \times 2$   $\therefore \mathbf{AB}$  cannot be found.

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$$\begin{aligned}
&\begin{pmatrix} k+2 & -1 & 0 \\ 3 & k^2-7 & 1 \end{pmatrix} \begin{pmatrix} k+1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix} \\
\therefore \begin{pmatrix} (k+2)(k+1) - 1 \times (-1) + 0 \times 3 \\ 3(k+1) + (k^2-7) \times (-1) + 1 \times 3 \end{pmatrix} &= \begin{pmatrix} 21 \\ 13 \end{pmatrix} \\
\therefore \begin{pmatrix} k^2 + 3k + 2 + 1 \\ 3k + 3 - k^2 + 7 + 3 \end{pmatrix} &= \begin{pmatrix} 21 \\ 13 \end{pmatrix} \\
\therefore \begin{pmatrix} k^2 + 3k + 3 \\ -k^2 + 3k + 13 \end{pmatrix} &= \begin{pmatrix} 21 \\ 13 \end{pmatrix}
\end{aligned}$$

Equating elements gives  $-k^2 + 3k + 13 = 13$

$$\therefore -k^2 + 3k = 0$$

$$\therefore -k(k-3) = 0$$

$$\therefore k = 0 \text{ or } 3$$



If  $k = 0$ , then  $k^2 + 3k + 3 = 3$  ✗

If  $k = 3$ , then  $k^2 + 3k + 3 = 9 + 9 + 3 = 21$  ✓

∴  $k = 3$

$$\mathbf{71} \quad \mathbf{P} = \begin{pmatrix} 5 & 2 \\ 6 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 1 & 0 \\ -1 & 4 \\ 2 & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{a} \quad -5\mathbf{Q} &= -5 \begin{pmatrix} 1 & 0 \\ -1 & 4 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 0 \\ 5 & -20 \\ -10 & -5 \end{pmatrix} \end{aligned}$$

$\mathbf{b} \quad \mathbf{Q}$  is  $3 \times 2$  and  $\mathbf{P}$  is  $2 \times 2$  ∴  $\mathbf{QP}$  is  $3 \times 2$ .

$$\begin{aligned} \mathbf{QP} &= \begin{pmatrix} 1 & 0 \\ -1 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 6 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 5 + 0 \times 6 & 1 \times 2 + 0 \times 4 \\ -1 \times 5 + 4 \times 6 & -1 \times 2 + 4 \times 4 \\ 2 \times 5 + 1 \times 6 & 2 \times 2 + 1 \times 4 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 2 \\ 19 & 14 \\ 16 & 8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \det \mathbf{P} &= 5 \times 4 - 2 \times 6 \\ &= 20 - 12 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \mathbf{P}^{-1} &= \frac{1}{\det \mathbf{P}} \begin{pmatrix} 4 & -2 \\ -6 & 5 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} 4 & -2 \\ -6 & 5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{5}{8} \end{pmatrix} \end{aligned}$$

$$\mathbf{72} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -3 & 4 \\ 2 & 2 & -3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -1 & 0 & 1 \\ -17 & -3 & 13 \\ -12 & -2 & 9 \end{pmatrix}$$

$$\begin{aligned} \mathbf{a} \quad \mathbf{BA} &= \begin{pmatrix} -1 & 0 & 1 \\ -17 & -3 & 13 \\ -12 & -2 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 3 & -3 & 4 \\ 2 & 2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \{\text{using technology}\} \end{aligned}$$

$$\mathbf{b} \quad \text{In matrix form, the system is} \quad \begin{pmatrix} 1 & 2 & -3 \\ 3 & -3 & 4 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix}$$

This has the form  $\mathbf{Ax} = \mathbf{b}$  where  $\mathbf{x}$  is a vector of unknowns and  $\mathbf{b}$  is a vector of constants.

Premultiplying by  $\mathbf{B}$ ,  $\mathbf{BAx} = \mathbf{Bb}$

$$\therefore \mathbf{Ix} = \mathbf{Bb}$$

$$\begin{aligned} \therefore \mathbf{x} &= \begin{pmatrix} -1 & 0 & 1 \\ -17 & -3 & 13 \\ -12 & -2 & 9 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \end{aligned}$$

∴  $x = 1$ ,  $y = -1$ , and  $z = -2$ .

$$\mathbf{73} \quad \mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$$

$$\begin{aligned} \mathbf{a} \quad \det \mathbf{A} &= 1 \times 7 - 3 \times 2 \\ &= 7 - 6 \\ &= 1 \end{aligned} \quad \therefore \mathbf{A}^{-1} = \frac{1}{1} \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$$



**b** In matrix form, the system is  $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \end{pmatrix}$ .

This has the form  $\mathbf{Ax} = \mathbf{b}$  where  $\mathbf{x}$  is a vector of unknowns and  $\mathbf{b}$  is a vector of constants.

Premultiplying by  $\mathbf{A}^{-1}$ ,  $\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$

$$\therefore \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 13 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

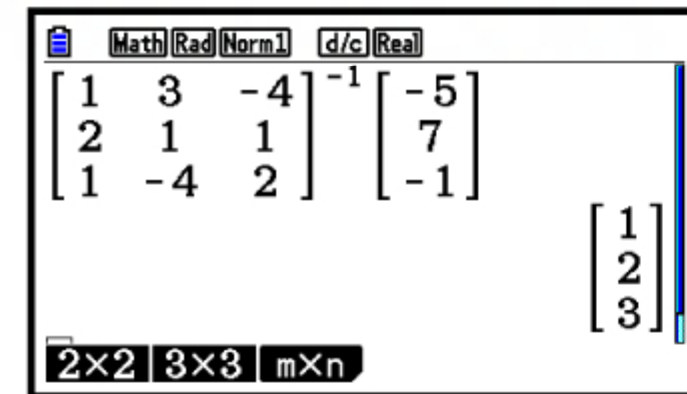
$\therefore x = 3$  and  $y = 1$ .

**74 a** In matrix form, the system is  $\begin{pmatrix} 1 & 3 & -4 \\ 2 & 1 & 1 \\ 1 & -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \\ -1 \end{pmatrix}$ .

**b**  $\begin{pmatrix} 1 & 3 & -4 \\ 2 & 1 & 1 \\ 1 & -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \\ -1 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & -4 \\ 2 & 1 & 1 \\ 1 & -4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -5 \\ 7 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \{\text{using technology}\}$$



$\therefore x = 1, y = 2,$  and  $z = 3$ .

**75**  $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -6 & -2 \end{pmatrix}$

**a**  $p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$

$$= \begin{vmatrix} \lambda - 3 & 1 \\ 6 & \lambda + 2 \end{vmatrix}$$

$$= (\lambda - 3)(\lambda + 2) - 6$$

$$= \lambda^2 - \lambda - 6 - 6$$

$$= \lambda^2 - \lambda - 12$$

**b**  $p(\lambda) = 0$  when  $\lambda^2 - \lambda - 12 = 0$

$$\therefore (\lambda - 4)(\lambda + 3) = 0$$

$$\therefore \lambda = 4 \text{ or } -3$$

$\therefore$  the eigenvalues are 4 and  $-3$ .

For  $\lambda = 4$ , consider  $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ .

$$\therefore \begin{pmatrix} 1 & 1 \\ 6 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore a + b = 0$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = -t$

$$\therefore \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form  $\begin{pmatrix} -1 \\ 1 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue 4.

For  $\lambda = -3$ , consider  $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ .

$$\therefore \begin{pmatrix} -6 & 1 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 6a - b = 0$$

Letting  $a = t$ ,  $t \neq 0$ , then  $b = 6t$

$$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form  $\begin{pmatrix} 1 \\ 6 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue  $-3$ .



**c i** Suppose  $\mathbf{x}$  is an eigenvector of  $\mathbf{A}$  corresponding to the eigenvalue  $\lambda$ .

$$\therefore \mathbf{Ax} = \lambda\mathbf{x}$$

$$\therefore 2\mathbf{Ax} = 2\lambda\mathbf{x}$$

$$\therefore (2\mathbf{A})\mathbf{x} = (2\lambda)\mathbf{x}$$

So,  $\mathbf{x}$  is also an eigenvector of  $2\mathbf{A}$  corresponding to the eigenvalue  $2\lambda$ .

Thus, using **b**:

- Any vector of the form  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}t$ ,  $t \neq 0$  is an eigenvector of  $2\mathbf{A}$  corresponding to the eigenvalue  $2(4) = 8$ .
- Any vector of the form  $\begin{pmatrix} 1 \\ 6 \end{pmatrix}t$ ,  $t \neq 0$  is an eigenvector of  $2\mathbf{A}$  corresponding to the eigenvalue  $2(-3) = -6$ .

**ii** Suppose  $\mathbf{x}$  is an eigenvector of  $\mathbf{A}$  corresponding to the eigenvalue  $\lambda$ .

$$\therefore \mathbf{Ax} = \lambda\mathbf{x}$$

$$\therefore \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}(\lambda\mathbf{x})$$

$$\therefore \mathbf{Ix} = \lambda(\mathbf{A}^{-1}\mathbf{x})$$

$$\therefore \mathbf{A}^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$$

So,  $\mathbf{x}$  is also an eigenvector of  $\mathbf{A}^{-1}$  corresponding to the eigenvalue  $\frac{1}{\lambda}$ .

Thus, using **b**:

- Any vector of the form  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}t$ ,  $t \neq 0$  is an eigenvector of  $\mathbf{A}^{-1}$  corresponding to the eigenvalue  $\frac{1}{4}$ .
- Any vector of the form  $\begin{pmatrix} 1 \\ 6 \end{pmatrix}t$ ,  $t \neq 0$  is an eigenvector of  $\mathbf{A}^{-1}$  corresponding to the eigenvalue  $-\frac{1}{3}$ .

$$\mathbf{76} \quad \mathbf{A} = \begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathbf{a} \quad \mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} -5 & 1 \\ 2 & -1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{P}^{-1}\mathbf{AP} &= \frac{1}{(-5)(-1) - 2(1)} \begin{pmatrix} -1 & -1 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -5 & 1 \\ 2 & -1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -4 & -10 \end{pmatrix} \begin{pmatrix} -5 & 1 \\ 2 & -1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -3 & 0 \\ 0 & 6 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

$\therefore$  the eigenvalues of  $\mathbf{A}$  are  $-1$  and  $2$ .

$$\mathbf{77} \quad \mathbf{A} = \begin{pmatrix} 3 & -4 \\ -2 & -1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{a} \quad p(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{A}) \\ &= \begin{vmatrix} \lambda - 3 & 4 \\ 2 & \lambda + 1 \end{vmatrix} \\ &= (\lambda - 3)(\lambda + 1) - 8 \\ &= \lambda^2 - 2\lambda - 3 - 8 \\ &= \lambda^2 - 2\lambda - 11 \end{aligned}$$



**b**  $p(\lambda) = 0$  when  $\lambda^2 - 2\lambda - 11 = 0$

$$\begin{aligned}\therefore \lambda &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2} \\ &= \frac{2 \pm \sqrt{48}}{2} \\ &= \frac{2 \pm 4\sqrt{3}}{2} \\ &= 1 \pm 2\sqrt{3}\end{aligned}$$

$\therefore$  the eigenvalues are  $1 \pm 2\sqrt{3}$ .

For  $\lambda = 1 + 2\sqrt{3}$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ .

$$\therefore \begin{pmatrix} -2 + 2\sqrt{3} & 4 \\ 2 & 2 + 2\sqrt{3} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 2a + (2 + 2\sqrt{3})b = 0$$

$$\therefore 2a + 2(1 + \sqrt{3})b = 0$$

$$\therefore a + (1 + \sqrt{3})b = 0$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = -(1 + \sqrt{3})t$

$$\therefore \mathbf{x} = \begin{pmatrix} -1 - \sqrt{3} \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form  $\begin{pmatrix} -1 - \sqrt{3} \\ 1 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue  $1 + 2\sqrt{3}$ .

For  $\lambda = 1 - 2\sqrt{3}$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ .

$$\therefore \begin{pmatrix} -2 - 2\sqrt{3} & 4 \\ 2 & 2 - 2\sqrt{3} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 2a + (2 - 2\sqrt{3})b = 0$$

$$\therefore 2a + 2(1 - \sqrt{3})b = 0$$

$$\therefore a + (1 - \sqrt{3})b = 0$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = -(1 - \sqrt{3})t$

$$\therefore \mathbf{x} = \begin{pmatrix} -1 + \sqrt{3} \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form  $\begin{pmatrix} -1 + \sqrt{3} \\ 1 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue  $1 - 2\sqrt{3}$ .

**c** From **b**:

- $\mathbf{x}_1 = \begin{pmatrix} -1 + \sqrt{3} \\ 1 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue  $1 - 2\sqrt{3}$ .

- $\mathbf{x}_2 = \begin{pmatrix} -1 - \sqrt{3} \\ 1 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue  $1 + 2\sqrt{3}$ .

$$\therefore \mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} -1 + \sqrt{3} & -1 - \sqrt{3} \\ 1 & 1 \end{pmatrix} \text{ diagonalises } \mathbf{A} \text{ with } \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 - 2\sqrt{3} & 0 \\ 0 & 1 + 2\sqrt{3} \end{pmatrix}.$$

**78**  $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$

**a** For  $\lambda = -2$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ .

$$\therefore \begin{pmatrix} -1 & -2 \\ -3 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -a - 2b = 0$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = -2t$

$$\therefore \mathbf{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form  $\begin{pmatrix} -2 \\ 1 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue  $-2$ .



For  $\lambda = 5$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ .

$$\therefore \begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -3a + b = 0$$

Letting  $a = t$ ,  $t \neq 0$ , then  $b = 3t$

$$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t, t \neq 0$$

Any vector of the form  $\begin{pmatrix} 1 \\ 3 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue 5.

**b** From **a**:

•  $\mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue  $-2$ .

•  $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue 5.

$$\therefore \mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix} \text{ diagonalises } \mathbf{A} \text{ with } \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}.$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{A}^{51} &= \mathbf{P} \begin{pmatrix} (-2)^{51} & 0 \\ 0 & 5^{51} \end{pmatrix} \mathbf{P}^{-1} \\ &= \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -2^{51} & 0 \\ 0 & 5^{51} \end{pmatrix} \left(-\frac{1}{7}\right) \begin{pmatrix} 3 & -1 \\ -1 & -2 \end{pmatrix} \\ &= -\frac{1}{7} \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \times 2^{51} & 2^{51} \\ -5^{51} & -2 \times 5^{51} \end{pmatrix} \\ &= -\frac{1}{7} \begin{pmatrix} 6 \times 2^{51} - 5^{51} & -2^{52} - 2 \times 5^{51} \\ -3 \times 2^{51} - 3 \times 5^{51} & 2^{51} - 6 \times 5^{51} \end{pmatrix} \end{aligned}$$

$$\mathbf{79} \quad \mathbf{a} \quad \mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} -2 & -7 \\ 1 & 3 \end{pmatrix} \text{ diagonalises } \mathbf{A} \text{ with } \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}.$$

$$\mathbf{b} \quad \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\begin{aligned} \therefore \mathbf{A} &= \mathbf{P} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{P}^{-1} \\ &= \begin{pmatrix} -2 & -7 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \left(\frac{1}{1}\right) \begin{pmatrix} 3 & 7 \\ -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -6 & -28 \\ 3 & 12 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 14 \\ -3 & -3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{A}^{30} &= \mathbf{P} \begin{pmatrix} 3^{30} & 0 \\ 0 & 4^{30} \end{pmatrix} \mathbf{P}^{-1} \\ &= \begin{pmatrix} -2 & -7 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3^{30} & 0 \\ 0 & 4^{30} \end{pmatrix} \begin{pmatrix} 3 & 7 \\ -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \times 3^{30} & -7 \times 4^{30} \\ 3^{30} & 3 \times 4^{30} \end{pmatrix} \begin{pmatrix} 3 & 7 \\ -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \times 3^{31} + 7 \times 4^{30} & -14 \times 3^{30} + 14 \times 4^{30} \\ 3^{31} - 3 \times 4^{30} & 7 \times 3^{30} - 6 \times 4^{30} \end{pmatrix} \end{aligned}$$



$$80 \quad \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$$

$$\begin{aligned} \mathbf{a} \quad \text{If } \det(\lambda \mathbf{I} - \mathbf{A}) = 0 \text{ then } \begin{vmatrix} \lambda - 1 & 1 \\ -2 & \lambda - 4 \end{vmatrix} &= 0 \\ \therefore (\lambda - 1)(\lambda - 4) + 2 &= 0 \\ \therefore \lambda^2 - 5\lambda + 4 + 2 &= 0 \\ \therefore \lambda^2 - 5\lambda + 6 &= 0 \\ \therefore (\lambda - 2)(\lambda - 3) &= 0 \\ \therefore \lambda &= 2 \text{ or } 3 \end{aligned}$$

$\therefore$  the eigenvalues of  $\mathbf{A}$  are 2 and 3.

For  $\lambda = 2$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ .

$$\begin{aligned} \therefore \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \therefore a + b &= 0 \end{aligned}$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = -t$

$$\therefore \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form  $\begin{pmatrix} -1 \\ 1 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue 2.

For  $\lambda = 3$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ .

$$\begin{aligned} \therefore \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \therefore 2a + b &= 0 \end{aligned}$$

Letting  $a = t$ ,  $t \neq 0$ , then  $b = -2t$

$$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form  $\begin{pmatrix} 1 \\ -2 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue 3.

$$\begin{aligned} \mathbf{b} \quad \mathbf{i} \quad \det \mathbf{A} &= (1)(4) - (-1)(2) \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \therefore \mathbf{A}^{-1} &= \frac{1}{6} \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad \mathbf{A}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} t &= t \begin{pmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= t \begin{pmatrix} -\frac{2}{3} + \frac{1}{6} \\ \frac{1}{3} + \frac{1}{6} \end{pmatrix} \\ &= t \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2} t \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned}$$

$\therefore$  any vector of the form  $\begin{pmatrix} -1 \\ 1 \end{pmatrix} t$ ,  $t \neq 0$  is also an eigenvector of  $\mathbf{A}^{-1}$  corresponding to the eigenvalue  $\frac{1}{2}$ .

$$\begin{aligned} \mathbf{A}^{-1} \begin{pmatrix} 1 \\ -2 \end{pmatrix} t &= t \begin{pmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= t \begin{pmatrix} \frac{2}{3} - \frac{1}{3} \\ -\frac{1}{3} - \frac{1}{3} \end{pmatrix} \\ &= t \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \\ &= \frac{1}{3} t \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}$$

$\therefore$  any vector of the form  $\begin{pmatrix} 1 \\ -2 \end{pmatrix} t$ ,  $t \neq 0$  is also an eigenvector of  $\mathbf{A}^{-1}$  corresponding to the eigenvalue  $\frac{1}{3}$ .



# TOPIC 2 SKILL BUILDER QUESTIONS

- 1 a** The line is parallel to  $2x - y = -3$  or  $y = 2x + 3$  which has gradient 2.

$\therefore$  the line has gradient 2 and passes through  $(5, 3)$ .

$\therefore$  the equation of the line is  $y - 3 = 2(x - 5)$

$$\therefore y - 3 = 2x - 10$$

$$\therefore y = 2x - 7$$

- 2 a** The gradient is  $\frac{y_1 - y_2}{x_1 - x_2} = \frac{10 - 4}{-1 - (-3)} = \frac{6}{2} = 3$

- b** The equation of the line is  $y = 3x + c$ .

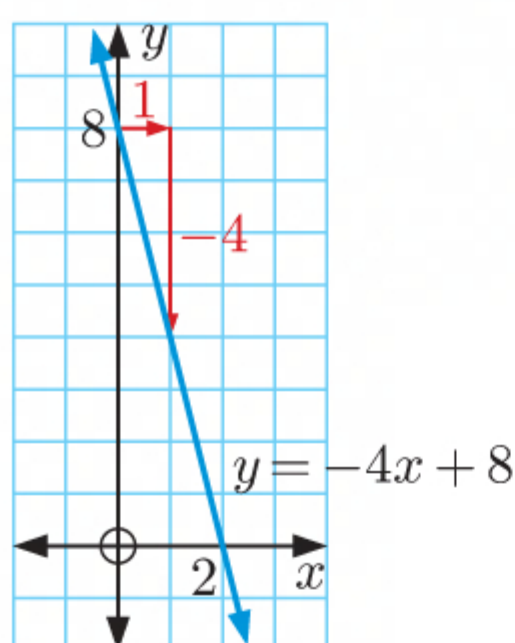
The line passes through  $(-3, 4)$ , so  $4 = 3(-3) + c$

$$\therefore c = 13$$

$\therefore$  the equation is  $y = 3x + 13$ .

- 3 a** For  $y = -4x + 8$ :

- the  $y$ -intercept is 8
- the gradient is  $-4$ .



- b** The line is perpendicular to  $y = -4x + 3$ , which has gradient  $-4$ .

$\therefore$  the line has gradient  $\frac{1}{4}$  and passes through  $(-1, 5)$ .

$\therefore$  the equation of the line is  $y - 5 = \frac{1}{4}(x - (-1))$

$$\therefore y - 5 = \frac{1}{4}(x + 1)$$

$$\therefore y - 5 = \frac{1}{4}x + \frac{1}{4}$$

$$\therefore y = \frac{1}{4}x + \frac{21}{4}$$

- c** When  $x = 0$ ,  $y = 13$   $\therefore$  the  $y$ -intercept is 13.

When  $y = 0$ ,  $3x + 13 = 0$

$$\therefore x = -\frac{13}{3}$$

$\therefore$  the  $x$ -intercept is  $-\frac{13}{3}$ .

- b** For  $7x + 4y = 14$ :

When  $x = 0$ ,  $4y = 14$

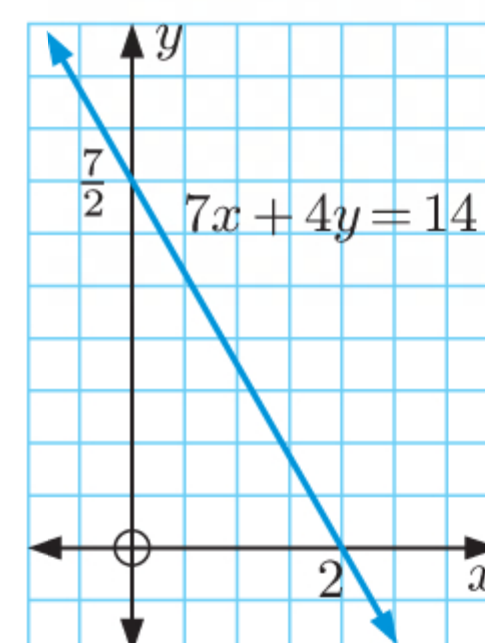
$$\therefore y = \frac{7}{2}$$

So, the  $y$ -intercept is  $\frac{7}{2}$ .

When  $y = 0$ ,  $7x = 14$

$$\therefore x = 2$$

So, the  $x$ -intercept is 2.



- 4 a**  $x$  adult tickets at \$30 each and  $y$  child tickets at \$15 costs \$120 in total.

$$\therefore 30x + 15y = 120$$

- c** When  $y = 0$ ,  $30x + 15(0) = 120$

$$\therefore 30x = 120$$

$$\therefore x = 4$$

$\therefore$  the  $x$ -intercept of the line  $30x + 15y = 120$  is 4.

If Tammy did not buy any child tickets, then she bought 4 adult tickets.

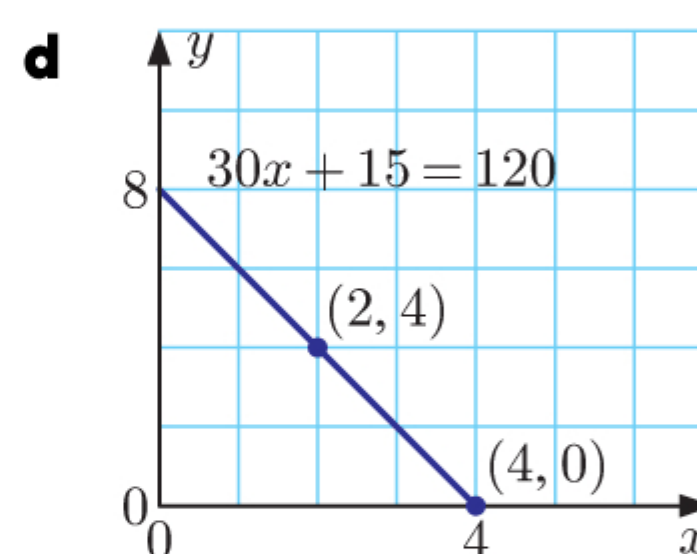
- b** When  $y = 4$ ,  $30x + 15(4) = 120$

$$\therefore 30x + 60 = 120$$

$$\therefore 30x = 60$$

$$\therefore x = 2$$

$\therefore$  Tammy bought 2 adult tickets.



- 5**  $y = \frac{3}{2}x + 1$  has gradient  $\frac{3}{2}$

$\therefore ax + by = 20$  has gradient  $-\frac{2}{3}$ .

$$\therefore -\frac{a}{b} = -\frac{2}{3}$$

$$\therefore 3a = 2b$$

$$\therefore a = \frac{2}{3}b \quad \dots (*)$$

Now  $ax + by = 20$  passes through  $(2, 2)$

$$\therefore a(2) + b(2) = 20$$

$$\therefore 2a + 2b = 20$$

$$\therefore a + b = 10$$

$$\therefore \frac{2}{3}b + b = 10 \quad \{\text{using } (*)\}$$

$$\therefore \frac{5}{3}b = 10$$

$$\therefore b = 6$$

Substituting  $b = 6$  into  $(*)$  gives  $a = \frac{2}{3}(6) = 4$ .

So,  $a = 4$  and  $b = 6$ .



**6 a**  $2x^2 - 9x = 0$   
 $\therefore x(2x - 9) = 0$   
 $\therefore x = 0$  or  $2x - 9 = 0$   
 $\therefore x = 0$  or  $\frac{9}{2}$

**c**  $4x^2 + 11x = 3$   
 $\therefore 4x^2 + 11x - 3 = 0$   
 $\therefore (4x - 1)(x + 3) = 0$   
 $\therefore x = \frac{1}{4}$  or  $-3$

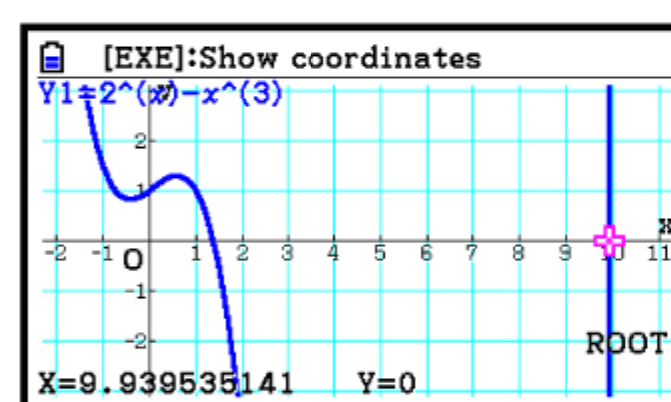
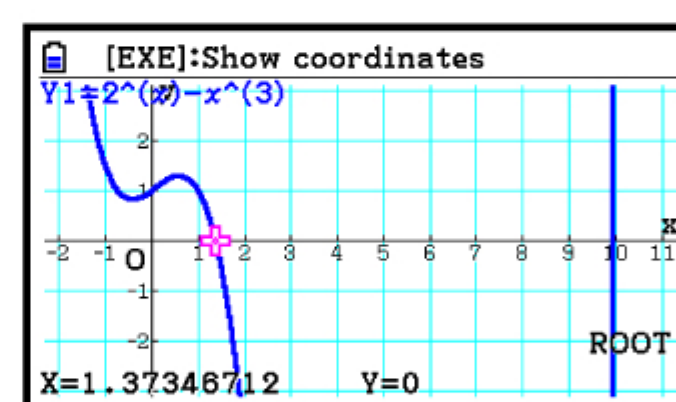
**7 a**  $x = -2$  is a solution to  $x^2 + bx + (b - 2) = 0$   
 $\therefore (-2)^2 + b(-2) + b - 2 = 0$   
 $\therefore 4 - 2b + b - 2 = 0$   
 $\therefore -b = -2$   
 $\therefore b = 2$

**8 a** We graph  $Y_1 = 2^x - X^3$ .  
The  $x$ -intercepts are  $\approx 1.37$  and  $9.94$ .  
 $\therefore$  the solutions are  $x \approx 1.37$  or  $9.94$ .

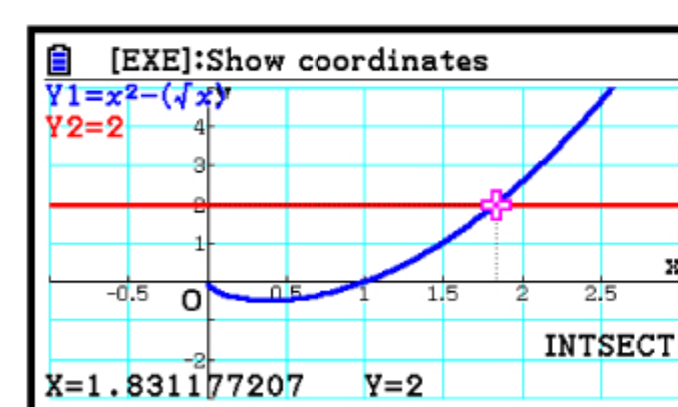
**b**  $x^2 + 8x - 20 = 0$   
 $\therefore (x + 10)(x - 2) = 0$   
 $\therefore x = -10$  or  $2$

**d**  $(x + 3)(1 - 2x) = -9$   
 $\therefore x - 2x^2 + 3 - 6x = -9$   
 $\therefore -2x^2 - 5x + 12 = 0$   
 $\therefore 2x^2 + 5x - 12 = 0$   
 $\therefore (2x - 3)(x + 4) = 0$   
 $\therefore x = \frac{3}{2}$  or  $-4$

**b**  $x^2 + 2x = 0$  {using **a**}  
 $\therefore x(x + 2) = 0$   
 $\therefore x = 0$  or  $-2$   
 $\therefore$  the other solution to the equation is  $x = 0$ .



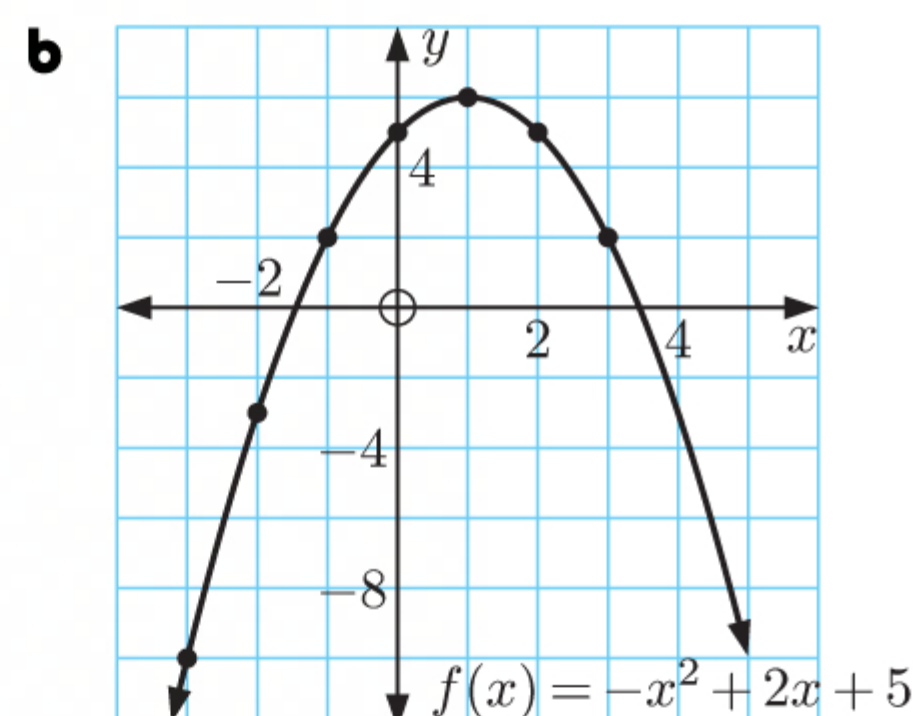
**b** We graph  $Y_1 = X^2 - \sqrt{X}$  and  $Y_2 = 2$  on the same set of axes.  
Using technology,  $x \approx 1.83$ .



**9**  $f(x) = -x^2 + 2x + 5$

**a**

$x$	-3	-2	-1	0	1	2	3
$f(x)$	-10	-3	2	5	6	5	2



**10 a**  $y = x^2 - x - 12$   
When  $y = 0$ ,  $x^2 - x - 12 = 0$   
 $\therefore (x - 4)(x + 3) = 0$   
 $\therefore x = 4$  or  $-3$   
 $\therefore$  the zeros are  $4$  and  $-3$ .

**c**  $y = 8x^2 - 2x - 3$   
When  $y = 0$ ,  $8x^2 - 2x - 3 = 0$   
 $\therefore (4x - 3)(2x + 1) = 0$   
 $\therefore x = \frac{3}{4}$  or  $-\frac{1}{2}$   
 $\therefore$  the zeros are  $\frac{3}{4}$  and  $-\frac{1}{2}$ .

**b**  $f(x) = 5x - x^2$   
When  $f(x) = 0$ ,  $5x - x^2 = 0$   
 $\therefore x(5 - x) = 0$   
 $\therefore x = 5$  or  $0$   
 $\therefore$  the zeros are  $5$  and  $0$ .



**11 a**  $y = (2x - 1)(x + 3)$

When  $x = 0$ ,  $y = (-1)(3) = -3$

$\therefore$  the  $y$ -intercept is  $-3$ .

When  $y = 0$ ,  $(2x - 1)(x + 3) = 0$

$\therefore x = \frac{1}{2}$  or  $-3$

$\therefore$  the  $x$ -intercepts are  $\frac{1}{2}$  and  $-3$ .

**c**  $y = 3x^2 + 4x - 4$

When  $x = 0$ ,  $y = -4$

$\therefore$  the  $y$ -intercept is  $-4$ .


When  $y = 0$ ,  $3x^2 + 4x - 4 = 0$

$\therefore (3x - 2)(x + 2) = 0$

$\therefore x = \frac{2}{3}$  or  $-2$

$\therefore$  the  $x$ -intercepts are  $\frac{2}{3}$  and  $-2$ .

**12 a**  $y = x^2 - 2x - 8$  has  $a = 1$ .

Since  $a > 0$ , the parabola has shape .

When  $x = 0$ ,  $y = -8$

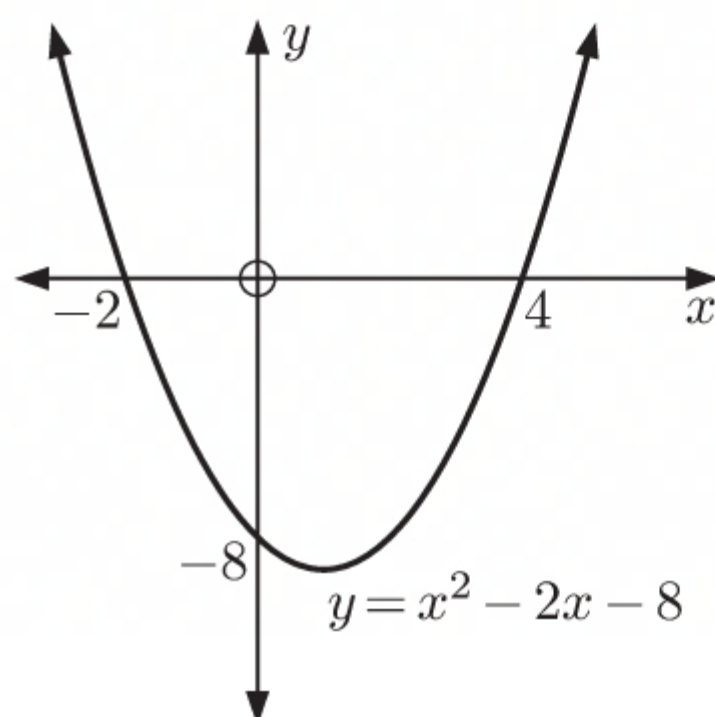
$\therefore$  the  $y$ -intercept is  $-8$ .

When  $y = 0$ ,  $x^2 - 2x - 8 = 0$

$\therefore (x - 4)(x + 2) = 0$

$\therefore x = 4$  or  $-2$


$\therefore$  the  $x$ -intercepts are  $4$  and  $-2$ .



**c**  $y = -\frac{1}{2}(x - 4)^2$

$= -\frac{1}{2}(x^2 - 8x + 16)$

$= -\frac{1}{2}x^2 + 4x - 8$  has  $a = -\frac{1}{2}$ .

Since  $a < 0$ , the parabola has shape .

When  $x = 0$ ,  $y = -\frac{1}{2}(-4)^2 = -8$

$\therefore$  the  $y$ -intercept is  $-8$ .

When  $y = 0$ ,  $-\frac{1}{2}(x - 4)^2 = 0$

$\therefore (x - 4)^2 = 0$

$\therefore x - 4 = 0$

$\therefore x = 4$

$\therefore$  the  $x$ -intercept is  $4$ .

**b**  $f(x) = (x + 1)^2$

$\therefore f(0) = (1)^2 = 1$

$\therefore$  the  $y$ -intercept is  $1$ .

When  $f(x) = 0$ ,  $(x + 1)^2 = 0$

$\therefore x + 1 = 0$


$\therefore x = -1$

$\therefore$  the  $x$ -intercept is  $-1$ .

**b**  $f(x) = -(2x + 1)(x - 3)$

$= -(2x^2 - 5x - 3)$

$= -2x^2 + 5x + 3$  has  $a = -2$ .

Since  $a < 0$ , the parabola has shape .

$f(0) = 3$

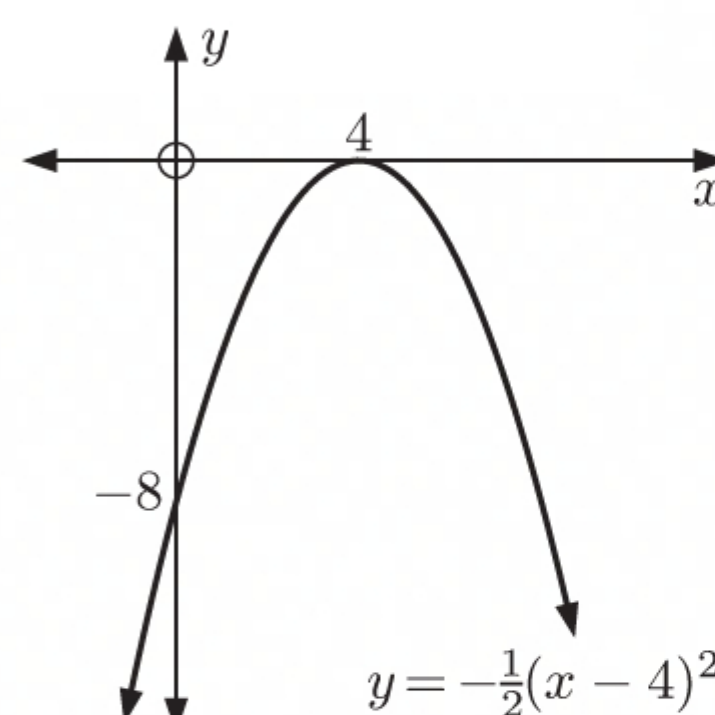
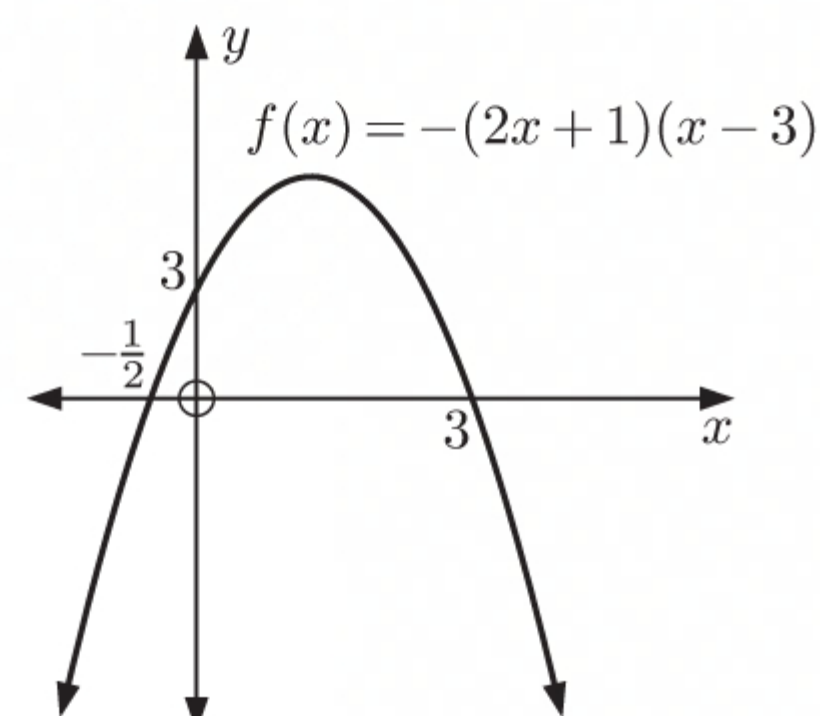
$\therefore$  the  $y$ -intercept is  $3$ .

When  $f(x) = 0$ ,  $-(2x + 1)(x - 3) = 0$

$\therefore (2x + 1)(x - 3) = 0$

$\therefore x = -\frac{1}{2}$  or  $3$

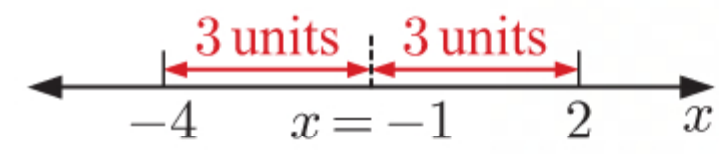
$\therefore$  the  $x$ -intercepts are  $-\frac{1}{2}$  and  $3$ .





- 13** The axis of symmetry  $x = -1$  lies midway between the  $x$ -intercepts.

$\therefore$  the other  $x$ -intercept is  $-4$ .



- 14 a**  $f(x) = 2x^2 + bx - 3$  has  $a = 2$ ,  $b = b$ , and  $c = -3$ .

The axis of symmetry is  $x = 6$ , so  $-\frac{b}{2a} = 6$

$$\therefore -\frac{b}{2(2)} = 6$$

$$\therefore b = -24$$

$$\begin{aligned} \mathbf{b} \quad f(6) &= 2(6)^2 - 24(6) - 3 \\ &= 72 - 144 - 3 \\ &= -75 \end{aligned}$$

So, the vertex is  $(6, -75)$ .

- 15 a**  $y = -(x-1)(x+3)$   
 $= -(x^2 + 2x - 3)$   
 $= -x^2 - 2x + 3$  has  $a = -1$ ,  $b = -2$ , and  $c = 3$ .

- i** When  $x = 0$ ,  $y = 3$

$\therefore$  the  $y$ -intercept is 3.

$$\text{When } y = 0, \quad -(x-1)(x+3) = 0$$


$$\therefore (x-1)(x+3) = 0$$

$$\therefore x = 1 \text{ or } -3$$

$\therefore$  the  $x$ -intercepts are 1 and  $-3$ .

- iii** When  $x = -1$ ,  $y = -(-2)(2) = 4$

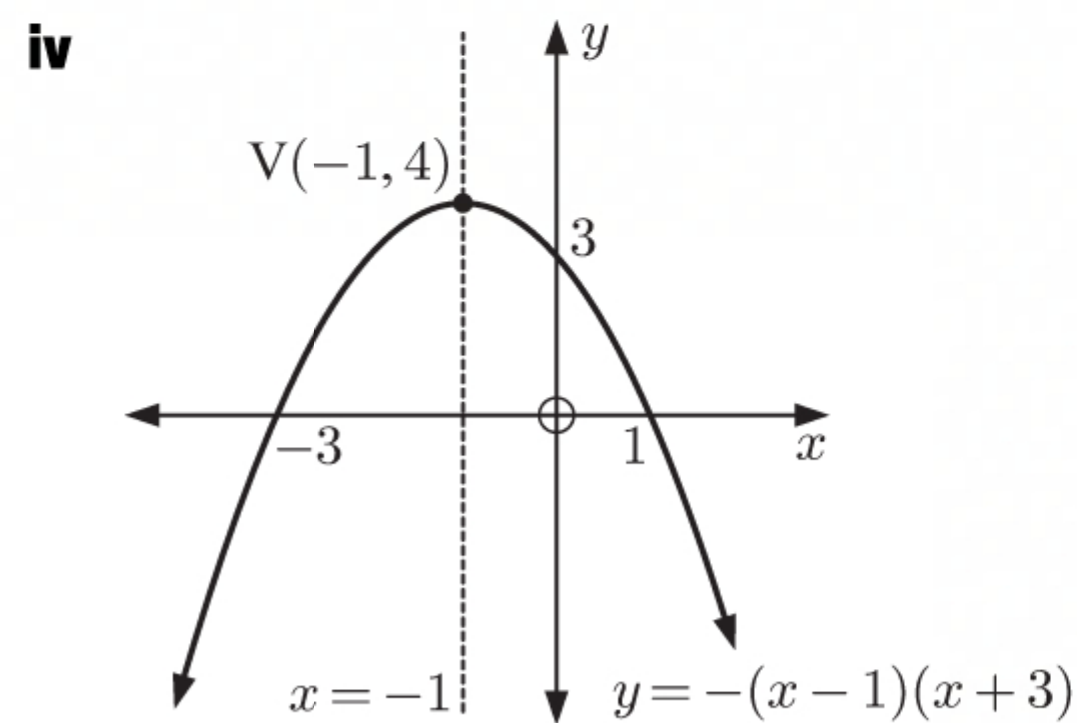
So, the vertex is  $(-1, 4)$ .

Since  $a < 0$ , the parabola has shape .

$\therefore$  the vertex is a maximum turning point.

- ii** The  $x$ -intercepts are 1 and  $-3$ .

$-1$  is halfway between 1 and  $-3$ , so the axis of symmetry is  $x = -1$ .



- v** The domain is  $\{x \mid x \in \mathbb{R}\}$ .

The range is  $\{y \mid y \leq 4\}$ .

- b**  $y = 2(x+7)(x-2)$   
 $= 2(x^2 + 5x - 14)$   
 $= 2x^2 + 10x - 28$  has  $a = 2$ ,  $b = 10$ , and  $c = -28$ .

- i** When  $x = 0$ ,  $y = -28$

$\therefore$  the  $y$ -intercept is  $-28$ .

$$\text{When } y = 0, \quad 2(x+7)(x-2) = 0$$


$$\therefore (x+7)(x-2) = 0$$

$$\therefore x = -7 \text{ or } 2$$

$\therefore$  the  $x$ -intercepts are  $-7$  and  $2$ .

- iii** When  $x = -\frac{5}{2}$ ,  $y = 2(-\frac{5}{2} + 7)(-\frac{5}{2} - 2)$   
 $= 2(\frac{9}{2})(-\frac{9}{2})$   
 $= -\frac{81}{2}$

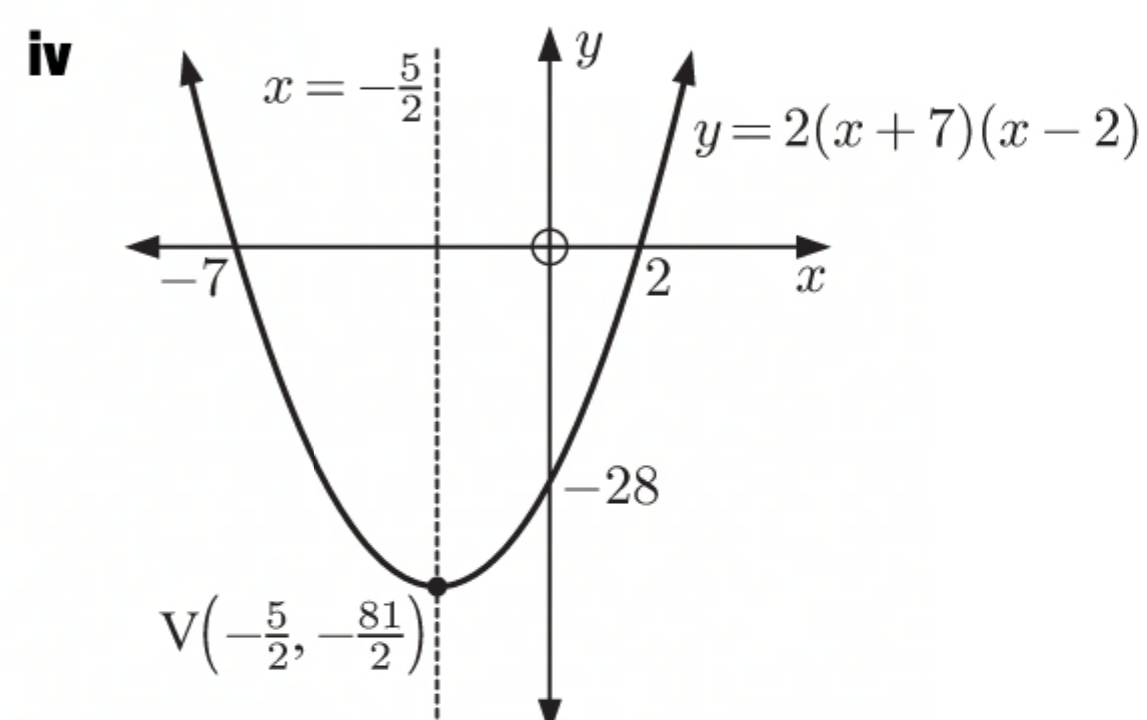
So, the vertex is  $(-\frac{5}{2}, -\frac{81}{2})$ .

Since  $a > 0$ , the parabola has shape .

$\therefore$  the vertex is a minimum turning point.

- ii** The  $x$ -intercepts are  $-7$  and  $2$ .

$-\frac{5}{2}$  is halfway between  $-7$  and  $2$ , so the axis of symmetry is  $x = -\frac{5}{2}$ .



- v** The domain is  $\{x \mid x \in \mathbb{R}\}$ .

The range is  $\{y \mid y \geq -\frac{81}{2}\}$ .



- 16 a** The  $x$ -intercepts are 2 and 5.

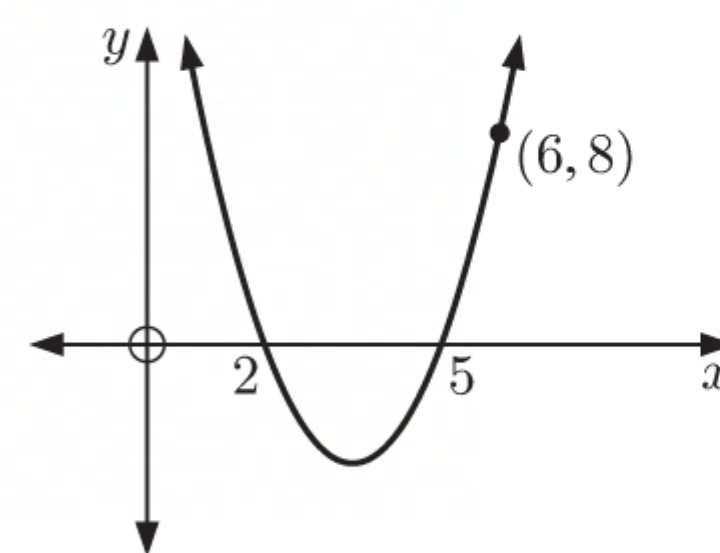
$\therefore$  the quadratic has the form  $y = a(x - 2)(x - 5)$  where  $a > 0$ .

But when  $x = 6$ ,  $y = 8$

$$\therefore 8 = a(4)(1)$$

$$\therefore a = 2$$

The quadratic is  $y = 2(x - 2)(x - 5)$ .



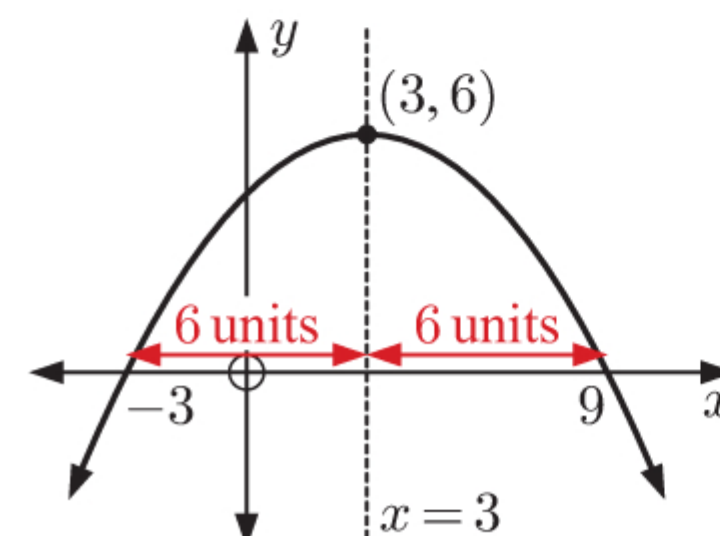
- b** The vertex is (3, 6).

$\therefore$  the axis of symmetry is  $x = 3$ .

The axis of symmetry  $x = 3$  lies midway between the  $x$ -intercepts.

$\therefore$  the other  $x$ -intercept is  $-3$ .

$\therefore$  the quadratic has the form  $y = a(x + 3)(x - 9)$  where  $a < 0$ .



But when  $x = 3$ ,  $y = 6$

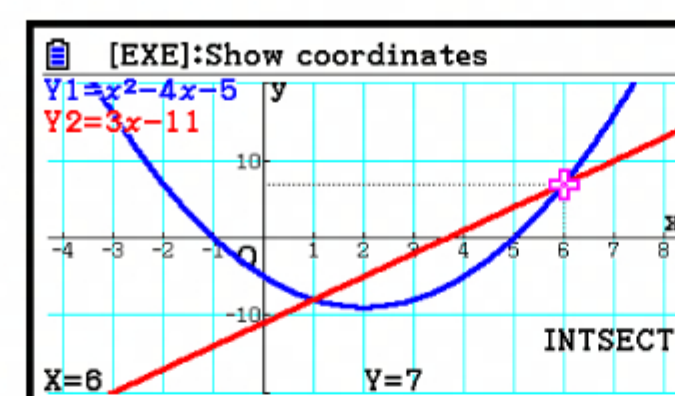
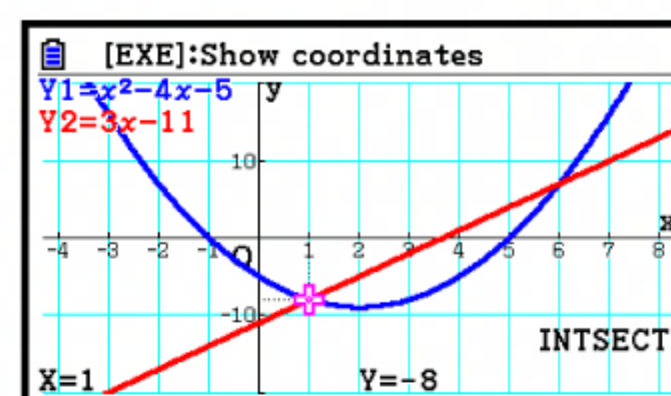
$$\therefore 6 = a(6)(-6)$$

$$\therefore a = -\frac{1}{6}$$

The quadratic is  $y = -\frac{1}{6}(x + 3)(x - 9)$ .

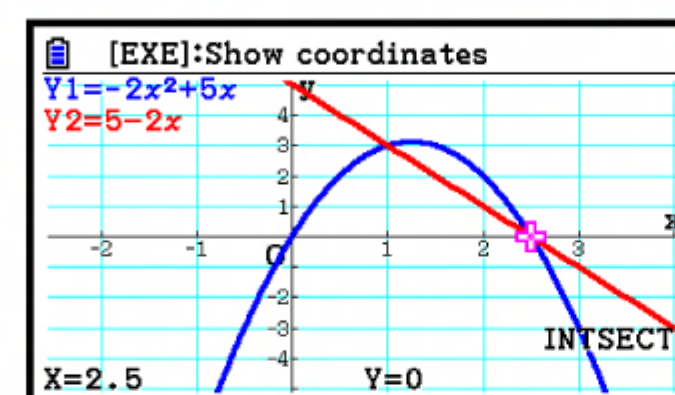
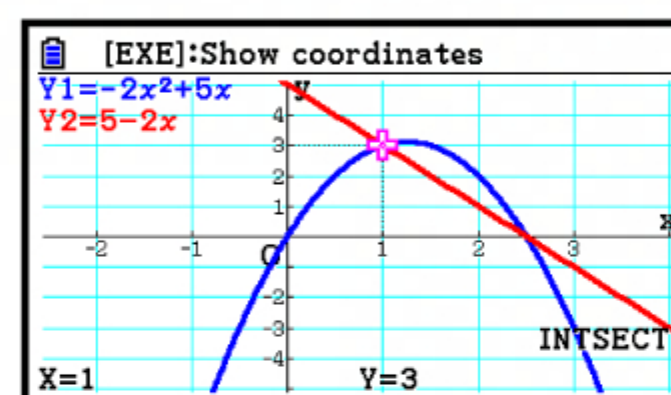
- 17 a** We graph  $Y_1 = X^2 - 4X - 5$  and  $Y_2 = 3X - 11$  on the same set of axes.

The graphs intersect at (1, -8) and (6, 7).



- b** We graph  $Y_1 = -2X^2 + 5X$  and  $Y_2 = 5 - 2X$  on the same set of axes.

The graphs intersect at (1, 3) and  $(\frac{5}{2}, 0)$ .



- 18**  $P = -0.05x^2 + 9x - 60$  dollars

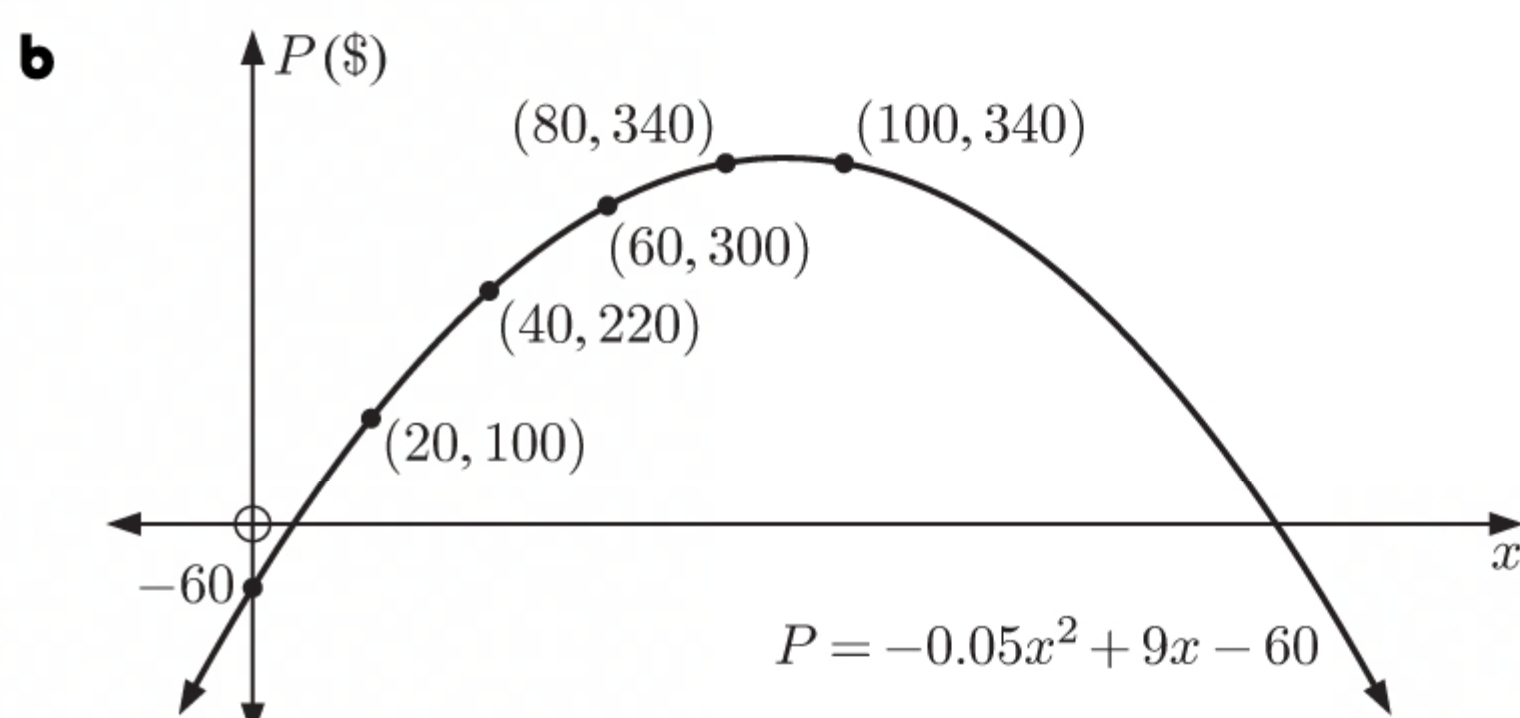
- a** When  $x = 0$ ,  $P = -0.05(0)^2 + 9(0) - 60$   
 $= -60$  dollars

When  $x = 40$ ,  $P = -0.05(40)^2 + 9(40) - 60$   
 $= 220$  dollars

When  $x = 80$ ,  $P = -0.05(80)^2 + 9(80) - 60$   
 $= 340$  dollars

So,

$x$	0	20	40	60	80	100
$P$	-60	100	220	300	340	340






**c**  $P$  has  $a = -0.05$ ,  $b = 9$ , and  $c = -60$

**i** Now  $-\frac{b}{2a} = -\frac{9}{2(-0.05)} = 90$

$\therefore$  the axis of symmetry is  $x = 90$ .

Since  $a < 0$ , the parabola has shape .

$\therefore$  the profit is maximised when 90 pies are sold.

**iii** When  $P = 200$ ,  $-0.05x^2 + 9x - 60 = 200$

$$\therefore -0.05x^2 + 9x - 260 = 0$$

$$\therefore x = \frac{-9 \pm \sqrt{9^2 - 4(-0.05)(-260)}}{2(-0.05)}$$

$$\therefore x \approx 143.9 \text{ or } 36.1$$

Now when  $x = 36$ ,  $P = 199.2$

$$x = 37, \quad P = 204.55$$

$$x = 143, \quad P = 204.55$$

and  $x = 144$ ,  $P = 199.2$

$\therefore$  between 37 and 143 pies must be sold to make a profit of \$200.

**iv** When  $x = 0$ ,  $P = -60$

$\therefore$  the baker loses \$60 if no pies are sold.

**19 a** Let the height of the equilateral triangle ends be  $h$  cm.

$$\therefore \left(\frac{x}{2}\right)^2 + h^2 = x^2 \quad \{\text{Pythagoras}\}$$

$$\therefore \frac{x^2}{4} + h^2 = x^2$$

$$\therefore h^2 = \frac{3x^2}{4}$$

$$\therefore h = \frac{\sqrt{3}}{2}x \quad \{h > 0\}$$

So, area of end  $= \frac{1}{2} \times x \times h$

$$= \frac{1}{2} \times x \times \frac{\sqrt{3}}{2}x$$

$$= \frac{\sqrt{3}}{4}x^2 \text{ cm}^2$$

**b** The sum of all side lengths of the prism must be 1.8 m or 180 cm.

$$\therefore 6x + 3y = 180$$

$$\therefore 3y = 180 - 6x$$

$$\therefore y = 60 - 2x$$

So, area of rectangular face  $= x \times y$

$$= x(60 - 2x)$$

$$= 60x - 2x^2 \text{ cm}^2$$


$\therefore$  the total surface area  $A = 2 \times \text{area of end} + 3 \times \text{area of rectangular face}$

$$= 2\left(\frac{\sqrt{3}}{4}x^2\right) + 3(60x - 2x^2) \quad \{\text{using a}\}$$

$$= \frac{\sqrt{3}}{2}x^2 + 180x - 6x^2$$

$$= \left(\frac{\sqrt{3}}{2} - 6\right)x^2 + 180x \text{ cm}^2$$

**c**  $A = \left(\frac{\sqrt{3}}{2} - 6\right)x^2 + 180x$  has  $a = \frac{\sqrt{3}}{2} - 6$ ,  $b = 180$ , and  $c = 0$

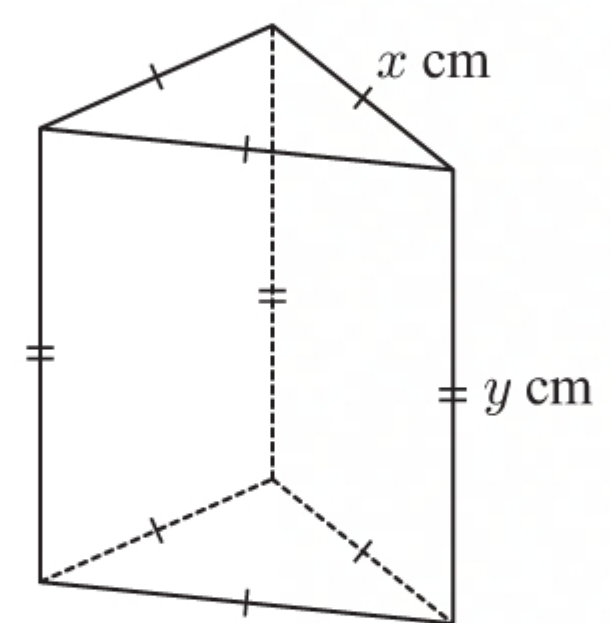
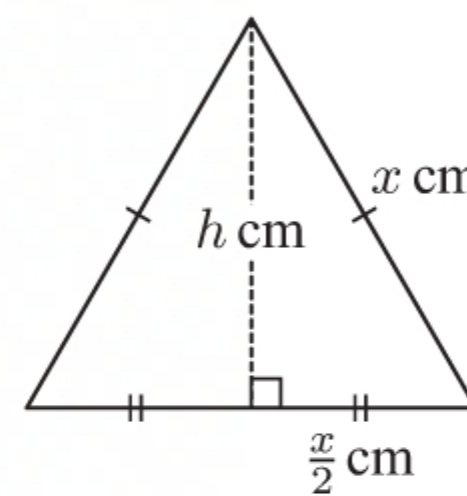
Since  $a < 0$ , the shape is .

The maximum surface area occurs when  $x = \frac{-b}{2a} = \frac{-180}{2\left(\frac{\sqrt{3}}{2} - 6\right)} \approx 17.5$

and  $y = 60 - 2x \approx 24.9$

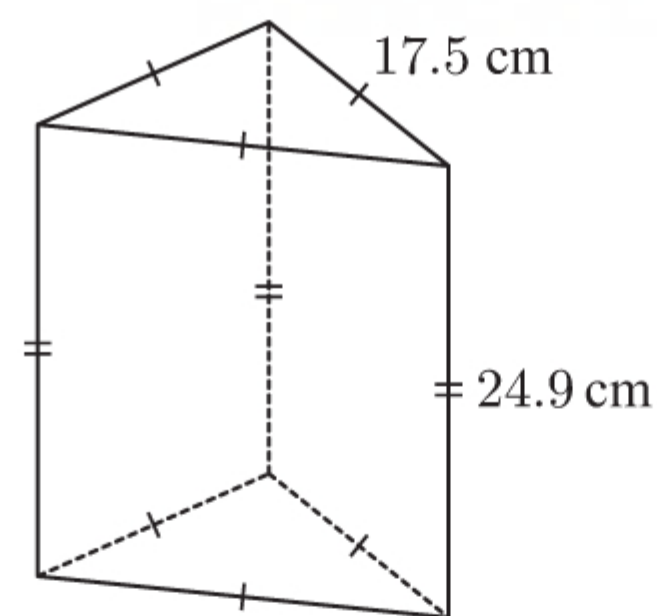
**ii** When  $x = 90$ ,  $P = -0.05(90)^2 + 9(90) - 60$   
 $= 345$  dollars

$\therefore$  the maximum possible daily profit is \$345.



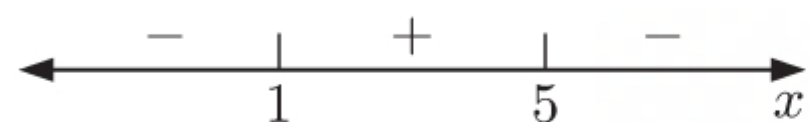


So, the dimensions Andreas should choose for the aquarium are shown alongside:



**20 a**  $(x - 1)(5 - x) \leq 0$

Sign diagram of LHS is

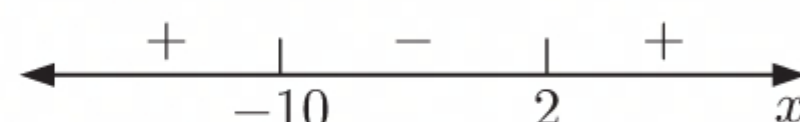


$\therefore x \leq 1$  or  $x \geq 5$

**b**  $x^2 + 8x - 20 < 0$

$\therefore (x + 10)(x - 2) < 0$

Sign diagram of LHS is



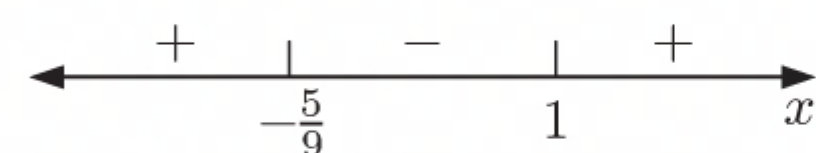
$\therefore -10 < x < 2$

**c**  $-9x^2 + 4x + 5 \geq 0$

$\therefore 9x^2 - 4x - 5 \leq 0$

$\therefore (9x + 5)(x - 1) \leq 0$

Sign diagram of LHS is



$\therefore -\frac{5}{9} \leq x \leq 1$

**21 a**  $x^2 > 9$

$\therefore x^2 - 9 > 0$

$\therefore (x + 3)(x - 3) > 0$

Sign diagram of LHS is



$\therefore x < -3$  or  $x > 3$

**b**  $x^2 - 15 \leq 2x$

$\therefore x^2 - 2x - 15 \leq 0$

$\therefore (x - 5)(x + 3) \leq 0$

Sign diagram of LHS is



$\therefore -3 \leq x \leq 5$

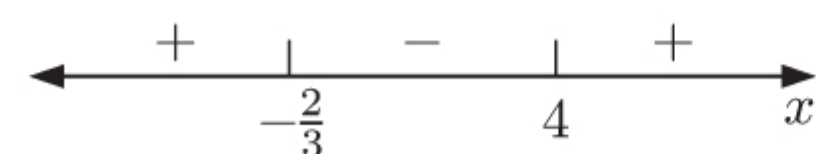
**c**  $3x^2 < 2(5x + 4)$

$\therefore 3x^2 < 10x + 8$

$\therefore 3x^2 - 10x - 8 < 0$

$\therefore (3x + 2)(x - 4) < 0$

Sign diagram of LHS is



$\therefore -\frac{2}{3} < x < 4$

**22**  $W(t) = 1000 - 0.5t$  litres

**a**  $W(0) = 1000$

The initial amount of water in the tank was 1000 L.

**b**  $W(t) = 700$ , so  $700 = 1000 - 0.5t$

$\therefore 0.5t = 300$

$\therefore t = 600$

After 600 hours, or 25 days, the amount of water in the tank is 700 L.

**c** The tank is empty when  $W(t) = 0$ .

This occurs when  $0.5t = 1000$

$\therefore t = 2000$

It will take 2000 hours, or 83 days, 8 hours, for the tank to empty.

**23** From the graph: Domain is  $\{x \mid -6 \leq x \leq 6 \text{ and } x \neq 3\}$

Range is  $\{y \mid 0 \leq y \leq 5\}$

**a**  $x = 0$  satisfies  $-6 \leq x \leq 6$  and  $x \neq 3$ .

$\therefore$  "0 is in the domain of  $f$ " is true.

**b**  $y = 0$  satisfies  $0 \leq y \leq 5$ .

$\therefore$  "0 is in the range of  $f$ " is true.

**c**  $y = 6$  does not satisfy  $0 \leq y \leq 5$ .

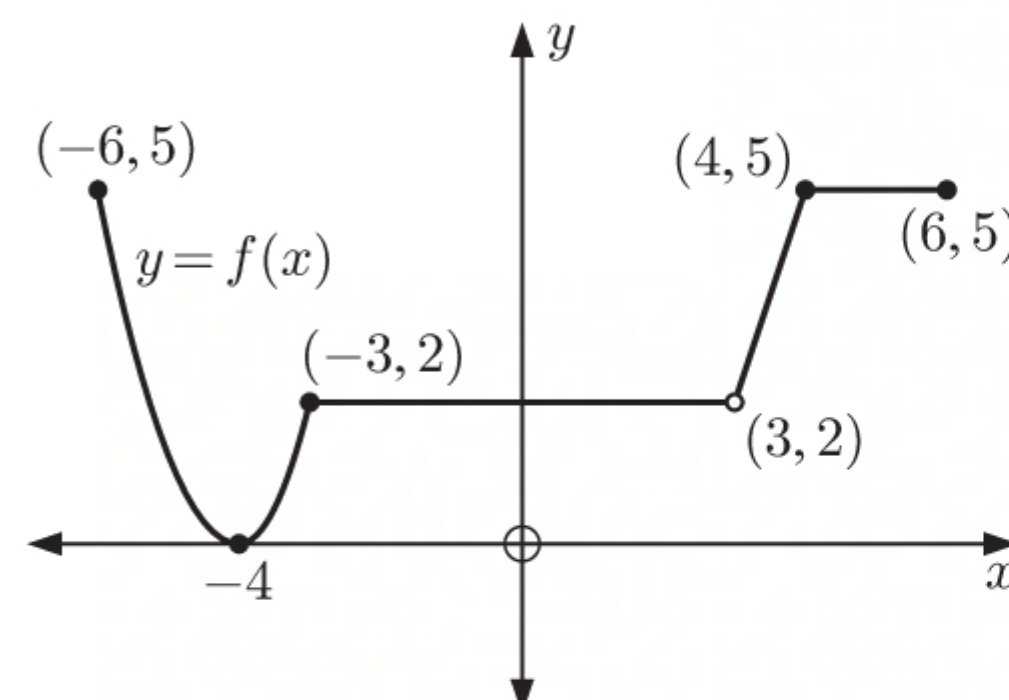
$\therefore$  "6 is in the range of  $f$ " is false.

**d**  $x = 3$  does not satisfy  $x \neq 3$ .

$\therefore$  "3 is in the domain of  $f$ " is false.

**e**  $y = 2$  satisfies  $0 \leq y \leq 5$ .

$\therefore$  "2 is in the range of  $f$ " is true.





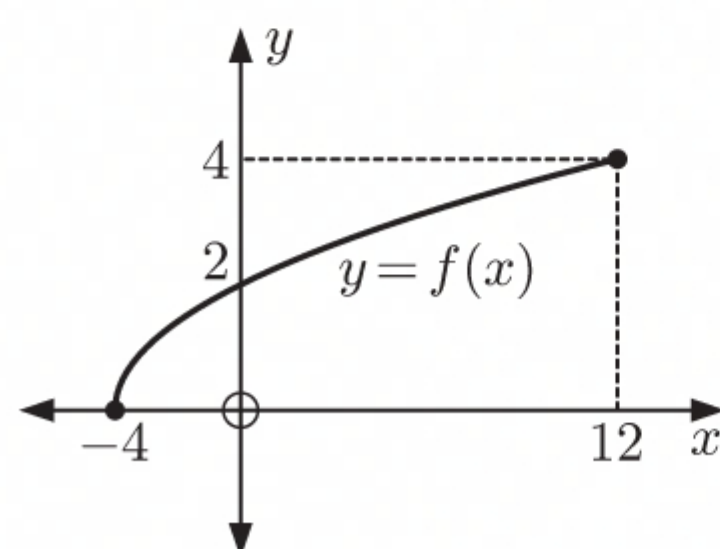
**24**  $f(x) = \sqrt{x+4}$ ,  $-4 \leq x \leq 12$

**a i**  $f(-4) = \sqrt{0} = 0$

**ii**  $f(0) = \sqrt{4} = 2$

**iii**  $f(12) = \sqrt{16} = 4$

**b**



**c** The range is  $\{y \mid 0 \leq y \leq 4\}$ .

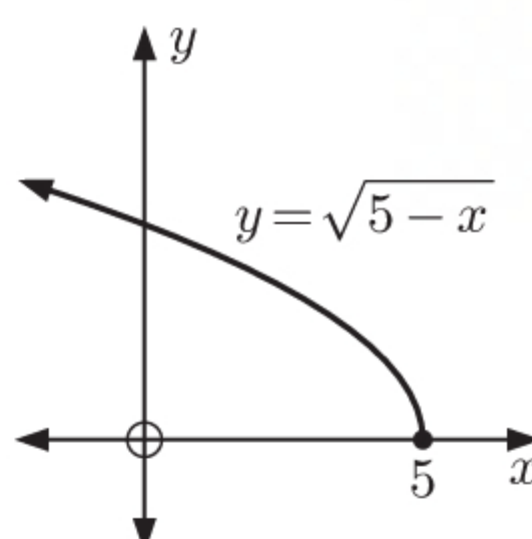
**25 a**  $f(x) = \sqrt{5-x}$

$\sqrt{5-x}$  is defined when  $5-x \geq 0$   
 $\therefore x \leq 5$

$\therefore$  the domain is  $\{x \mid x \leq 5\}$ .

A square root cannot be negative.

$\therefore$  the range is  $\{y \mid y \geq 0\}$ .



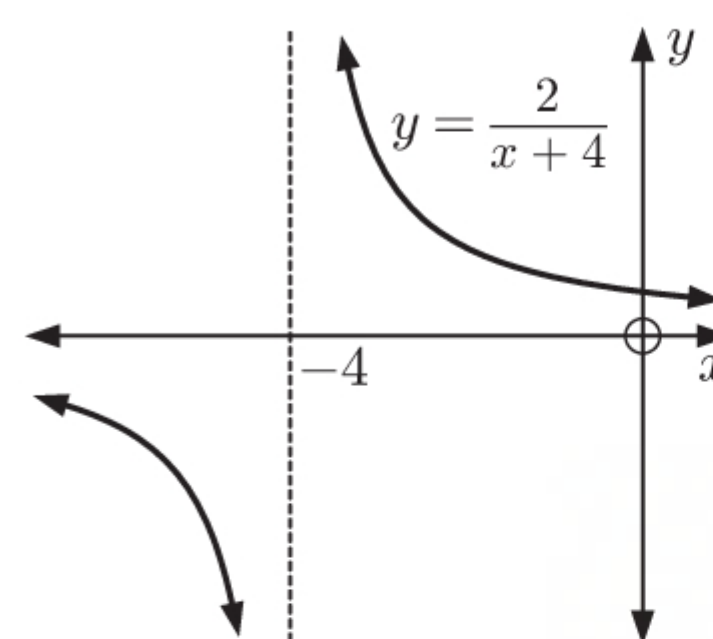
**b**  $f(x) = \frac{2}{x+4}$

$\frac{1}{x+4}$  is defined when  $x+4 \neq 0$   
 $\therefore x \neq -4$

$\therefore$  the domain is  $\{x \mid x \neq -4\}$ .

No matter how large or small  $x$  is,  $y = f(x)$  is never zero.

$\therefore$  the range is  $\{y \mid y \neq 0\}$ .



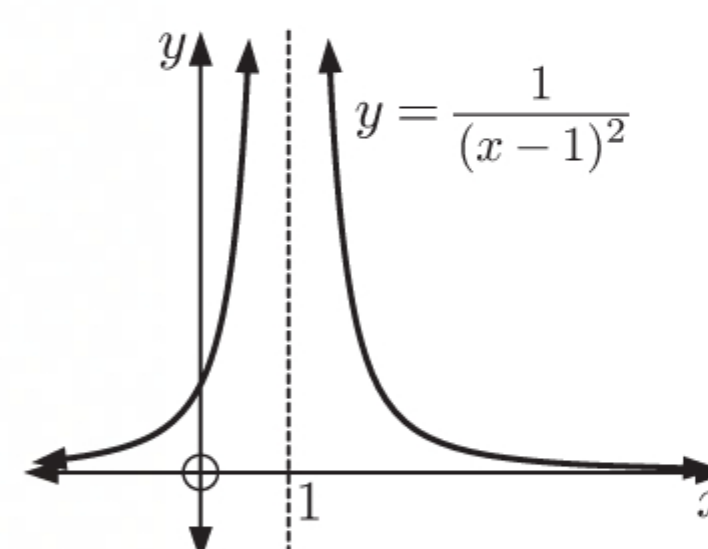
**c**  $f(x) = \frac{1}{(x-1)^2}$

$\frac{1}{(x-1)^2}$  is defined when  $(x-1)^2 \neq 0$   
 $\therefore x-1 \neq 0$   
 $\therefore x \neq 1$

$\therefore$  the domain is  $\{x \mid x \neq 1\}$ .

$y = f(x)$  is always positive and never zero.

$\therefore$  the range is  $\{y \mid y > 0\}$ .

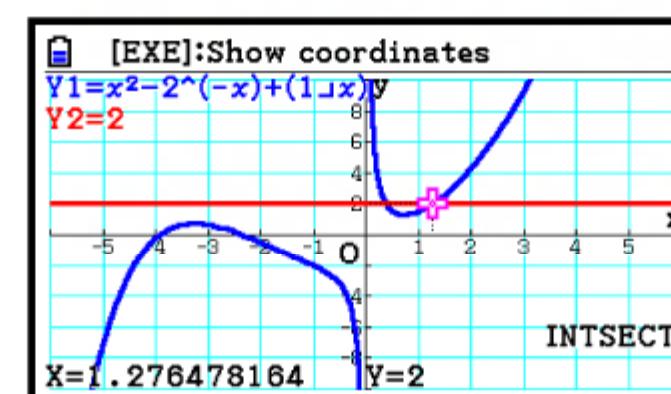
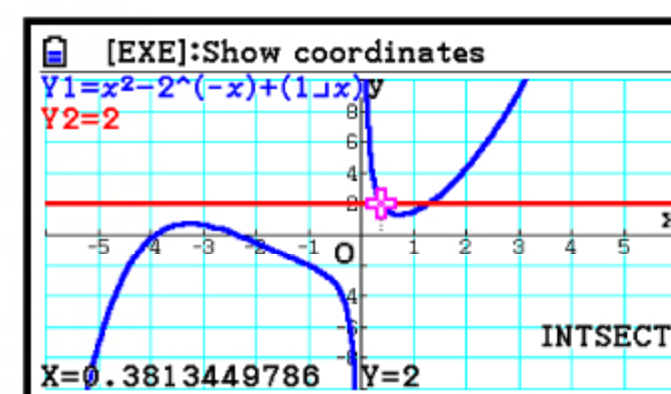


**26**  $h(x) = x^2 - 2^{-x} + \frac{1}{x}$

**a**  $h(-2) = (-2)^2 - 2^{-(-2)} + \frac{1}{-2}$   
 $= 4 - 4 - \frac{1}{2} = -\frac{1}{2}$

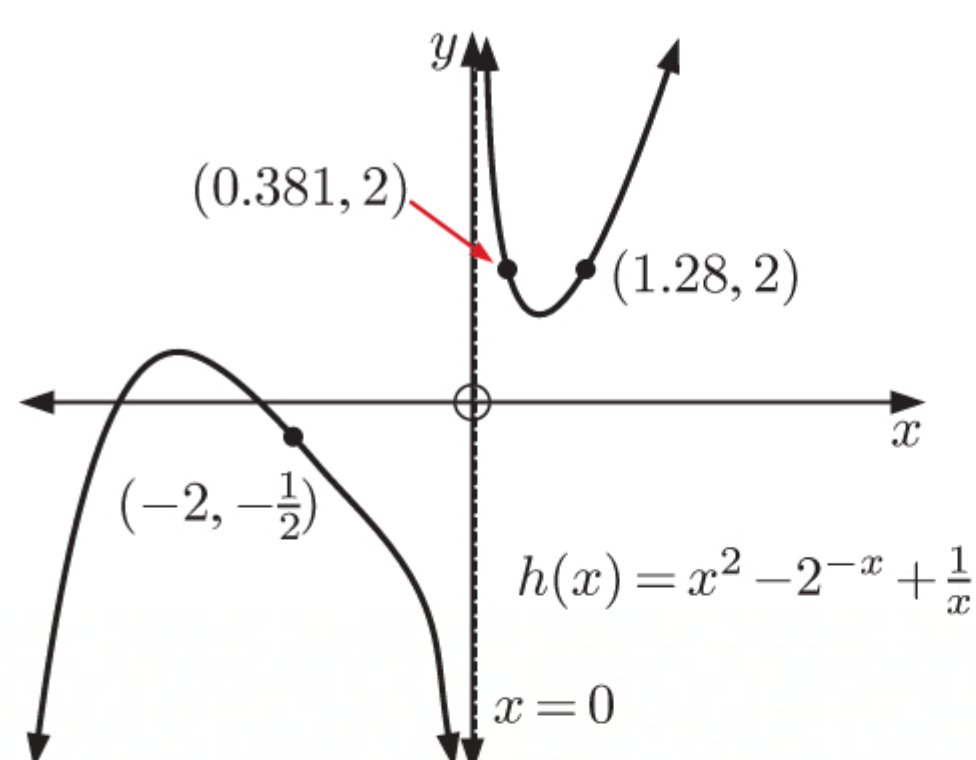
**b** We graph  $Y_1 = X^2 - 2^{-X} + \frac{1}{X}$  and  $Y_2 = 2$  on the same set of axes, and find their point of intersection.

The solution to  $h(x) = 2$  is  $x \approx 0.381$  or  $1.28$ .



**c** The vertical asymptote is  $x = 0$ .

**d**



**e** Using technology, the range is

$\{y \mid y < 0.741 \text{ or } y > 1.30, y \in \mathbb{R}\}$ .



$$27 \quad f(x) = \frac{1}{x-1} + \sqrt{x+1}, \quad g(x) = x^2$$

$$\mathbf{a} \quad \frac{1}{x-1} \text{ is defined when } x-1 \neq 0 \\ \therefore x \neq 1$$

$$\sqrt{x+1} \text{ is defined when } x+1 \geq 0 \\ \therefore x \geq -1$$

$\therefore$  the domain of  $f$  is  $\{x \mid x \geq -1 \text{ and } x \neq 1\}$ .

$\mathbf{c}$  The domain of  $g$  is  $\{x \mid x \in \mathbb{R}\}$ .

$$\text{Now } \frac{1}{x^2-1} \text{ is defined when } x^2-1 \neq 0 \\ \therefore (x+1)(x-1) \neq 0 \\ \therefore x \neq -1 \text{ or } 1$$

and  $x^2+1 \geq 0$ , so  $\sqrt{x^2+1}$  is defined for every value of  $x$ .

$\therefore$  the domain of  $(f \circ g)$  is  $\{x \mid x \neq -1 \text{ and } x \neq 1\}$ .

This is different to the domain of  $f$  and  $g$  since  $(f \circ g)$  is defined using  $g$  whose range is  $\{y \mid y \geq 0\}$ .

$$28 \quad f(x) = \sqrt{x+4}, \quad g(x) = x^2 - 3$$

$$\mathbf{a} \quad (f \circ g)(x) = f(g(x)) \\ = f(x^2 - 3) \\ = \sqrt{x^2 - 3 + 4} \\ = \sqrt{x^2 + 1}$$

$x^2 + 1 \geq 1$ , so  $\sqrt{x^2 + 1}$  is defined for every value of  $x$ .

$\therefore$  the domain is  $\{x \mid x \in \mathbb{R}\}$ .

$$x^2 + 1 \geq 1$$

$$\therefore \sqrt{x^2 + 1} \geq 1$$

$\therefore$  the range is  $\{y \mid y \geq 1\}$ .

$$\mathbf{b} \quad (g \circ f)(x) = g(f(x)) \\ = g(\sqrt{x+4}) \\ = (\sqrt{x+4})^2 - 3 \\ = x + 1$$

$$\sqrt{x+4} \text{ is defined when } x+4 \geq 0 \\ \therefore x \geq -4$$

$\therefore$  the domain is  $\{x \mid x \geq -4\}$ .

$$x \geq -4$$

$$\therefore x + 1 \geq -3$$

$\therefore$  the range is  $\{y \mid y \geq -3\}$ .

29  $\mathbf{a}$  The tank is being filled at 600 mL per minute or 0.6 L per minute.

$$\therefore \frac{dV}{dt} = 0.6$$

$$\therefore V = \int 0.6 dt \\ = 0.6t + c$$

The tank initially contains 400 L of water.

$$\therefore \text{ when } t = 0, \quad V = 400$$

$$\therefore c = 400$$

$$\text{So, } V = 0.6t + 400$$

$$\mathbf{c} \quad (h \circ V)(t) = h(0.6t + 400) \quad \{\text{using } \mathbf{a}\} \\ = \frac{0.6t + 400}{1000\pi}$$

This is the height of water in the tank after  $t$  minutes.

$\mathbf{b}$  The tank is cylindrical with radius  $r = 1$  m.

Let the height of the tank be  $h$  m.

$$\therefore \text{ the volume of the tank } = \pi \times 1^2 \times h \\ = \pi h \text{ m}^3$$

$$\text{Now } 1 \text{ m}^3 \equiv 1000 \text{ L}$$

$$\therefore \pi h \times 1000 = V$$

$$\therefore h = \frac{V}{1000\pi}$$

$\mathbf{d}$  1 hour  $\equiv$  60 minutes

$$(h \circ V)(60) = \frac{0.6(60) + 400}{1000\pi} \\ \approx 0.139$$

$\therefore$  after 1 hour, the height of the water in the tank was about 0.139 m or 13.9 cm.

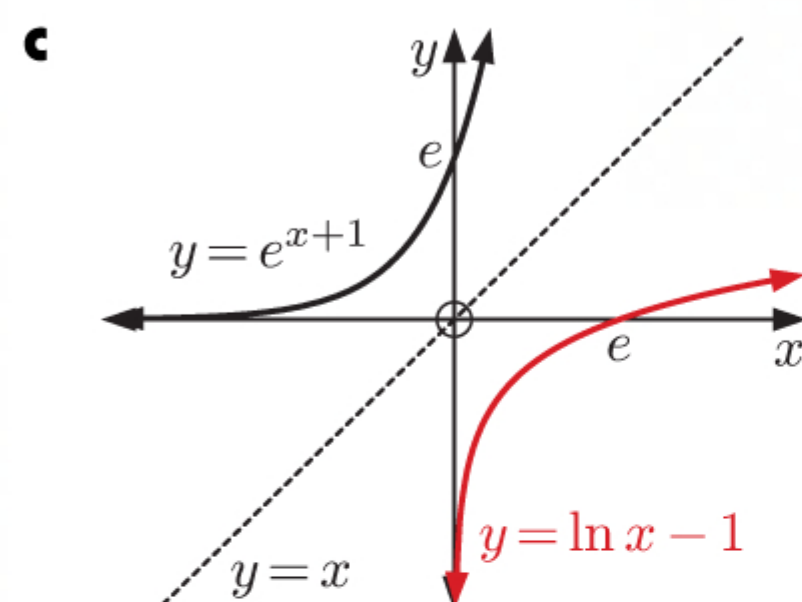


**30**  $f: x \mapsto e^{x+1}$ ,  $g: x \mapsto \ln x - 1$

**a**  $(f \circ g)(x) = f(g(x))$   
 $= f(\ln x - 1)$   
 $= e^{\ln x - 1 + 1}$   
 $= e^{\ln x}$   
 $= x$

The domain is  $\{x \mid x > 0\}$ .

The range is  $\{y \mid y > 0\}$ .



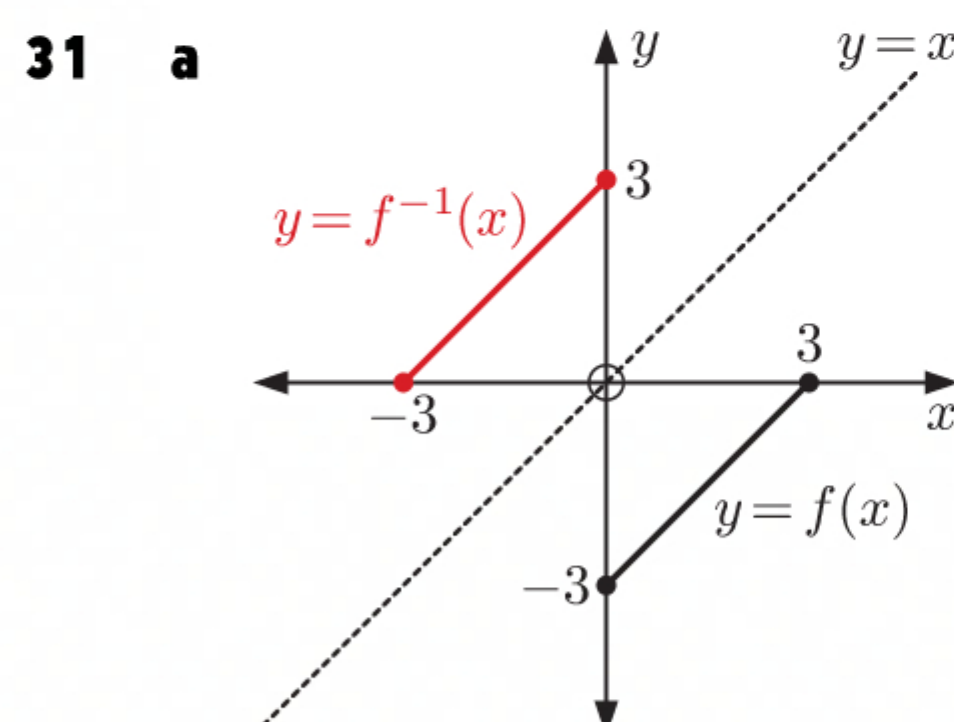
**b**  $(g \circ f)(x) = g(f(x))$   
 $= g(e^{x+1})$   
 $= \ln(e^{x+1}) - 1$   
 $= x + 1 - 1$   
 $= x$

The domain is  $\{x \mid x \in \mathbb{R}\}$ .

The range is  $\{y \mid y \in \mathbb{R}\}$ .

**d** The graph of  $y = g(x)$  is a reflection of the graph of  $y = f(x)$  in the line  $y = x$ .

$\therefore f$  and  $g$  are inverses of one another.

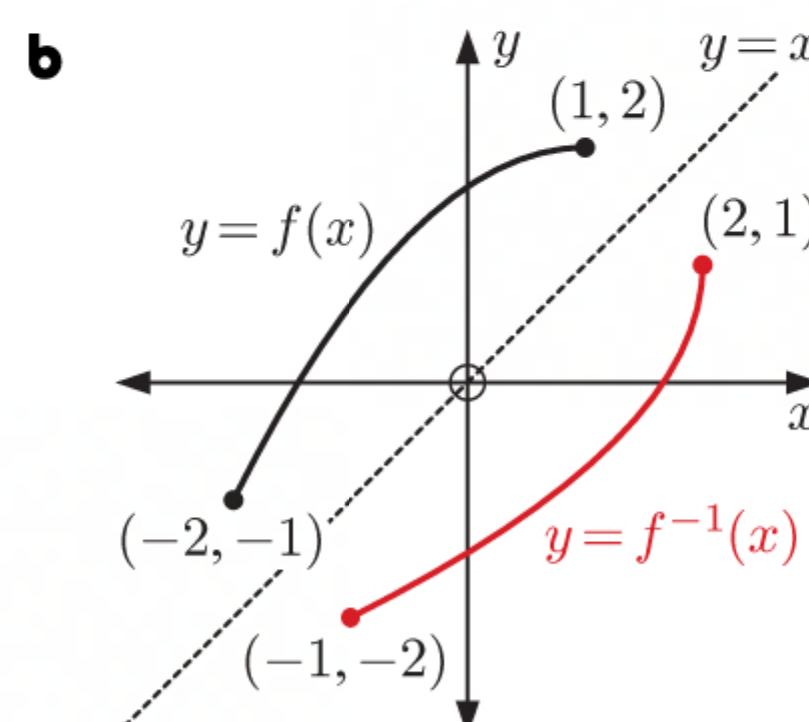


Domain of  $f$ :  $\{x \mid 0 \leq x \leq 3\}$

Range of  $f$ :  $\{y \mid -3 \leq y \leq 0\}$

Domain of  $f^{-1}$ :  $\{x \mid -3 \leq x \leq 0\}$

Range of  $f^{-1}$ :  $\{y \mid 0 \leq y \leq 3\}$

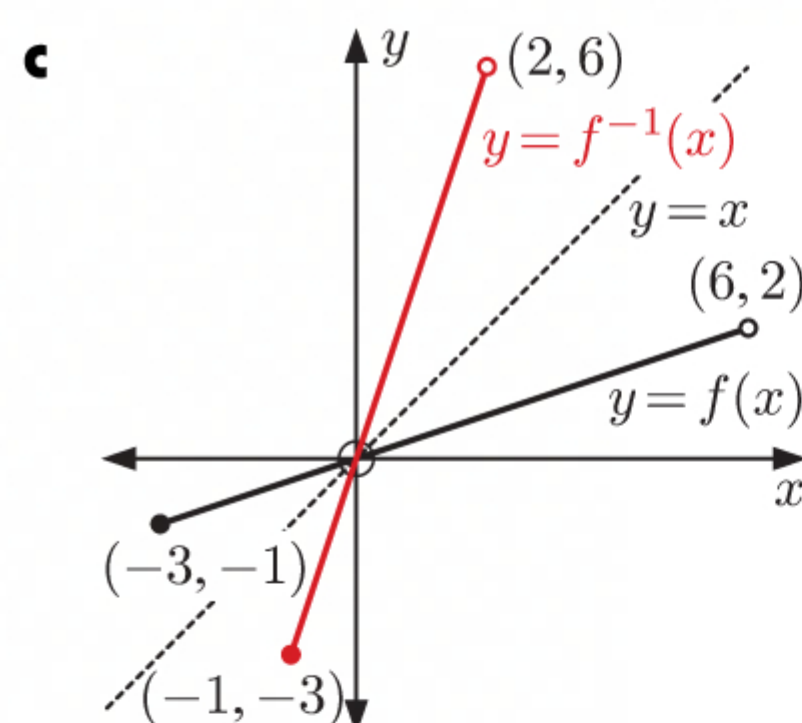


Domain of  $f$ :  $\{x \mid -2 \leq x \leq 1\}$

Range of  $f$ :  $\{y \mid -1 \leq y \leq 2\}$

Domain of  $f^{-1}$ :  $\{x \mid -1 \leq x \leq 2\}$

Range of  $f^{-1}$ :  $\{y \mid -2 \leq y \leq 1\}$



Domain of  $f$ :  $\{x \mid -3 \leq x < 6\}$

Range of  $f$ :  $\{y \mid -1 \leq y < 2\}$

Domain of  $f^{-1}$ :  $\{x \mid -1 \leq x < 2\}$

Range of  $f^{-1}$ :  $\{y \mid -3 \leq y < 6\}$

**32**  $f: x \mapsto 3x + 1$ ,  $g: x \mapsto 4 - x$

**a**  $f(g(x)) = f(4 - x)$   
 $= 3(4 - x) + 1$   
 $= 12 - 3x + 1$   
 $= 13 - 3x$

**b**  $(g \circ f)(-4) = g(f(-4))$   
 $= g(3(-4) + 1)$   
 $= g(-11)$   
 $= 4 - (-11)$   
 $= 15$

**c**  $f$  is  $y = 3x + 1$   
 $\therefore f^{-1}$  is  $x = 3y + 1$   
 $\therefore 3y = x - 1$   
 $\therefore y = \frac{1}{3}x - \frac{1}{3}$   
 So,  $f^{-1}(x) = \frac{1}{3}x - \frac{1}{3}$   
 $\therefore f^{-1}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{1}{2}\right) - \frac{1}{3}$   
 $= \frac{1}{6} - \frac{1}{3}$   
 $= -\frac{1}{6}$



**33**  $f: x \mapsto \ln x, \quad g: x \mapsto 3 + x$

**a**  $f$  is  $y = \ln x$   $g$  is  $y = 3 + x$

$\therefore f^{-1}$  is  $x = \ln y$   $\therefore g^{-1}$  is  $x = 3 + y$

$\therefore y = e^x$   $\therefore y = x - 3$

So,  $f^{-1}(x) = e^x$  So,  $g^{-1}(x) = x - 3$

$\therefore f^{-1}(2) \times g^{-1}(2) = e^2(2 - 3) = -e^2$

**b**  $(f \circ g)(x) = f(g(x))$   $f \circ g$  is  $y = \ln(3 + x)$

$= f(3 + x)$   $\therefore (f \circ g)^{-1}$  is  $x = \ln(3 + y)$

$= \ln(3 + x)$   $\therefore e^x = 3 + y$

$\therefore y = e^x - 3$

So,  $(f \circ g)^{-1}(x) = e^x - 3$

$\therefore (f \circ g)^{-1}(2) = e^2 - 3$

**34**  $f: x \mapsto x + 5, \quad g: x \mapsto 7 - 3x$

**a i**  $f$  is  $y = x + 5$

$\therefore f^{-1}$  is  $x = y + 5$

$\therefore y = x - 5$

So,  $f^{-1}(x) = x - 5$

**ii**  $g$  is  $y = 7 - 3x$

$\therefore g^{-1}$  is  $x = 7 - 3y$

$\therefore 3y = 7 - x$

$\therefore y = \frac{7}{3} - \frac{1}{3}x$

So,  $g^{-1}(x) = \frac{7}{3} - \frac{1}{3}x$

**iii**  $(f \circ g)(x) = f(g(x))$

$= f(7 - 3x)$

$= (7 - 3x) + 5$

$= 12 - 3x$

**b**  $f \circ g$  is  $y = 12 - 3x$  {using **a iii**}

$\therefore (f \circ g)^{-1}$  is  $x = 12 - 3y$

$\therefore 3y = 12 - x$

$\therefore y = 4 - \frac{1}{3}x$

So,  $(f \circ g)^{-1}(x) = 4 - \frac{1}{3}x$

$(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x))$

$= g^{-1}(x - 5)$  {using **a i**}

$= \frac{7}{3} - \frac{1}{3}(x - 5)$  {using **a ii**}

$= \frac{7}{3} - \frac{1}{3}x + \frac{5}{3}$

$= 4 - \frac{1}{3}x$

$= (f \circ g)^{-1}(x)$

**35**  $f$  is  $y = x^2 + 2x, \quad x \leq -1$

$\therefore f^{-1}$  is  $x = y^2 + 2y, \quad y \leq -1$

$\therefore x + 1 = y^2 + 2y + 1$

$\therefore x + 1 = (y + 1)^2$

$\therefore y + 1 = -\sqrt{x + 1} \quad \{y + 1 \leq 0\}$

$\therefore y = -1 - \sqrt{x + 1}$

$\therefore f^{-1}(x) = -1 - \sqrt{x + 1}$

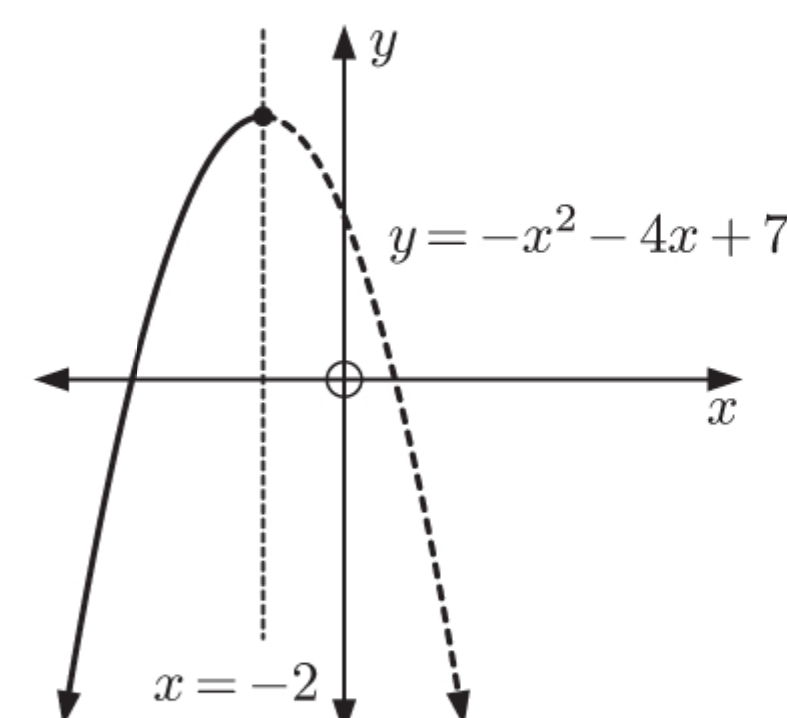
$f(x)$  has domain  $\{x \mid x \leq -1\}$  and range  $\{y \mid y \geq -1\}$

$\therefore f^{-1}(x)$  has domain  $\{x \mid x \geq -1\}$  and range  $\{y \mid y \leq -1\}$ .

**36**  $f(x) = -x^2 - 4x + 7, \quad x \leq k$

**a** The largest value of  $k$  such that  $f^{-1}(x)$  exists corresponds to the axis of symmetry of  $y = -x^2 - 4x + 7$ .

$\therefore k = \frac{-(-4)}{2(-1)} = \frac{4}{-2} = -2$





**b i**  $f$  is  $y = -x^2 - 4x + 7, \quad x \leq -2$   
 $\therefore f^{-1}$  is  $x = -y^2 - 4y + 7, \quad y \leq -2$   
 $\therefore x = -(y^2 + 4y - 7)$   
 $\therefore x = -[(y + 2)^2 - 11]$   
 $\therefore x = -(y + 2)^2 + 11$   
 $\therefore x - 11 = -(y + 2)^2$   
 $\therefore 11 - x = (y + 2)^2$   
 $\therefore -\sqrt{11 - x} = y + 2 \quad \{y \leq -2\}$   
 $\therefore -\sqrt{11 - x} - 2 = y$   
 $\therefore f^{-1}(x) = -\sqrt{11 - x} - 2$

**ii** Domain =  $\{x \mid x \leq 11\}$   
Range =  $\{y \mid y \leq -2\}$

**37 a** The moon completes one full orbit in about 28 days.

$\therefore$  it completes  $\frac{t}{28}$  of an orbit after  $t$  days.

So, after  $t$  days the moon has travelled through  $\frac{t}{28} \times 360^\circ = \left(\frac{90t}{7}\right)^\circ$ .

$\therefore$  the distance  $D$  km travelled by the moon in  $t$  days is given by

$$D = \frac{\frac{90t}{7}}{360} \times 2\pi \times (3.84 \times 10^5) \quad \{\text{arc length formula}\}$$

$$= \frac{192\,000\pi}{7} t$$

**b** We have assumed that:

- the moon has a circular orbit
- the moon moves at a constant speed along its orbit.

**c i** One full orbit is about 28 days.

$$\text{When } t = 28, \quad D = \frac{192\,000\pi}{7} \times 28$$

$$\approx 2.41 \times 10^6$$

$\therefore$  the moon travels about  $2.41 \times 10^6$  km in one full orbit.

**d** In our model, the time it takes for the moon to complete one full orbit is an overestimate.

$\therefore$  the fraction of an orbit it has travelled after  $t$  days is an underestimate.

$\therefore$  our model is an underestimate of the actual distance travelled by the moon in  $t$  days.

So, our answers in **c** are underestimates.

**38 a i** Isabelita can prepare 600 spring rolls in  $\frac{600}{120} = 5$  hours.

**ii** Arturo can prepare 600 spring rolls in  $\frac{600}{100} = 6$  hours.

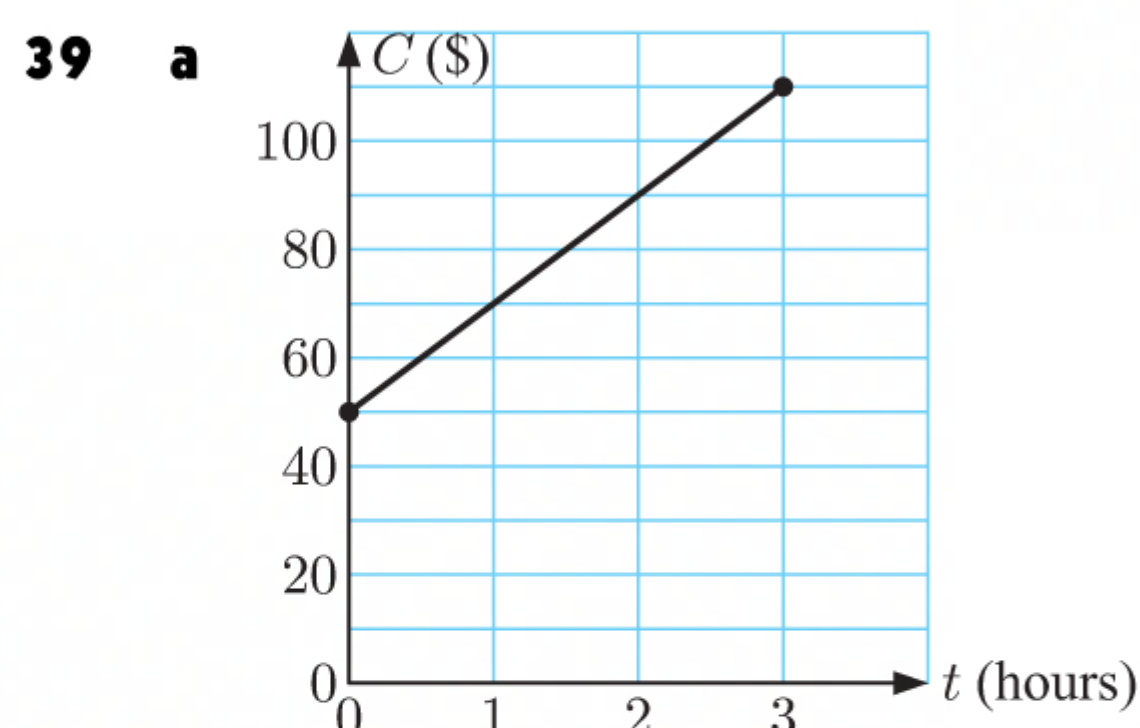
**iii** Isabelita can complete  $\frac{120}{600} = \frac{1}{5}$  of the job each hour, and Arturo can complete  $\frac{100}{600} = \frac{1}{6}$  of the job each hour.

So, working together they can complete  $\frac{1}{5} + \frac{1}{6} = \frac{11}{30}$  of the job each hour.

$\therefore$  it would take them  $\frac{30}{11} \approx 2.73$  hours  $\approx 2$  hours 44 minutes to complete the job together.

**b** In **a i** and **a ii**, we assume that Isabelita and Arturo can work at a constant rate over many hours. It is likely that they would slow down or need a rest, so this assumption may not be reasonable.

In **a iii**, we assume that Isabelita and Arturo can work without getting in each other's way. This assumption seems reasonable.



**b** The  $C$ -intercept is 50.

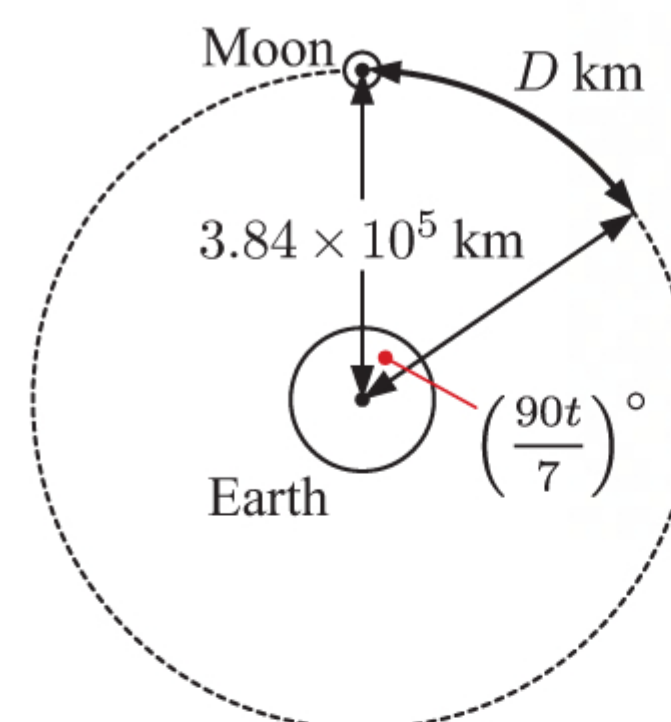
$$\text{The gradient is } \frac{70 - 50}{1 - 0} = 20.$$

$$\therefore C = 20t + 50$$

**c** When  $t = 1\frac{1}{2} = \frac{3}{2}$ ,  $C = 20\left(\frac{3}{2}\right) + 50$

$$= 80$$

$\therefore$  \$80 is charged for a  $1\frac{1}{2}$  hour appointment.





**d** When  $C = 118$ ,  $118 = 20t + 50$

$$\therefore 20t = 68$$

$$\therefore t = 3.4$$

$\therefore$  the appointment lasted 3.4 hours = 3 hours 24 minutes.

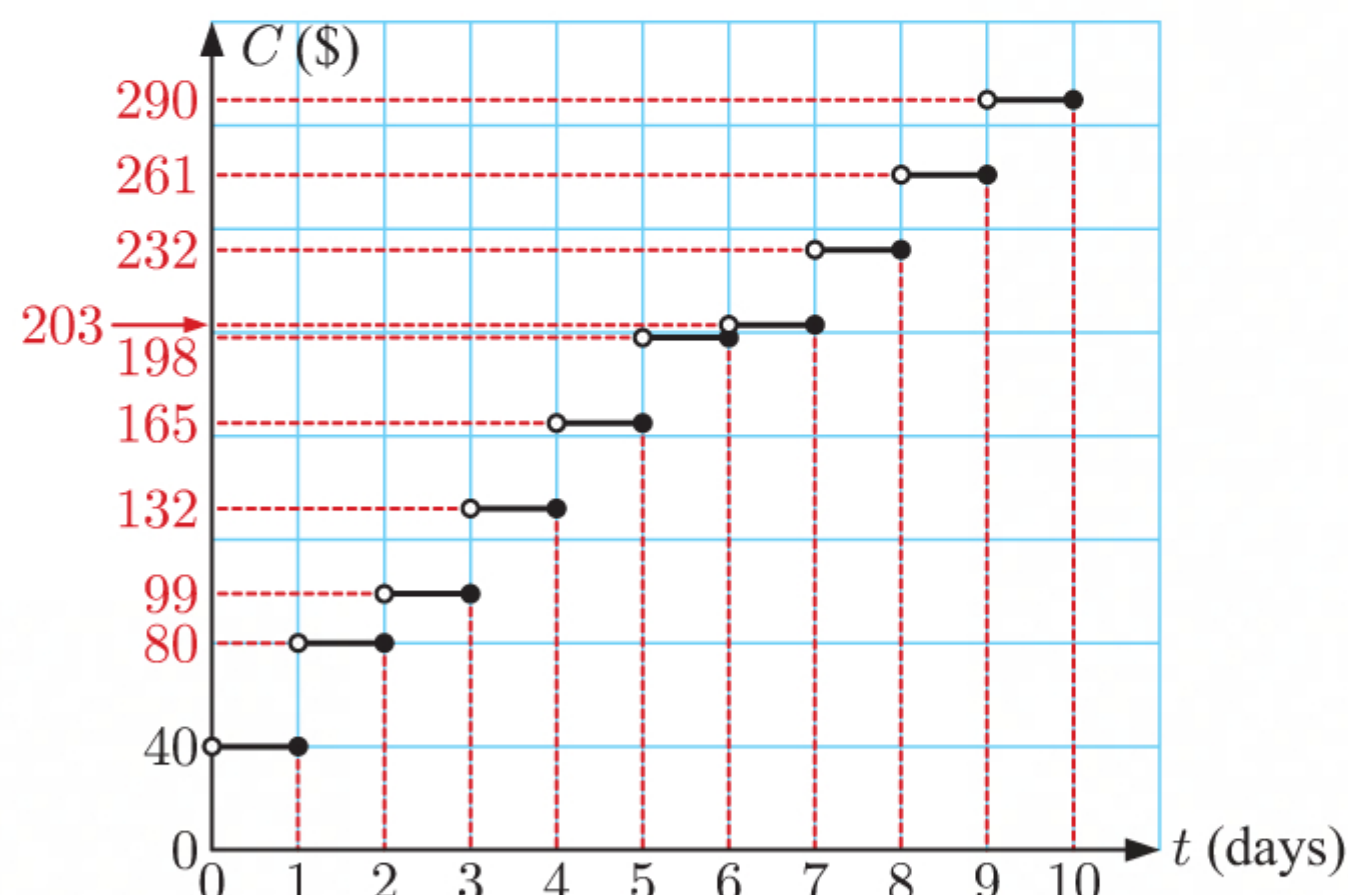
12:24 pm is 3 hours and 24 minutes after 9:00 am.

$\therefore$  the technician completed the repairs at 12:24 pm.

**40**

Hire period ( $t$ days)	Cost
1 - 2	\$40 per day
3 - 6	\$33 per day
7+	\$29 per day

**a**



**b i** From the graph, it costs \$80 to hire a car for 2 days.

**ii** From the graph, it costs \$165 to hire a car for 5 days.

**iii** From the graph, it costs \$261 to hire a car for 9 days.

**c** If they hire a car for the first 2 days, and the last 3 days, they will spend  $\$80 + \$99 = \$179$ .

If they hire a car for the whole 8 days, they will spend \$232.

$\therefore$  it is cheaper for Georgia and Tim to hire one car for the first 2 days, and a separate car for the last 3 days.

**41** 
$$T(t) = \begin{cases} 260 - 240 \times 2^{-\frac{t}{a}}, & 0 \leq t < 10 \\ 200, & 10 \leq t < 30 \\ 30 + b \times 5^{\frac{30-t}{10}}, & t \geq 30 \end{cases}$$

**a**  $T(0) = 260 - 240 \times 2^0 \quad \{0 \leq t < 10\}$   
 $= 20$

$\therefore$  the temperature of the casserole when it is placed in the oven is  $20^\circ\text{C}$ .

**b** When  $t = 10$ ,  $260 - 240 \times 2^{-\frac{t}{a}} = 200$

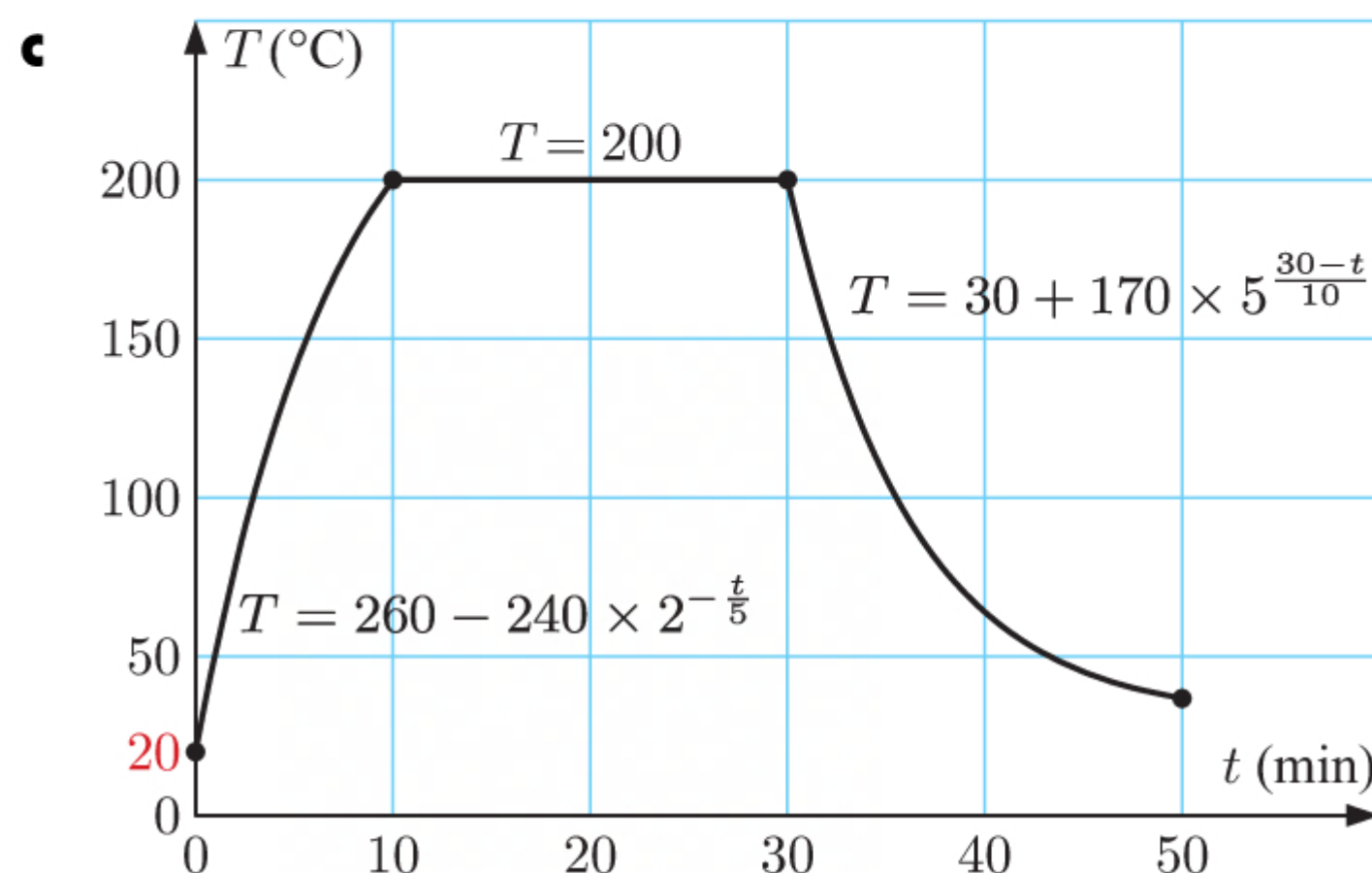
$$\therefore -240 \times 2^{-\frac{10}{a}} = -60$$

$$\therefore a = 5 \quad \{\text{technology}\}$$

When  $t = 30$ ,  $200 = 30 + b \times 5^{\frac{30-t}{10}}$

$$\therefore 170 = b \times 5^0$$

$$\therefore b = 170$$



**d** From the graph, the temperature of the casserole begins to fall after  $t = 30$  minutes.

$\therefore$  the casserole was in the oven for 30 minutes.



**e**  $T = 45$  when  $30 + 170 \times 5^{\frac{30-t}{10}} = 45 \quad \{t \geq 30\}$   
 $t \approx 45.1 \quad \{\text{technology}\}$

$\therefore$  after being removed from the oven, it took about  $45.1 - 30 = 15.1$  minutes for the temperature of the casserole to fall to  $45^\circ\text{C}$ .

**42 a** When  $t = 0$ ,  $y = 49$

$\therefore d = 49$

**b** When  $t = 1$ ,  $y = 44$

$\therefore a(1)^3 + b(1)^2 + c(1) + 49 = 44$

$\therefore a + b + c = -5$

When  $t = 2$ ,  $y = 57$

$\therefore a(2)^3 + b(2)^2 + c(2) + 49 = 57$

$\therefore 8a + 4b + 2c = 8$

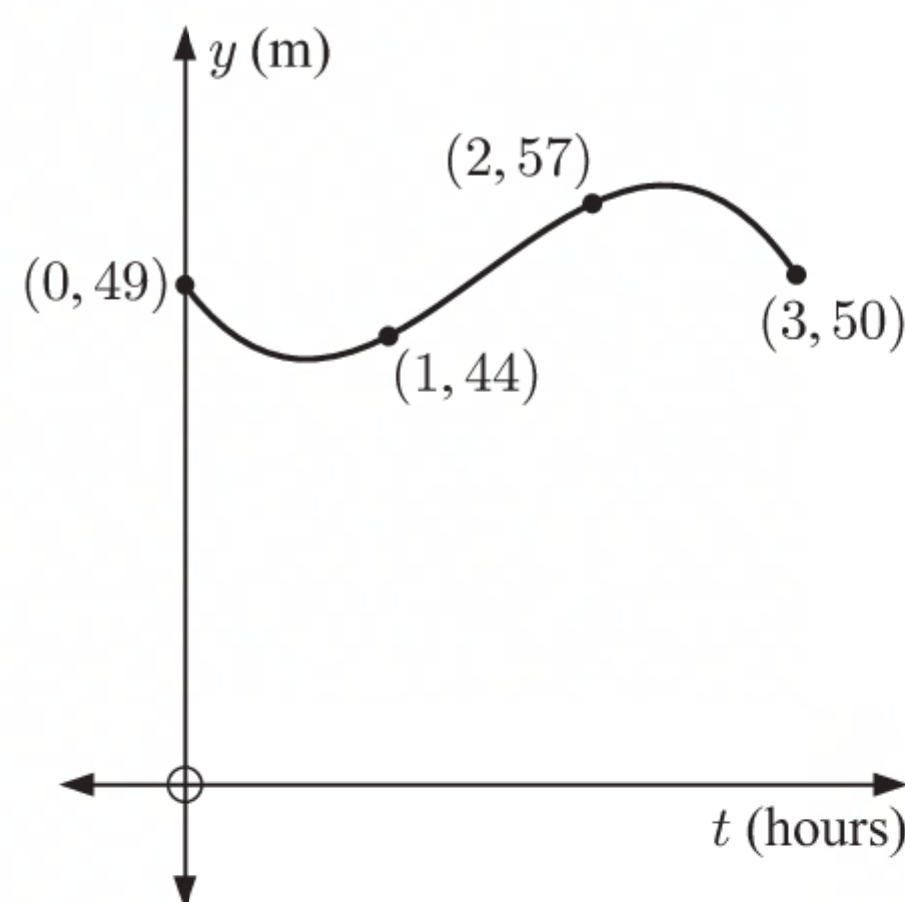
When  $t = 3$ ,  $y = 50$

$\therefore a(3)^3 + b(3)^2 + c(3) + 49 = 50$

$\therefore 27a + 9b + 3c = 1$

So, we have the system of equations 
$$\begin{cases} a + b + c = -5 \\ 8a + 4b + 2c = 8 \\ 27a + 9b + 3c = 1 \end{cases}$$

Solving these equations simultaneously using technology, we find that  $a = -\frac{19}{3}$ ,  $b = 28$ , and  $c = -\frac{80}{3}$ .



	a	b	c	d
1	1	1	1	-5
2	8	4	2	8
3	27	9	3	1

	X	Y	Z
1	-6.333	28	-26.66
2			
3			

**c**  $y = -\frac{19}{3}t^3 + 28t^2 - \frac{80}{3}t + 49 \quad \{\text{from a and b}\}$

When  $t = 2\frac{1}{2} = \frac{5}{2}$ ,

$y = -\frac{19}{3}\left(\frac{5}{2}\right)^3 + 28\left(\frac{5}{2}\right)^2 - \frac{80}{3}\left(\frac{5}{2}\right) + 49$   
 $= 58.375$

$\therefore$  we estimate that Stephen and Hugh's elevation after  $2\frac{1}{2}$  hours is 58.375 m.

**d** Percentage error  $= \frac{|V_A - V_E|}{V_E} \times 100\%$   
 $= \frac{|58.375 - 60|}{60} \times 100\% \quad \{\text{from c}\}$   
 $= \frac{1.625}{60} \times 100\%$   
 $\approx 2.71\%$

**43 a**

$z$	9	2
$w$	27	

$z$  is multiplied by  $\frac{2}{9}$

$\therefore w$  is multiplied by  $\frac{2}{9} \quad \{\text{as } w \propto z\}$

$\therefore w = 27 \times \frac{2}{9} = 6$

**b**

$z$	9	
$w$	27	45

$w$  is multiplied by  $\frac{45}{27} = \frac{5}{3}$

$\therefore z$  is multiplied by  $\frac{5}{3} \quad \{\text{as } w \propto z\}$

$\therefore z = 9 \times \frac{5}{3} = 15$

**44**  $V \propto I$ , so  $V = kI$  where  $k$  is the proportionality constant.

**a** When  $V = 9$  volts,  $I = 0.01$  amps, so  $9 = k(0.01)$

$\therefore k = \frac{9}{0.01} = 900$

**b** When  $V = 12$  volts,  $12 = 900I$

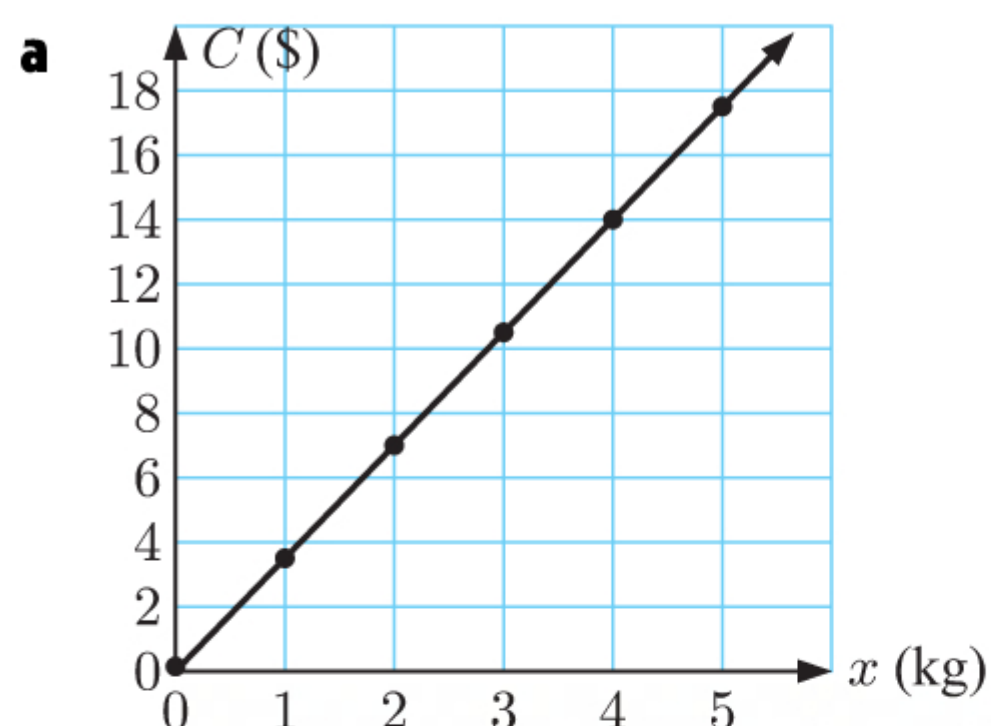
$\therefore I \approx 0.0133$  amps

**c** When  $I = 0.018$  amps,  $V = 900(0.018)$   
 $= 16.2$  volts



45

Weight ( $x$ kg)	0	1	2	3	4	5
Cost (\$ $C$ )	0	3.5	7	10.5	14	17.5



**b** The graph of  $C$  against  $x$  is a straight line which passes through the origin.

$\therefore C$  and  $x$  are directly proportional.

**c i** The gradient of the line  $= \frac{3.5 - 0}{1 - 0} = 3.5$   
 $\therefore C = 3.5x$

**ii** When  $x = 10$ ,  $C = 3.5(10) = 35$   
 $\therefore$  purchasing 10 kg of tomatoes costs \$35.

**46** The mass  $m$  g of an orange is directly proportional to the cube of its diameter  $d$ , so  $m \propto d^3$ .

If  $d$  is increased by 14.6%, then

$d$  is multiplied by 1.146

$\therefore d^3$  is multiplied by  $(1.146)^3$

$\therefore m$  is multiplied by  $(1.146)^3 \approx 1.505$  {as  $m \propto d^3$ }

$\therefore m$  is increased by about 50.5%

So, orange A is about 50.5% heavier than orange B.

**47**  $y \propto \frac{1}{x}$

**a** If  $x$  is tripled, then

$x$  is multiplied by 3

$\therefore y$  is multiplied by  $\frac{1}{3}$

$\therefore y$  is divided by 3.

**c** If  $x$  is multiplied by  $\frac{5}{3}$ , then

$y$  is multiplied by  $\frac{3}{5}$ .

**b** If  $x$  is divided by 4, then

$x$  is multiplied by  $\frac{1}{4}$

$\therefore y$  is multiplied by 4.

**d** If  $x$  is decreased by 60%, then

$x$  is multiplied by  $1 - 0.6 = 0.4$

$\therefore y$  is multiplied by  $\frac{1}{0.4} = 2.5$

$\therefore y$  is increased by 150%.

**48** The pressure  $P$  is inversely proportional to the area  $A$ , so  $P = \frac{k}{A}$  where  $k$  is a constant.

When  $A = 2 \text{ m}^2$ ,  $P = 300 \text{ Pa}$ , so  $300 = \frac{k}{2}$

$\therefore k = 600$

So,  $P = \frac{600}{k}$ .

**a** When  $A = 0.8 \text{ m}^2$ ,  $P = \frac{600}{0.8}$   
 $= 750 \text{ Pa}$

**b** If  $P$  is reduced by 15%, then

$P$  is multiplied by  $1 - 0.15 = 0.85$

$\therefore A$  is multiplied by  $\frac{1}{0.85} \approx 1.176$  {as  $P \propto \frac{1}{A}$ }

$\therefore A$  is increased by about 17.6%.



49 a

$$\begin{array}{|c|c|c|} \hline r & 2 & 5 \\ \hline A & 7 & \\ \hline \end{array}$$

$\times \frac{5}{2}$

 $r$  is multiplied by  $\frac{5}{2}$ 

$$\therefore r^3 \text{ is multiplied by } \left(\frac{5}{2}\right)^3 = \frac{125}{8}$$

$$\therefore A \text{ is multiplied by } \frac{8}{125} \quad \left\{ \text{as } A \propto \frac{1}{r^3} \right\}$$

$$\therefore A = 7 \times \frac{8}{125} = \frac{56}{125} = 0.448$$

b

$$\begin{array}{|c|c|c|} \hline r & 2 & \\ \hline A & 7 & 16 \\ \hline \end{array}$$

$\times \frac{16}{7}$

 $A$  is multiplied by  $\frac{16}{7}$ 

$$\therefore r^3 \text{ is multiplied by } \frac{7}{16} \quad \left\{ \text{as } A \propto \frac{1}{r^3} \right\}$$

$$\therefore r \text{ is multiplied by } \sqrt[3]{\frac{7}{16}}$$

$$\therefore r = 2 \times \sqrt[3]{\frac{7}{16}} \approx 1.52$$

50 a The volume of a cone  $V = \frac{1}{3}\pi r^2 h$ 

$$\therefore h = \frac{3V}{\pi r^2}$$

$$\therefore h \propto \frac{1}{r^2} \quad \left\{ \text{as } V \text{ is a constant} \right\}$$

b If the radius of the cone is increased to 3.2 cm, then  $r$  is multiplied by  $\frac{3.2}{2.8} = \frac{8}{7}$ 

$$\therefore r^2 \text{ is multiplied by } \left(\frac{8}{7}\right)^2 = \frac{64}{49}$$

$$\therefore h \text{ is multiplied by } \frac{49}{64} \quad \left\{ \text{as } h \propto \frac{1}{r^2} \right\}$$

So, the height of the cone is decreased to  $14.3 \times \frac{49}{64} \approx 10.9$  cm.c If the height of the cone is decreased to 10.8 cm, then  $h$  is multiplied by  $\frac{10.8}{14.3} = \frac{108}{143}$ 

$$\therefore r^2 \text{ is multiplied by } \frac{143}{108} \quad \left\{ \text{as } h \propto \frac{1}{r^2} \right\}$$

$$\therefore r \text{ is multiplied by } \sqrt{\frac{143}{108}} \quad \left\{ \text{as } r > 0 \right\}$$

So, the radius of the cone is increased to  $2.8 \times \sqrt{\frac{143}{108}} \approx 3.22$  cm.

d The cones would otherwise be too thin or too wide to be practical to eat.

51 a

$x$	1	2	4
$y$	2	$\frac{1}{4}$	$\frac{1}{32}$
$x^2y$	2	1	$\frac{1}{2}$
$x^3y$	2	2	2
$x^4y$	2	4	8


b  $x^3y = 2$  for each point.
$$\therefore x^3y = 2 \text{ or } y = \frac{2}{x^3} \text{ is the correct model.}$$
c When  $x = 5$ ,  $y = \frac{2}{5^3} = \frac{2}{125}$ 

52

Diameter ( $d$ m)	0.77	1.22	1.69	2.25
Mass ( $m$ kg)	0.97	2.44	4.68	8.30

a As the diameter of a rug increases, we expect the mass of the rug to increase.

 $\therefore$  we expect direct variation between the variables.b The correlation coefficient  $r$  is very close to 1, so the fit is excellent.The power is very close to 2, so it is reasonable to conclude that  $m$  is directly proportional to  $d^2$ .The model is  $m \approx 1.64d^2$ .c When  $d = 1.5$ ,  $m \approx 1.64(1.5)^2 \approx 3.69$ So, a rug with diameter 1.5 m has mass  $\approx 3.69$  kg.

	Des	Norm1	d/c	Real
PowerReg				
a = 1.63744203				
b = 2.0017764				
r = 0.99999982				
r <sup>2</sup> = 0.99999964				
MSe = 4.4849 × 10 <sup>-7</sup>				
y = a · x <sup>b</sup>				
				COPY DRAW



53 a

$x$	2	3	5	6
$y$	1.13	8.60	111	275

The correlation coefficient  $r$  is very close to 1, so the fit is excellent.

The power is very close to 5, so it is reasonable to conclude that  $y$  is directly proportional to  $x^5$ .

The model is  $y \approx 0.0353x^5$ .

[2nd] [Norm] [d/c] [Real] <b>PowerReg</b> $a = 0.035267$ $b = 5.00283583$ $r = 0.99999966$ $r^2 = 0.99999932$ $MSe = 6.2892 \times 10^{-6}$ $y = a \cdot x^b$	[COPY] [DRAW]
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b

$x$	1	4	5	7	8
$y$	72.1	4.6	2.9	1.5	1.1

The correlation coefficient  $r$  is very close to  $-1$ , so the fit is excellent.

The power is very close to  $-2$ , so it is reasonable to conclude that  $y$  is inversely proportional to  $x^2$ .

The model is  $y \approx \frac{72.6}{x^2}$ .

[2nd] [Norm] [d/c] [Real] <b>PowerReg</b> $a = 72.5921348$ $b = -2.001783$ $r = -0.9999416$ $r^2 = 0.99988329$ $MSe = 4.3086 \times 10^{-4}$ $y = a \cdot x^b$	[COPY] [DRAW]
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 54  $f(x) = 2 \times 3^{-x}$ 

a  $f(0) = 2 \times 3^0$   
 $= 2$

b  $f(1) = 2 \times 3^{-1}$   
 $= \frac{2}{3}$

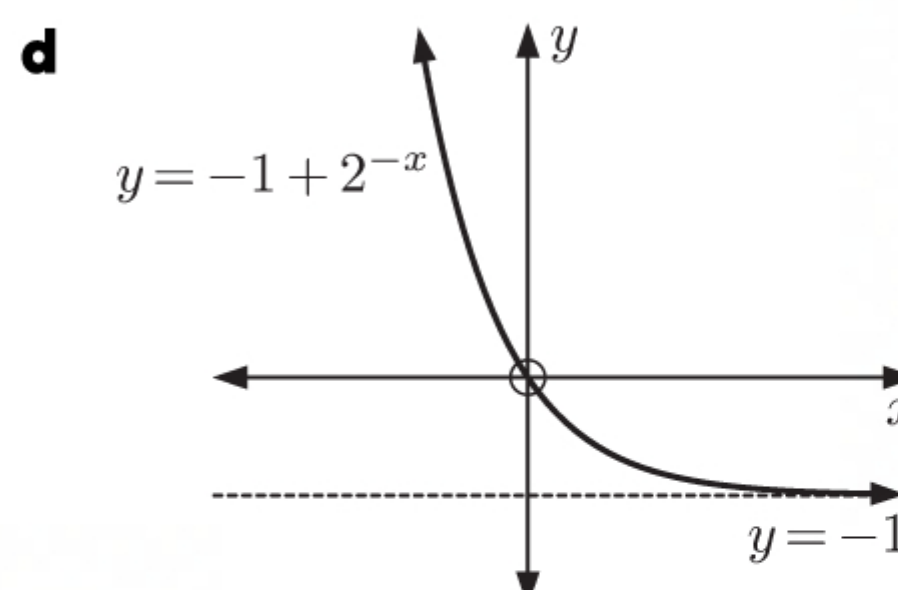
c  $f(-2) = 2 \times 3^2$   
 $= 18$

 55  $y = -1 + 2^{-x}$ 

a When  $x = 0$ ,  $y = -1 + 1 = 0$   
 $\therefore$  the  $y$ -intercept is 0, and the  $x$ -intercept is 0.

c The domain is  $\{x \mid x \in \mathbb{R}\}$ .  
The range is  $\{y \mid y > -1\}$ .

b The horizontal asymptote is  $y = -1$ .


 56  $y = a \times 2^x + b$ 

$x$	0	1	2	3
$y$	20	$p$	35	$q$

a When  $x = 0$ ,  $y = 20$   
 $\therefore 20 = a \times 2^0 + b$   
 $\therefore a + b = 20 \quad \dots (1)$

When  $x = 2$ ,  $y = 35$   
 $\therefore 35 = a \times 2^2 + b$   
 $\therefore 4a + b = 35 \quad \dots (2)$

b Using (1),  $b = 20 - a \quad \dots (3)$

Substituting  $b = 20 - a$  into (2) gives

$$4a + 20 - a = 35$$

$$\therefore 3a = 15$$

$$\therefore a = 5$$

Substituting  $a = 5$  into (3) gives  $b = 20 - 5 = 15$

$\therefore a = 5$  and  $b = 15$

c Using b,  $y = 5 \times 2^x + 15$

When  $x = 1$ ,  $y = p$   
 $\therefore p = 5 \times 2^1 + 15$   
 $\therefore p = 25$

When  $x = 3$ ,  $y = q$   
 $\therefore q = 5 \times 2^3 + 15$   
 $\therefore q = 55$

 57  $f(x) = 2 \times \left(\frac{1}{3}\right)^x + 1$ 

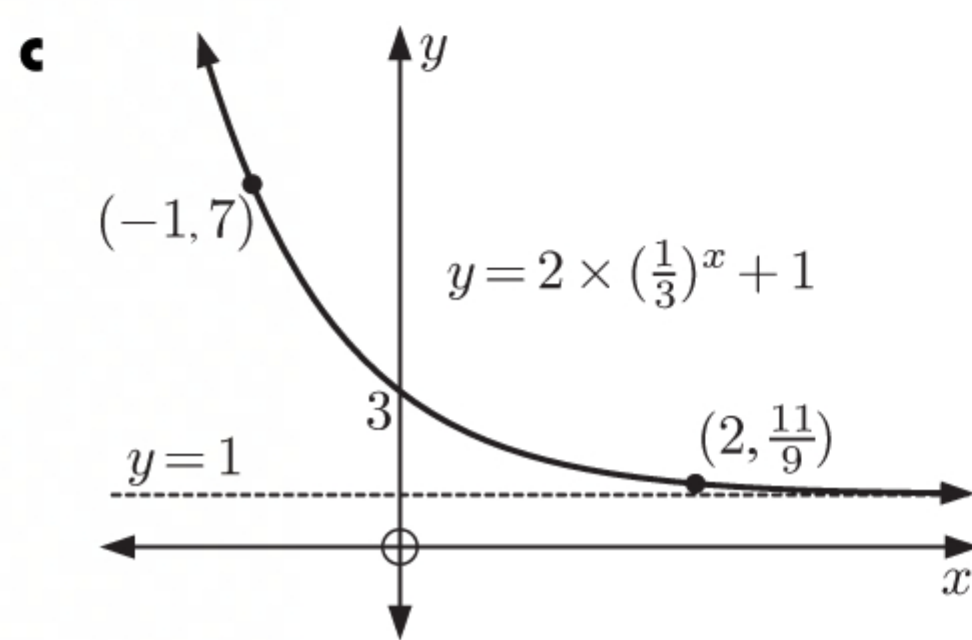
a i  $f(0) = 2 + 1$   
 $= 3$

ii  $f(2) = 2 \times \left(\frac{1}{3}\right)^2 + 1$   
 $= \frac{2}{9} + 1$   
 $= \frac{11}{9}$

iii  $f(-1) = 2 \times \left(\frac{1}{3}\right)^{-1} + 1$   
 $= 2 \times 3 + 1$   
 $= 7$

b The horizontal asymptote is  $y = 1$ .



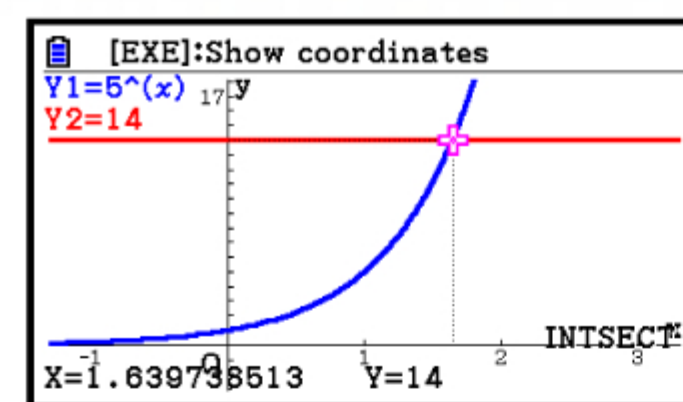


**d** The domain is  $\{x \mid x \in \mathbb{R}\}$ .

The range is  $\{y \mid y > 1\}$ .

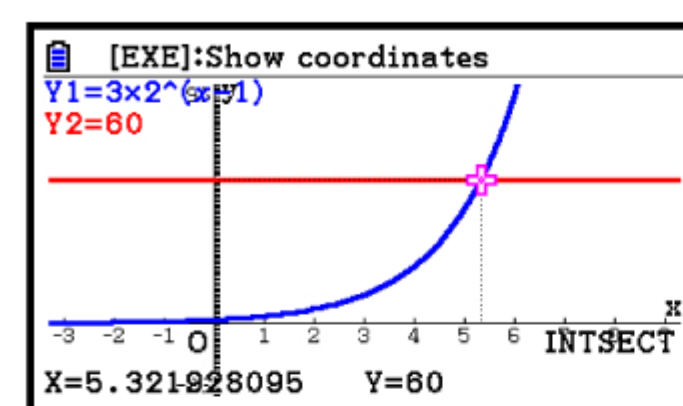
- 58 a** We graph  $Y_1 = 5^x$  and  $Y_2 = 14$  on the same set of axes, and find their point of intersection.

The solution is  $x \approx 1.64$ .



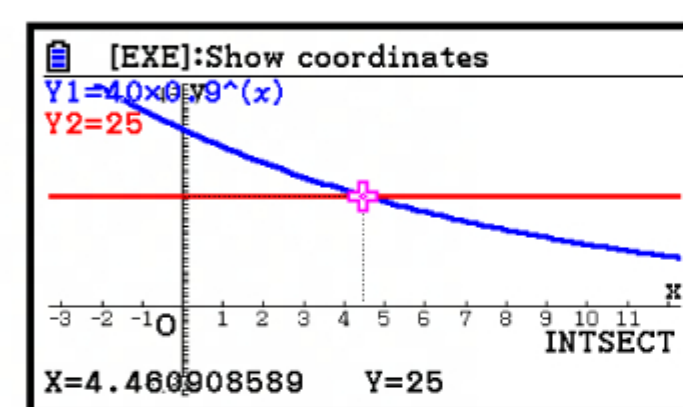
- b** We graph  $Y_1 = 3 \times 2^{x-1}$  and  $Y_2 = 60$  on the same set of axes, and find their point of intersection.

The solution is  $x \approx 5.32$ .



- c** We graph  $Y_1 = 40 \times (0.9)^x$  and  $Y_2 = 25$  on the same set of axes, and find their point of intersection.

The solution is  $x \approx 4.46$ .



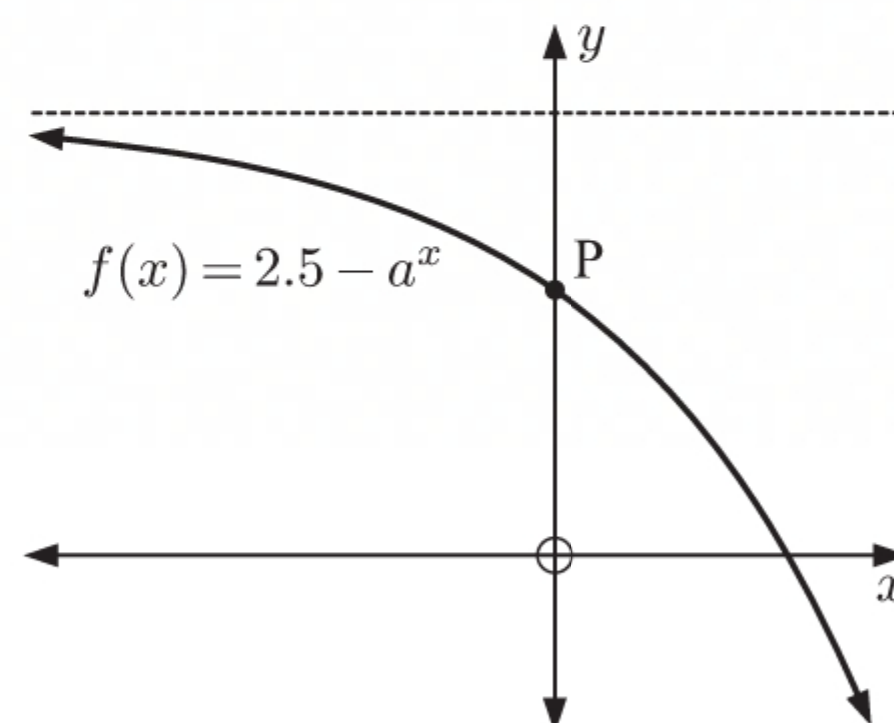
- 59 a**  $f(0) = 2.5 - a^0$   
 $= 2.5 - 1 \quad \{a > 0\}$   
 $= 1.5$

$\therefore$  P has coordinates  $(0, 1.5)$ .

- b** The point  $(3, -5.5)$  lies on the graph

$$\begin{aligned}\therefore f(3) &= -5.5 \\ \therefore -5.5 &= 2.5 - a^3 \\ \therefore a^3 &= 8 \\ \therefore a &= 2\end{aligned}$$

- c** The horizontal asymptote has equation  $y = 2.5$ .



- 60 a**  $P(t) = a(0.95)^t + b$

Now  $P(0) = 2500$

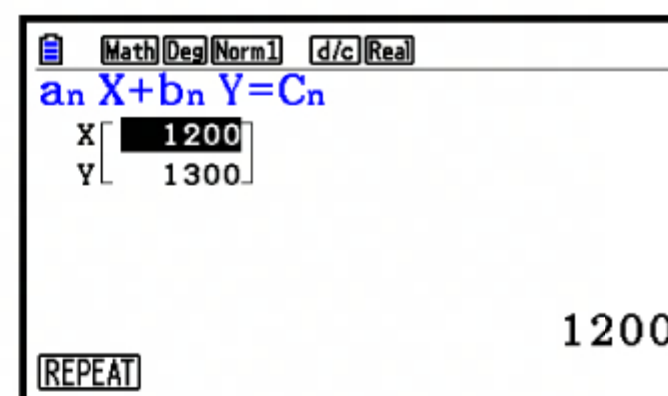
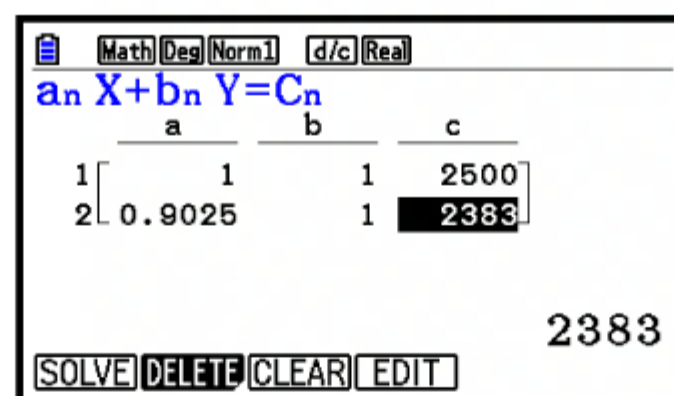
and  $P(2) = 2383$

$$\therefore 2500 = a(0.95)^0 + b$$

$$\therefore 2383 = a(0.95)^2 + b$$

$$\therefore a + b = 2500 \quad \dots (1)$$

$$\therefore 0.9025a + b = 2383 \quad \dots (2)$$



Solving (1) and (2) simultaneously, we find:

**i**  $a = 1200$

**ii**  $b = 1300$

- b**  $P(t) = 1200(0.95)^t + 1300 \quad \{\text{using a}\}$

**i**  $P(3) = 1200(0.95)^3 + 1300$   
 $= 2328.85$

After 3 weeks, there are about 2330 bees.

- c** The horizontal asymptote is  $P = 1300$ .

As  $t \rightarrow \infty$ , the population approaches 1300 bees.

**ii**  $P(5) = 1200(0.95)^5 + 1300$   
 $\approx 2228.54$

After 5 weeks, there are about 2230 bees.



**61 a i** When  $t = 4$ ,  $P \approx 60$

$\therefore$  there is about 60% of Carbon-14 remaining after 4000 years.

**ii** When  $P = 50$ ,  $t \approx 5.5$

$\therefore$  it will take approximately 5500 years for the percentage of Carbon-14 to fall to 50%.

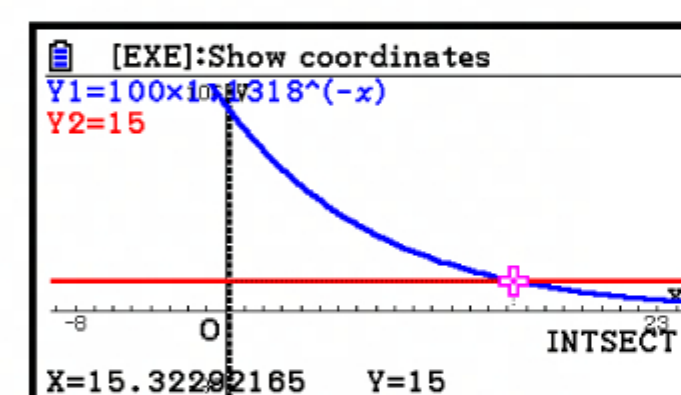
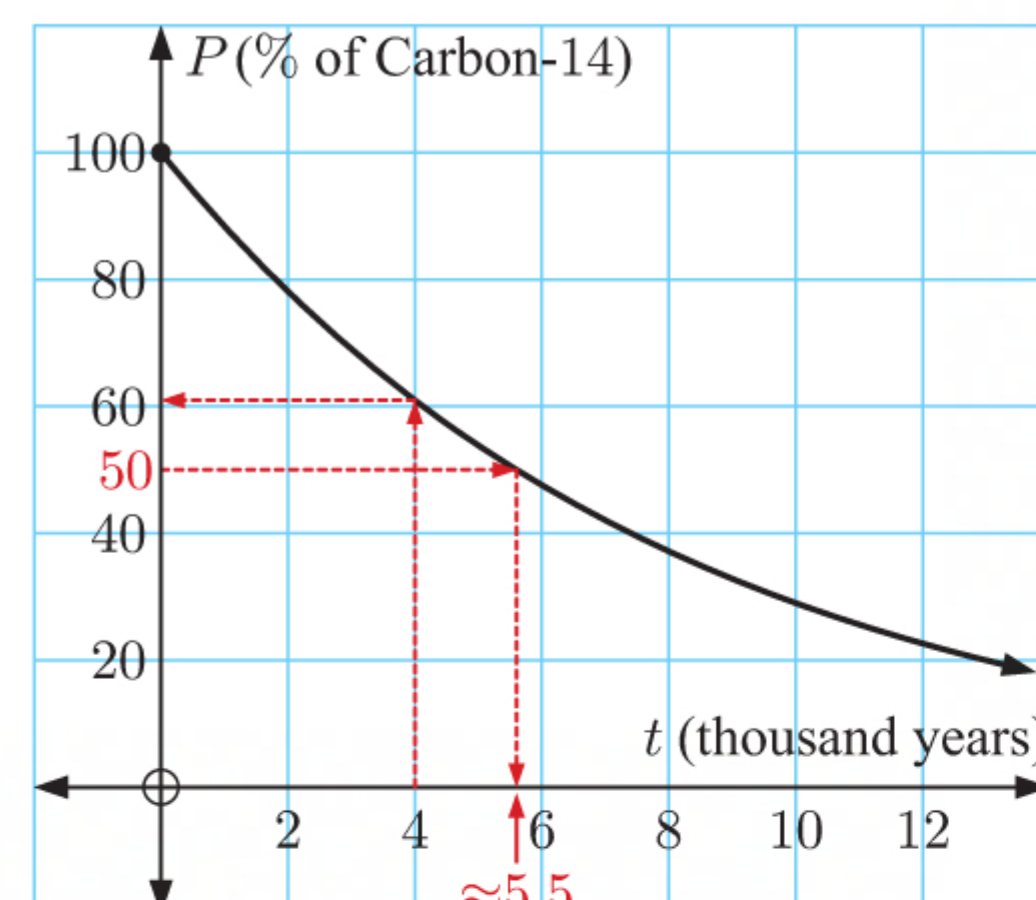
**b**  $P = 100 \times (1.1318)^{-t}$ ,  $t \geq 0$

**i** When  $t = 8$ ,  $P = 100 \times (1.1318)^{-8}$   
 $\approx 37.1$

$\therefore$  there is about 37.1% of Carbon-14 remaining after 8000 years.

**ii** When  $P = 15$ ,  $15 = 100 \times (1.1318)^{-t}$   
 $\therefore t \approx 15.3$  {using technology}

$\therefore$  it will take approximately 15 300 years for the percentage of Carbon-14 to fall to 15%.



**62**  $N = 120 \times (1.04)^t$

**a** When  $t = 0$ ,  $N = 120$

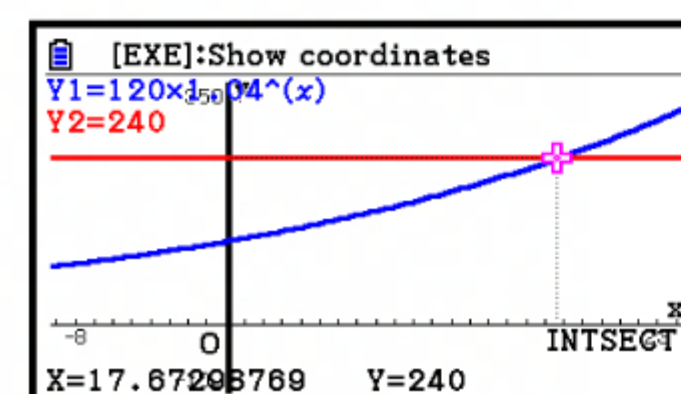
$\therefore$  there were 120 people who started the settlement.

**b** When  $t = 4$ ,  $N = 120 \times (1.04)^4$   
 $\approx 140$

$\therefore$  there were about 140 people on the island after 4 years.

**c** When  $N = 120 \times 2 = 240$ ,  $240 = 120 \times (1.04)^t$   
 $\therefore t \approx 17.7$  {using technology}

$\therefore$  it will take about 17.7 years for the number of people to double.



**63 a**  $W(t) = 100 \times a^t$   
 $\therefore W(0) = 100 \times a^0$   
 $= 100$   $\{a > 0\}$

The initial weight of the sample is 100 mg.

**b** After 4 days, the sample is  $\frac{1}{2} \times 100 = 50$  mg.

$\therefore W(4) = 50$   
 $\therefore 50 = 100 \times a^4$   
 $\therefore a^4 = \frac{1}{2}$   
 $\therefore a = \frac{1}{\sqrt[4]{2}}$   $\{a > 0\}$   
 $\approx 0.8409$

The weight of the sample on any given day is reduced to about 84.09% of this weight the following day.

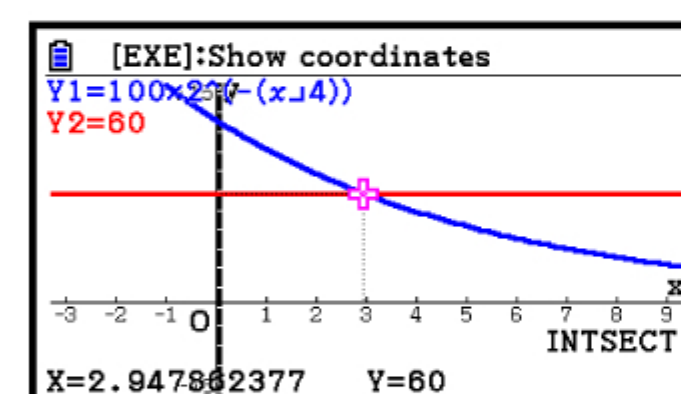
**c**  $W(t) = 100 \times \left(\frac{1}{\sqrt[4]{2}}\right)^t$  {from **b**}

$\therefore W(6) = 100 \times \left(\frac{1}{\sqrt[4]{2}}\right)^6$   
 $\approx 35.4$

The weight of the sample after 6 days is about 35.4 mg.

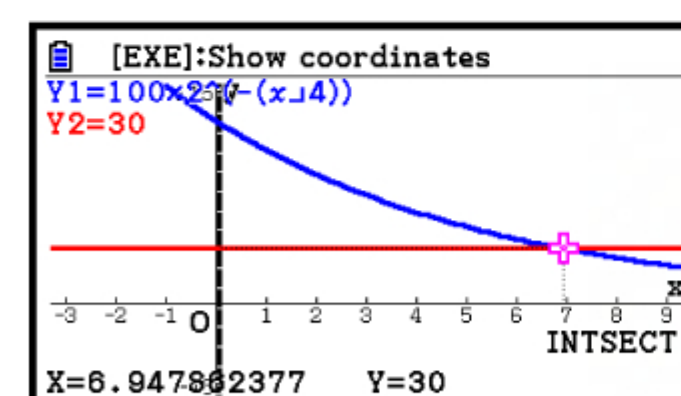
**d i** When  $W(t) = 60$ ,  $60 = 100 \times 2^{-\frac{t}{4}}$   
 $\therefore t \approx 2.95$  {using technology}

It will take about 2.95 days for the weight to fall to 60 mg.



**ii** When  $W(t) = 30$ ,  $30 = 100 \times 2^{-\frac{t}{4}}$   
 $\therefore t \approx 6.95$  {using technology}

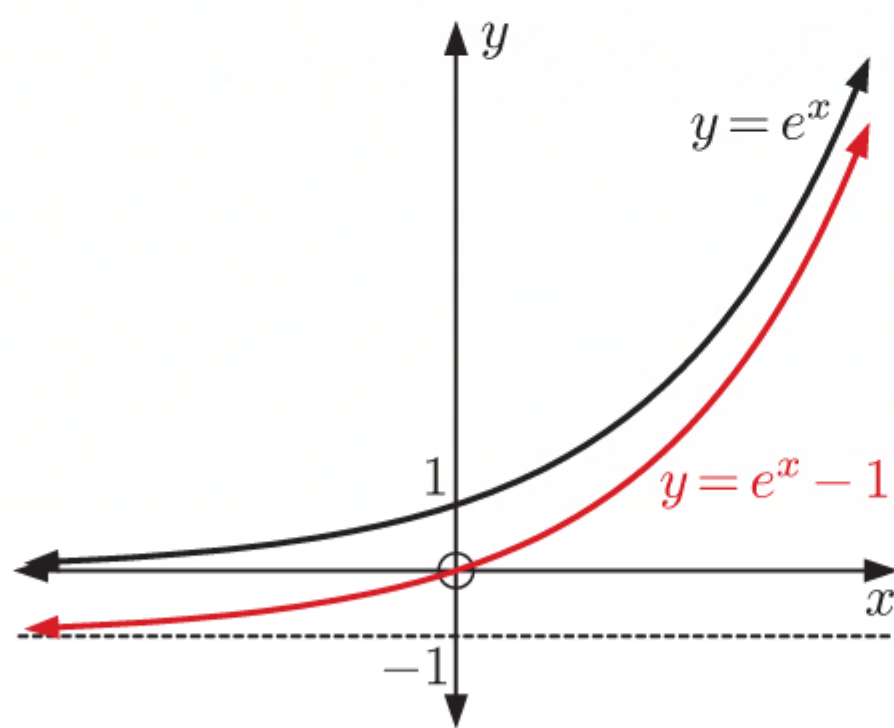
It will take about 6.95 days for the weight to fall to 30 mg.





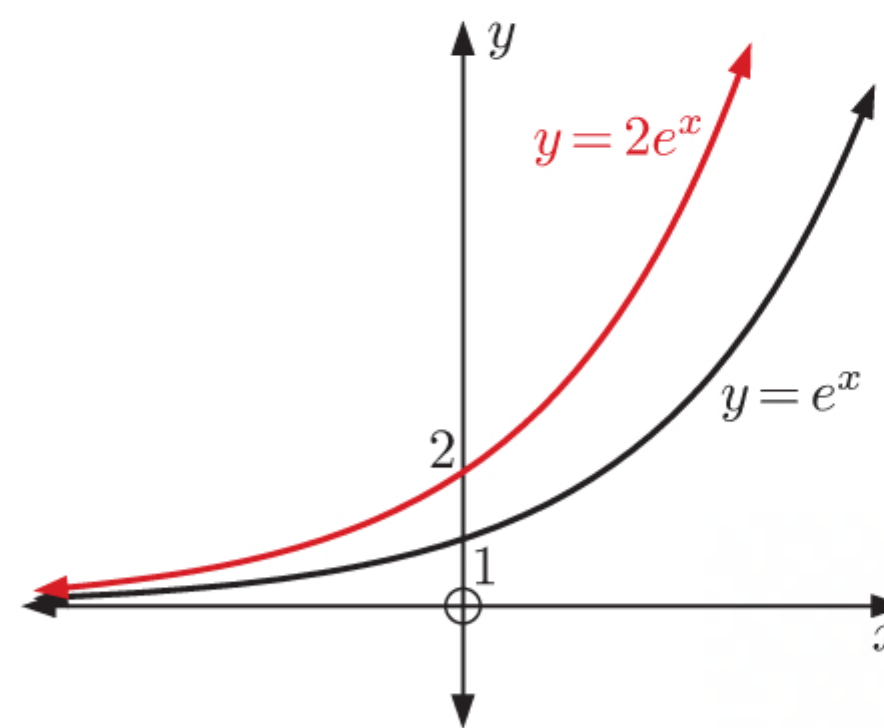
- 64 a**  $y = e^x - 1$  is a vertical translation of  $y = e^x$  by 1 unit downwards.

The  $y$ -intercept is 0, and the horizontal asymptote is  $y = -1$ .



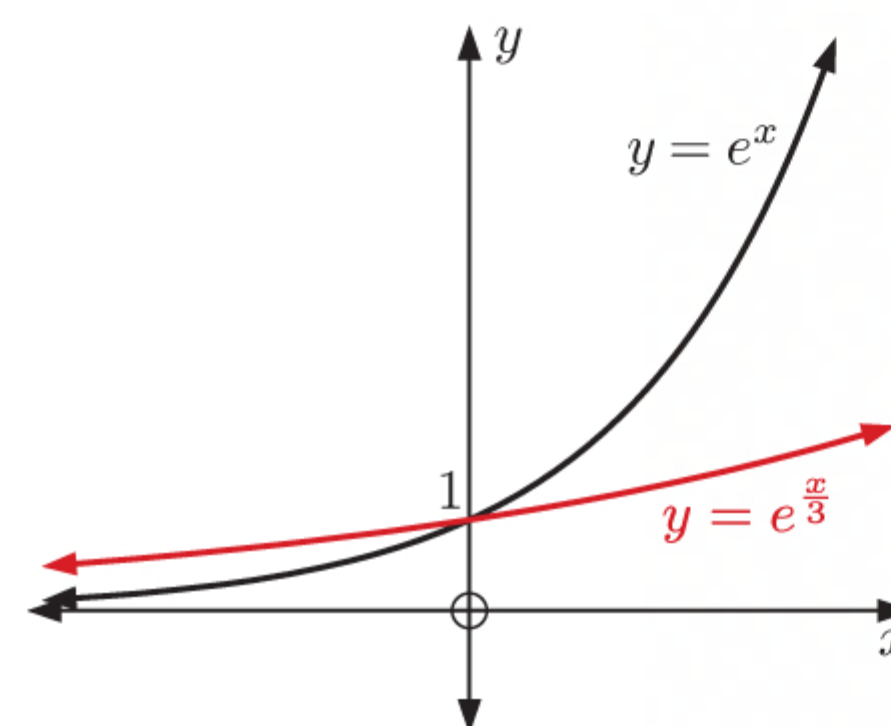
- b**  $y = 2e^x$  is a vertical stretch of  $y = e^x$  with scale factor 2.

The  $y$ -intercept is 2, and the horizontal asymptote is  $y = 0$ .

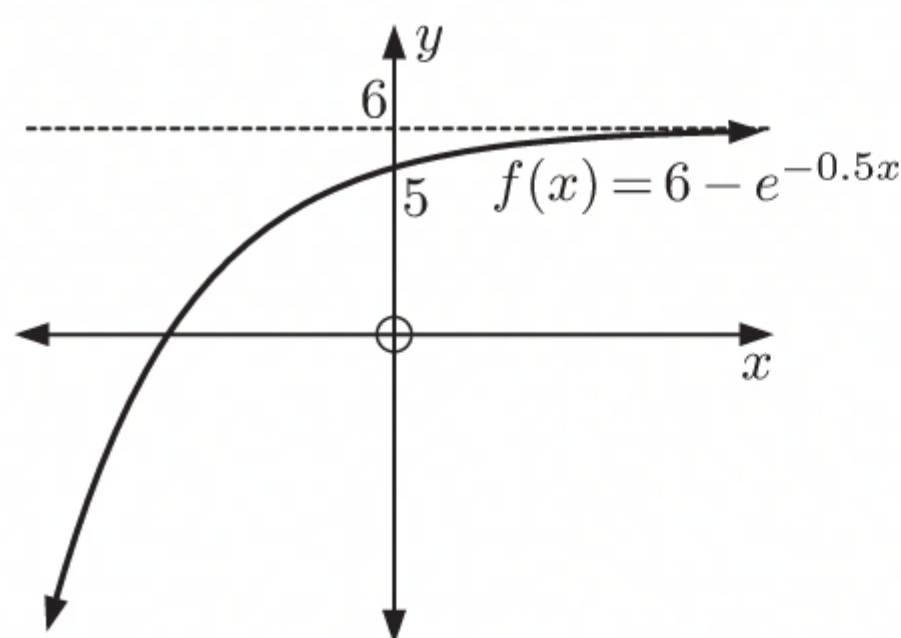


- c**  $y = e^{\frac{x}{3}}$  is a horizontal stretch of  $y = e^x$  with scale factor 3.

The  $y$ -intercept is 1, and the horizontal asymptote is  $y = 0$ .



**65 a**



- b** The domain of  $f(x)$  is  $\{x \mid x \in \mathbb{R}\}$ .

The range of  $f(x)$  is  $\{y \mid y < 6\}$ .

- c** As  $x \rightarrow -\infty$ ,  $y = f(x) \rightarrow -\infty$ .

As  $x \rightarrow \infty$ ,  $y = f(x) \rightarrow 6$ .

- d** From the graph in **a**:

**i**  $f(x) = k$  has one solution if  $k < 6$ .

**ii**  $f(x) = k$  has no solutions if  $k \geq 6$ .

**66 a**  $P(t) = \frac{2000}{1 + Ce^{-0.2t}}$

Now  $P(0) = 500$

$$\therefore 500 = \frac{2000}{1 + C}$$

$$\therefore 1 + C = \frac{2000}{500} = 4$$

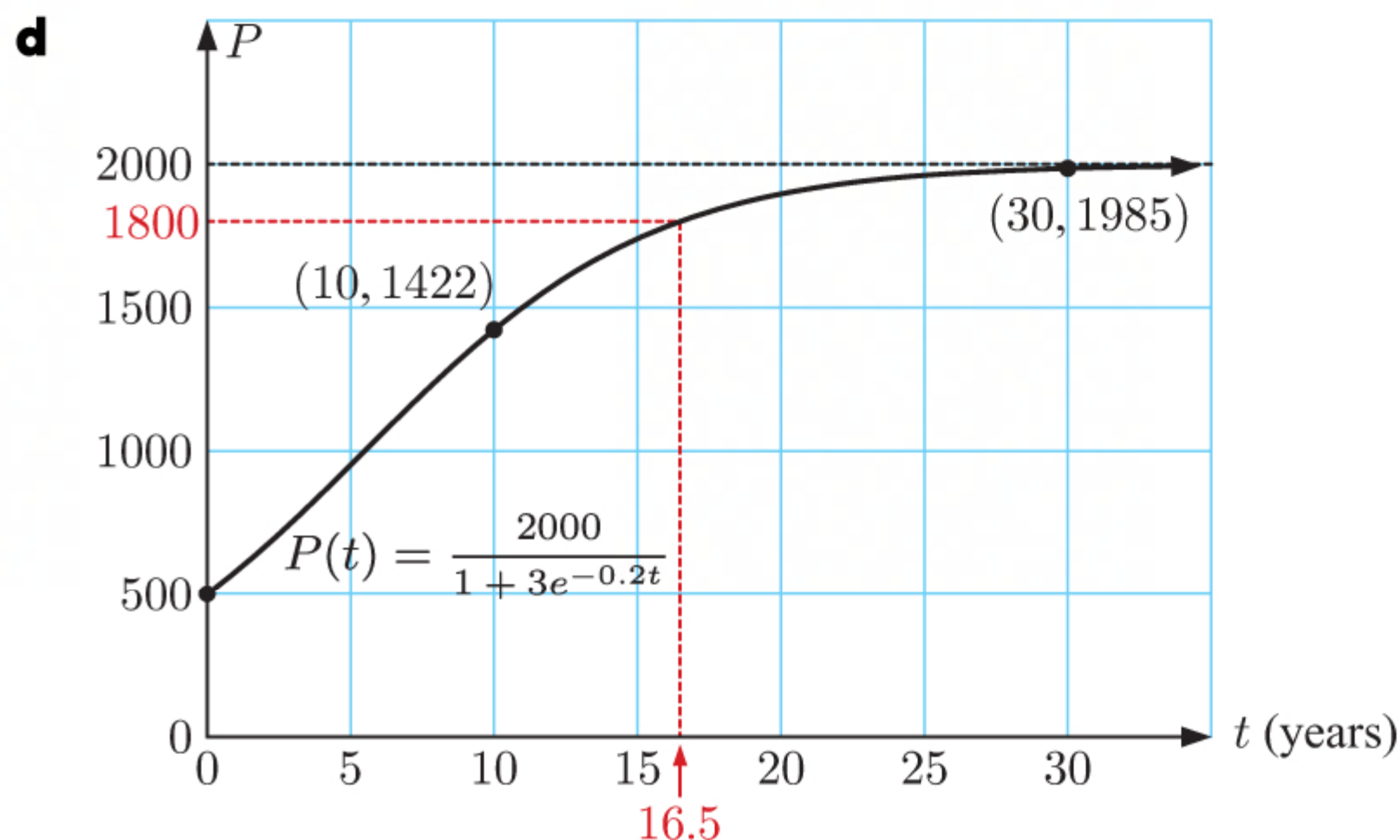
$$\therefore C = 3$$

**c i**  $P(10) = \frac{2000}{1 + 3e^{-0.2(10)}} \approx 1422$

$\therefore$  after 10 years, the population was about 1422.

**ii**  $P(30) = \frac{2000}{1 + 3e^{-0.2(30)}} \approx 1985$

$\therefore$  after 30 years, the population was about 1985.



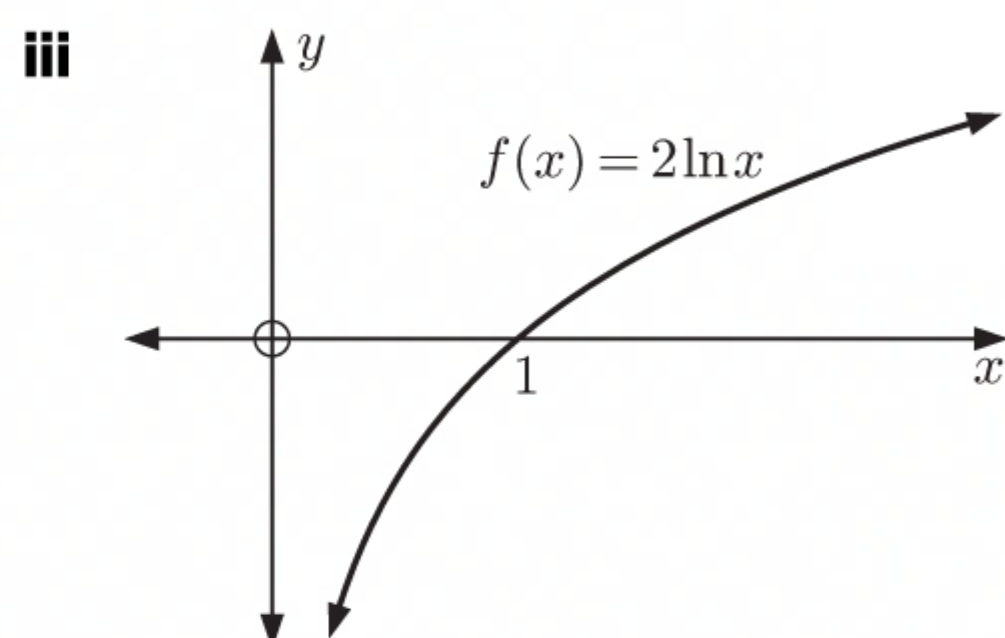
- e** From the graph in **d**, it took about 16.5 years for the population to reach 1800.



**67 a**  $f(x) = 2 \ln x$

**i** Domain is  $\{x \mid x > 0\}$ .

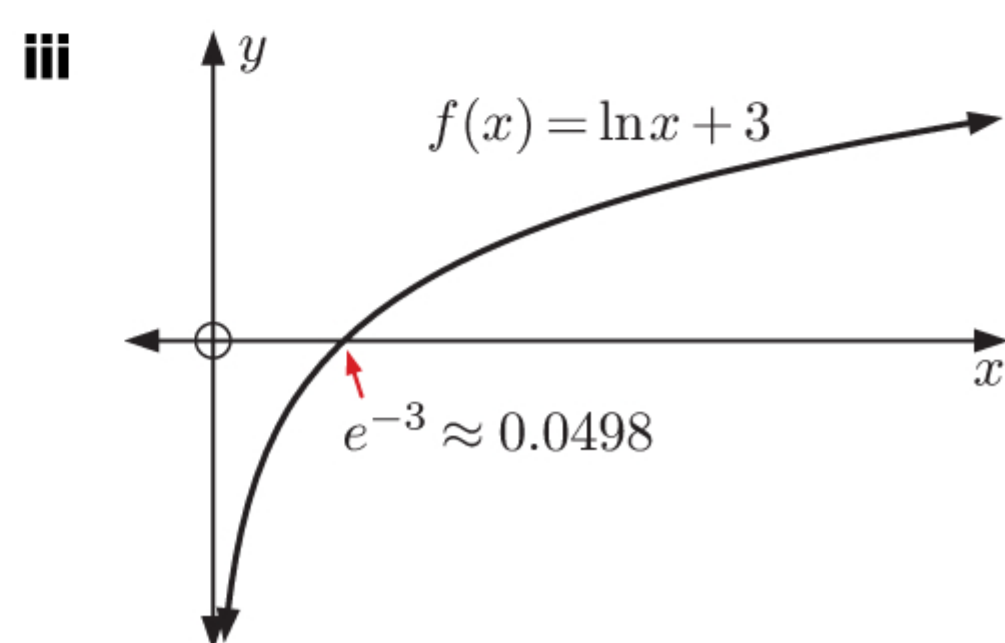
Range is  $\{y \mid y \in \mathbb{R}\}$ .



**b**  $f(x) = \ln x + 3$

**i** Domain is  $\{x \mid x > 0\}$ .

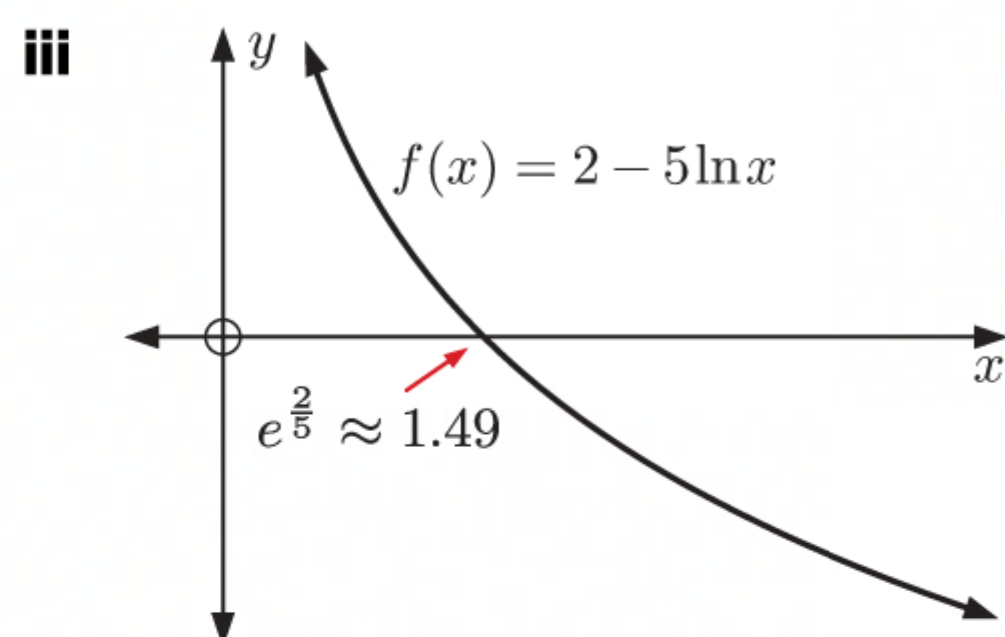
Range is  $\{y \mid y \in \mathbb{R}\}$ .



**c**  $f(x) = 2 - 5 \ln x$

**i** Domain is  $\{x \mid x > 0\}$ .

Range is  $\{y \mid y \in \mathbb{R}\}$ .



**68 a**  $\ln\left(\frac{e^2}{x^3}\right) = \ln(e^2) - \ln(x^3)$   
 $= 2 \ln e - 3 \ln x$   
 $= 2 - 3 \ln x$

**b** When  $y = 0$ ,  $2 - 3 \ln x = 0$

$$\therefore \ln x = \frac{2}{3}$$

$$\therefore x = e^{\frac{2}{3}} \approx 1.95$$

So, the  $x$ -intercept is  $e^{\frac{2}{3}}$ .

**ii** When  $f(x) = 0$ ,  $2 \ln x = 0$

$$\therefore \ln x = 0$$

$$\therefore x = 1$$

So, the  $x$ -intercept is 1.

**iv**  $f$  is  $y = 2 \ln x$

$$\therefore f^{-1} \text{ is } x = 2 \ln y$$

$$\therefore \frac{x}{2} = \ln y$$

$$\therefore y = e^{\frac{x}{2}}$$

$$\therefore f^{-1}(x) = e^{\frac{x}{2}}$$

**ii** When  $f(x) = 0$ ,  $\ln x + 3 = 0$

$$\therefore \ln x = -3$$

$$\therefore x = e^{-3} \approx 0.0498$$

So, the  $x$ -intercept is  $e^{-3}$ .

**iv**  $f$  is  $y = \ln x + 3$

$$\therefore f^{-1} \text{ is } x = \ln y + 3$$

$$\therefore \ln y = x - 3$$

$$\therefore y = e^{x-3}$$

$$\therefore f^{-1}(x) = e^{x-3}$$

**ii** When  $f(x) = 0$ ,  $2 - 5 \ln x = 0$

$$\therefore \ln x = \frac{2}{5}$$

$$\therefore x = e^{\frac{2}{5}} \approx 1.49$$

So, the  $x$ -intercept is  $e^{\frac{2}{5}}$ .

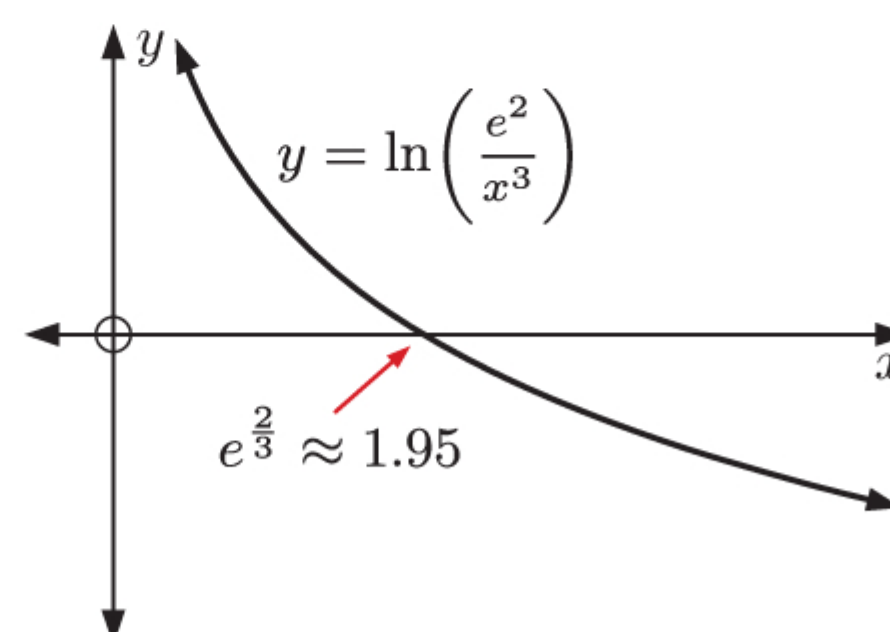
**iv**  $f$  is  $y = 2 - 5 \ln x$

$$\therefore f^{-1} \text{ is } x = 2 - 5 \ln y$$

$$\therefore \ln y = \frac{2-x}{5}$$

$$\therefore y = e^{\frac{2-x}{5}}$$

$$\therefore f^{-1}(x) = e^{\frac{2-x}{5}}$$





**69 a**  $f(x) = x^2 - 5x + 6$

The graph of  $y = g(x)$  is found by translating  $y = f(x)$  8 units upwards.

$$\therefore g(x) = f(x) + 8$$

$$\therefore g(x) = (x^2 - 5x + 6) + 8$$

$$\therefore g(x) = x^2 - 5x + 14$$

**70 a**  $f(x) = \frac{1}{\sqrt{x-4}} + 3$  is defined when  $\sqrt{x-4} > 0$   
 $\therefore x - 4 > 0$   
 $\therefore x > 4$

$\therefore$  the domain is  $\{x \mid x > 4\}$ .

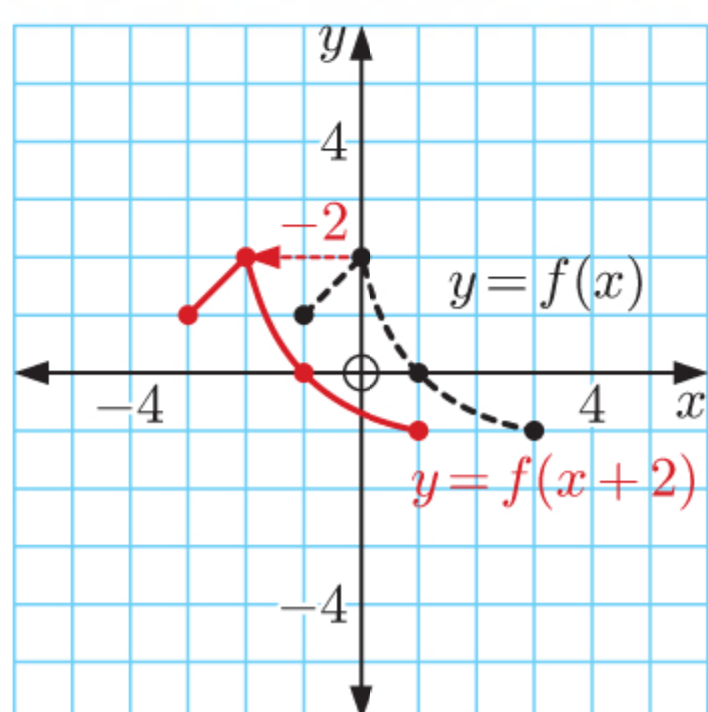
Now,  $\frac{1}{\sqrt{x-4}} > 0$  {for  $x > 4$ }

$$\therefore \frac{1}{\sqrt{x-4}} + 3 > 3$$

$$\therefore f(x) > 3$$

$\therefore$  the range is  $\{y \mid y > 3\}$ .

**71 a** The graph of  $y = f(x+2)$  is found by translating  $y = f(x)$  2 units to the left.



**b**  $f(x) = -2x^2 + x + 3$

The graph of  $y = g(x)$  is found by translating  $y = f(x)$  1 unit to the right.

$$\therefore g(x) = f(x-1)$$

$$\therefore g(x) = -2(x-1)^2 + (x-1) + 3$$

$$\therefore g(x) = -2(x^2 - 2x + 1) + x - 1 + 3$$

$$\therefore g(x) = -2x^2 + 4x - 2 + x + 2$$

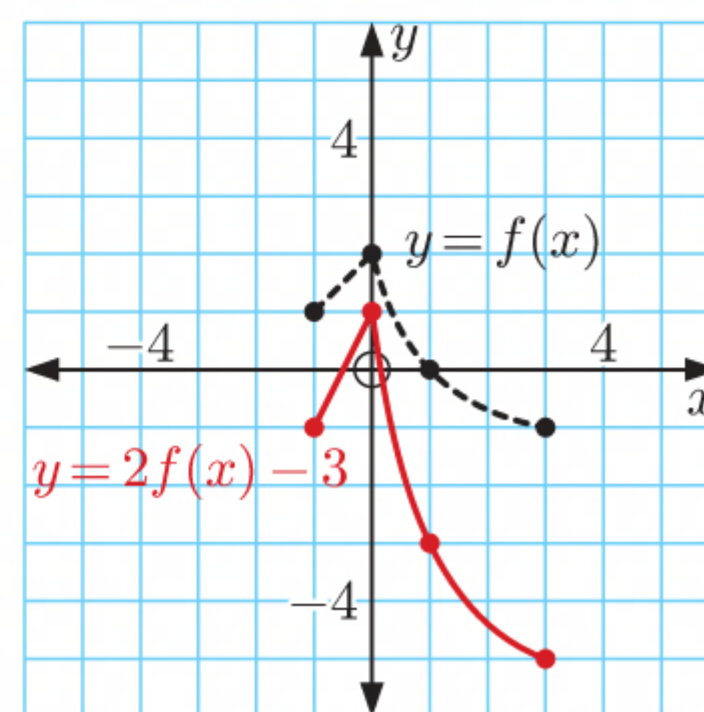
$$\therefore g(x) = -2x^2 + 5x$$

**b** A translation through  $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$  maps  $y = \frac{1}{\sqrt{x}}$  onto  $f$ .

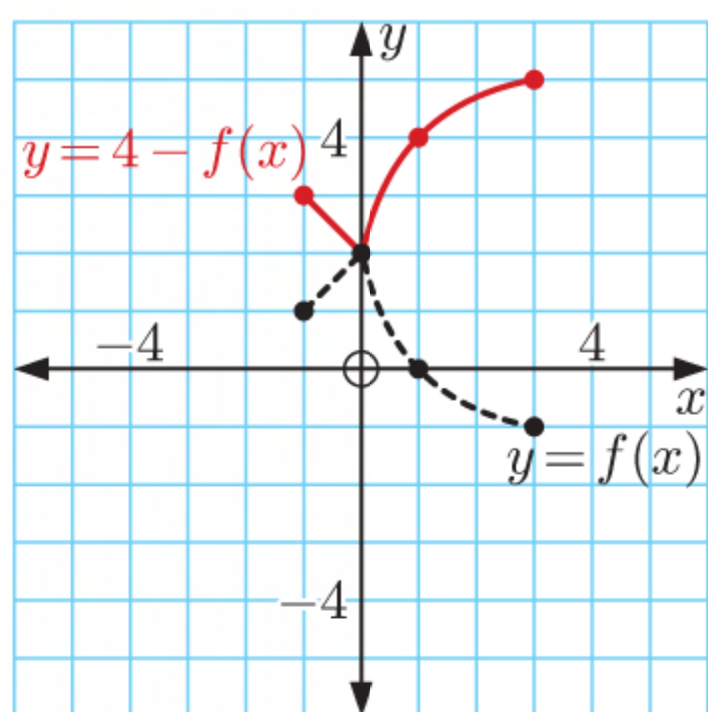
**c**  $y = \frac{1}{\sqrt{x}}$  has vertical asymptote  $x = 0$   
 and horizontal asymptote  $y = 0$

$\therefore y = f(x)$  has vertical asymptote  $x = 4$   
 and horizontal asymptote  $y = 3$ .

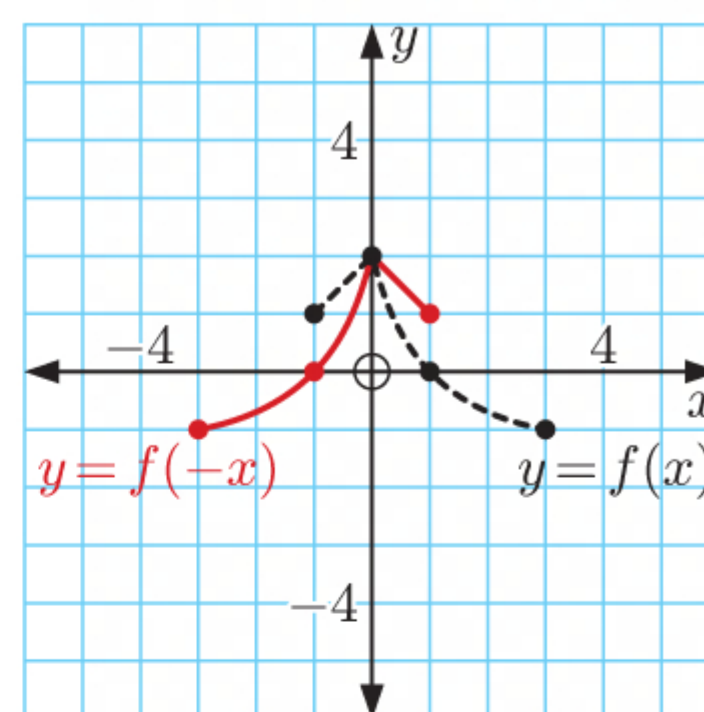
**b** The graph of  $y = 2f(x) - 3$  is a vertical stretch of  $y = f(x)$  with scale factor 2, followed by a translation 3 units downwards.



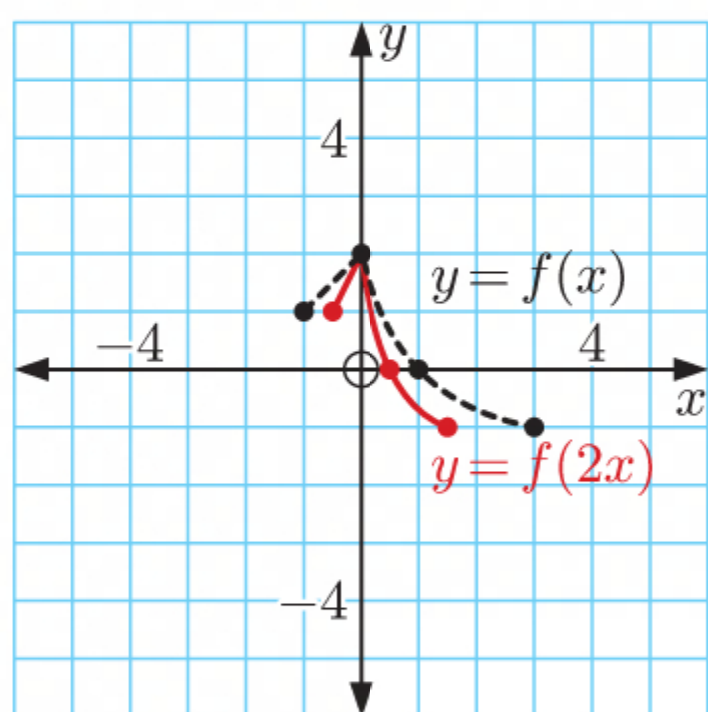
**c** The graph of  $y = 4 - f(x)$  is found by reflecting  $y = f(x)$  in the  $x$ -axis, then translating 4 units upwards.



**d** The graph of  $y = f(-x)$  is found by reflecting  $y = f(x)$  in the  $y$ -axis.



**e** The graph of  $y = f(2x)$  is a horizontal stretch of  $y = f(x)$  with scale factor  $\frac{1}{2}$ .





**72** Let  $y = f(x) = \frac{2}{x}$

- a** The image when  $y = f(x)$  is reflected in the  $y$ -axis has equation

$$\begin{aligned} y &= f(-x) \\ \therefore y &= \frac{2}{-x} \\ \therefore y &= -\frac{2}{x} \end{aligned}$$

- c** The image when  $y = f(x)$  is stretched horizontally with scale factor 3 has equation

$$\begin{aligned} y &= f\left(\frac{1}{3}x\right) \\ \therefore y &= \frac{2}{\frac{1}{3}x} \\ \therefore y &= \frac{6}{x} \end{aligned}$$

**73 a** For the function  $f(x) = \sin 4x$ :

- the amplitude is  $|a| = 1$
- the principal axis is  $y = 0$
- the period is  $\frac{2\pi}{b} = \frac{2\pi}{4} = \frac{\pi}{2}$ .

**74 a i** The amplitude is  $|a| = 2$ .

**iii** The period is  $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$ .

**b i** The amplitude is  $|a| = 1$ .

**iii** The period is  $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$ .

**c i** The amplitude is  $|a| = 3$ .

**iii** The period is  $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$ .

**d i** The amplitude is  $|a| = 1$ .

**iii** The period is  $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$ .

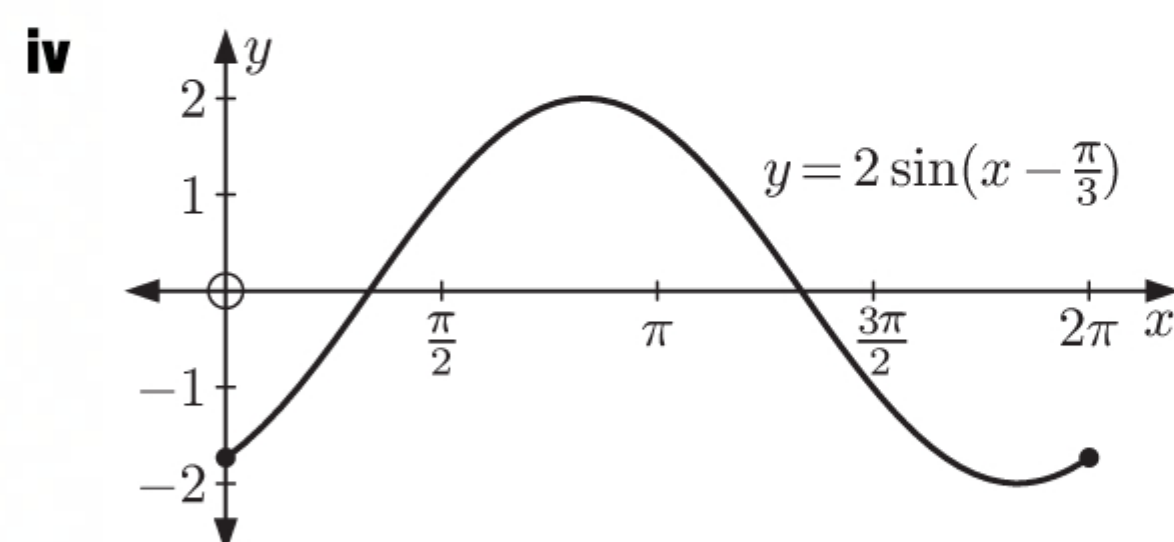
- b** The image when  $y = f(x)$  is translated through  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$  has equation

$$\begin{aligned} y &= f(x+1) + 2 \\ \therefore y &= \frac{2}{x+1} + 2 \end{aligned}$$

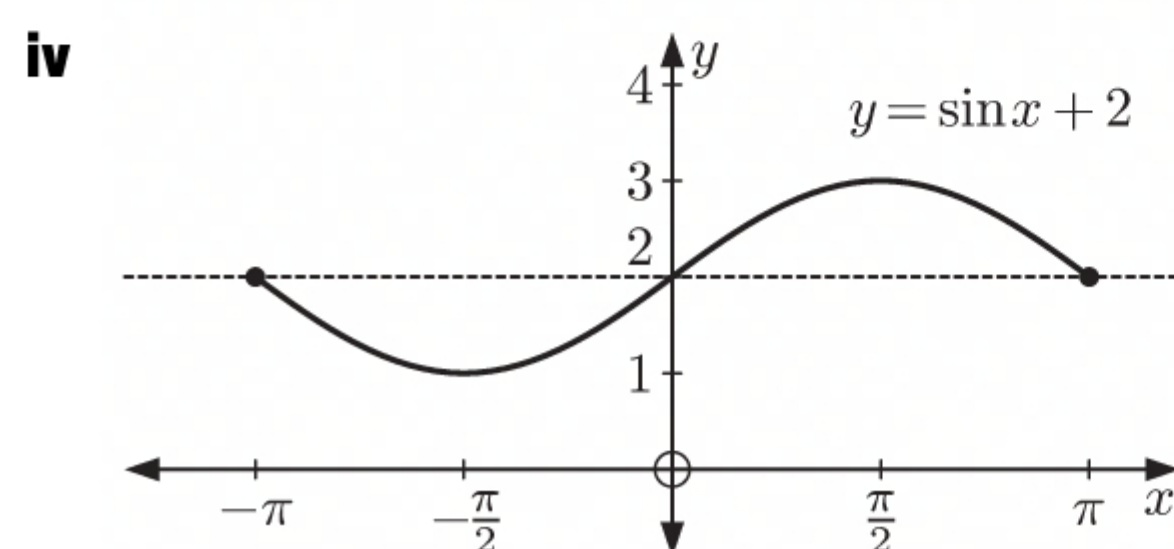
**b** For the function  $f(x) = -2 \sin \frac{x}{2} - 1$ :

- the amplitude is  $|a| = |-2| = 2$
- the principal axis is  $y = -1$
- the period is  $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$ .

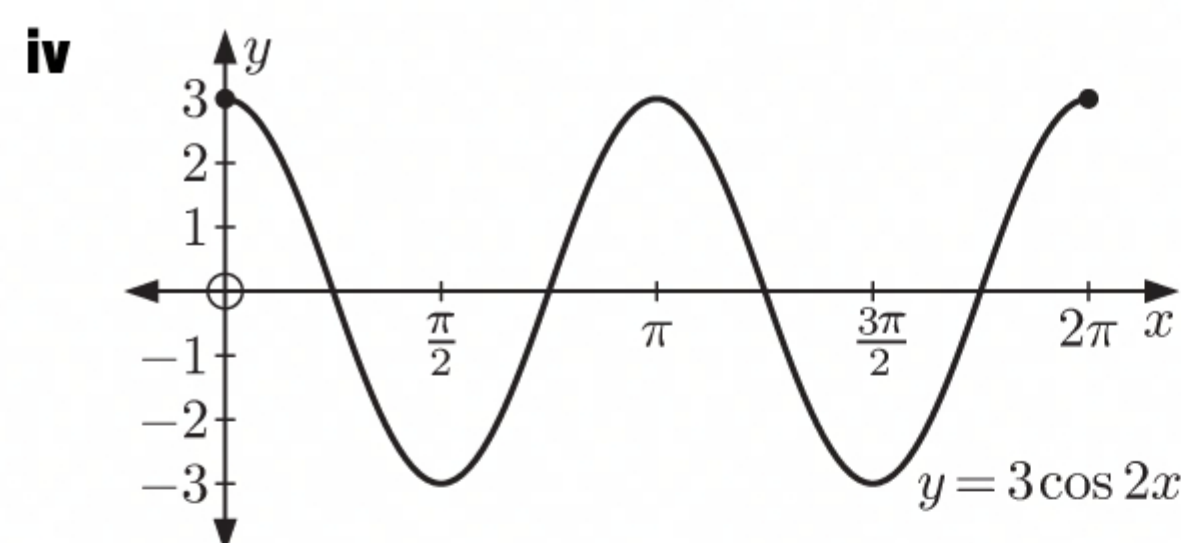
**ii** The principal axis is  $y = 0$ .



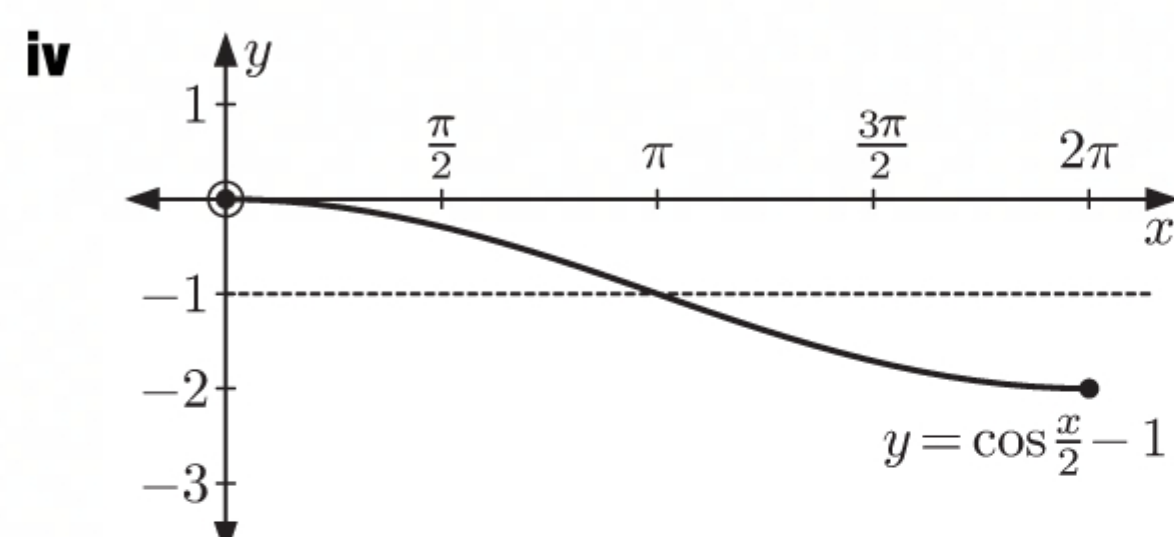
**ii** The principal axis is  $y = 2$ .



**ii** The principal axis is  $y = 0$ .



**ii** The principal axis is  $y = -1$ .

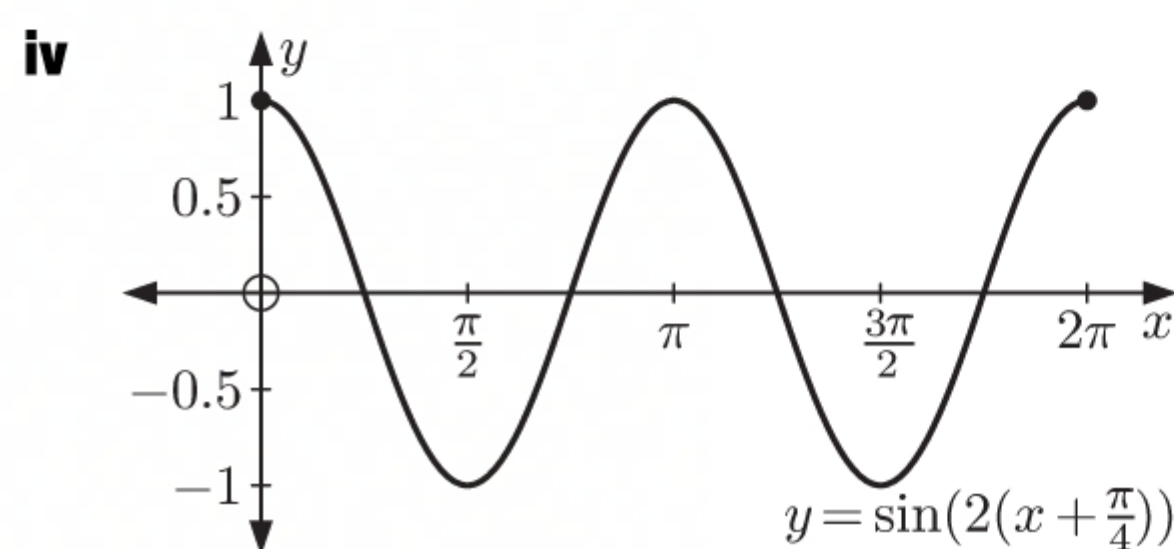




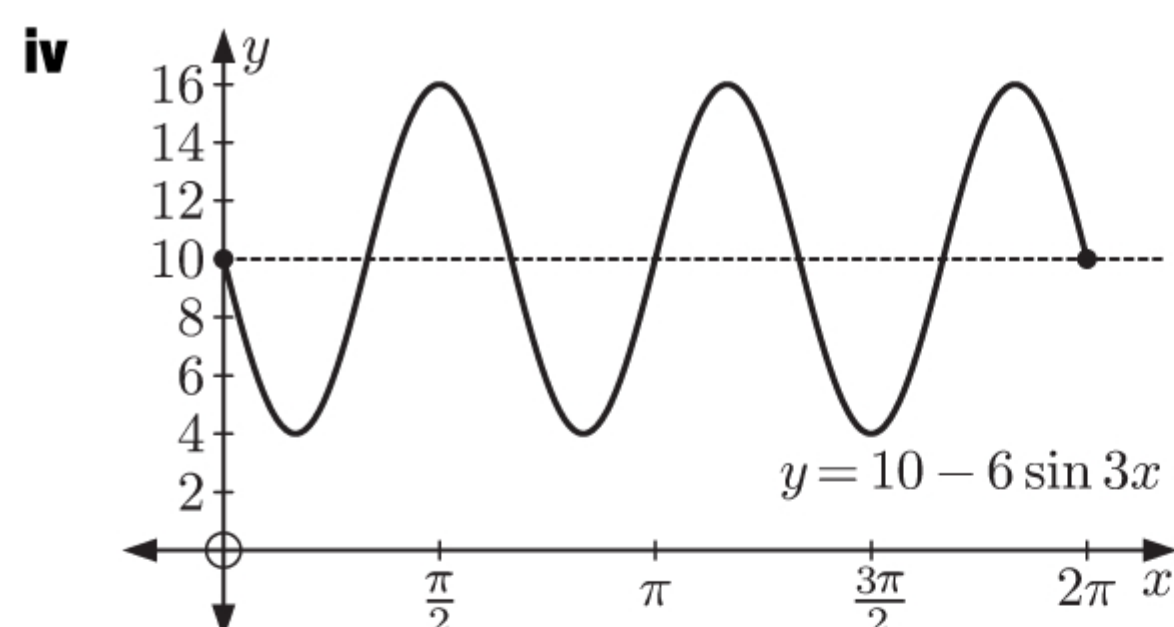
- e** **i** The amplitude is  $|a| = 1$ .  
**iii** The period is  $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$ .

- f** **i** The amplitude is  $|a| = |-6| = 6$ .  
**iii** The period is  $\frac{2\pi}{b} = \frac{2\pi}{3}$ .

- ii** The principal axis is  $y = 0$ .



- ii** The principal axis is  $y = 10$ .

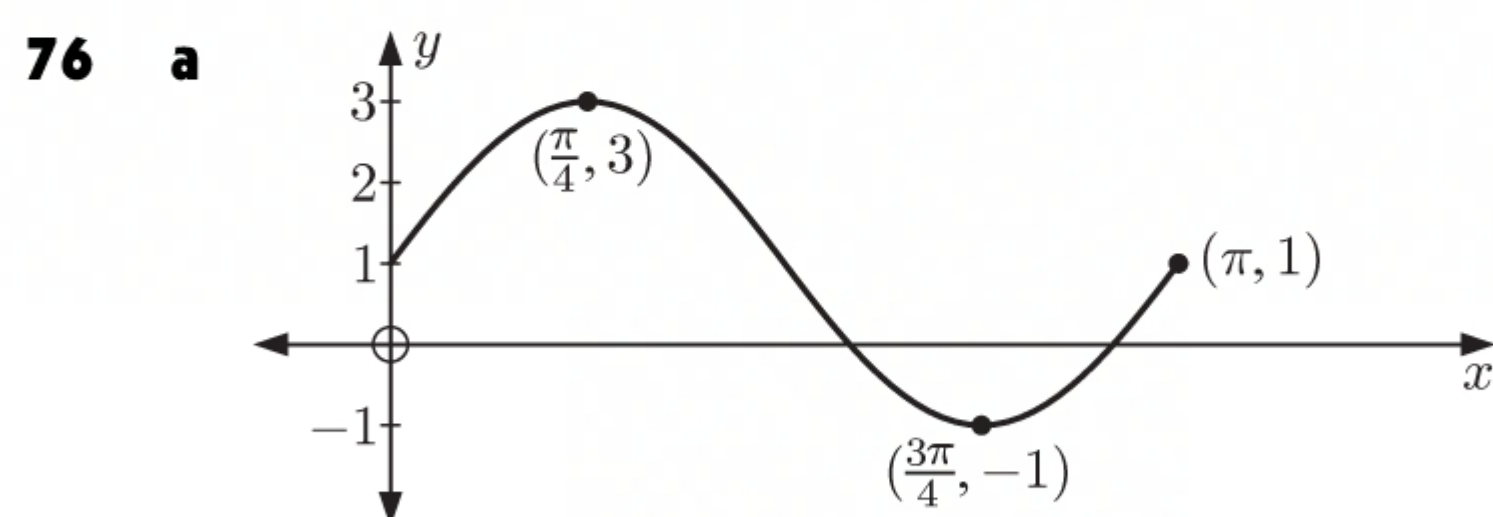


**75 a**  $\sin x \xrightarrow[\text{vertical stretch}]{\text{scale factor 2}} 2 \sin x \xrightarrow[\text{horizontal stretch}]{\text{scale factor 3}} 2 \sin \frac{x}{3}$

A vertical stretch with scale factor 2, then a horizontal stretch with scale factor 3 maps  $y = \sin x$  onto  $y = 2 \sin \frac{x}{3}$ .

**b**  $\sin x \xrightarrow[\text{translation}]{\begin{pmatrix} -\frac{\pi}{3} \\ -4 \end{pmatrix}} \sin\left(x + \frac{\pi}{3}\right) - 4$

A translation  $\frac{\pi}{3}$  units left and 4 units downwards maps  $y = \sin x$  onto  $y = \sin\left(x + \frac{\pi}{3}\right) - 4$ .



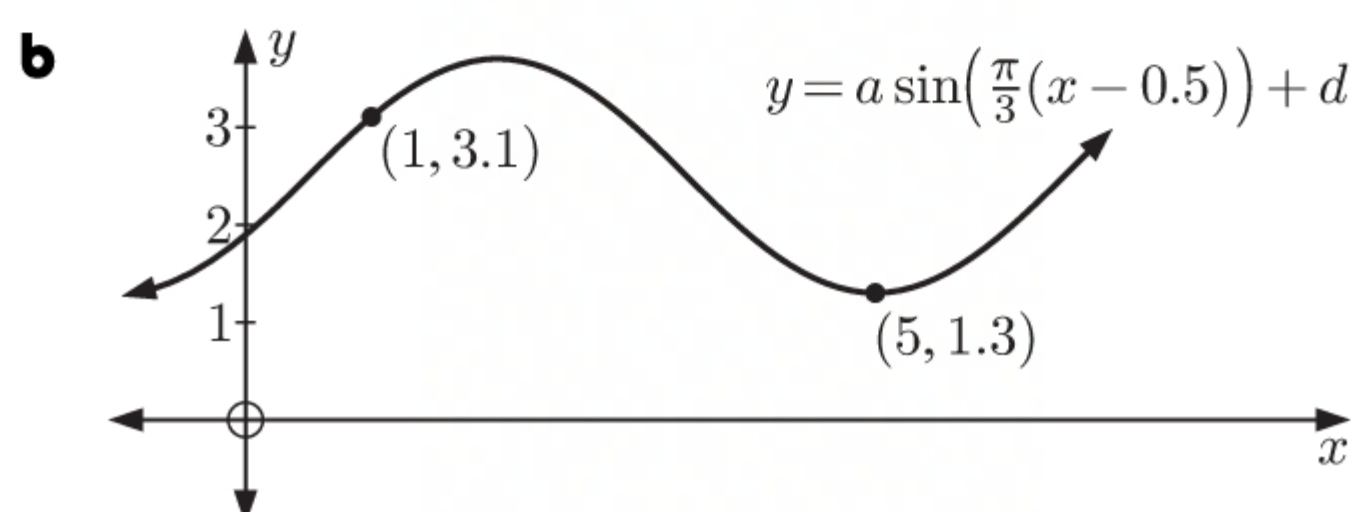
The amplitude is 2, so  $a = 2$ .

The period is  $\pi$ , so  $\frac{2\pi}{b} = \pi$  and  $\therefore b = 2$ .

There is no horizontal translation, so  $c = 0$ .

The principal axis is  $y = 1$ , so  $d = 1$ .

$\therefore$  the equation of the function is  $y = 2 \sin 2x + 1$ .



When  $x = 1$ ,  $y = 3.1$

$$\therefore a \sin\left(\frac{\pi}{3}(1 - 0.5)\right) + d = 3.1$$

$$\therefore a \sin\left(\frac{\pi}{3} \times 0.5\right) + d = 3.1$$

$$\therefore a \sin \frac{\pi}{6} + d = 3.1$$

$$\therefore \frac{1}{2}a + d = 3.1 \quad \dots (1)$$

$$-\frac{1}{2}a - d = -3.1 \quad \{(1) \times -1\}$$

$$-a + d = 1.3$$

Adding,  $-\frac{3}{2}a = -1.8$

$$\therefore a = 1.2$$

and when  $x = 5$ ,  $y = 1.3$

$$\therefore a \sin\left(\frac{\pi}{3}(5 - 0.5)\right) + d = 1.3$$

$$\therefore a \sin\left(\frac{\pi}{3} \times 4.5\right) + d = 1.3$$

$$\therefore a \sin \frac{3\pi}{2} + d = 1.3$$

$$\therefore -a + d = 1.3 \quad \dots (2)$$

Substituting  $a = 1.2$  into (2) gives  $-1.2 + d = 1.3$

$$\therefore d = 2.5$$

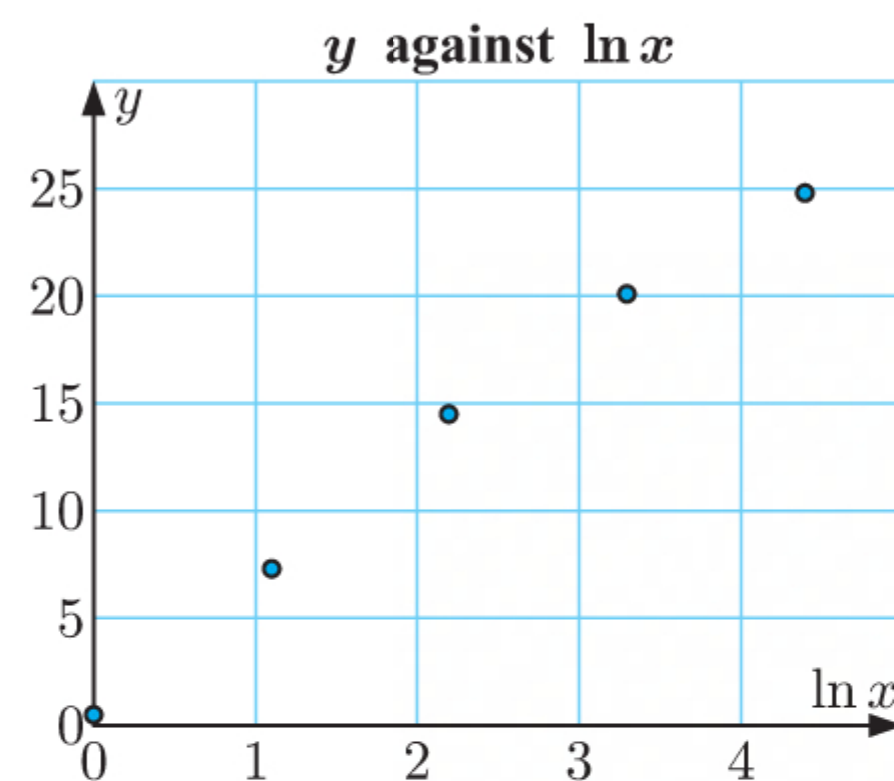
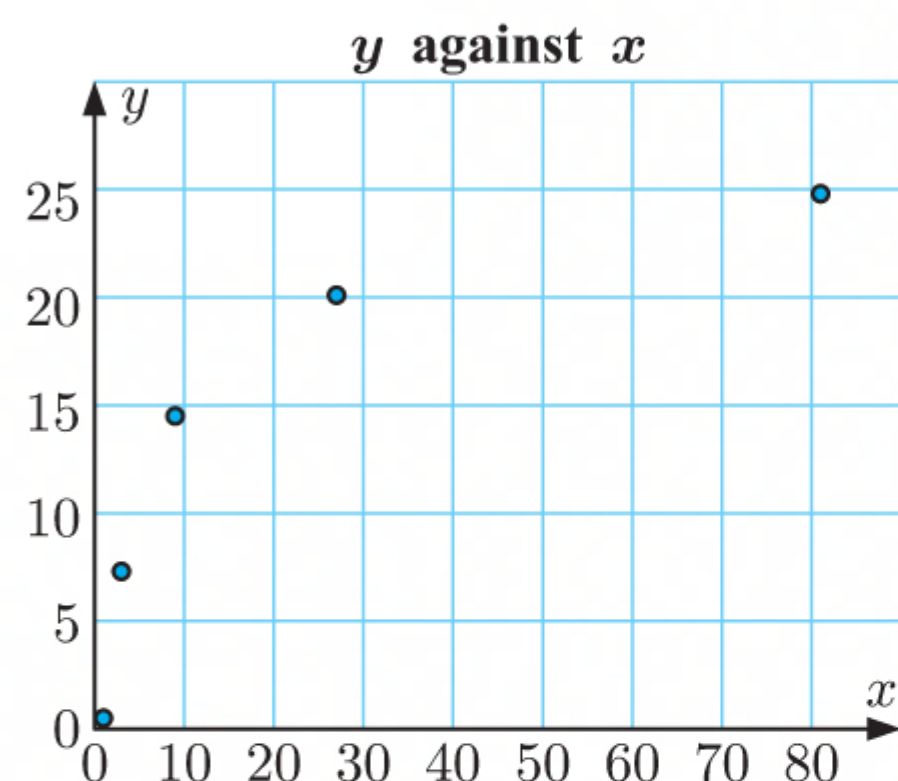
$\therefore$  the equation of the sine function is  $y = 1.2 \sin\left(\frac{\pi}{3}(x - 0.5)\right) + 2.5$ .



77

$x$	1	3	9	27	81
$y$	0.5	7.3	14.5	20.1	24.8
$\ln x$	0	1.10	2.20	3.30	4.39

a



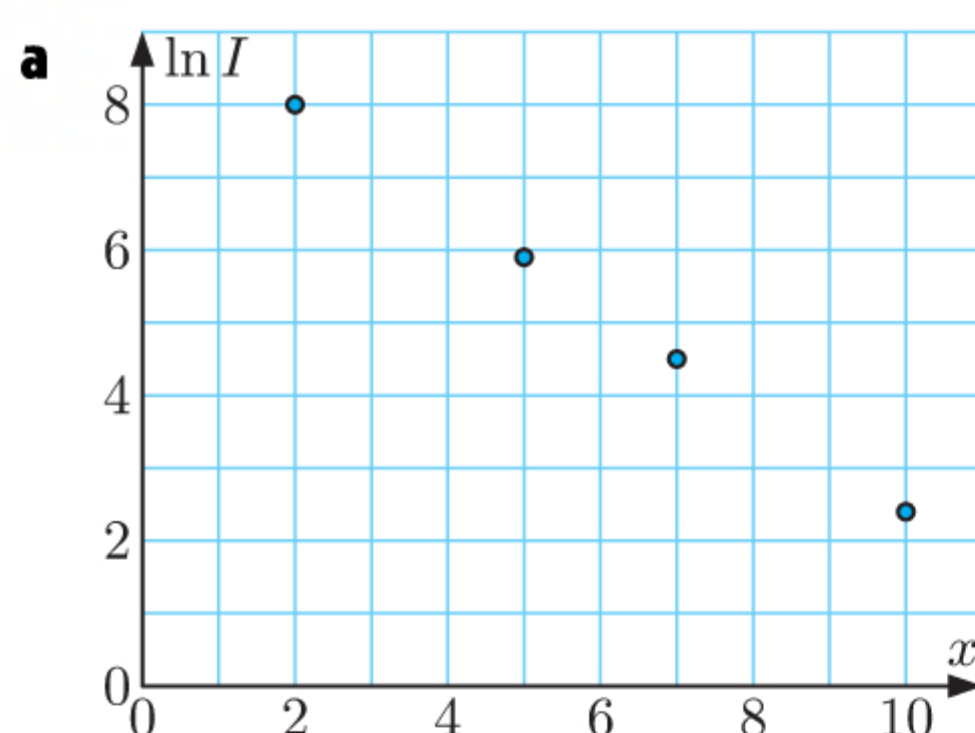
b The graph of  $y$  against  $\ln x$  appears linear, so a logarithmic model is appropriate.



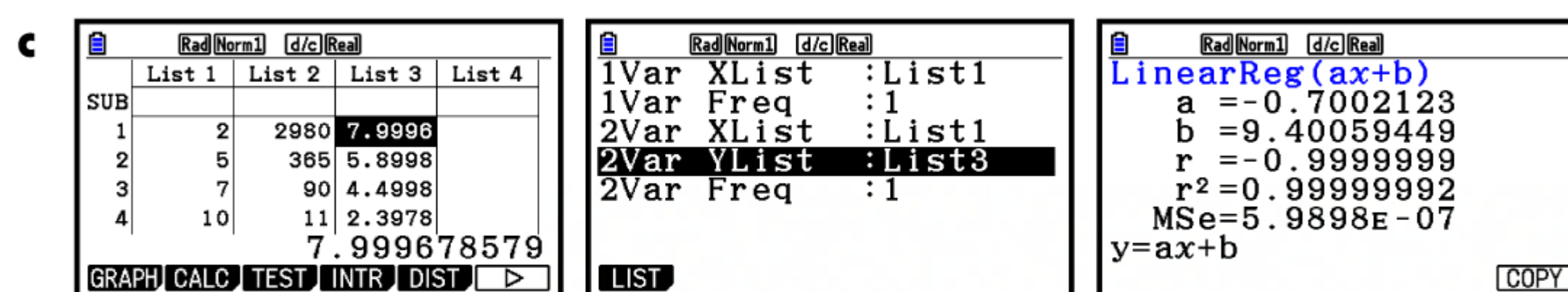
Using technology, the equation connecting  $y$  and  $x$  is  $y = 1.16 + 5.59 \ln x$ .

78

$x$ (metres)	2	5	7	10
$I$ (units)	2980	365	90	11
$\ln I$	8.00	5.90	4.50	2.40



b The graph of  $\ln I$  against  $x$  appears linear, so an exponential model is appropriate.



Using technology, the linear model connecting  $\ln I$  and  $x$  is  $\ln I \approx -0.700x + 9.40$

$$\therefore I \approx e^{-0.700x+9.40}$$

$$\therefore I \approx e^{9.40} \times (e^{-0.700})^x$$

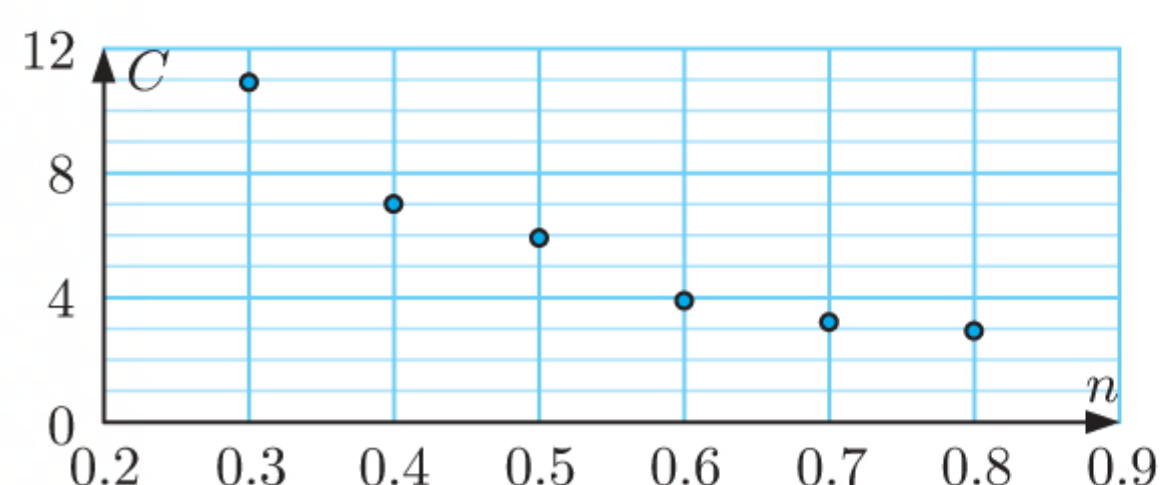
$$\therefore I \approx 12\,100 \times (0.496)^x$$

d  $I = 1000$  when  $12\,100 \times (0.496)^x \approx 1000$

$$\therefore x \approx 3.56 \quad \{\text{technology}\}$$

$\therefore$  the intensity is 1000 units at about 3.56 m below the surface of the lake.

79

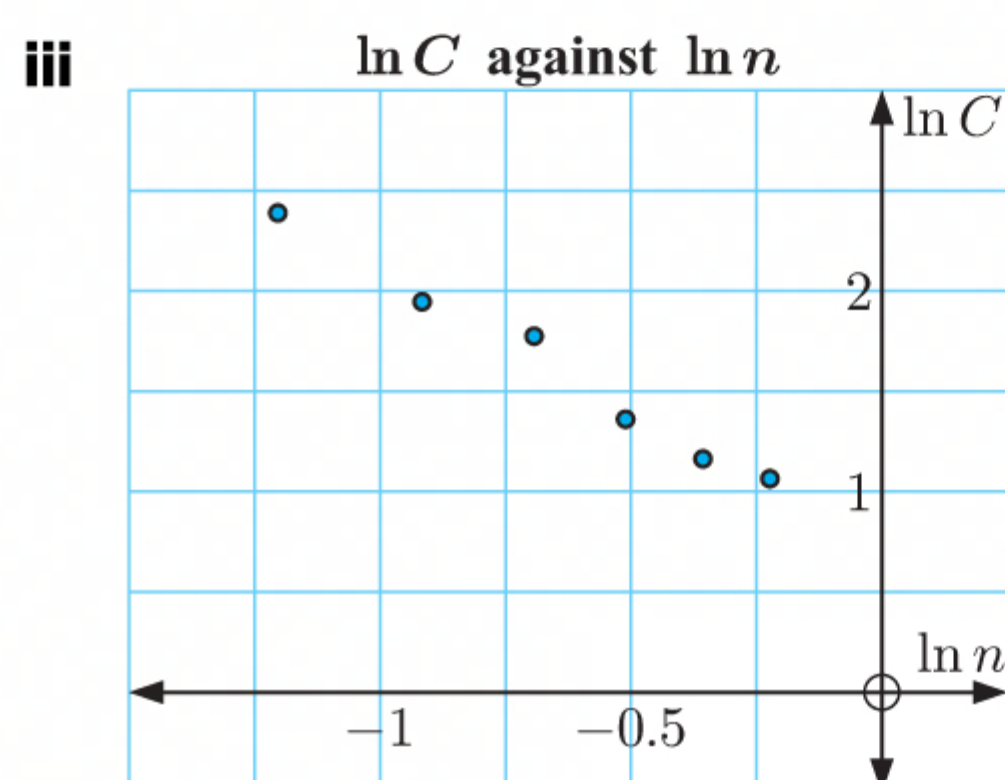
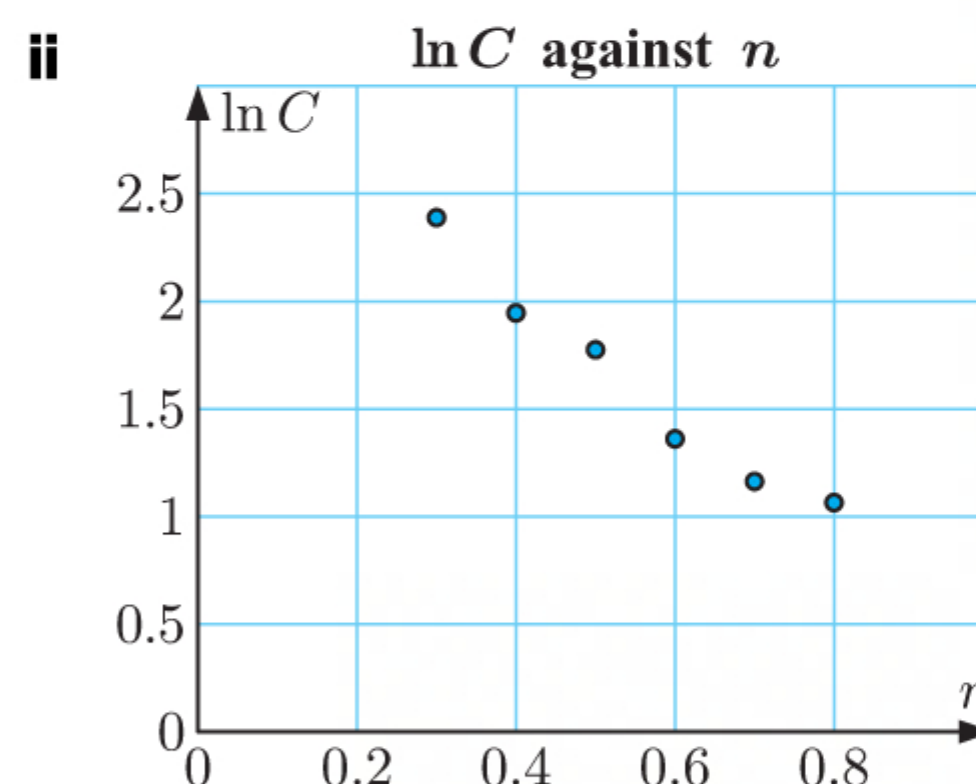
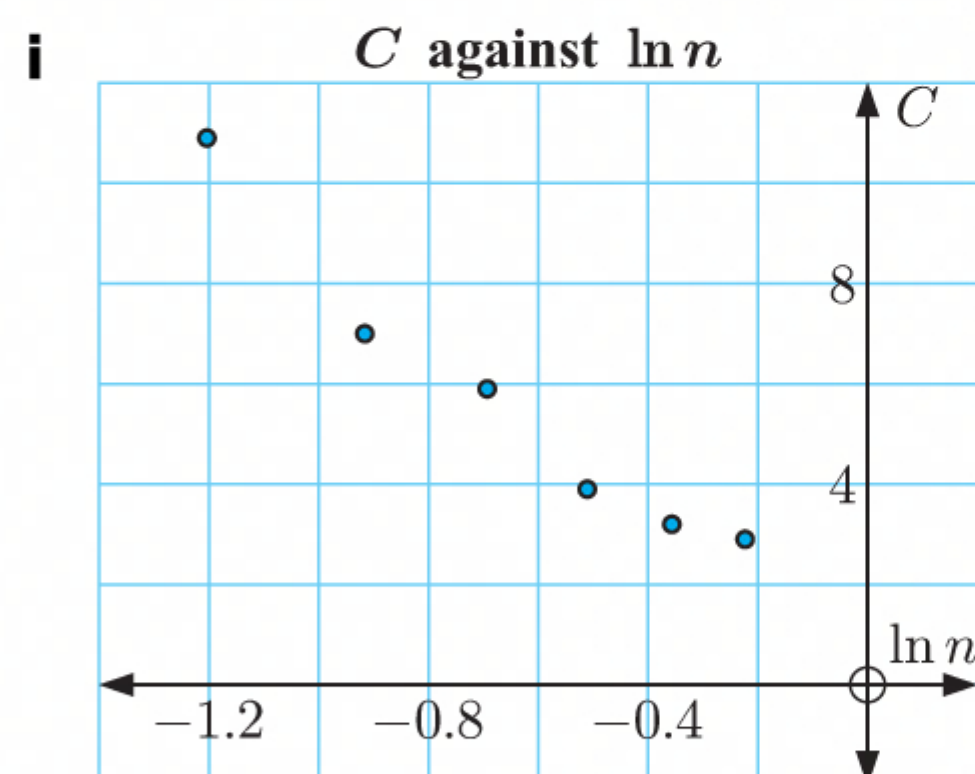


a The graph of  $C$  against  $n$  appears to be a curve, so a linear model is not appropriate.



**b**

$n$	0.3	0.4	0.5	0.6	0.7	0.8
$C$	10.9	7.0	5.9	3.9	3.2	2.9
$\ln n$	-1.20	-0.916	-0.693	-0.511	-0.357	-0.223
$\ln C$	2.39	1.95	1.77	1.36	1.16	1.06



**c** The graph of  $\ln C$  against  $\ln n$  appears to be the most linear, so a power model is appropriate.

**d**

	List 1	List 2	List 3	List 4
SUB				
1	0.3	10.9	-1.203	2.3887
2	0.4	7	-0.916	1.9459
3	0.5	5.9	-0.693	1.7749
4	0.6	3.9	-0.511	1.3609
				2.388762789

GRAPH CALC TEST INTR DIST

1Var XList	:List1
1Var Freq	:1
2Var XList	:List3
2Var YList	:List4
2Var Freq	:1

LIST

LinearReg(ax+b)
a = -1.3884078
b = 0.71300719
r = -0.9926897
r <sup>2</sup> = 0.98543285
MSe = 4.7536E-03
y = ax + b

COPY

Using technology, the linear model connecting  $\ln C$  and  $\ln n$  is  $\ln C \approx -1.39 \ln n + 0.713$

$$\therefore C \approx e^{0.713 - 1.39 \ln n}$$

$$\therefore C \approx e^{0.713} \times e^{-1.39 \ln n}$$

$$\therefore C \approx 2.04 \times n^{-1.39}$$

**e** When  $n = 0.1$ ,  $C \approx 50.0$

$\therefore$  if there are 0.1 sheep per hectare grazing on the pasture, about 50.0% of the ground is covered by pasture.

**f**  $C = 9$  when  $2.04 \times n^{-1.39} \approx 9$

$$\therefore n \approx 0.344 \quad \{\text{technology}\}$$

$\therefore$  the owner should allow about 0.344 sheep per hectare to ensure that 9% of the ground is covered by pasture.



# TOPIC 3 SKILL BUILDER QUESTIONS

- 1 a Perimeter =  $2 \times$  line segment length + inner arc length + outer arc length

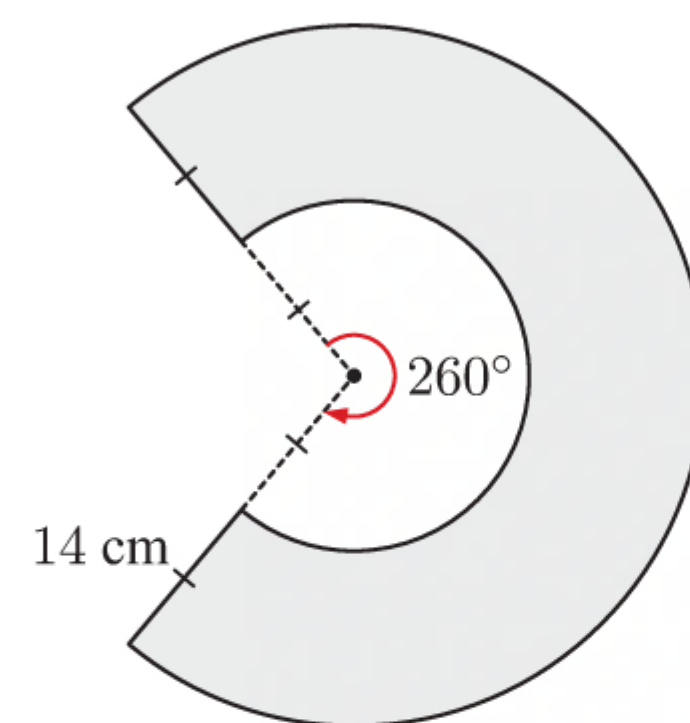
$$= 2 \times 14 + \frac{260}{360} \times 2\pi \times 14 + \frac{260}{360} \times 2\pi \times 28$$

$$\approx 219 \text{ cm}$$

- b Area = outer sector area – inner sector area

$$= \frac{260}{360} \times \pi \times 28^2 - \frac{260}{360} \times \pi \times 14^2$$

$$\approx 1330 \text{ cm}^2$$



- 2 a Let the height of the triangles be  $h$  cm.

$$\text{Now } h^2 = 20^2 + 5^2$$

$$\therefore h = \sqrt{20^2 + 5^2} = 5\sqrt{17}$$

Surface area

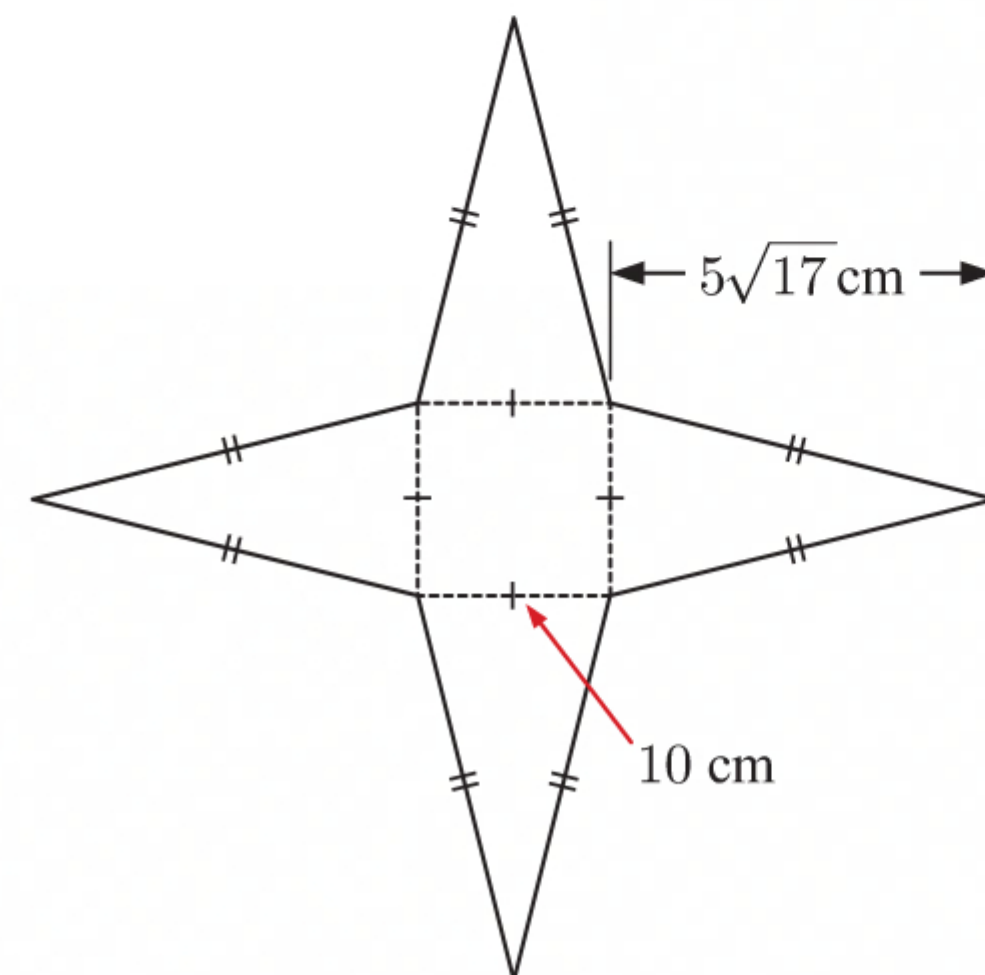
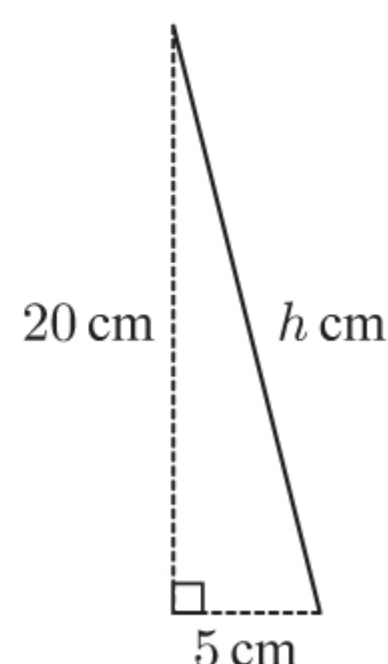
$$= \text{area of square} + 4 \times \text{area of triangle}$$

$$= 10^2 + 4 \left( \frac{1}{2} \times 10 \times 5\sqrt{17} \right)$$

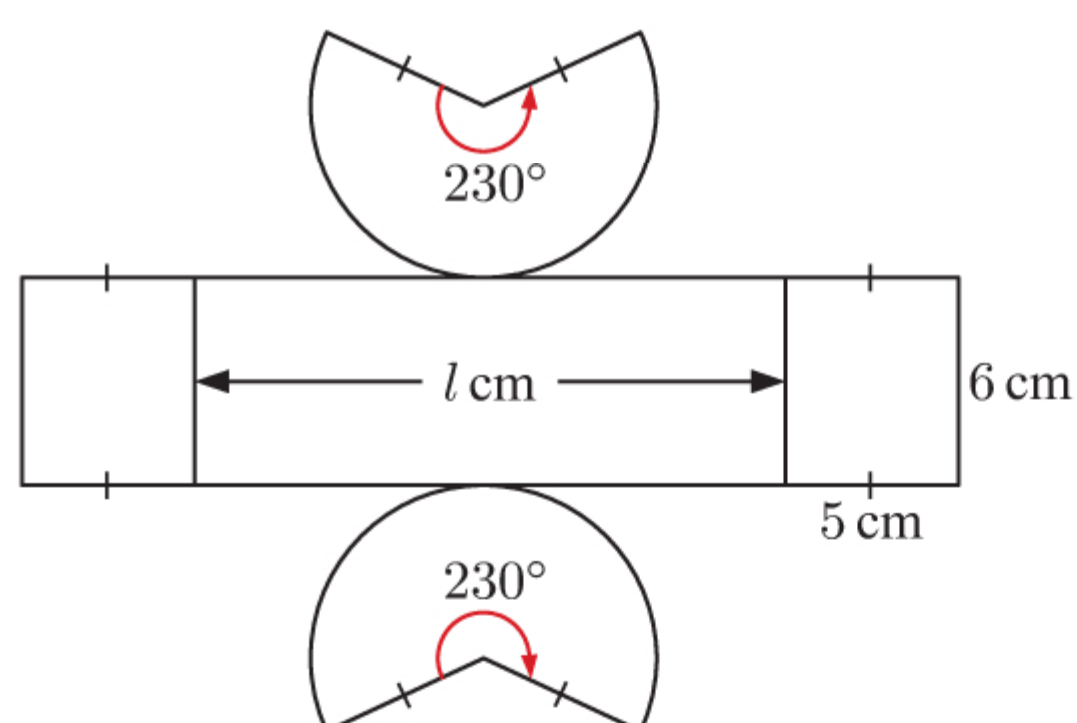
$$= 100 + 4(25\sqrt{17})$$

$$= 100 + 100\sqrt{17} \text{ cm}^2$$

$$\approx 512 \text{ cm}^2$$



- b



$$l = 2\pi r \times \frac{230}{360}$$

$$= 2\pi(5) \times \frac{23}{36}$$

$$= \frac{115\pi}{18}$$

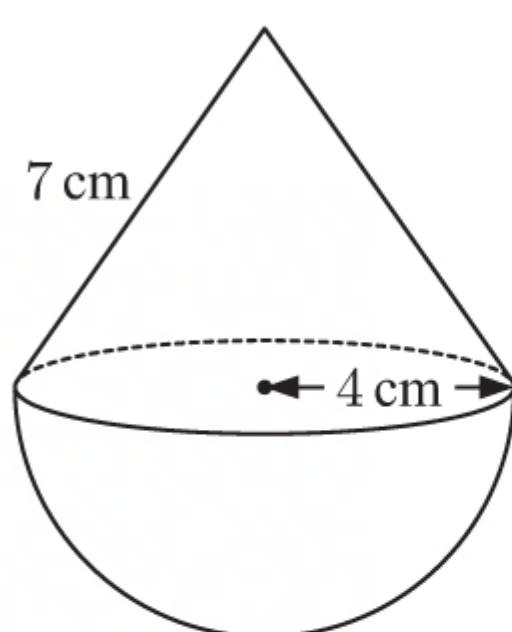
Surface area =  $2 \times$  area of sector + area of curved surface  
+  $2 \times$  area of rectangle

$$= 2 \times \frac{230}{360} \times \pi(5)^2 + \frac{115\pi}{18} \times 6 + 2 \times 6 \times 5$$

$$= \frac{575\pi}{18} + \frac{115\pi}{3} + 60$$

$$\approx 281 \text{ cm}^2$$

- c



$$\text{Surface area} = \frac{1}{2}4\pi r^2 + \pi rs$$

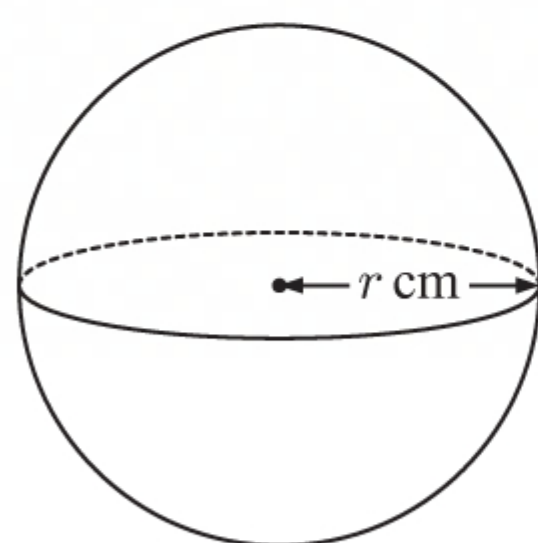
$$= \frac{1}{2} \times 4\pi(4)^2 + \pi(4)(7)$$

$$= 32\pi + 28\pi$$

$$= 60\pi \text{ cm}^2$$

$$\approx 188 \text{ cm}^2$$

- 3



Let the beach ball have radius  $r$  cm.

$$\text{Surface area} = 2800 \text{ cm}^2$$

$$\therefore 4\pi r^2 = 2800$$

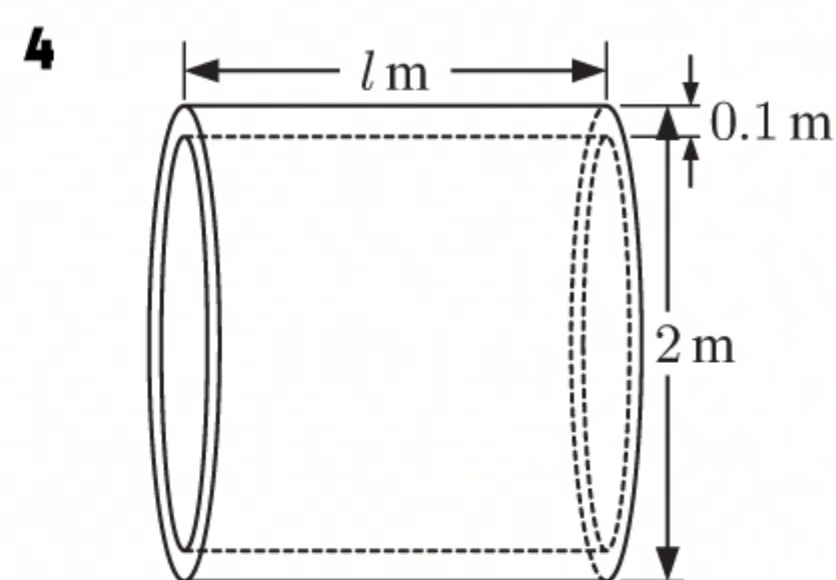
$$\therefore r^2 = \frac{700}{\pi}$$

$$\therefore r = \sqrt{\frac{700}{\pi}} \quad \{r > 0\}$$

$$\therefore r \approx 14.9$$

$\therefore$  the radius of the beach ball is approximately 14.9 cm.





Let the pipe have length  $l$  m.

Now volume of concrete = volume of whole cylinder – volume of hollow section

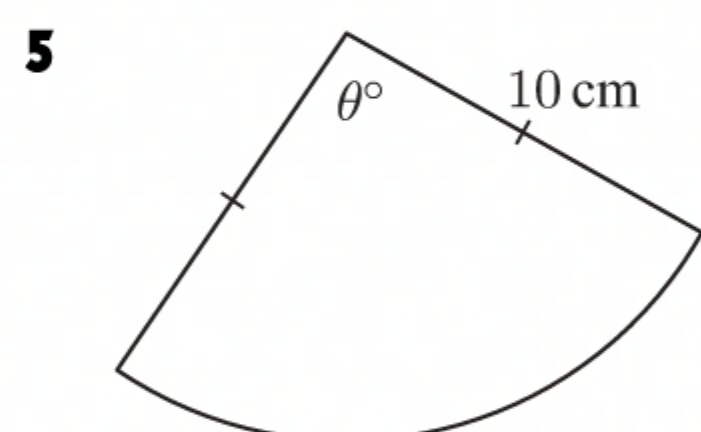
$$\therefore 3 = \pi(1)^2 \times l - \pi(0.9)^2 \times l$$

$$\therefore 3 = \pi l - 0.81\pi l$$

$$\therefore 3 = 0.19\pi l$$

$$\therefore l = \frac{3}{0.19\pi} \approx 5.03$$

$\therefore$  the pipe is approximately 5.03 m long.



**a** Perimeter =  $2r + \text{arc length}$

$$\therefore 40 = 2(10) + \text{arc length}$$

$$\therefore \text{arc length} = 20 \text{ cm}$$

**b** Now, arc length =  $\frac{\theta}{360} \times 2\pi r$

$$\therefore 20 = \frac{\theta}{360} \times 2\pi(10) \quad \{\text{from a}\}$$

$$\therefore 20 = \frac{\theta\pi}{18}$$

$$\therefore \theta = \frac{360}{\pi}$$

$$\begin{aligned} \therefore \text{area} &= \frac{\theta}{360} \times \pi r^2 \\ &= \frac{\left(\frac{360}{\pi}\right)}{360} \times \pi(10)^2 \\ &= 100 \text{ cm}^2 \end{aligned}$$

**6 a** Total height = hemisphere radius + cone height

$$\therefore 7 = \text{hemisphere radius} + 4$$

$$\therefore \text{hemisphere radius} = 3 \text{ m}$$

$$\therefore \text{cone radius} = 3 \text{ m} \quad \{\text{hemisphere radius} = \text{cone radius}\}$$

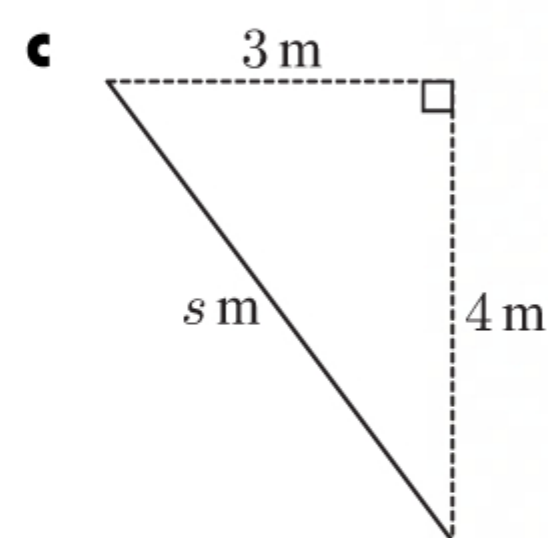
**b** Volume = volume of hemisphere + volume of cone

$$= \frac{1}{2} \times \frac{4}{3}\pi r^3 + \frac{1}{3} \times \pi r^2 \times h$$

$$= \frac{1}{2} \times \frac{4}{3}\pi(3)^3 + \frac{1}{3} \times \pi(3)^2 \times 4$$

$$= 18\pi + 12\pi$$

$$= 30\pi \approx 94.2 \text{ m}^3$$



Let the slant height of the cone be  $s$  m.

$$\therefore s^2 = 4^2 + 3^2 \quad \{\text{Pythagoras}\}$$

$$\therefore s^2 = 16 + 9$$

$$\therefore s^2 = 25$$

$$\therefore s = 5 \quad \{s > 0\}$$

$\therefore$  the slant height of the cone is 5 m.

**d** Surface area =  $\frac{1}{2} \times 4\pi r^2 + \pi r s$

$$= 2\pi(3)^2 + \pi(3)(5)$$

$$= 18\pi + 15\pi$$

$$= 33\pi \approx 104 \text{ m}^2$$

**e** Weight = surface area  $\times$  weight of polymer per  $\text{m}^2$

$$= 33\pi \times 1.23$$

$$\approx 128 \text{ kg}$$

**7 a** Arc length =  $\theta r$

$$\therefore \pi = 3\theta$$

$$\therefore \theta = \frac{\pi}{3}$$

**b** Shaded area =  $\frac{1}{2} \left(2\pi - \frac{\pi}{3}\right)(3)^2$

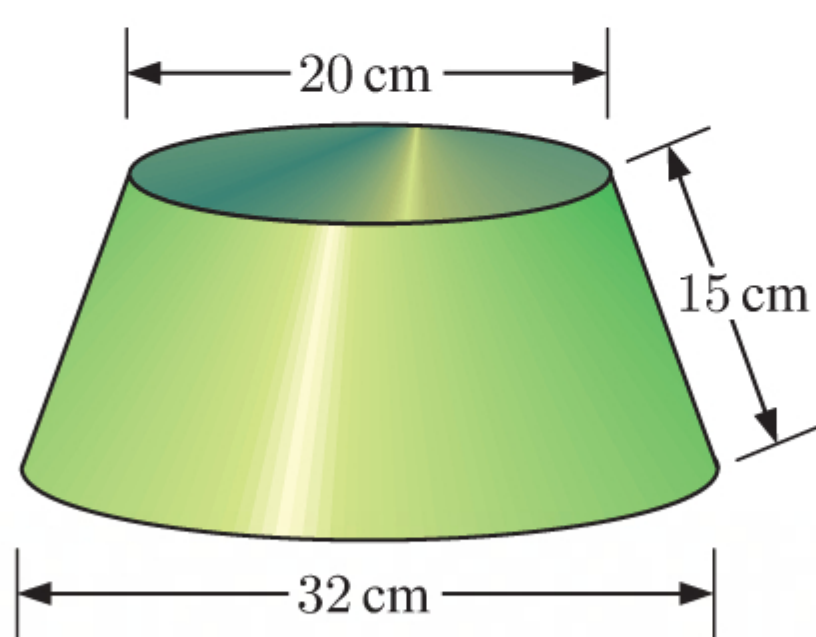
$$= \frac{1}{2} \times \frac{5\pi}{3} \times 9$$

$$= \frac{15\pi}{2}$$

$$\approx 23.6 \text{ cm}^2$$



8



$$\text{The shorter arc length} = 2\pi\left(\frac{20}{2}\right) = 20\pi \text{ cm}$$

$$\text{The longer arc length} = 2\pi\left(\frac{32}{2}\right) = 32\pi \text{ cm}$$

Now, the shorter arc length  $= \theta r$

$$\therefore 20\pi = \theta r \quad \dots (*)$$

and the longer arc length  $= \theta(r + 15)$

$$\therefore 32\pi = \theta r + 15\theta$$

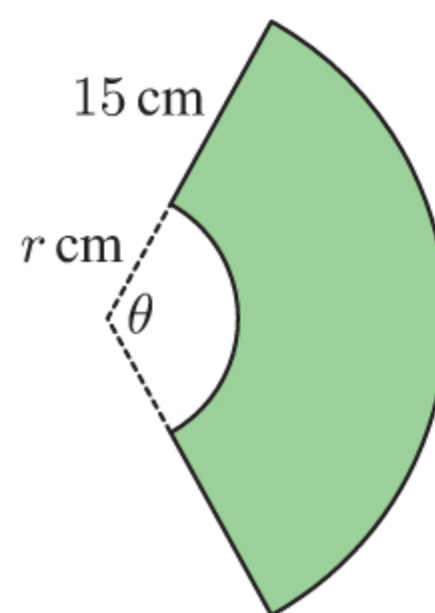
$$\therefore 32\pi = 20\pi + 15\theta \quad \{\text{using } (*)\}$$

$$\therefore 15\theta = 12\pi$$

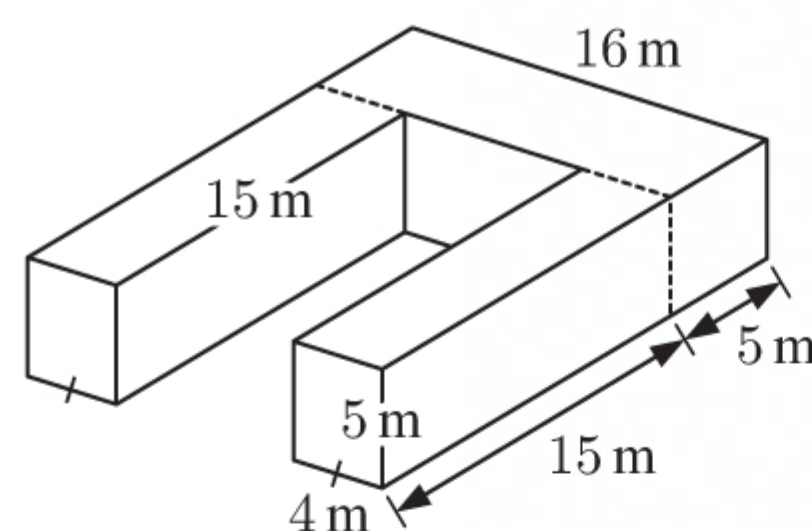
$$\therefore \theta = \frac{4\pi}{5}$$

Substituting  $\theta = \frac{4\pi}{5}$  into  $(*)$  gives  $20\pi = \frac{4\pi}{5}r$

$$\therefore r = 25$$



9 a Volume  $= 2 \times (15 \times 5 \times 4) + (16 \times 5 \times 5)$   
 $= 1000 \text{ m}^3$

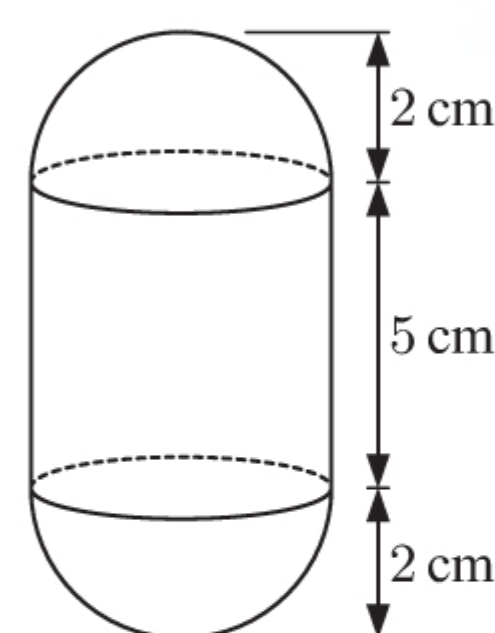


b Volume = volume of sphere + volume of cylinder

$$= \frac{4}{3}\pi(2)^3 + \pi(2)^2 \times 5$$

$$= \frac{32\pi}{3} + 20\pi$$

$$\approx 96.3 \text{ cm}^3$$



c Let the height of the triangular cross-section be  $h$  cm.

$$\therefore h^2 + \left(\frac{15}{2}\right)^2 = 20^2 \quad \{\text{Pythagoras}\}$$

$$\therefore h^2 + \frac{225}{4} = 400$$

$$\therefore h^2 = \frac{1375}{4}$$

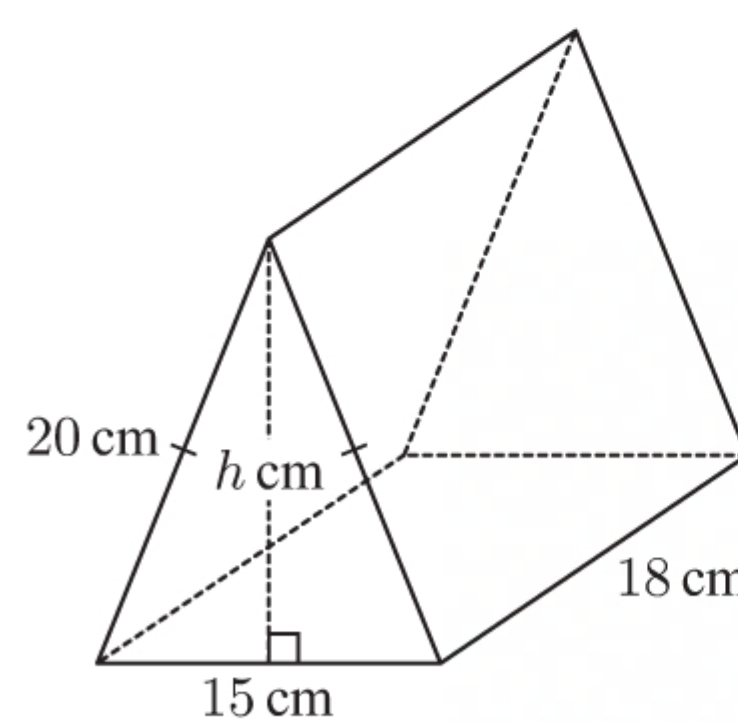
$$\therefore h = \frac{\sqrt{1375}}{2} \quad \{h > 0\}$$

$\therefore$  volume = cross-sectional area  $\times$  length

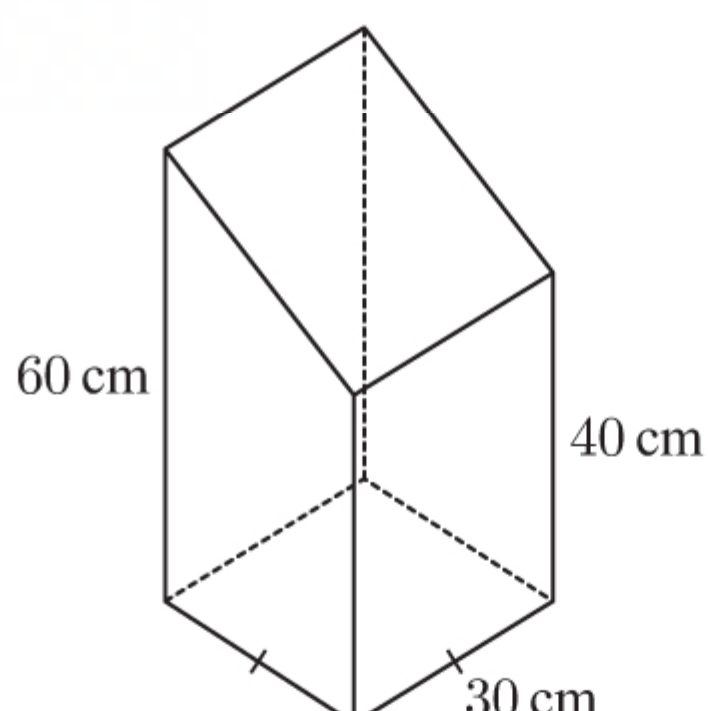
$$= \left(\frac{1}{2} \times h \times 15\right) \times 18$$

$$= \frac{1}{2} \times \frac{\sqrt{1375}}{2} \times 15 \times 18$$

$$\approx 2500 \text{ cm}^3$$



10



Volume = cross-sectional area  $\times$  length

$$= \left(\frac{60 + 40}{2}\right) \times 30 \times 30$$

$$= 50 \times 30 \times 30$$

$$= 45\,000 \text{ cm}^3$$

$$\therefore \text{capacity} = 45\,000 \text{ mL} = 45 \text{ L}$$

$$\therefore \text{number of filled containers} = \frac{300 \text{ L}}{45 \text{ L}} = 6\frac{2}{3}$$

$\therefore$  6 petrol containers can be completely filled with 300 L of petrol.



**11 a**  $A(2, 4, 1)$  and  $B(4, 0, 7)$ 

$$\begin{aligned}
 \text{i } AB &= \sqrt{(4-2)^2 + (0-4)^2 + (7-1)^2} \\
 &= \sqrt{2^2 + (-4)^2 + 6^2} \\
 &= \sqrt{4 + 16 + 36} \\
 &= \sqrt{56} \text{ units}
 \end{aligned}$$

**ii** The midpoint is  $\left(\frac{2+4}{2}, \frac{4+0}{2}, \frac{1+7}{2}\right)$ ,  
which is  $(3, 2, 4)$ .

**b**  $A(3, -5, 2)$  and  $B(-1, 2, -3)$ 

$$\begin{aligned}
 \text{i } AB &= \sqrt{(-1-3)^2 + (2-(-5))^2 + (-3-2)^2} \\
 &= \sqrt{(-4)^2 + 7^2 + (-5)^2} \\
 &= \sqrt{16 + 49 + 25} \\
 &= \sqrt{90} \text{ units}
 \end{aligned}$$

**ii** The midpoint is  $\left(\frac{3+(-1)}{2}, \frac{-5+2}{2}, \frac{2+(-3)}{2}\right)$ ,  
which is  $\left(1, -\frac{3}{2}, -\frac{1}{2}\right)$ .

**c**  $A(-6, 0, 5)$  and  $B(-3, -3, 1)$ 

$$\begin{aligned}
 \text{i } AB &= \sqrt{(-3-(-6))^2 + (-3-0)^2 + (1-5)^2} \\
 &= \sqrt{3^2 + (-3)^2 + (-4)^2} \\
 &= \sqrt{9 + 9 + 16} \\
 &= \sqrt{34} \text{ units}
 \end{aligned}$$

**ii** The midpoint is  $\left(\frac{-6+(-3)}{2}, \frac{0+(-3)}{2}, \frac{5+1}{2}\right)$ ,  
which is  $\left(-\frac{9}{2}, -\frac{3}{2}, 3\right)$ .

**12**  $P(k, 6, -5)$  and  $Q(2, -1, -8)$ 

$$\begin{aligned}
 PQ &= \sqrt{(2-k)^2 + (-1-6)^2 + (-8-(-5))^2} \\
 &= \sqrt{(2-k)^2 + (-7)^2 + (-3)^2} \\
 &= \sqrt{4 - 4k + k^2 + 49 + 9} \\
 &= \sqrt{k^2 - 4k + 62}
 \end{aligned}$$

Now  $PQ = 9$  units

$$\therefore \sqrt{k^2 - 4k + 62} = 9$$

$$\therefore k^2 - 4k + 62 = 81$$

$$\therefore k^2 - 4k - 19 = 0$$

$$\begin{aligned}
 \therefore k &= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-19)}}{2} \\
 &= \frac{4 \pm \sqrt{92}}{2} \\
 &= \frac{4 \pm 2\sqrt{23}}{2} = 2 \pm \sqrt{23}
 \end{aligned}$$

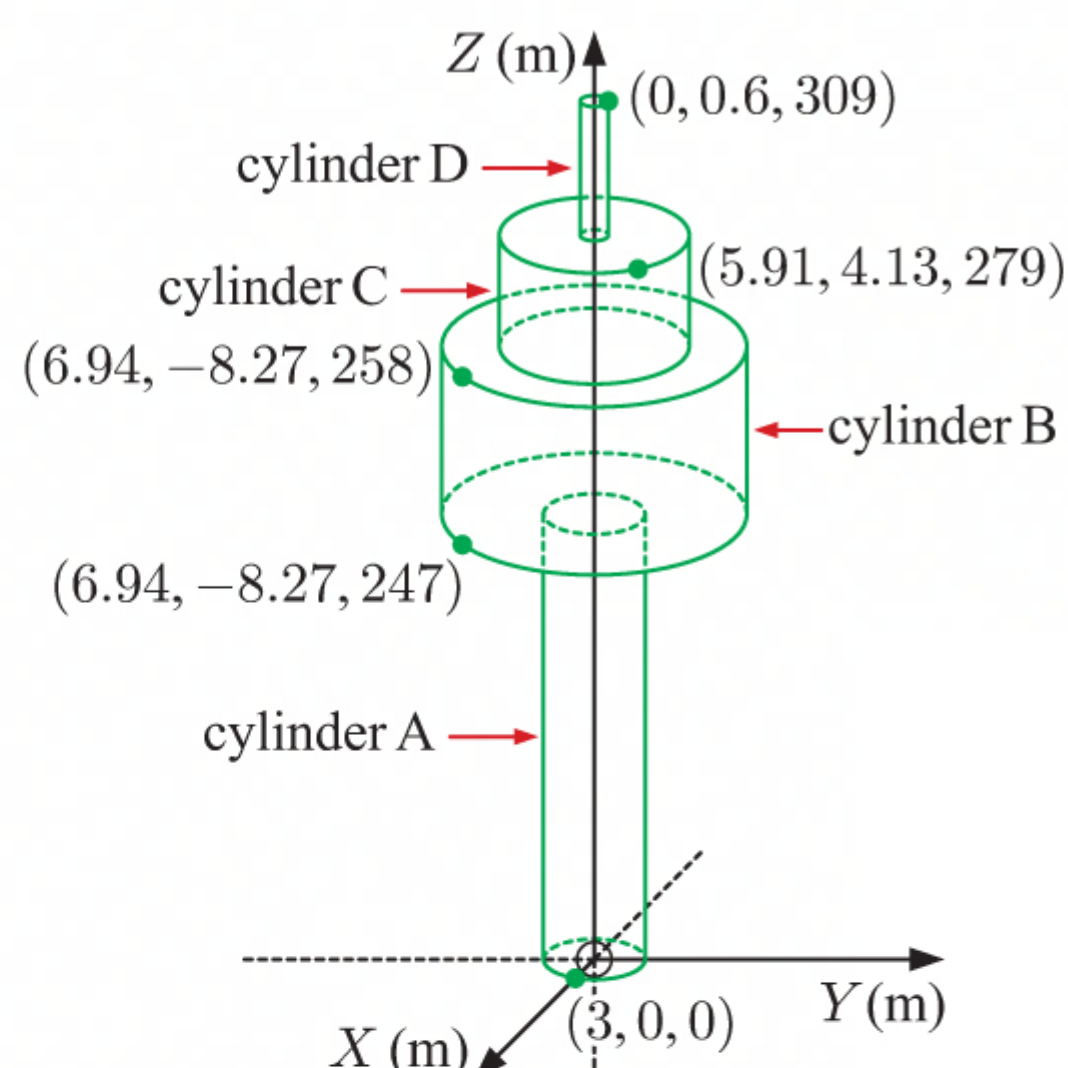
**13**  $A$  is  $(3, 4, -6)$  and  $M$  is  $\left(-\frac{1}{2}, 9, -7\right)$ .Let  $B$  have coordinates  $(x, y, z)$ .

$$\therefore \left(\frac{x+3}{2}, \frac{y+4}{2}, \frac{z-6}{2}\right) = \left(-\frac{1}{2}, 9, -7\right)$$

$$\therefore \frac{x+3}{2} = -\frac{1}{2}, \quad \frac{y+4}{2} = 9, \quad \text{and} \quad \frac{z-6}{2} = -7$$

$$\therefore x+3 = -1, \quad y+4 = 18, \quad \text{and} \quad z-6 = -14$$

$$\therefore x = -4, \quad y = 14, \quad \text{and} \quad z = -8$$

So,  $B$  has coordinates  $(-4, 14, -8)$ .**14**

Height of cylinder A = 247 m

Radius of cylinder A = 3 m

Height of cylinder B =  $258 - 247 = 11$  m

$$\begin{aligned}
 \text{Radius of cylinder B} &= \sqrt{6.94^2 + (-8.27)^2} \\
 &= \sqrt{116.5565} \text{ m}
 \end{aligned}$$

Height of cylinder C =  $279 - 258 = 21$  m

$$\begin{aligned}
 \text{Radius of cylinder C} &= \sqrt{5.91^2 + 4.13^2} \\
 &= \sqrt{51.985} \text{ m}
 \end{aligned}$$

Height of cylinder D =  $309 - 279 = 30$  m

Radius of cylinder D = 0.6 m

Volume of tower = volume of cylinder A + volume of cylinder B + volume of cylinder C + volume of cylinder D

$$= \pi(3)^2 \times 247 + \pi(\sqrt{116.5565})^2 \times 11 + \pi(\sqrt{51.985})^2 \times 21 + \pi(0.6)^2 \times 30$$

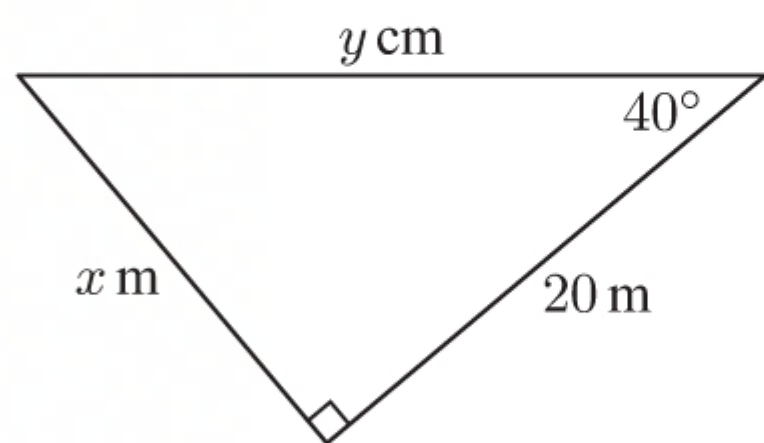
$$= 2223\pi + 1282.1215\pi + 1091.685\pi + 10.8\pi$$

$$= 4607.6065\pi$$

$$\approx 14475.22 \approx 14500 \text{ m}^3$$



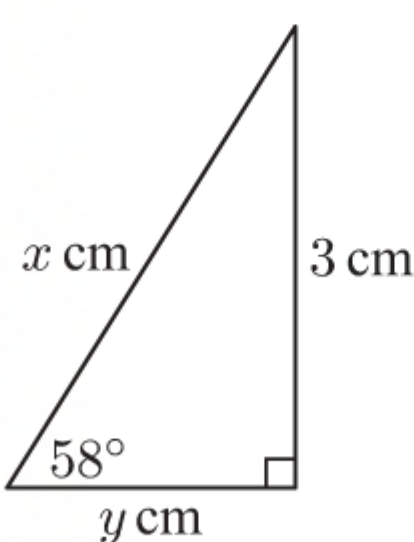
15 a



$$\begin{aligned}\tan 40^\circ &= \frac{x}{20} \\ \therefore x &= 20 \tan 40^\circ \\ &\approx 16.8\end{aligned}$$

$$\begin{aligned}\cos 40^\circ &= \frac{20}{y} \\ \therefore y &= \frac{20}{\cos 40^\circ} \\ &\approx 26.1\end{aligned}$$

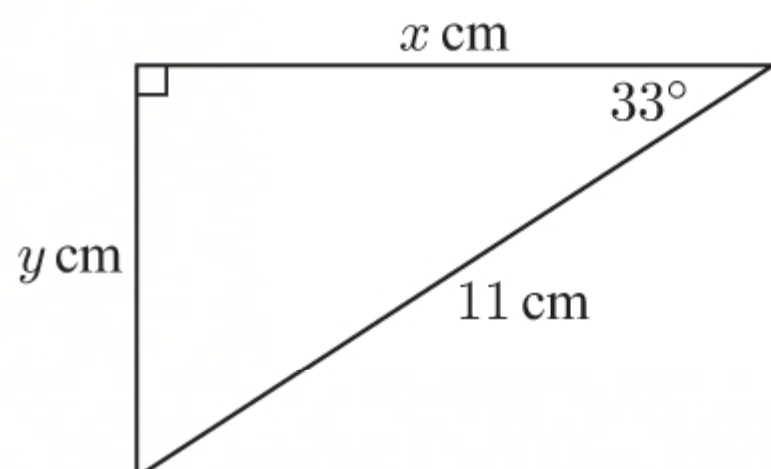
b



$$\begin{aligned}\sin 58^\circ &= \frac{3}{x} \\ \therefore x &= \frac{3}{\sin 58^\circ} \\ &\approx 3.54\end{aligned}$$

$$\begin{aligned}\tan 58^\circ &= \frac{3}{y} \\ \therefore y &= \frac{3}{\tan 58^\circ} \\ &\approx 1.87\end{aligned}$$

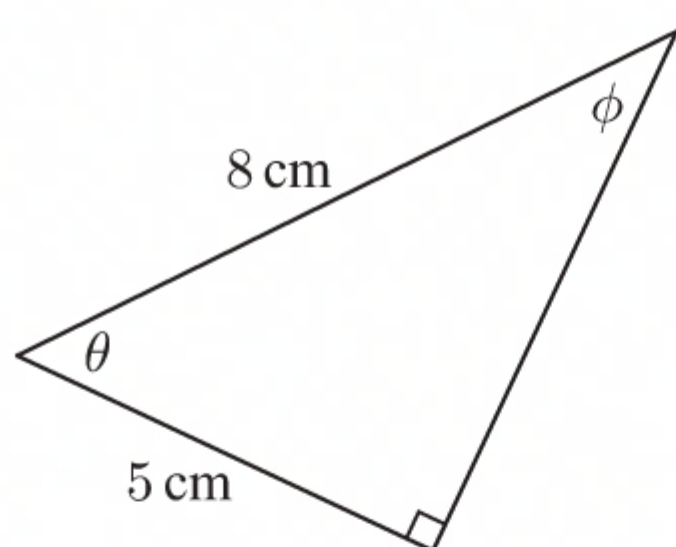
c



$$\begin{aligned}\cos 33^\circ &= \frac{x}{11} \\ \therefore x &= 11 \cos 33^\circ \\ &\approx 9.23\end{aligned}$$

$$\begin{aligned}\sin 33^\circ &= \frac{y}{11} \\ \therefore y &= 11 \sin 33^\circ \\ &\approx 5.99\end{aligned}$$

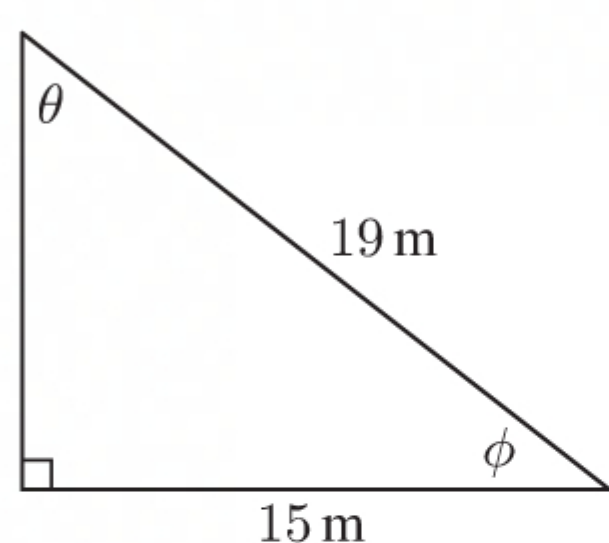
16 a



$$\begin{aligned}\cos \theta &= \frac{5}{10} \\ \therefore \theta &= \cos^{-1}\left(\frac{5}{10}\right) \\ &\approx 51.3^\circ\end{aligned}$$

$$\begin{aligned}\sin \phi &= \frac{5}{10} \\ \therefore \phi &= \sin^{-1}\left(\frac{5}{10}\right) \\ &\approx 38.7^\circ\end{aligned}$$

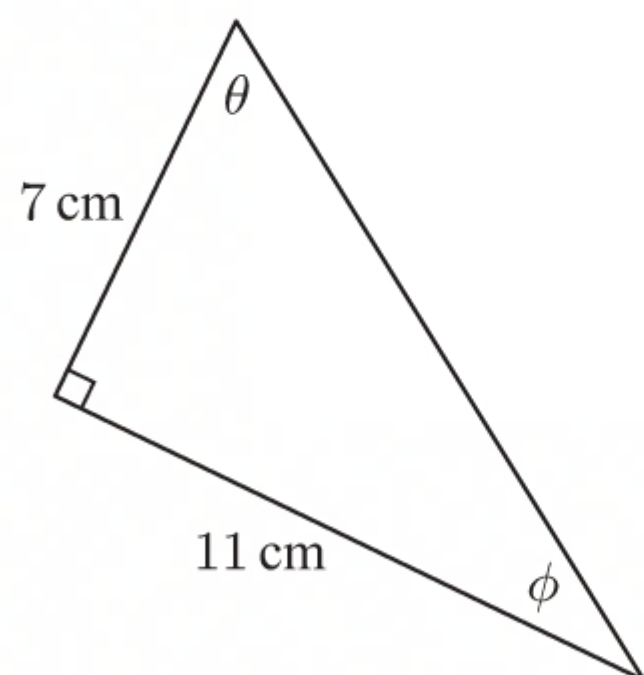
b



$$\begin{aligned}\sin \theta &= \frac{15}{19} \\ \therefore \theta &= \sin^{-1}\left(\frac{15}{19}\right) \\ &\approx 52.1^\circ\end{aligned}$$

$$\begin{aligned}\cos \phi &= \frac{15}{19} \\ \therefore \phi &= \cos^{-1}\left(\frac{15}{19}\right) \\ &\approx 37.9^\circ\end{aligned}$$

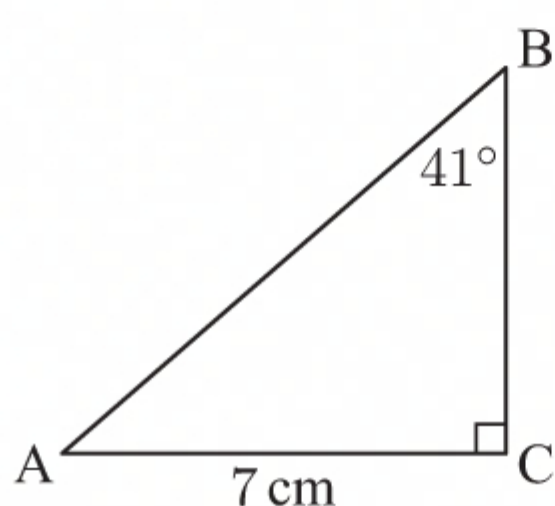
c



$$\begin{aligned}\tan \theta &= \frac{7}{11} \\ \therefore \theta &= \tan^{-1}\left(\frac{7}{11}\right) \\ &\approx 57.5^\circ\end{aligned}$$

$$\begin{aligned}\tan \phi &= \frac{7}{11} \\ \therefore \phi &= \tan^{-1}\left(\frac{7}{11}\right) \\ &\approx 32.5^\circ\end{aligned}$$

17 a



$$\begin{aligned}\sin 41^\circ &= \frac{7}{AB} \\ \therefore AB &= \frac{7}{\sin 41^\circ}\end{aligned}$$

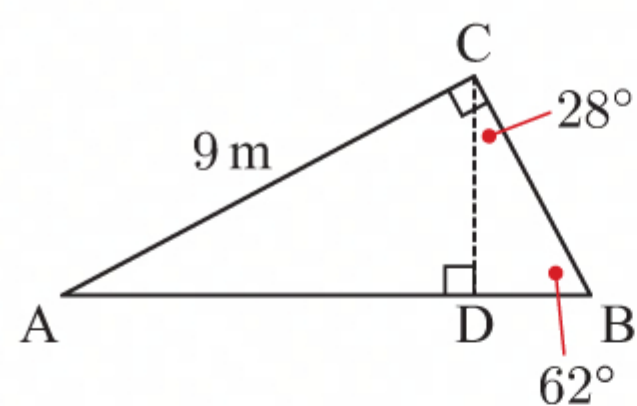
and

$$\begin{aligned}\tan 41^\circ &= \frac{7}{BC} \\ \therefore BC &= \frac{7}{\tan 41^\circ}\end{aligned}$$

$$\begin{aligned}\text{Perimeter} &= AB + BC + AC \\ &= \frac{7}{\sin 41^\circ} + \frac{7}{\tan 41^\circ} + 7 \\ &\approx 25.7 \text{ cm}\end{aligned}$$

$$\begin{aligned}\text{Area} &= \frac{1}{2} \times AC \times BC \\ &= \frac{1}{2} \times 7 \times \frac{7}{\tan 41^\circ} \\ &\approx 28.2 \text{ cm}^2\end{aligned}$$



**b**

$$\widehat{ABC} = 90^\circ - 28^\circ = 62^\circ \quad \{\text{angles in triangle BCD}\}$$

Now in triangle ABC:

$$\tan 62^\circ = \frac{9}{BC}$$

$$\therefore BC = \frac{9}{\tan 62^\circ}$$

$$\text{and } \sin 62^\circ = \frac{9}{AB}$$

$$\therefore AB = \frac{9}{\sin 62^\circ}$$

$$\text{Perimeter} = AB + BC + AC$$

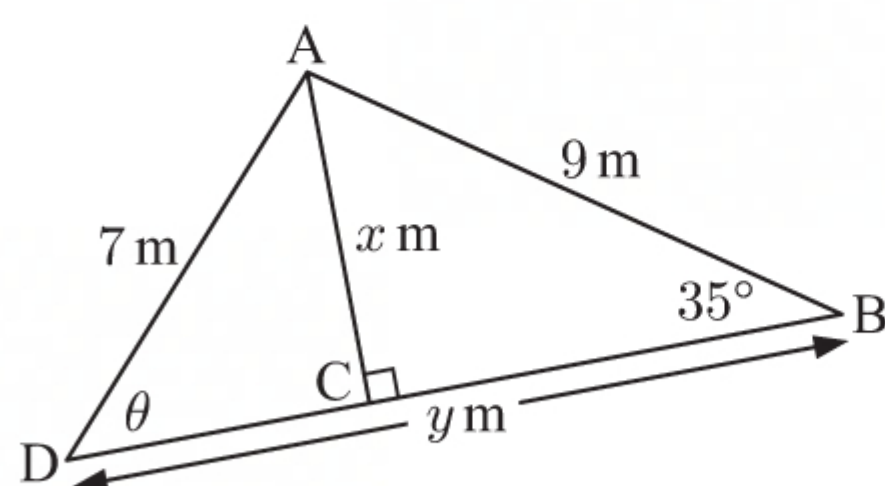
$$= \frac{9}{\sin 62^\circ} + \frac{9}{\tan 62^\circ} + 9$$

$$\approx 24.0 \text{ m}$$

$$\text{Area} = \frac{1}{2} \times AC \times BC$$

$$= \frac{1}{2} \times 9 \times \frac{9}{\tan 62^\circ}$$

$$\approx 21.5 \text{ m}^2$$

**18 a**

$$\text{In triangle ABC, } \sin 35^\circ = \frac{x}{9}$$

$$\therefore x = 9 \sin 35^\circ \approx 5.16$$

$$\text{In triangle ACD, } \sin \theta = \frac{x}{7}$$

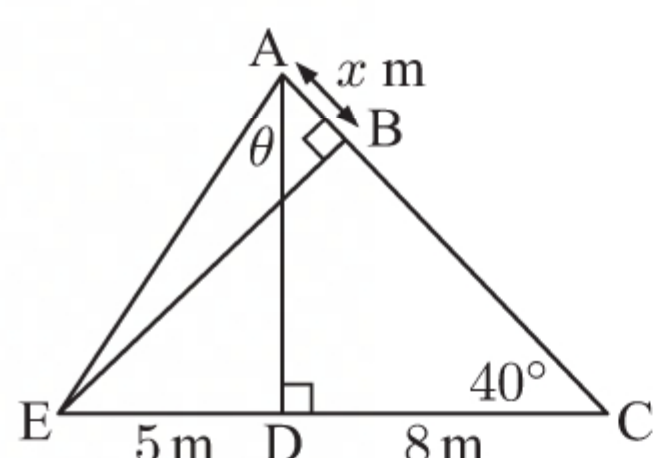
$$= \frac{9 \sin 35^\circ}{7}$$

$$\therefore \theta = \sin^{-1}\left(\frac{9 \sin 35^\circ}{7}\right) \approx 47.5^\circ$$

$$\widehat{DAB} \approx 180^\circ - 35^\circ - 47.5^\circ \quad \{\text{angles in triangle DAB}\}$$

$$\approx 97.5^\circ$$

$$\text{Using the cosine rule in triangle DAB, } y \approx \sqrt{7^2 + 9^2 - 2(7)(9) \cos 97.5^\circ} \approx 12.1$$

**b**

$$\text{In triangle ACD, } \tan 40^\circ = \frac{AD}{8}$$

$$\therefore AD = 8 \tan 40^\circ$$

$$\text{In triangle ADE, } \tan \theta = \frac{5}{AD}$$

$$= \frac{5}{8 \tan 40^\circ}$$

$$\therefore \theta = \tan^{-1}\left(\frac{5}{8 \tan 40^\circ}\right)$$

$$\approx 36.68^\circ$$

$$\approx 36.7^\circ$$

$$\text{and } \sin 36.68^\circ \approx \frac{5}{AE}$$

$$\therefore AE \approx \frac{5}{\sin 36.68^\circ}$$

$$\text{Now } \widehat{DAC} = 90^\circ - 40^\circ \quad \{\text{angles in triangle ACD}\}$$

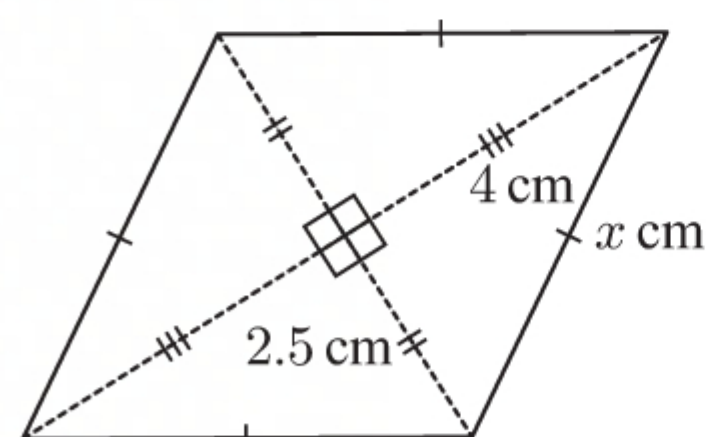
$$= 50^\circ$$

$$\therefore \widehat{EAB} \approx 50^\circ + 36.68^\circ \approx 86.68^\circ$$

$$\text{In triangle ABE, } \cos \widehat{EAB} = \frac{x}{AE}$$

$$\therefore \cos 86.68^\circ \approx \frac{x}{\left(\frac{5}{\sin 36.68^\circ}\right)}$$

$$\therefore x \approx \frac{5 \cos 86.68^\circ}{\sin 36.68^\circ} \approx 0.485$$

**19 a**

**b** Let  $x$  cm be the side length of the rhombus.

$$\therefore x^2 = 4^2 + (2.5)^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x^2 = 22.25$$

$$\therefore x = \sqrt{22.25} \quad \{x > 0\}$$

$$\therefore x \approx 4.72$$

$\therefore$  the length of the rhombus' sides are approximately 4.72 cm.



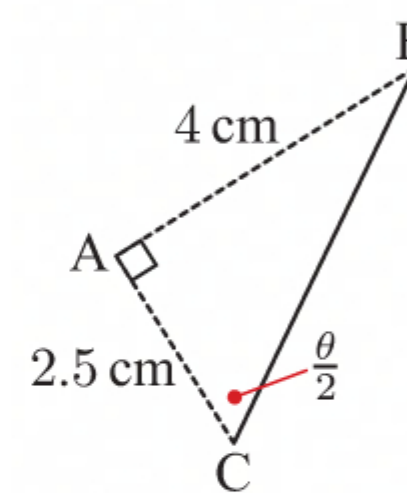
- c** Let  $\theta$  be the larger angle in the rhombus.

$$\therefore \widehat{ACB} = \frac{\theta}{2} \quad \{\text{diagonals of a rhombus bisect its angles}\}$$

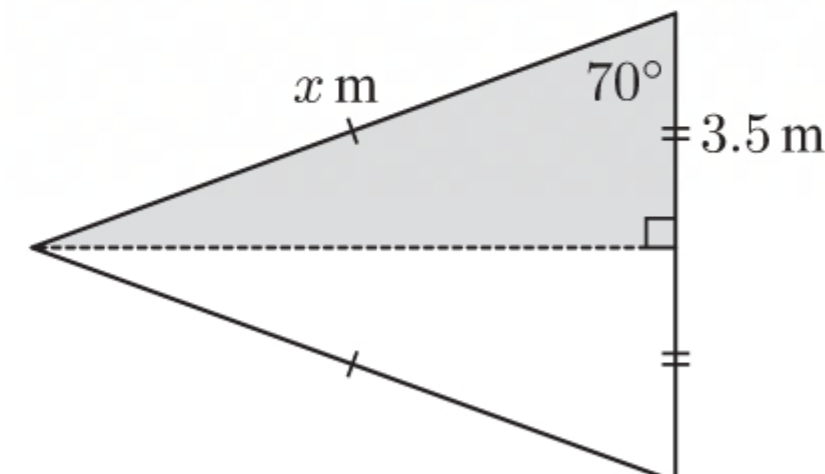
$$\therefore \tan \frac{\theta}{2} = \frac{4}{2.5}$$

$$\therefore \frac{\theta}{2} = \tan^{-1}\left(\frac{4}{2.5}\right)$$

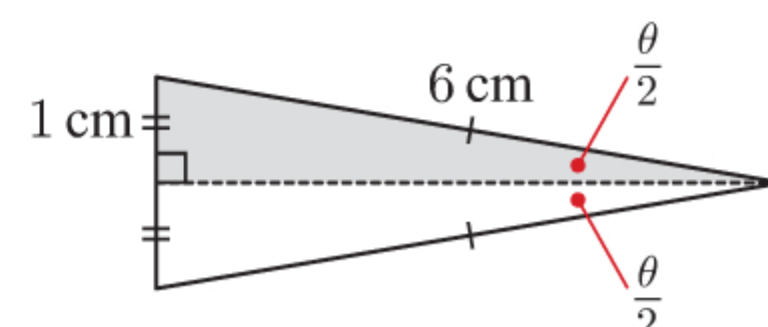
$$\therefore \theta = 2 \tan^{-1}\left(\frac{4}{2.5}\right) \approx 116^\circ$$



- 20 a** In the shaded right angled triangle,  $\cos 70^\circ = \frac{3.5}{x}$   
 $\therefore x = \frac{3.5}{\cos 70^\circ} \approx 10.2$



- b** In the shaded right angled triangle,  $\sin \frac{\theta}{2} = \frac{1}{6}$   
 $\therefore \frac{\theta}{2} = \sin^{-1}\left(\frac{1}{6}\right)$   
 $\therefore \theta = 2 \sin^{-1}\left(\frac{1}{6}\right) \approx 19.2^\circ$



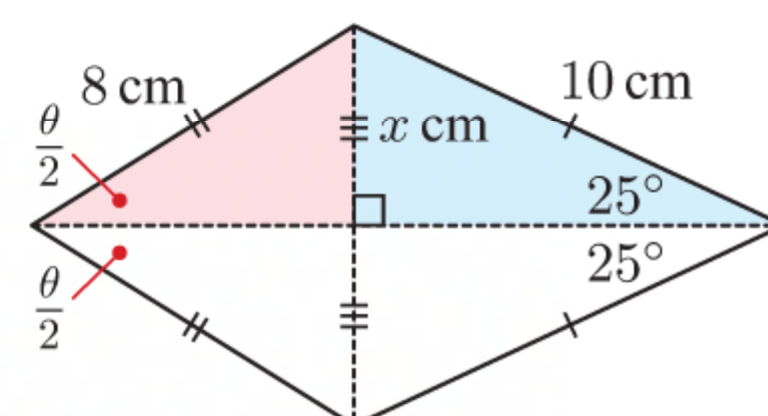
- c** In the blue right angled triangle,  $\sin 25^\circ = \frac{x}{10}$   
 $\therefore x = 10 \sin 25^\circ$

In the pink right angled triangle,  $\sin \frac{\theta}{2} = \frac{x}{8}$

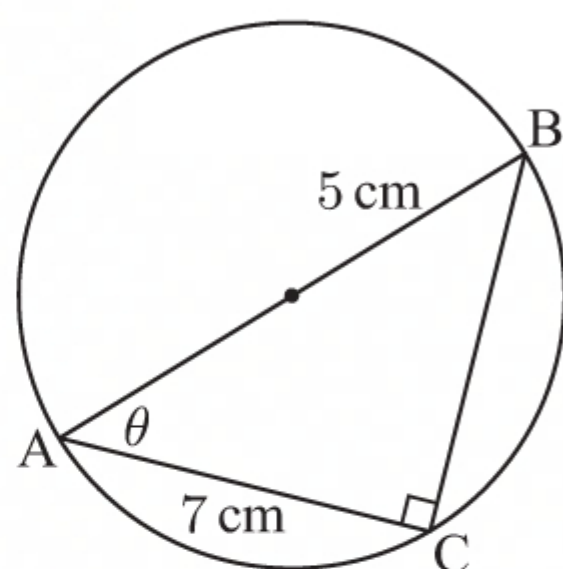
$$\therefore \sin \frac{\theta}{2} = \frac{10 \sin 25^\circ}{8}$$

$$\therefore \frac{\theta}{2} = \sin^{-1}\left(\frac{10 \sin 25^\circ}{8}\right)$$

$$\therefore \theta = 2 \sin^{-1}\left(\frac{10 \sin 25^\circ}{8}\right) \approx 63.8^\circ$$



- 21 a**



$$\widehat{ACB} = 90^\circ \quad \{\text{angle in a semi-circle}\}$$

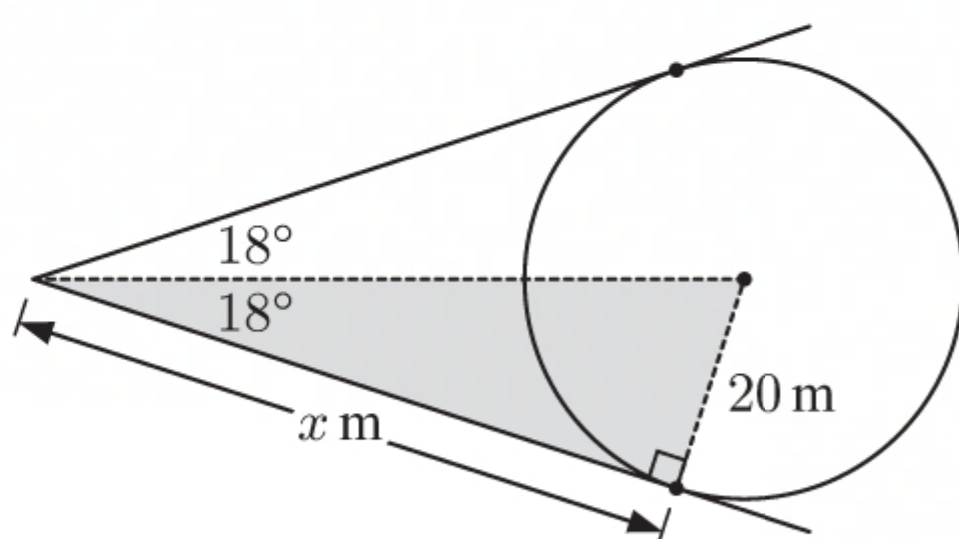
$$\therefore \triangle ABC \text{ is right angled at } C.$$

$$\therefore \cos \theta = \frac{7}{AB}$$

$$\therefore \cos \theta = \frac{7}{10}$$

$$\therefore \theta = \cos^{-1}\left(\frac{7}{10}\right) \approx 45.6^\circ$$

- b**



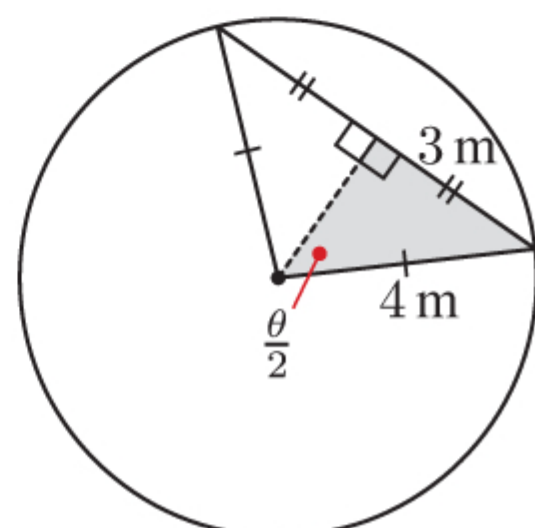
We construct the right angled triangle as shown.

For the shaded triangle,  $\tan 18^\circ = \frac{20}{x}$

$$\therefore x = \frac{20}{\tan 18^\circ}$$

$$\therefore x \approx 61.6$$

- c**



We construct the altitude as shown.

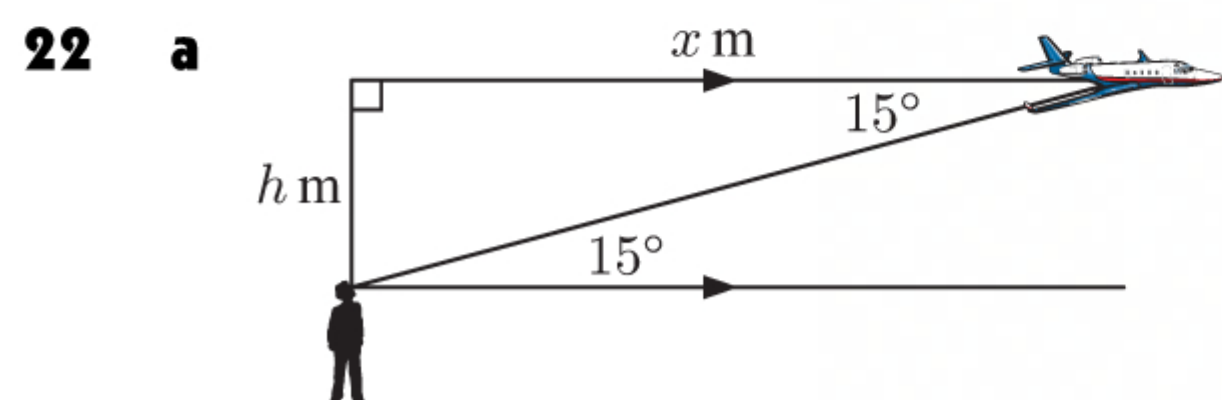
For the shaded triangle,  $\sin \frac{\theta}{2} = \frac{3}{4}$

$$\therefore \frac{\theta}{2} = \sin^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \theta = 2 \sin^{-1}\left(\frac{3}{4}\right)$$

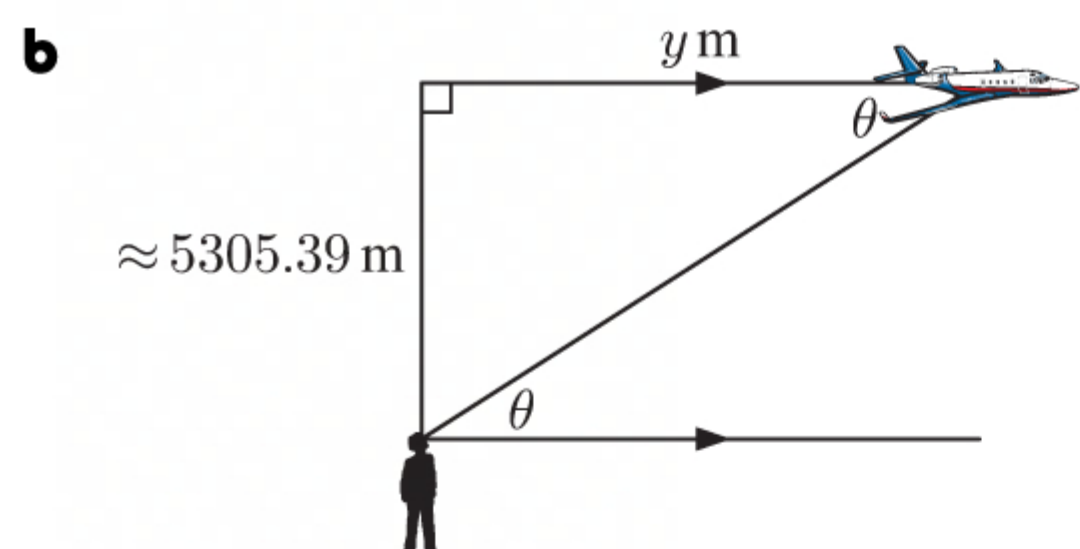
$$\therefore \theta \approx 97.2^\circ$$





$$\begin{aligned}
 x &= \text{speed} \times \text{time} \\
 &= 110 \times 180 \quad \{3 \text{ minutes} = 180 \text{ seconds}\} \\
 &= 19\,800 \\
 \therefore \tan 15^\circ &= \frac{h}{19\,800} \\
 \therefore h &= 19\,800 \tan 15^\circ \\
 \therefore h &\approx 5305.39 \approx 5310
 \end{aligned}$$

$\therefore$  the plane is approximately  $5310 \text{ m} \approx 5.31 \text{ km}$  above the ground.



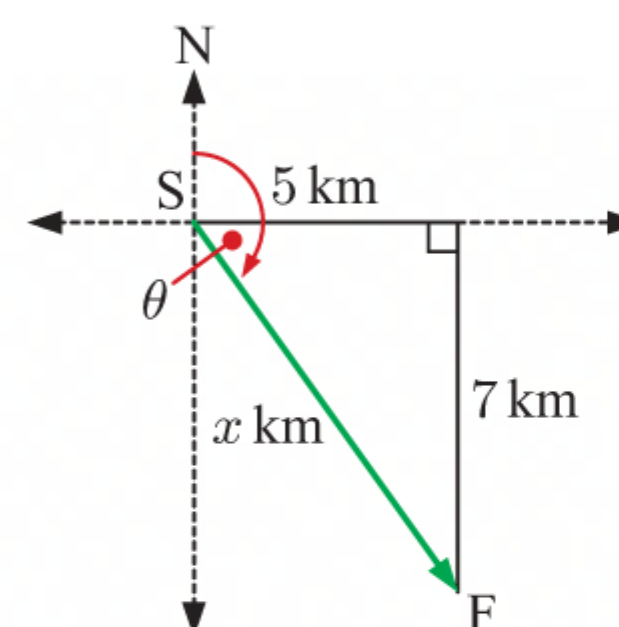
$$\begin{aligned}
 y &= \text{speed} \times \text{time} \\
 &= 110 \times 420 \quad \{7 \text{ minutes} = 420 \text{ seconds}\} \\
 &= 46\,200 \\
 \therefore \tan \theta &\approx \frac{5305.39}{46\,200} \\
 \therefore \theta &\approx \tan^{-1}\left(\frac{5305.39}{46\,200}\right) \\
 \therefore \theta &\approx 6.55^\circ
 \end{aligned}$$

$\therefore$  the angle of elevation of the plane at 2:42 pm is approximately  $6.55^\circ$ .

- 23 a** Suppose the helicopter starts at S and lands at F.

$$\begin{aligned}
 \text{Now } x^2 &= 7^2 + 5^2 \quad \{\text{Pythagoras}\} \\
 &= 74 \\
 \therefore x &= \sqrt{74} \quad \{x > 0\} \\
 &\approx 8.60
 \end{aligned}$$

$\therefore$  the helicopter is about  $8.60 \text{ km}$  from its starting point.

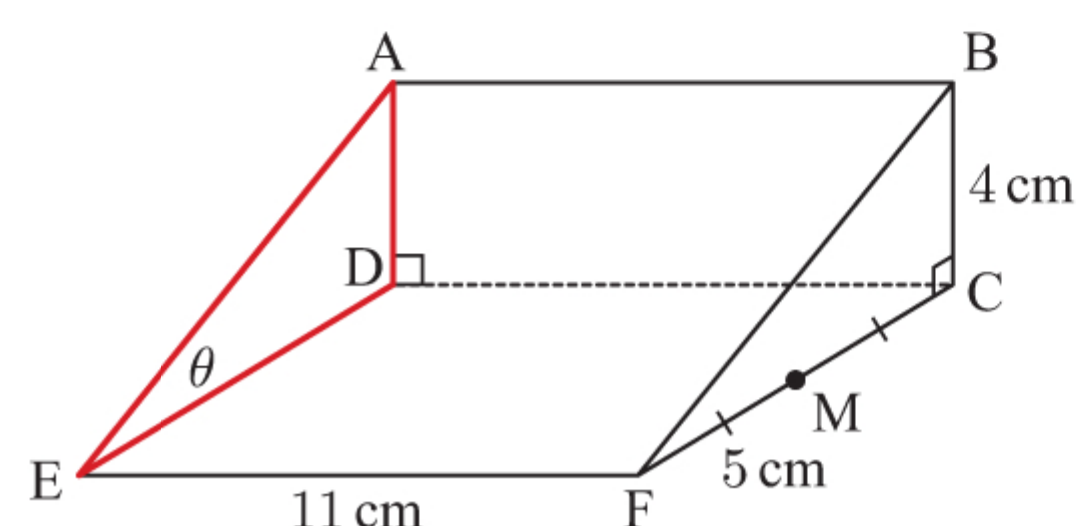


**b**  $\tan \theta = \frac{7}{5}$   
 $\therefore \theta = \tan^{-1}\left(\frac{7}{5}\right) \approx 54.5^\circ$   
 So, the bearing  $\approx 90^\circ + 54.5^\circ$   
 $\approx 144.5^\circ$

- 24 a** The projection of [AE] onto the base plane is [DE].

$\therefore$  the required angle is  $\widehat{AED}$ .

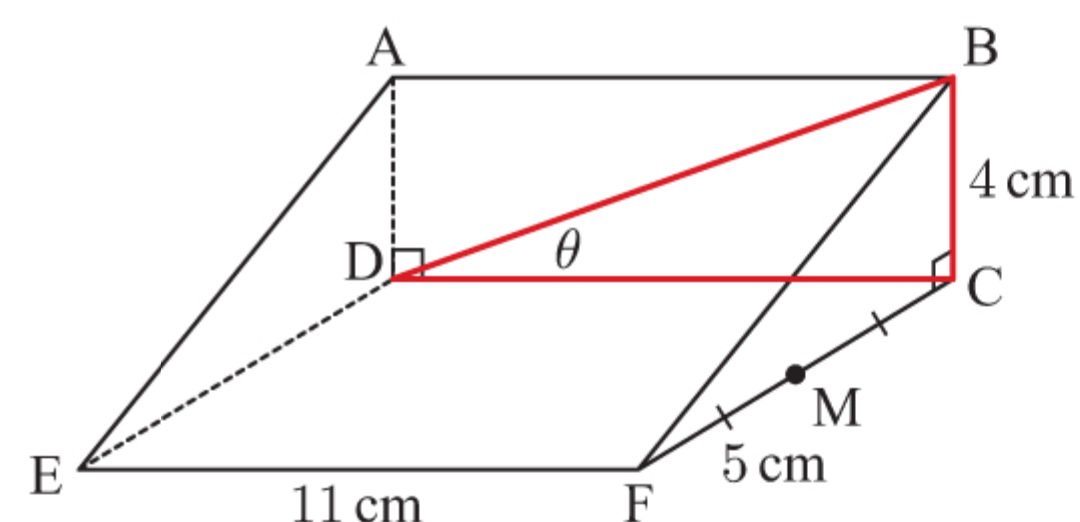
$$\begin{aligned}
 \tan \theta &= \frac{4}{10} \\
 \therefore \theta &= \tan^{-1}\left(\frac{4}{10}\right) \approx 21.8^\circ \\
 \text{The angle is about } 21.8^\circ.
 \end{aligned}$$



- b** The projection of [BD] onto the base plane is [CD].

$\therefore$  the required angle is  $\widehat{BDC}$ .

$$\begin{aligned}
 \tan \theta &= \frac{4}{11} \\
 \therefore \theta &= \tan^{-1}\left(\frac{4}{11}\right) \approx 20.0^\circ \\
 \text{The angle is about } 20.0^\circ.
 \end{aligned}$$

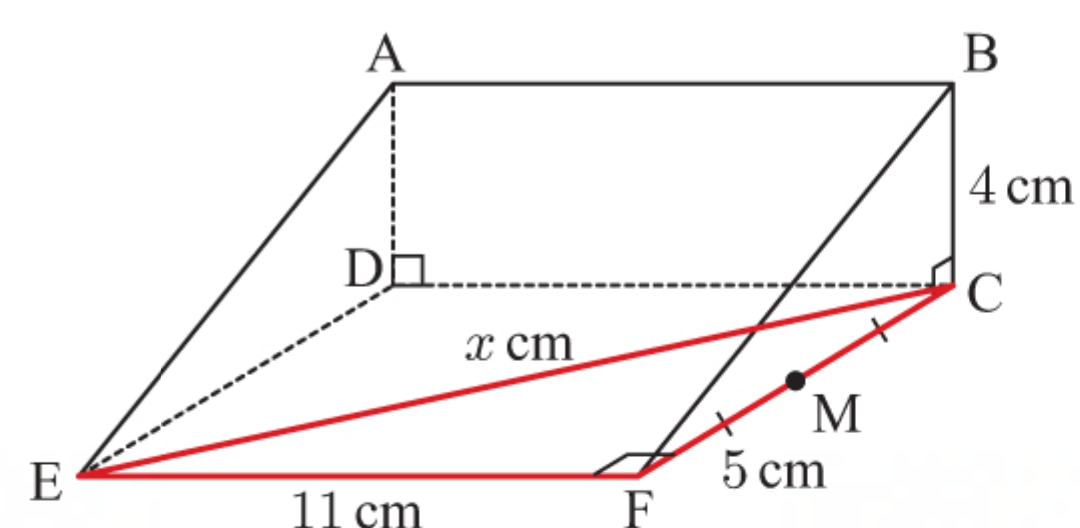


- c** The projection of [BE] onto the base plane is [CE].

$\therefore$  the required angle is  $\widehat{BEC}$ .

Let CE be  $x \text{ cm}$ .

$$\begin{aligned}
 \text{Using Pythagoras in } \triangle ECF, \quad x^2 &= 10^2 + 11^2 \\
 \therefore x^2 &= 221 \\
 \therefore x &= \sqrt{221} \quad \{x > 0\}
 \end{aligned}$$





Let  $\widehat{BEC}$  be  $\alpha$ .

$$\therefore \tan \alpha = \frac{4}{\sqrt{221}}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{4}{\sqrt{221}}\right) \approx 15.1^\circ$$

The angle is about  $15.1^\circ$ .

- d** The projection of  $[AM]$  onto the base plane is  $[DM]$ .

$\therefore$  the required angle is  $\widehat{AMD}$ .

Let  $DM$  be  $x$  cm.

Using Pythagoras in  $\triangle DCM$ ,  $x^2 = 11^2 + 5^2$

$$\therefore x^2 = 146$$

$$\therefore x = \sqrt{146} \quad \{x > 0\}$$

Let  $\widehat{AMD}$  be  $\alpha$ .

$$\therefore \tan \alpha = \frac{4}{\sqrt{146}}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{4}{\sqrt{146}}\right) \approx 18.3^\circ$$

The angle is about  $18.3^\circ$ .

- 25 a** The projection of  $[CD]$  onto the base plane is  $[CO]$ .

$\therefore$  the required angle is  $\widehat{OCD}$ .

$$\tan \theta = \frac{6}{7}$$

$$\therefore \theta = \tan^{-1}\left(\frac{6}{7}\right) \approx 40.6^\circ$$

The angle is about  $40.6^\circ$ .

- b** The projection of  $[OF]$  onto the base plane is  $[OB]$ .

$\therefore$  the required angle is  $\widehat{BOF}$ .

Now  $FB = 6$  units

$$\text{and } OB = \sqrt{(5-0)^2 + (7-0)^2 + (0-0)^2}$$

$$= \sqrt{25 + 49}$$

$$= \sqrt{74} \text{ units}$$

$$\therefore \tan \theta = \frac{6}{\sqrt{74}}$$

$$\therefore \theta = \tan^{-1}\left(\frac{6}{\sqrt{74}}\right) \approx 34.9^\circ$$

The angle is about  $34.9^\circ$ .

- c** The projection of  $[AG]$  onto the base plane is  $[AC]$ .

$\therefore$  the required angle is  $\widehat{GAC}$ .

Now  $GC = 6$  units

$$\text{and } AC = \sqrt{(0-5)^2 + (7-0)^2 + (0-0)^2}$$

$$= \sqrt{25 + 49}$$

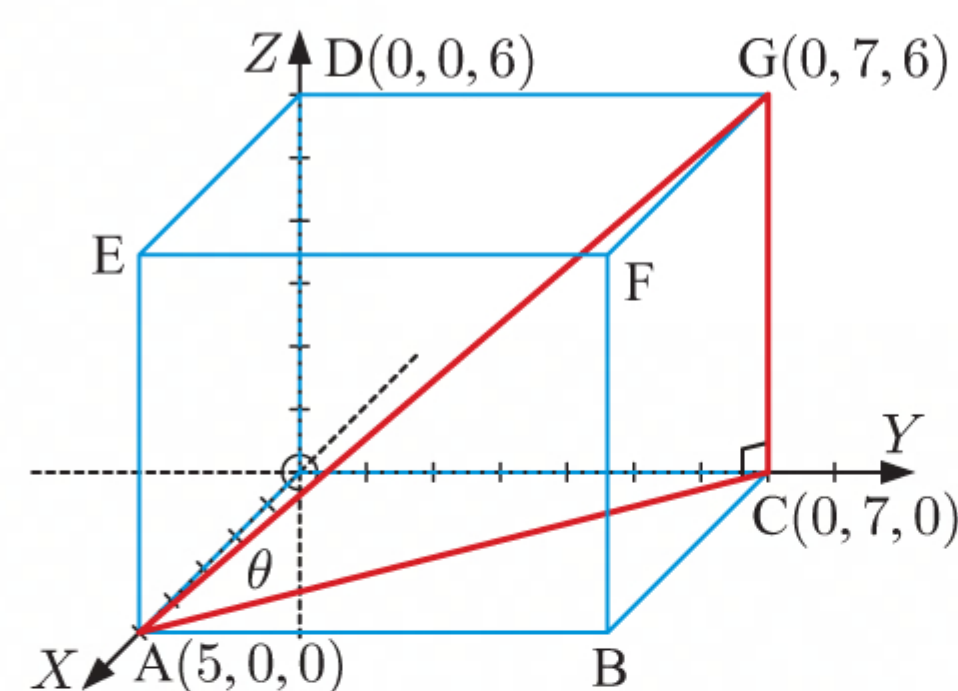
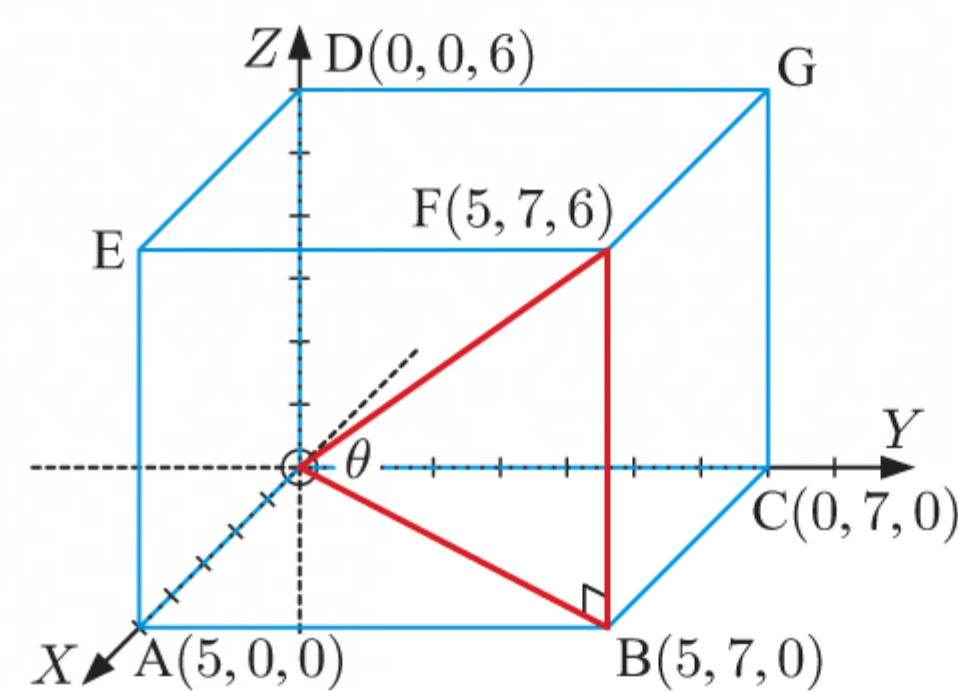
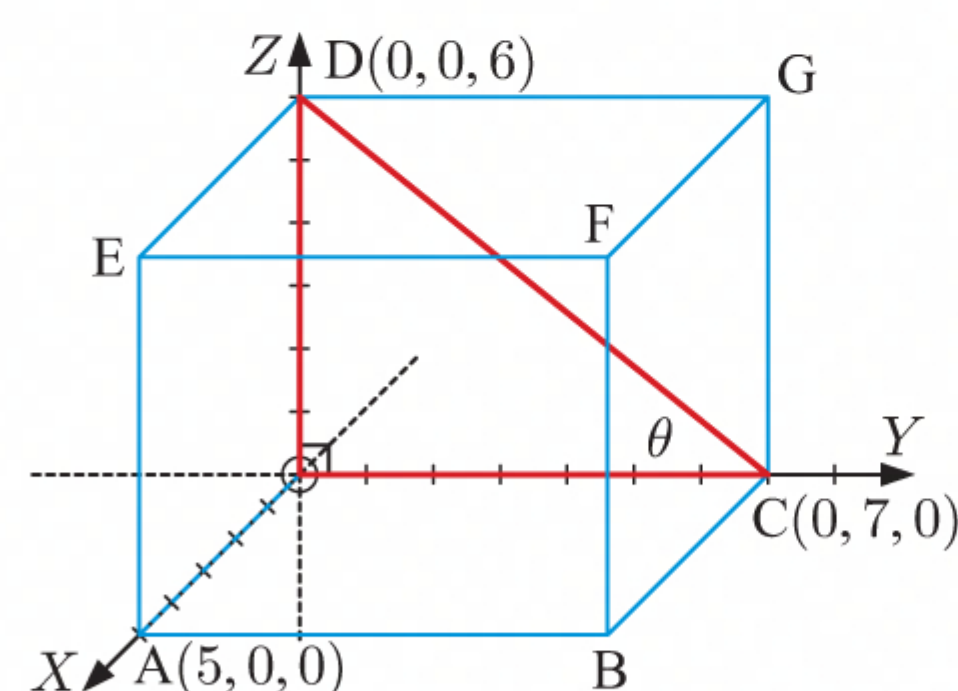
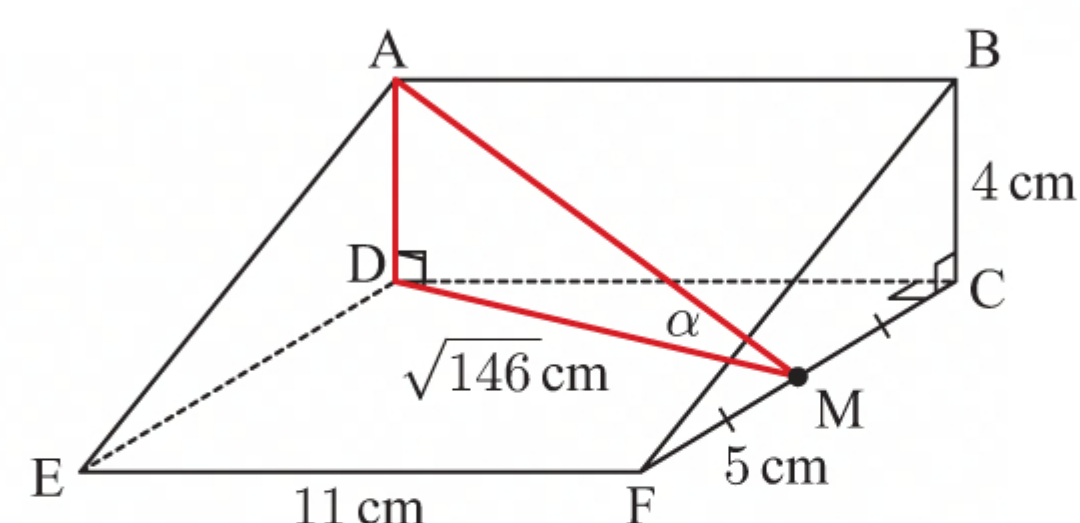
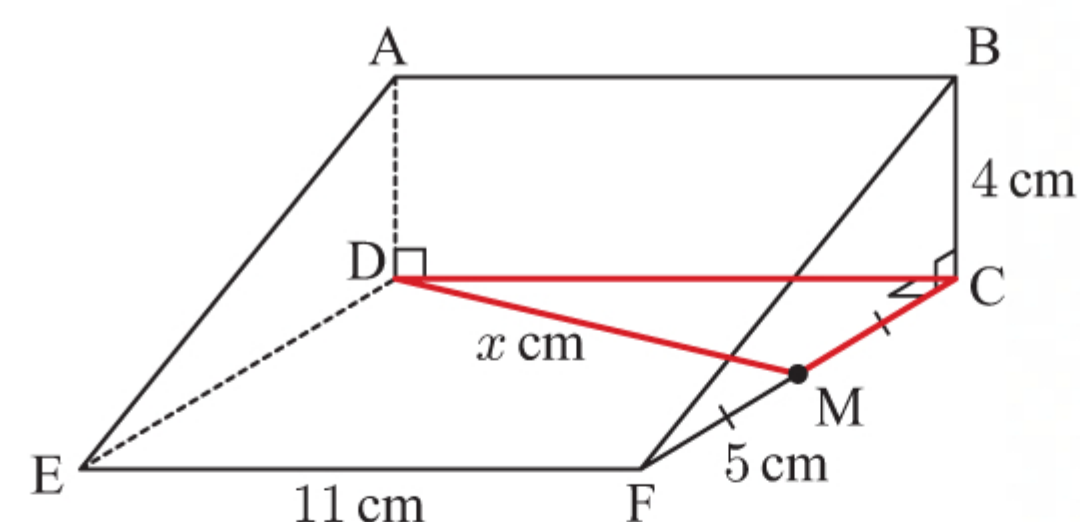
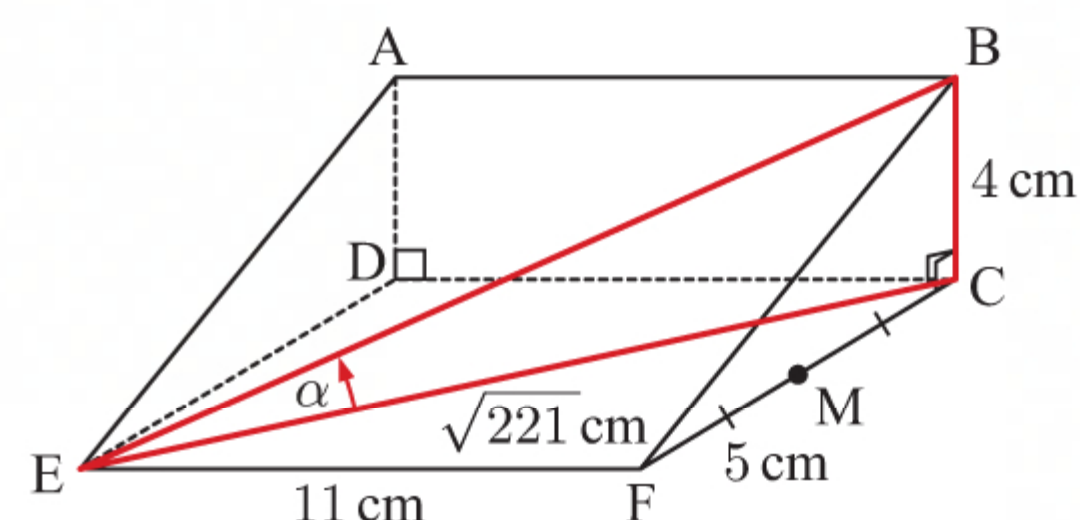
$$= \sqrt{74} \text{ units}$$

$$\therefore \tan \theta = \frac{6}{\sqrt{74}}$$

$$\therefore \theta = \tan^{-1}\left(\frac{6}{\sqrt{74}}\right) \approx 34.9^\circ$$

The angle is about  $34.9^\circ$ .

- 26 a**  $M$  is the midpoint of  $[AD]$  which is  $\left(\frac{0+(-10)}{2}, \frac{4+4}{2}, \frac{6+6}{2}\right)$  or  $(-5, 4, 6)$ .





**b** Let  $\widehat{CMD}$  be  $\theta$ .

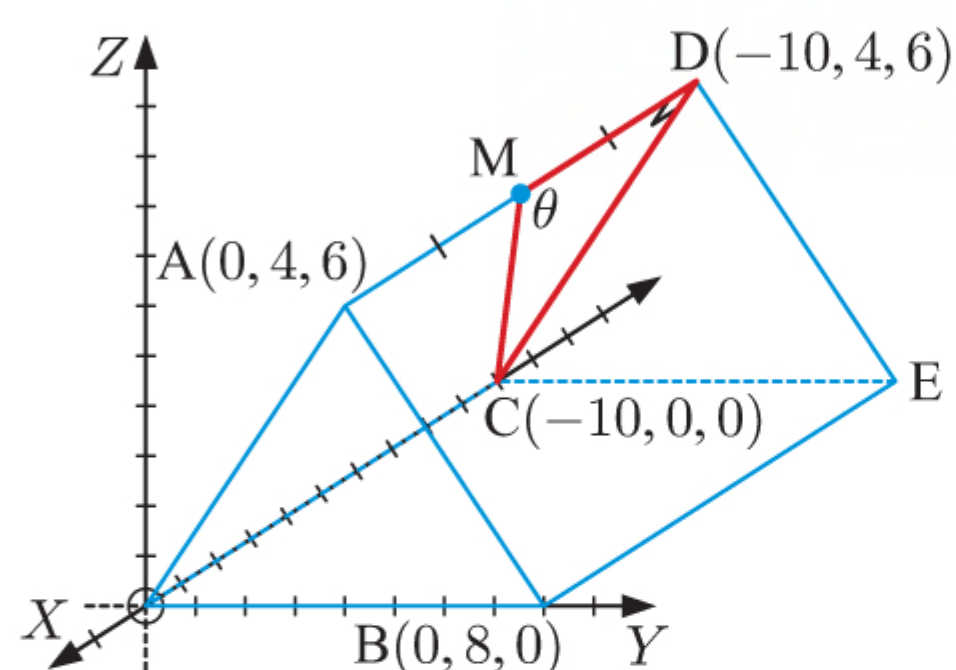
Now  $DM = 5$  units

$$\begin{aligned}\text{and } CD &= \sqrt{(-10 - (-10))^2 + (4 - 0)^2 + (6 - 0)^2} \\ &= \sqrt{0^2 + 4^2 + 6^2} \\ &= \sqrt{52} \text{ units}\end{aligned}$$

$$\therefore \tan \theta = \frac{\sqrt{52}}{5}$$

$$\therefore \theta = \tan^{-1}\left(\frac{\sqrt{52}}{5}\right) \approx 55.3^\circ$$

$$\therefore \widehat{CMD} \approx 55.3^\circ$$



**c i** The required angle is  $\widehat{DON}$ , where N has coordinates  $(-10, 4, 0)$ .

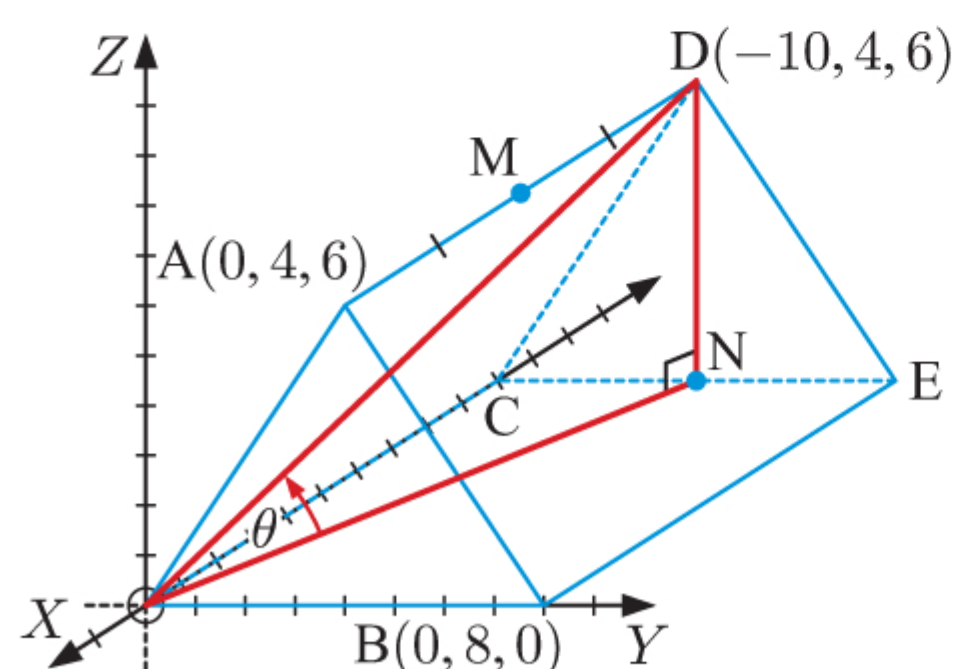
Now  $DN = 6$  units

$$\begin{aligned}\text{and } NO &= \sqrt{(-10 - 0)^2 + (4 - 0)^2 + (0 - 0)^2} \\ &= \sqrt{(-10)^2 + 4^2 + 0^2} \\ &= \sqrt{116} \text{ units}\end{aligned}$$

$$\therefore \tan \theta = \frac{6}{\sqrt{116}}$$

$$\therefore \theta = \tan^{-1}\left(\frac{6}{\sqrt{116}}\right) \approx 29.1^\circ$$

The angle is about  $29.1^\circ$ .



**ii** The required angle is  $\widehat{MEP}$ , where P has coordinates  $(-5, 4, 0)$ .

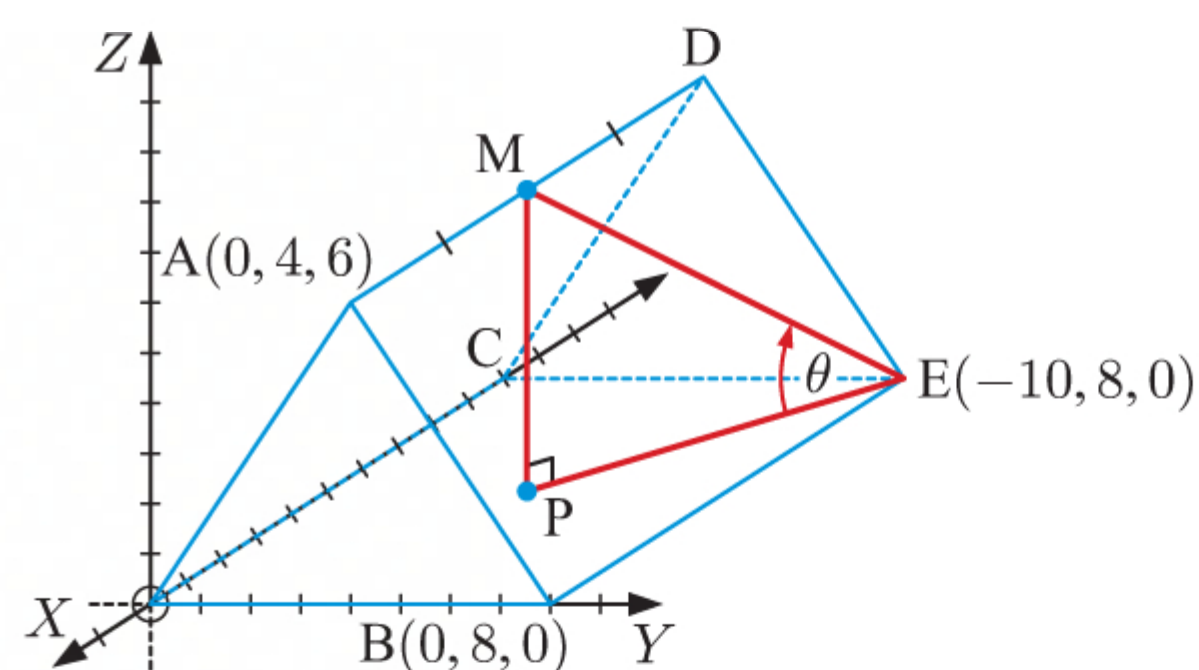
Now  $MP = 6$  units

$$\begin{aligned}\text{and } PE &= \sqrt{(-10 - (-5))^2 + (8 - 4)^2 + (0 - 0)^2} \\ &= \sqrt{(-5)^2 + 4^2 + 0^2} \\ &= \sqrt{41} \text{ units}\end{aligned}$$

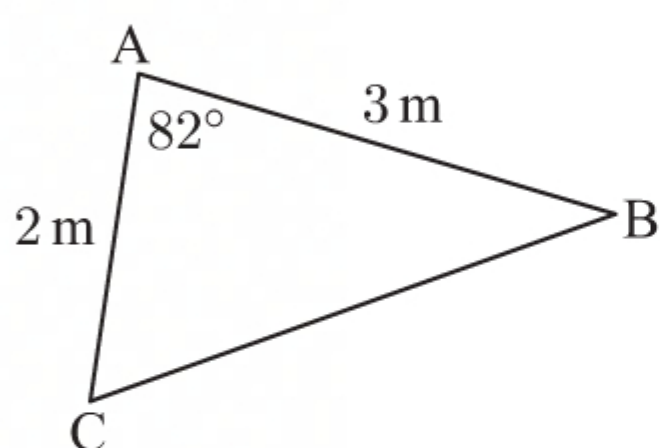
$$\therefore \tan \theta = \frac{6}{\sqrt{41}}$$

$$\therefore \theta = \tan^{-1}\left(\frac{6}{\sqrt{41}}\right) \approx 43.1^\circ$$

The angle is about  $43.1^\circ$ .

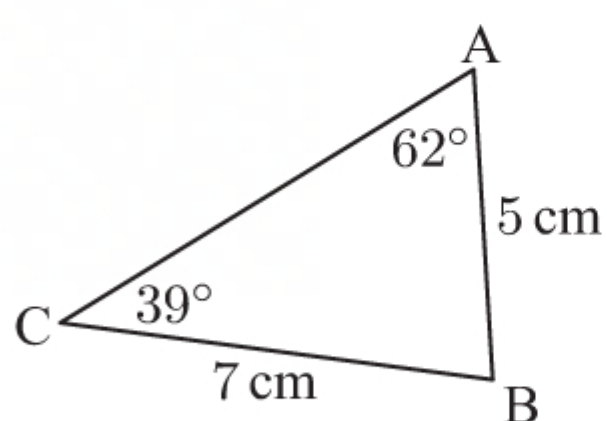


**27 a**



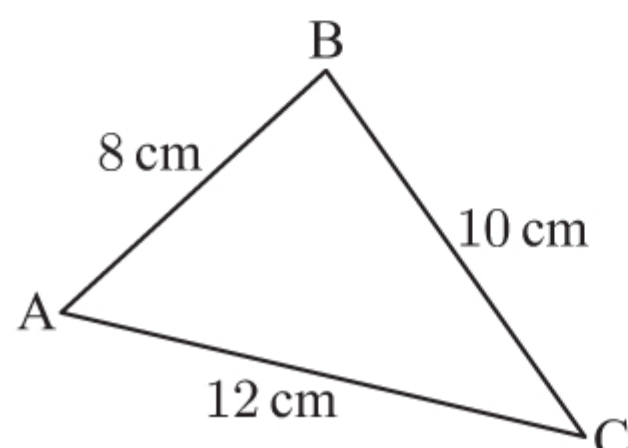
$$\begin{aligned}\text{Area} &= \frac{1}{2}bc \sin A \\ &= \frac{1}{2} \times 2 \times 3 \times \sin 82^\circ \\ &\approx 2.97 \text{ m}^2\end{aligned}$$

**b**



$$\begin{aligned}\widehat{ABC} &= 180^\circ - 62^\circ - 39^\circ \quad \{\text{angles in a triangle}\} \\ &= 79^\circ \\ \therefore \text{area} &= \frac{1}{2}ac \sin B \\ &= \frac{1}{2} \times 7 \times 5 \times \sin 79^\circ \\ &\approx 17.2 \text{ cm}^2\end{aligned}$$

**28 a**



**b** The smallest angle in triangle ABC is opposite the shortest side.

$\therefore \widehat{BCA}$  is the smallest angle.

By the cosine rule:

$$\cos \widehat{BCA} = \frac{12^2 + 10^2 - 8^2}{2 \times 12 \times 10}$$

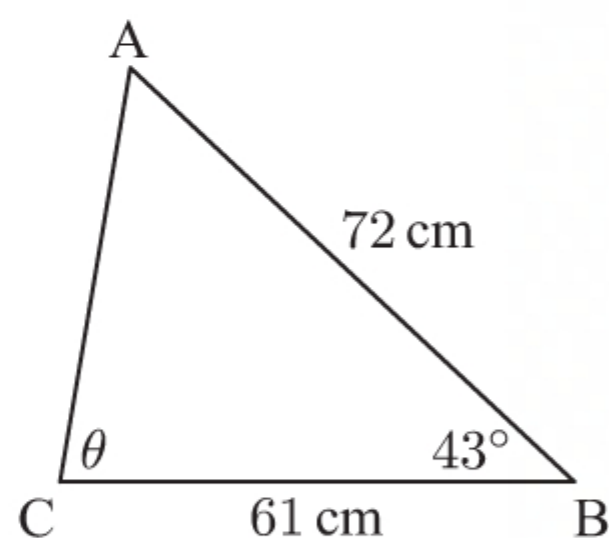
$$\therefore \cos \widehat{BCA} = \frac{180}{240} = \frac{3}{4}$$

$$\therefore \widehat{BCA} = \cos^{-1}\left(\frac{3}{4}\right) \approx 41.4^\circ$$



$$\begin{aligned} \text{c Area} &= \frac{1}{2}ab \sin C \\ &\approx \frac{1}{2} \times 10 \times 12 \times \sin 41.4^\circ \\ &\approx 39.7 \text{ cm}^2 \end{aligned}$$

29 a



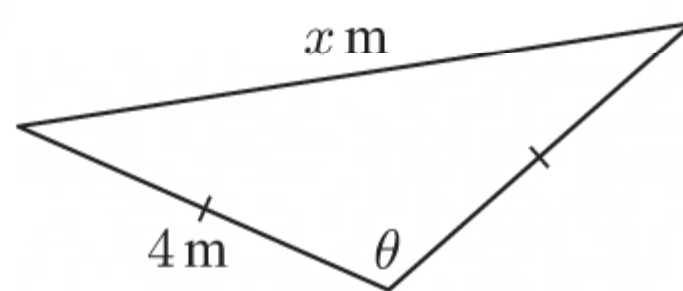
$$\begin{aligned} \text{Using the cosine rule, } AC^2 &= 61^2 + 72^2 - 2 \times 61 \times 72 \times \cos 43^\circ \\ &\approx 2480.8 \\ \therefore AC &\approx 49.808 \\ &\approx 49.8 \text{ cm} \quad \{\text{as } AC > 0\} \end{aligned}$$

b Let  $\hat{ACB} = \theta$

$$\begin{aligned} \text{Using the sine rule, } \frac{\sin \theta}{72} &= \frac{\sin 43^\circ}{AC} \\ \therefore \sin \theta &\approx \frac{72 \sin 43^\circ}{49.808} \\ &\approx 0.9859 \\ \therefore \theta &\approx 80.4^\circ \\ \text{and so } \hat{ACB} &\approx 80.4^\circ \end{aligned}$$

30 a

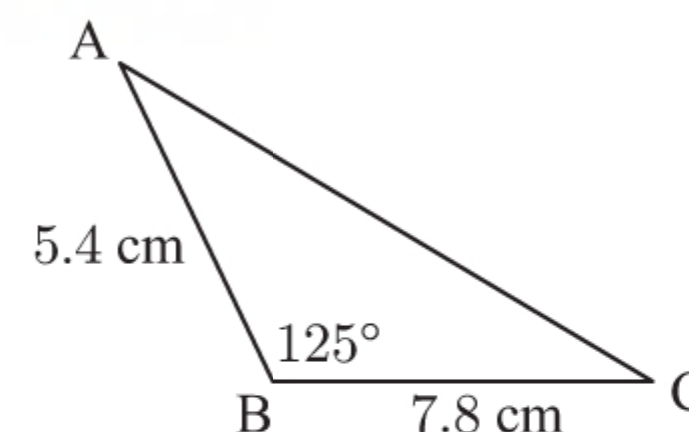
$$\begin{aligned} \text{Area} &= \frac{1}{2}ab \sin \theta \\ &= \frac{1}{2} \times 4 \times 4 \times \sin \theta \\ \therefore 4 &= 8 \sin \theta \\ \therefore \sin \theta &= \frac{1}{2} \\ \therefore \theta &= 150^\circ \quad \{\text{since } \theta \text{ is obtuse}\} \end{aligned}$$



$$\begin{aligned} \text{b Using the cosine rule, } x^2 &= 4^2 + 4^2 - 2 \times 4 \times 4 \times \cos 150^\circ \\ \therefore x &= \sqrt{4^2 + 4^2 - 2 \times 4 \times 4 \times \cos 150^\circ} \\ \therefore x &\approx 7.73 \end{aligned}$$

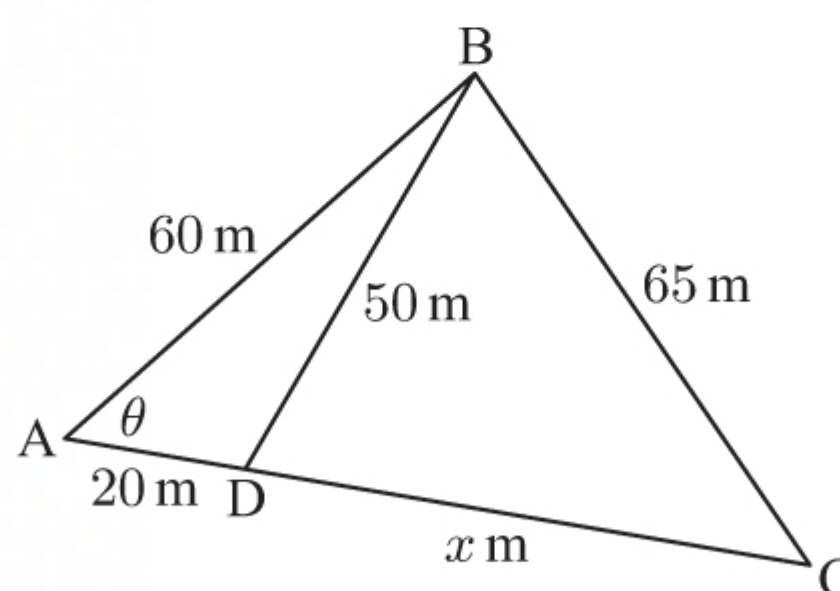
$$\begin{aligned} \text{31 a Area} &= \frac{1}{2} \times 5.4 \times 7.8 \times \sin 125^\circ \\ &\approx 17.3 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{b Using the cosine rule, } AC^2 &= 5.4^2 + 7.8^2 - 2 \times 5.4 \times 7.8 \times \cos 125^\circ \\ \therefore AC &= \sqrt{5.4^2 + 7.8^2 - 2 \times 5.4 \times 7.8 \times \cos 125^\circ} \\ \therefore AC &\approx 11.8 \text{ cm} \end{aligned}$$



32 a By the cosine rule in  $\triangle BAD$ :

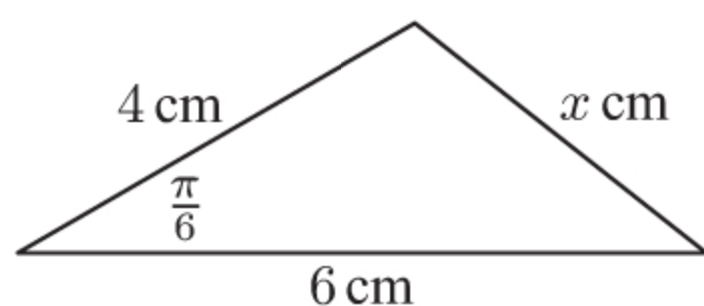
$$\begin{aligned} \cos \theta &= \frac{60^2 + 20^2 - 50^2}{2 \times 60 \times 20} \\ \therefore \cos \theta &= \frac{1500}{2400} \\ \therefore \cos \theta &= \frac{5}{8} \end{aligned}$$



b By the cosine rule in  $\triangle ABC$ :

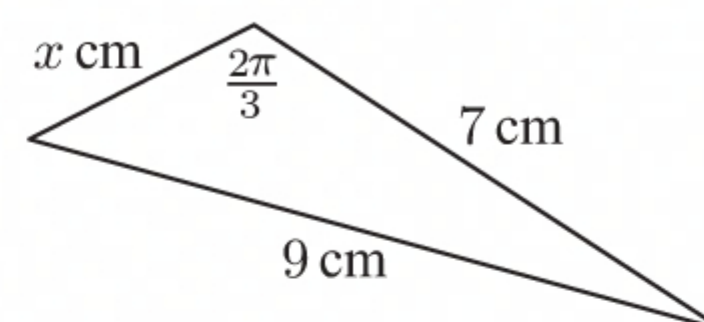
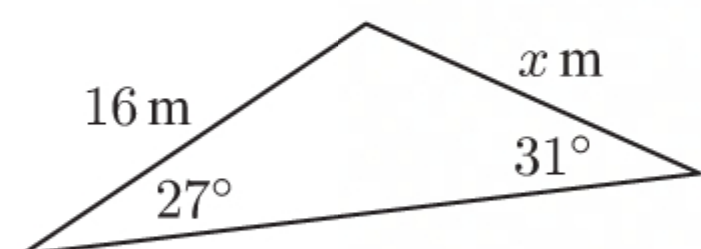
$$\begin{aligned} 65^2 &= 60^2 + (20 + x)^2 - 2(60)(20 + x) \cos \theta \\ \therefore 65^2 &= 60^2 + (20 + x)^2 - 2(60)(20 + x) \left(\frac{5}{8}\right) \quad \{\text{using a}\} \\ \therefore 4225 &= 3600 + 400 + 40x + x^2 - 1500 - 75x \\ \therefore x^2 - 35x - 1725 &= 0 \\ \therefore x &= \frac{35 \pm \sqrt{(-35)^2 - 4(1)(-1725)}}{2} \\ &= \frac{35 \pm \sqrt{8125}}{2} \\ &= \frac{35 + 25\sqrt{13}}{2} \quad \{x > 0\} \end{aligned}$$



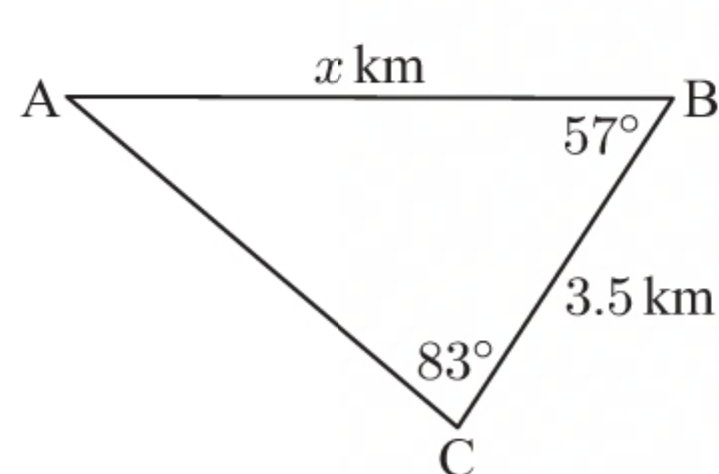
**33 a**

By the cosine rule:  $x^2 = 4^2 + 6^2 - 2(4)(6) \cos \frac{\pi}{6}$   
 $\therefore x^2 = 16 + 36 - 48\left(\frac{\sqrt{3}}{2}\right)$   
 $\therefore x^2 = 52 - 24\sqrt{3}$   
 $\therefore x = \sqrt{52 - 24\sqrt{3}} \quad \{x > 0\}$   
 $\therefore x \approx 3.23$

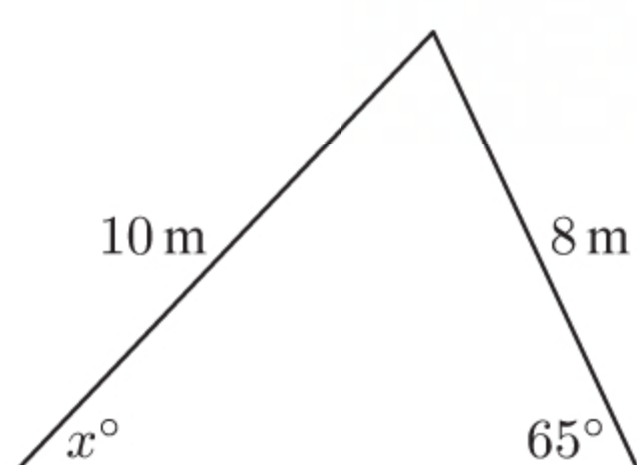
**b** By the cosine rule:  $9^2 = x^2 + 7^2 - 2(x)(7) \cos \frac{2\pi}{3}$   
 $\therefore 81 = x^2 + 49 - 14\left(-\frac{1}{2}\right)x$   
 $\therefore 81 = x^2 + 49 + 7x$   
 $\therefore x^2 + 7x - 32 = 0$   
 $\therefore x = \frac{-7 \pm \sqrt{7^2 - 4(1)(-32)}}{2}$   
 $= \frac{-7 \pm \sqrt{177}}{2}$   
 $= \frac{-7 + \sqrt{177}}{2} \quad \{x > 0\}$

**34 a**

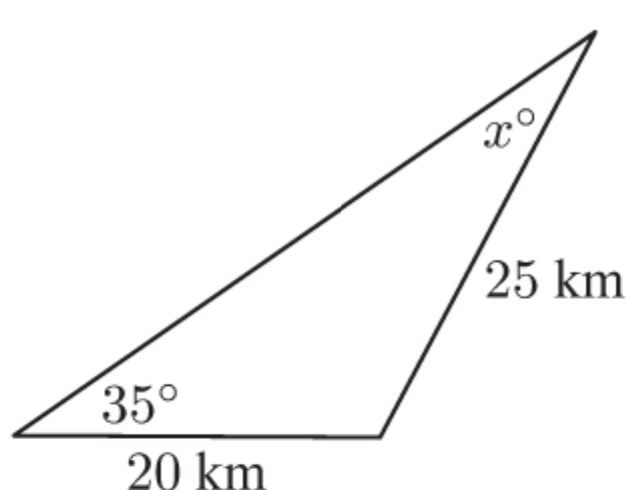
By the sine rule:  $\frac{x}{\sin 27^\circ} = \frac{16}{\sin 31^\circ}$   
 $\therefore x = \frac{16 \sin 27^\circ}{\sin 31^\circ}$   
 $\therefore x \approx 14.1$

**b**

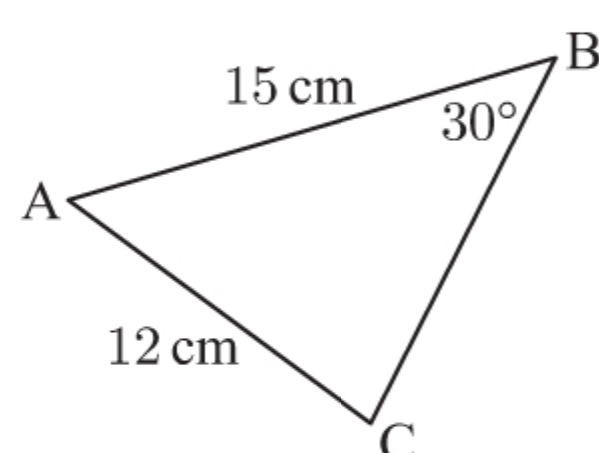
$\widehat{BAC} = 180^\circ - 57^\circ - 83^\circ \quad \{\text{angles in a triangle}\}$   
 $= 40^\circ$   
 Now by the sine rule:  $\frac{x}{\sin 83^\circ} = \frac{3.5}{\sin 40^\circ}$   
 $\therefore x = \frac{3.5 \sin 83^\circ}{\sin 40^\circ}$   
 $\therefore x \approx 5.40$

**35 a**

By the sine rule:  $\frac{\sin x^\circ}{8} = \frac{\sin 65^\circ}{10}$   
 $\therefore \sin x^\circ = \frac{8 \sin 65^\circ}{10}$   
 $\therefore x = \sin^{-1}\left(\frac{8 \sin 65^\circ}{10}\right)$   
 $\therefore x \approx 46.5$

**b**

By the sine rule:  $\frac{\sin x^\circ}{20} = \frac{\sin 35^\circ}{25}$   
 $\therefore \sin x^\circ = \frac{20 \sin 35^\circ}{25}$   
 $\therefore x = \sin^{-1}\left(\frac{20 \sin 35^\circ}{25}\right)$   
 $\therefore x \approx 27.3$

**36 a**

$\frac{\sin \widehat{ACB}}{15} = \frac{\sin 30^\circ}{12} \quad \{\text{sine rule}\}$   
 $\therefore \sin \widehat{ACB} = \frac{15 \sin 30^\circ}{12}$   
 $\therefore \widehat{ACB} = \sin^{-1}\left(\frac{15 \sin 30^\circ}{12}\right) \quad \text{or its supplement}$   
 $\therefore \widehat{ACB} \approx 38.7^\circ \text{ or } 180^\circ - 38.7^\circ$   
 $\therefore \widehat{ACB} \approx 38.7^\circ \text{ or } 141.3^\circ$

**b**  $\widehat{BAC}$  is acute if  $\widehat{ACB}$  is obtuse.

$\therefore \widehat{BAC} \approx 180^\circ - 30^\circ - 141.3^\circ \quad \{\text{using a}\}$   
 $\approx 8.7^\circ$



**37** By the cosine rule in  $\triangle ABD$ :

$$\cos \theta = \frac{8^2 + 4^2 - 5^2}{2 \times 8 \times 4}$$

$$\therefore \cos \theta = \frac{55}{64}$$

$$\therefore \theta = \cos^{-1}\left(\frac{55}{64}\right) \approx 30.8^\circ$$

By the cosine rule in  $\triangle DBC$ :

$$\cos \phi = \frac{8^2 + 6^2 - 6^2}{2 \times 8 \times 6}$$

$$\therefore \cos \phi = \frac{64}{96} = \frac{2}{3}$$

$$\therefore \phi = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.2^\circ$$

$$\text{Now } \widehat{ABC} = \theta + \phi$$

$$\approx 30.8^\circ + 48.2^\circ$$

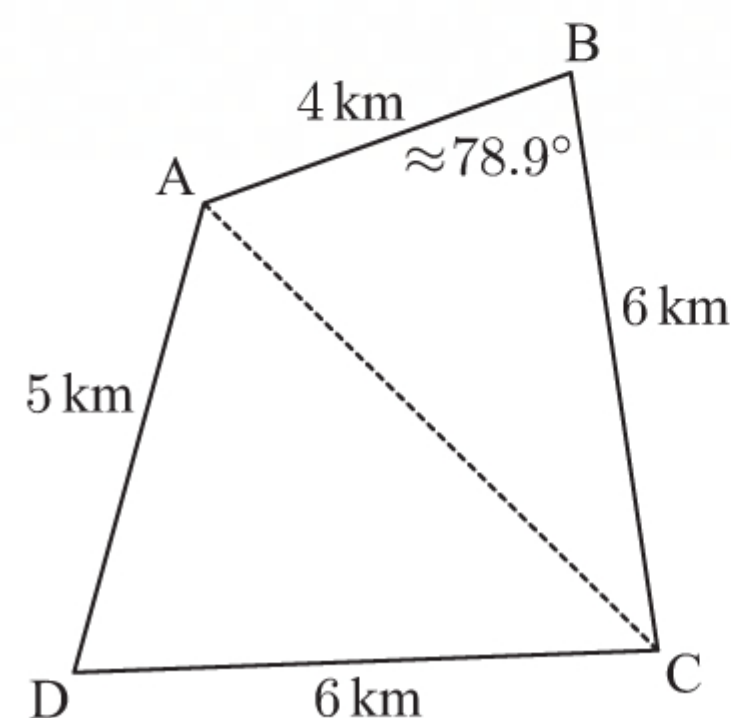
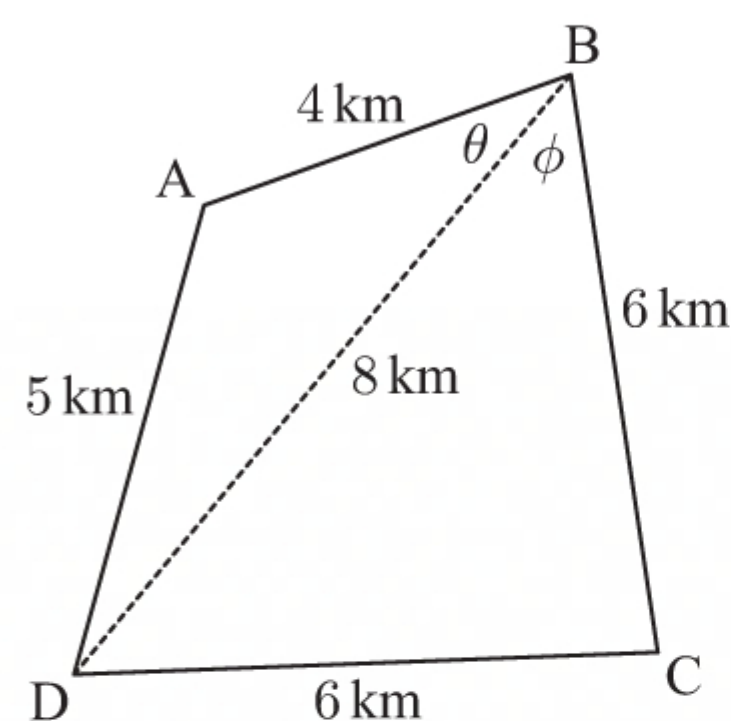
$$\approx 78.9^\circ$$

By the cosine rule in  $\triangle ABC$ :

$$AC^2 \approx 4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 78.9^\circ$$

$$\therefore AC \approx \sqrt{4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 78.9^\circ}$$

$$\therefore AC \approx 6.54 \text{ cm}$$

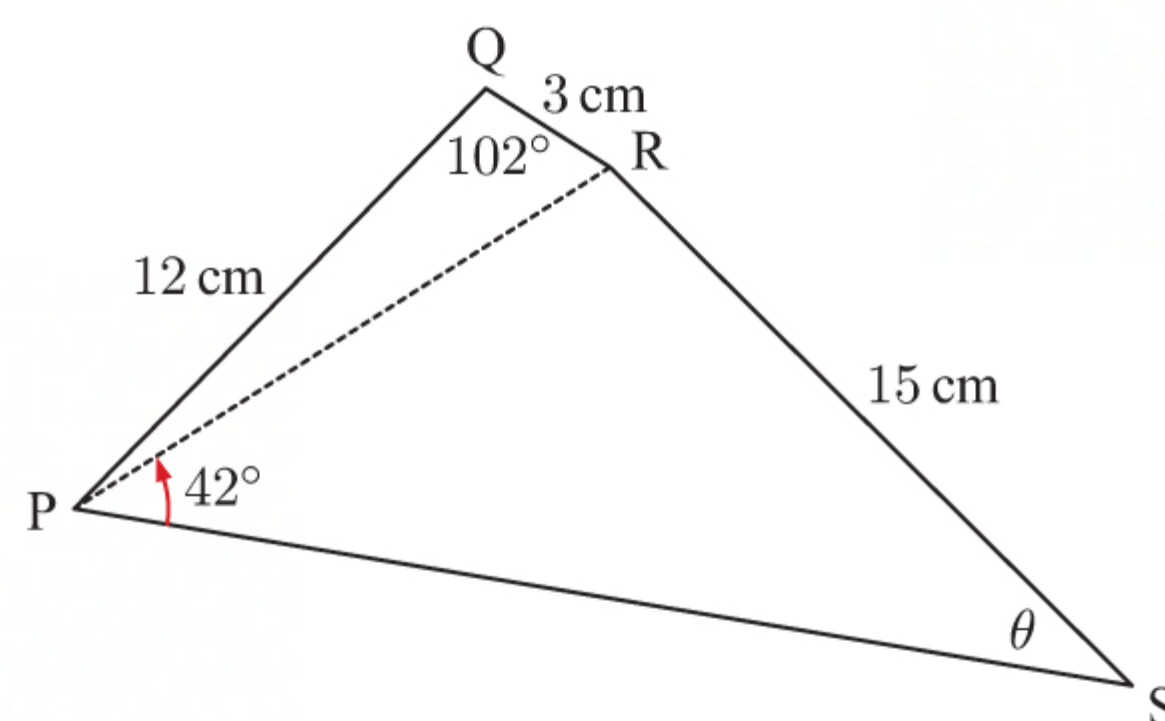


**38 a** By the cosine rule in  $\triangle PQR$ :

$$PR^2 = 3^2 + 12^2 - 2(3)(12) \cos 102^\circ$$

$$\therefore PR = \sqrt{3^2 + 12^2 - 2(3)(12) \cos 102^\circ}$$

$$\therefore PR \approx 12.96 \text{ cm} \approx 13.0 \text{ cm}$$

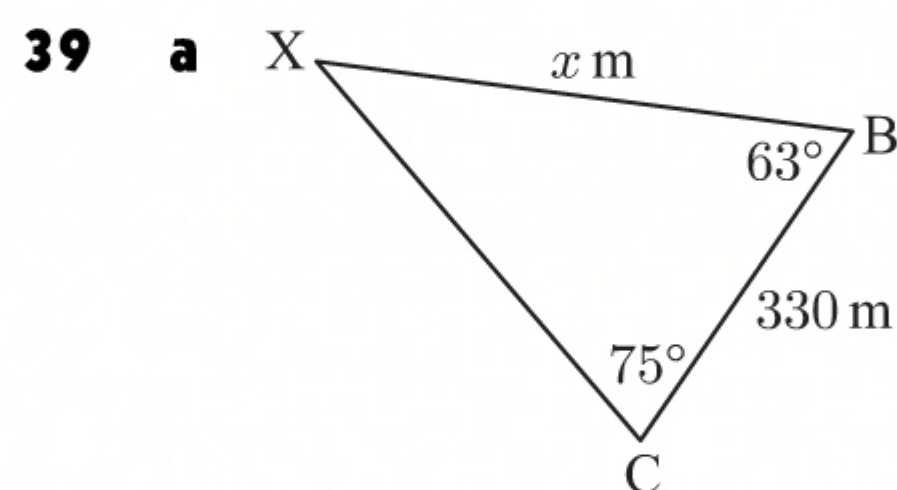


**b** By the sine rule in  $\triangle PRS$ :  $\frac{\sin \theta}{PR} = \frac{\sin 42^\circ}{15}$

$$\therefore \sin \theta = \frac{PR \sin 42^\circ}{15}$$

$$\therefore \sin \theta \approx \frac{12.96 \sin 42^\circ}{15} \quad \{\text{from a}\}$$

$$\therefore \theta \approx \sin^{-1}\left(\frac{12.96 \sin 42^\circ}{15}\right) \approx 35.3^\circ$$



**b** Let XB be  $x$  m.

$$\widehat{CXB} = 180^\circ - 63^\circ - 75^\circ \quad \{\text{angles in a triangle}\}$$

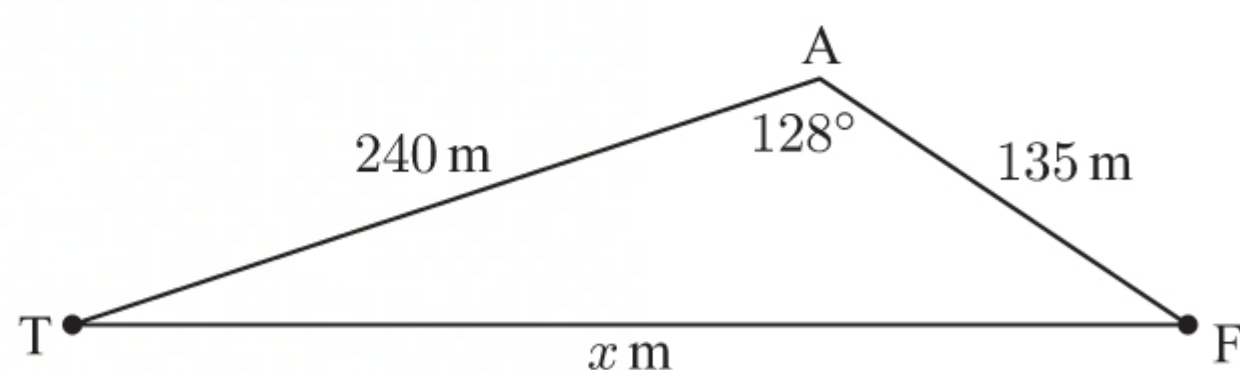
$$= 42^\circ$$

$$\therefore \frac{x}{\sin 75^\circ} = \frac{330}{\sin 42^\circ} \quad \{\text{sine rule}\}$$

$$\therefore x = \frac{330 \sin 75^\circ}{\sin 42^\circ} \approx 476$$

The distance between the monument and B is about 476 m.

**40**



**a** Let the distance from T to F be  $x$  m.

Using the cosine rule,

$$x^2 = 240^2 + 135^2 - 2 \times 240 \times 135 \cos 128^\circ$$

$$\therefore x \approx 340.18$$

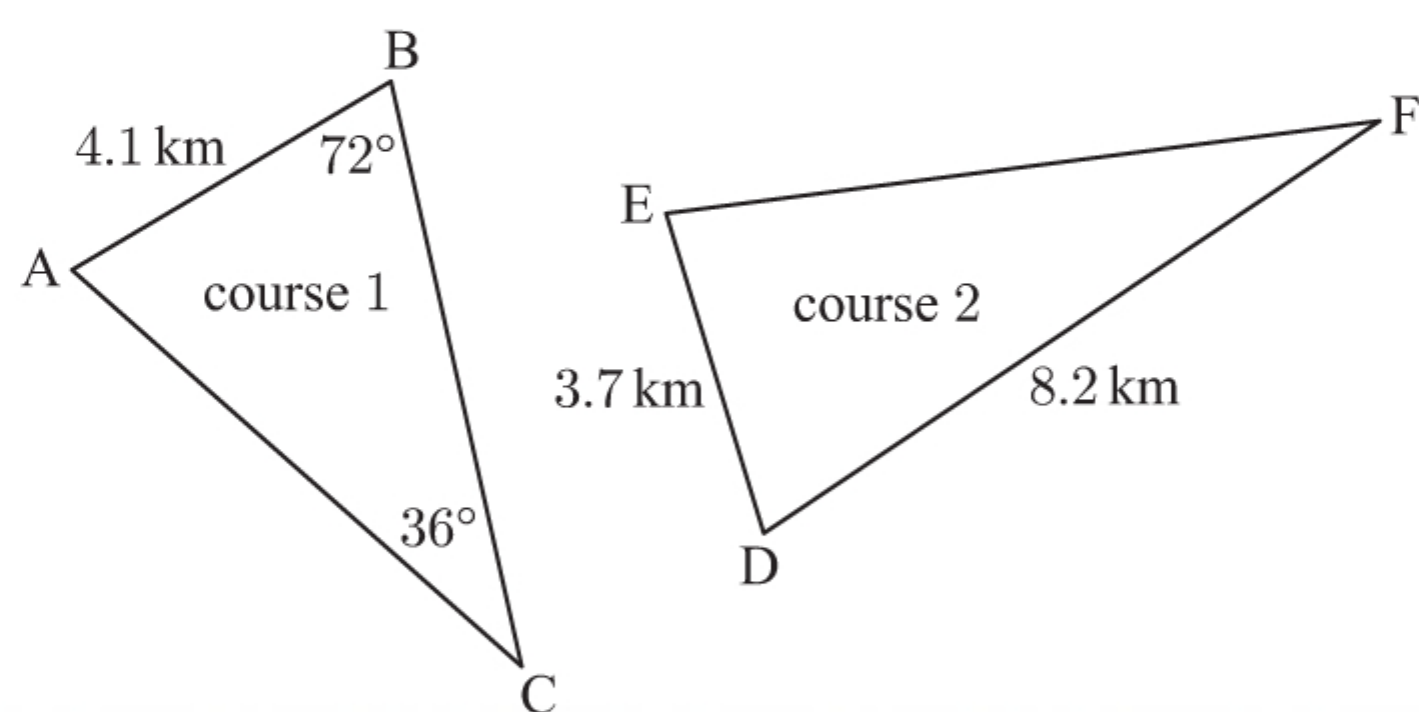
So, the distance is about 340 m.

**b** Using the sine rule,  $\frac{\sin \widehat{ATF}}{135} = \frac{\sin 128^\circ}{340.18}$

$$\therefore \widehat{ATF} \approx \sin^{-1}\left(\frac{135 \sin 128^\circ}{340.18}\right) \approx 18.2^\circ$$



41



**a** By the sine rule in  $\triangle ABC$ :

$$\frac{AC}{\sin 72^\circ} = \frac{4.1}{\sin 36^\circ}$$

$$\therefore AC = \frac{4.1 \sin 72^\circ}{\sin 36^\circ}$$

Now  $[EF]$  is 20% longer than  $[AC]$ .

$$\begin{aligned}\therefore EF &= 1.2 \times AC \\ &= 1.2 \times \frac{4.1 \sin 72^\circ}{\sin 36^\circ} \\ &\approx 7.96 \text{ km}\end{aligned}$$

**c** Area of course 2  $\approx \frac{1}{2} \times 3.7 \times 7.96 \times \sin 80.4^\circ$   
 $\approx 14.5 \text{ km}^2$

**b** By the cosine rule in  $\triangle DEF$ :

$$\cos \widehat{DEF} \approx \frac{3.7^2 + 7.96^2 - 8.2^2}{2 \times 3.7 \times 7.96}$$

$$\therefore \widehat{DEF} \approx \cos^{-1} \left( \frac{3.7^2 + 7.96^2 - 8.2^2}{2 \times 3.7 \times 7.96} \right)$$

$$\therefore \widehat{DEF} \approx 80.4^\circ$$

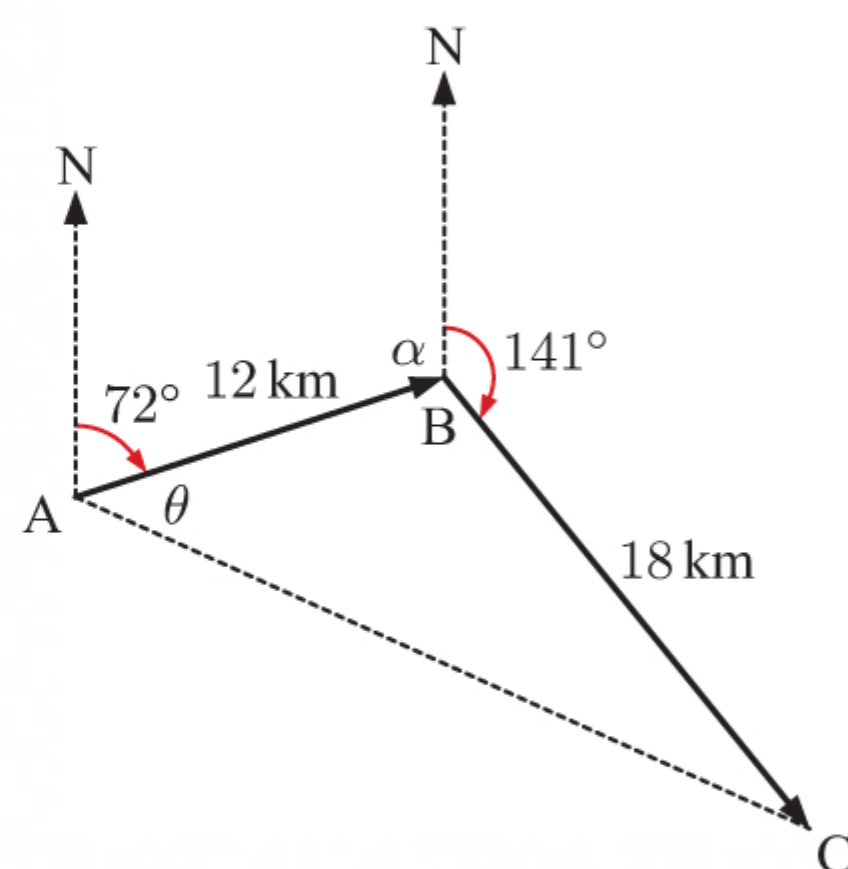
**d** From **a**,  $AC = \frac{4.1 \sin 72^\circ}{\sin 36^\circ} \approx 6.63 \text{ km}$

$$\begin{aligned}\widehat{BAC} &= 180^\circ - 72^\circ - 36^\circ \quad \{\text{angles in a triangle}\} \\ &= 72^\circ \\ &= \widehat{ABC}\end{aligned}$$

$\therefore \triangle ABC$  is an isosceles triangle.

$$\therefore BC = AC$$

$$\begin{aligned}\therefore \text{total length of course 1} &\approx 2 \times 6.63 + 4.1 \\ &\approx 17.4 \text{ km}\end{aligned}$$

42 **a**

**b**  $\alpha = 180^\circ - 72^\circ \quad \{\text{co-interior angles}\}$   
 $= 108^\circ$

$$\begin{aligned}\therefore \widehat{ABC} &= 360^\circ - 141^\circ - 108^\circ \quad \{\text{angles at a point}\} \\ &= 111^\circ\end{aligned}$$

Using the cosine rule in  $\triangle ABC$ :  $AC^2 = 12^2 + 18^2 - 2 \times 12 \times 18 \times \cos 111^\circ$

$$\begin{aligned}\therefore AC &= \sqrt{12^2 + 18^2 - 2 \times 12 \times 18 \times \cos 111^\circ} \quad \{AC > 0\} \\ \therefore AC &\approx 24.9563 \text{ km}\end{aligned}$$

By the cosine rule:  $\cos \theta \approx \frac{12^2 + 24.9563^2 - 18^2}{2 \times 12 \times 24.9563}$

$$\begin{aligned}\therefore \theta &\approx \cos^{-1} \left( \frac{12^2 + 24.9563^2 - 18^2}{2 \times 12 \times 24.9563} \right) \\ &\approx 42.3^\circ\end{aligned}$$

So, the boat is about 25.0 km from its starting position, on the bearing  $72^\circ + 42.3^\circ \approx 114.3^\circ$ .



**43**  $a = \sin 20^\circ$ ,  $b = \tan 50^\circ$

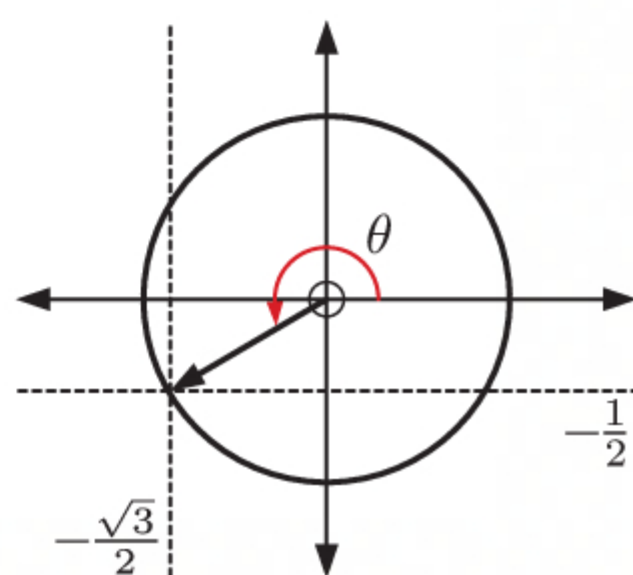
**a**  $\sin 160^\circ = \sin(180^\circ - 20^\circ)$   
 $= \sin 20^\circ$   
 $= a$

**b**  $\tan(-50^\circ) = \frac{\sin(-50^\circ)}{\cos(-50^\circ)}$   
 $= \frac{-\sin 50^\circ}{\cos 50^\circ}$   
 $= -\tan 50^\circ$   
 $= -b$

**c**  $\cos 70^\circ = \cos(90^\circ - 20^\circ)$   
 $= \sin 20^\circ$   
 $= a$

**d**  $\tan 20^\circ = \frac{\sin 20^\circ}{\cos 20^\circ}$   
 $= \frac{\sin 20^\circ}{\sqrt{1 - \sin^2 20^\circ}} \quad \{0^\circ < 20^\circ < 90^\circ, \text{ so } \cos 20^\circ > 0\}$   
 $= \frac{a}{\sqrt{1 - a^2}}$

**44 a**

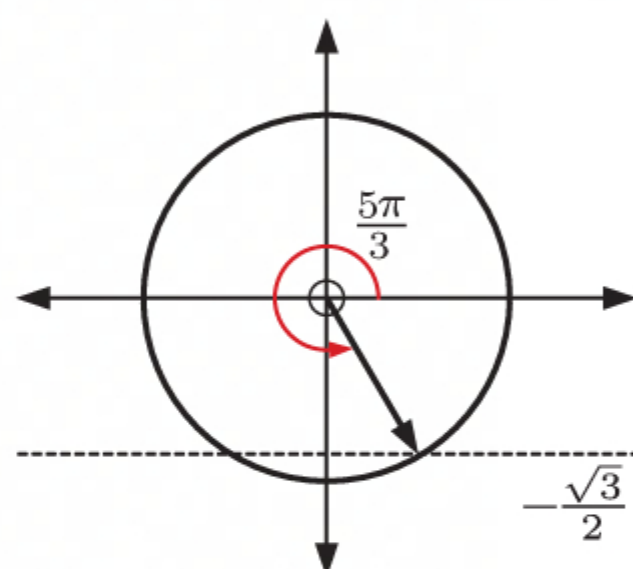


$\sin \theta = -\frac{1}{2}$  and  $\cos \theta = -\frac{\sqrt{3}}{2}$   
 $\therefore \theta = 210^\circ$

**b**  $\tan \theta = \frac{\sin \theta}{\cos \theta}$   
 $= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$   
 $= \frac{1}{\sqrt{3}}$

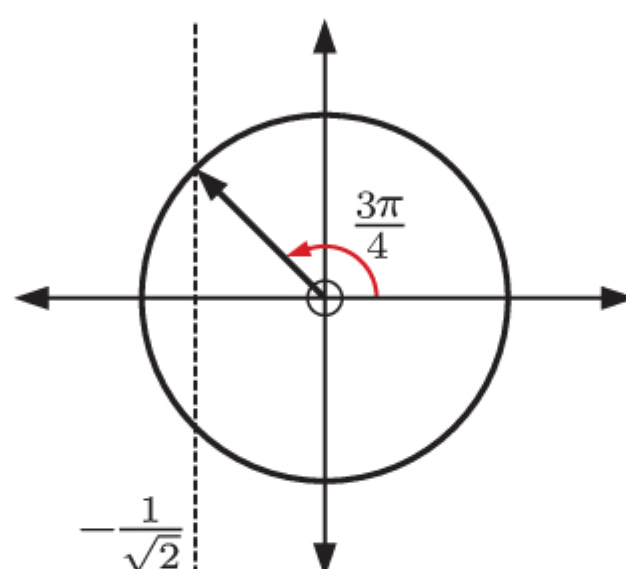
**c**  $\tan 2\theta = \tan 420^\circ$   
 $= \frac{\sin 420^\circ}{\cos 420^\circ}$   
 $= \frac{\sin 60^\circ}{\cos 60^\circ}$   
 $= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$   
 $= \sqrt{3}$

**45 a**



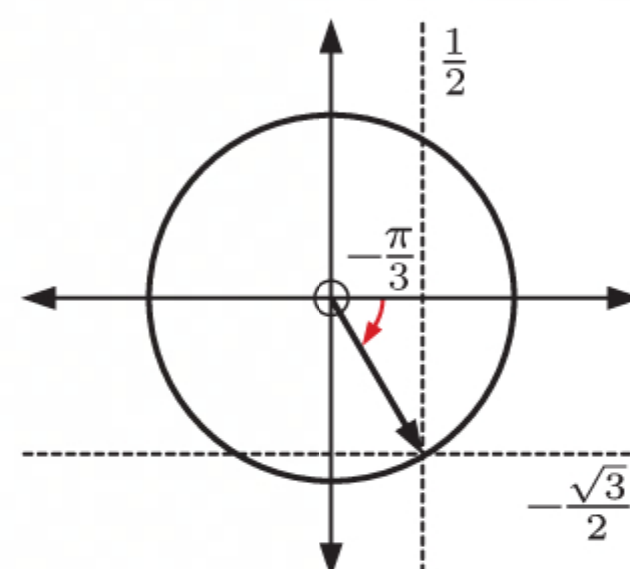
$\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$

**b**



$\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

**c**



$\tan\left(-\frac{\pi}{3}\right) = \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)}$   
 $= \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$   
 $= -\sqrt{3}$

**46 a**

$\sin \frac{\pi}{3} \cos \frac{\pi}{4}$   
 $= \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right)$   
 $= \frac{\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$   
 $= \frac{\sqrt{6}}{4}$

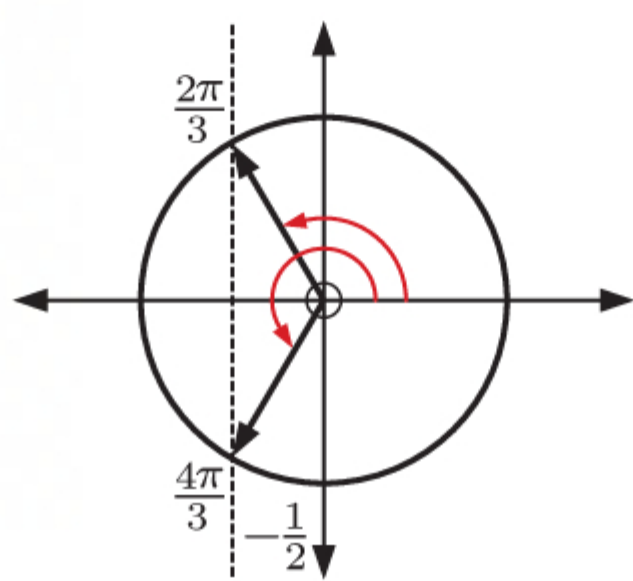
**b**  $2 \tan^2\left(\frac{2\pi}{3}\right) + 1$   
 $= 2 \left(\frac{\sin \frac{2\pi}{3}}{\cos \frac{2\pi}{3}}\right)^2 + 1$   
 $= 2 \left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)^2 + 1$   
 $= 2(-\sqrt{3})^2 + 1$   
 $= 2(3) + 1$   
 $= 7$

**c**

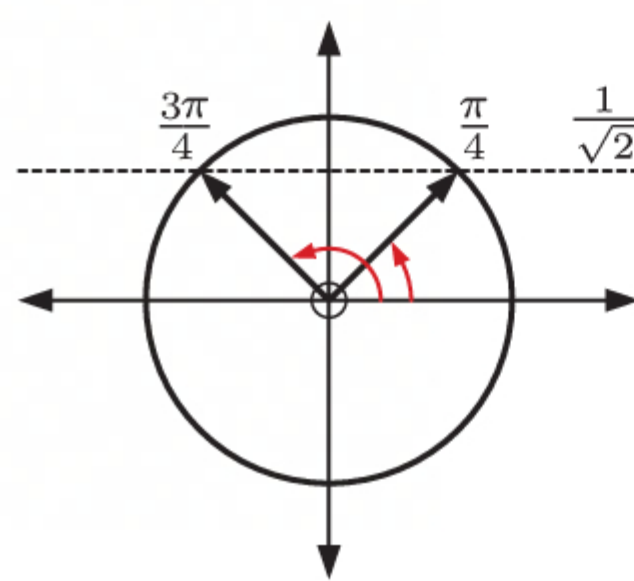
$\frac{\cos \frac{5\pi}{6} \tan^2\left(\frac{3\pi}{4}\right)}{\sin\left(-\frac{\pi}{3}\right)}$   
 $= \frac{\cos \frac{5\pi}{6} \left(\frac{\sin \frac{3\pi}{4}}{\cos \frac{3\pi}{4}}\right)^2}{\sin\left(-\frac{\pi}{3}\right)}$   
 $= \frac{\left(-\frac{\sqrt{3}}{2}\right) \left(\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right)^2}{\left(-\frac{\sqrt{3}}{2}\right)}$   
 $= (-1)^2$   
 $= 1$



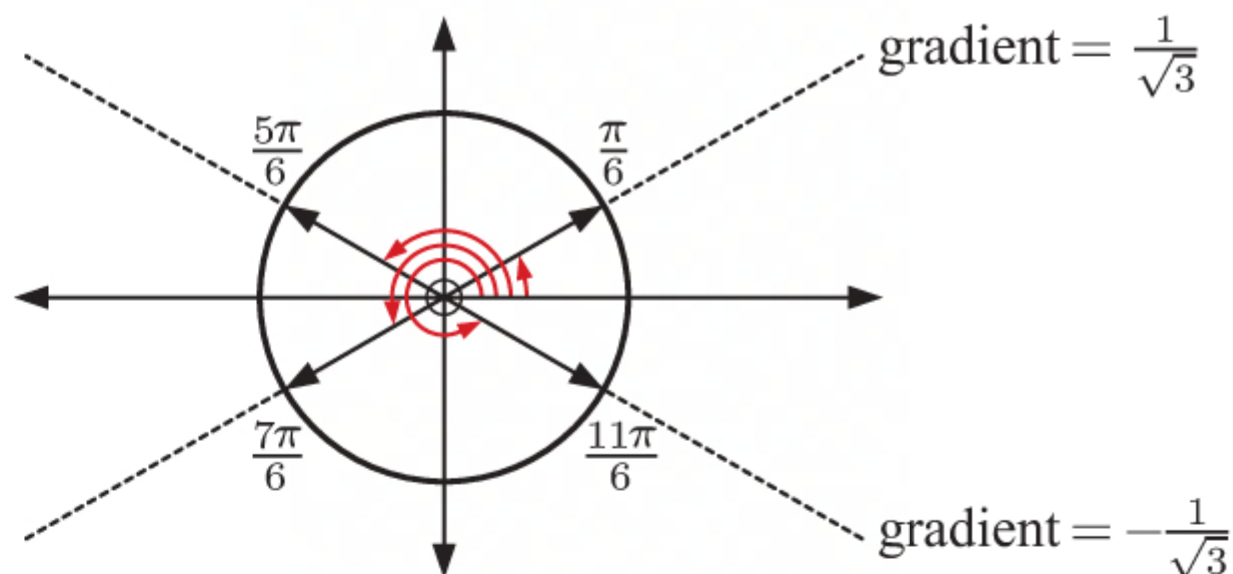
**47 a**  $\cos \theta = -\frac{1}{2}$   
 $\therefore \theta = \frac{2\pi}{3}$  or  $\frac{4\pi}{3}$



**b**  $\sin \theta = \frac{1}{\sqrt{2}}$   
 $\therefore \theta = \frac{\pi}{4}$  or  $\frac{3\pi}{4}$



**c**  $\tan^2 \theta = \frac{1}{3}$   
 $\therefore \tan \theta = \pm \frac{1}{\sqrt{3}}$   
 $\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}$



**48 a**  $\sin \theta = \frac{4}{5}$

Now  $\cos^2 \theta + \sin^2 \theta = 1$

$\therefore \cos^2 \theta + \left(\frac{4}{5}\right)^2 = 1$

$\therefore \cos^2 \theta + \frac{16}{25} = 1$

$\therefore \cos^2 \theta = \frac{9}{25}$

$\therefore \cos \theta = \pm \frac{3}{5}$

**b**  $\cos \theta = -\frac{2}{7}$

Now  $\cos^2 \theta + \sin^2 \theta = 1$

$\therefore \left(-\frac{2}{7}\right)^2 + \sin^2 \theta = 1$

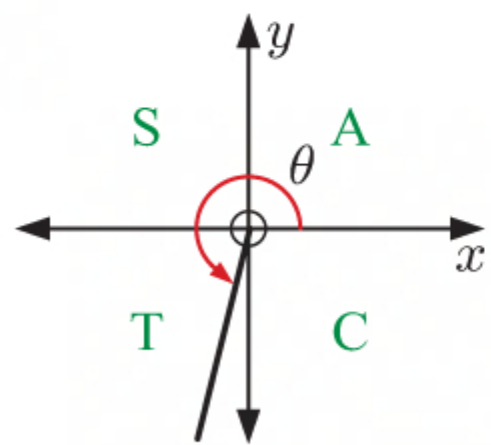
$\therefore \frac{4}{49} + \sin^2 \theta = 1$

$\therefore \sin^2 \theta = \frac{45}{49}$

$\therefore \sin \theta = \pm \frac{\sqrt{45}}{7}$

**49 a**  $\cos \theta = -\frac{1}{4}$  and  $\pi < \theta < \frac{3\pi}{2}$

$\theta$  is in quadrant 3, so  $\sin \theta$  is negative.



Now  $\cos^2 \theta + \sin^2 \theta = 1$

$\therefore \left(-\frac{1}{4}\right)^2 + \sin^2 \theta = 1$

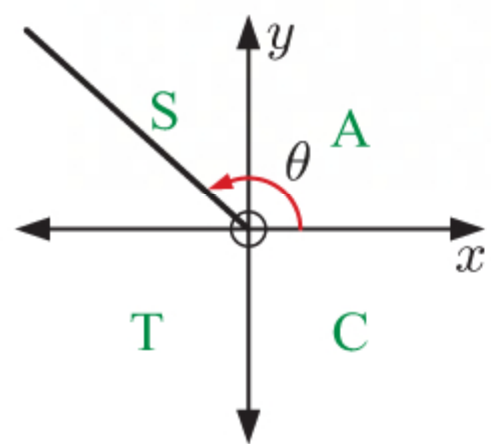
$\therefore \frac{1}{16} + \sin^2 \theta = 1$

$\therefore \sin^2 \theta = \frac{15}{16}$

$\therefore \sin \theta = \frac{-\sqrt{15}}{4} \quad \{\sin \theta < 0\}$

**b**  $\sin \theta = \frac{2}{3}$  and  $\frac{\pi}{2} < \theta < \pi$

$\theta$  is in quadrant 2, so  $\cos \theta$  is negative.



Now  $\cos^2 \theta + \sin^2 \theta = 1$

$\therefore \cos^2 \theta + \left(\frac{2}{3}\right)^2 = 1$

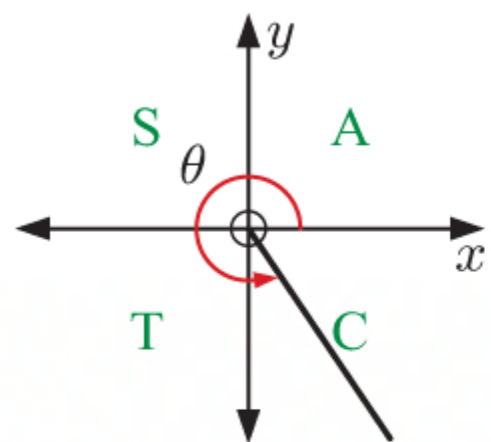
$\therefore \cos^2 \theta + \frac{4}{9} = 1$

$\therefore \cos^2 \theta = \frac{5}{9}$

$\therefore \cos \theta = \frac{-\sqrt{5}}{3} \quad \{\cos \theta < 0\}$

**c**  $\sin \theta = -\frac{5}{6}$  and  $\frac{3\pi}{2} < \theta < 2\pi$

$\theta$  is in quadrant 4, so  $\cos \theta$  is positive.



Now  $\cos^2 \theta + \sin^2 \theta = 1$

$\therefore \cos^2 \theta + \left(-\frac{5}{6}\right)^2 = 1$

$\therefore \cos^2 \theta + \frac{25}{36} = 1$

$\therefore \cos^2 \theta = \frac{11}{36}$

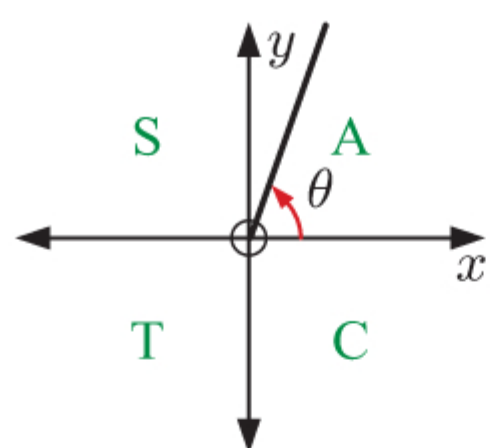
$\therefore \cos \theta = \frac{\sqrt{11}}{6} \quad \{\cos \theta > 0\}$

So,  $\tan \theta = \frac{\sin \theta}{\cos \theta}$   
 $= \frac{-\frac{5}{6}}{\frac{\sqrt{11}}{6}}$   
 $= -\frac{5}{\sqrt{11}}$



**d**  $\cos \theta = \frac{1}{3}$  and  $0 < \theta < \frac{\pi}{2}$

$\theta$  is in quadrant 1, so  $\sin \theta$  is positive.



Now  $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \left(\frac{1}{3}\right)^2 + \sin^2 \theta = 1$$

$$\therefore \frac{1}{9} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{8}{9}$$

$$\therefore \sin \theta = \frac{\sqrt{8}}{3} \quad \{\sin \theta > 0\}$$

So,  $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$$= \frac{\frac{\sqrt{8}}{3}}{\frac{1}{3}}$$

$$= \sqrt{8}$$

**50 a**  $\tan \theta = -\frac{1}{3}$  and  $\frac{\pi}{2} < \theta < \pi$

$\theta$  is in quadrant 2, so  $\sin \theta > 0$  and  $\cos \theta < 0$ .

Now  $\tan \theta = -\frac{1}{3}$

$$\therefore \frac{\sin \theta}{\cos \theta} = -\frac{1}{3}$$

$$\therefore \cos \theta = -3 \sin \theta \quad \dots (*)$$

So,  $\cos^2 \theta + \sin^2 \theta = 1$

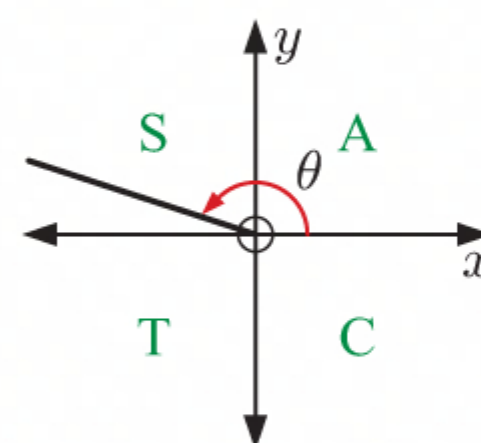
$$\therefore 9 \sin^2 \theta + \sin^2 \theta = 1 \quad \{\text{using } (*)\}$$

$$\therefore 10 \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{1}{10}$$

$$\therefore \sin \theta = \frac{1}{\sqrt{10}} \quad \{\sin \theta > 0\}$$

$$\therefore \cos \theta = -\frac{3}{\sqrt{10}}$$



**b**  $\tan \theta = \frac{1}{\sqrt{2}}$  and  $\pi < \theta < \frac{3\pi}{2}$

$\theta$  is in quadrant 3, so  $\sin \theta < 0$  and  $\cos \theta < 0$ .

Now  $\tan \theta = \frac{1}{\sqrt{2}}$

$$\therefore \frac{\sin \theta}{\cos \theta} = \frac{1}{\sqrt{2}}$$

$$\therefore \cos \theta = \sqrt{2} \sin \theta \quad \dots (*)$$

So,  $\cos^2 \theta + \sin^2 \theta = 1$

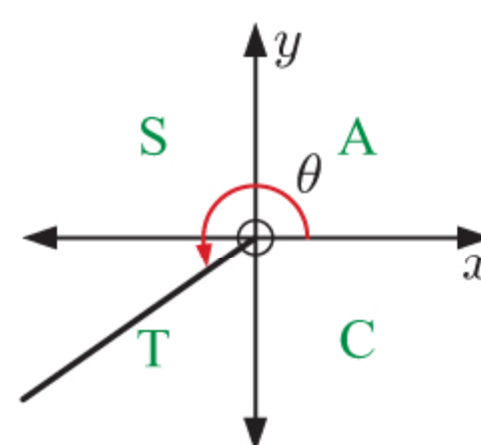
$$\therefore 2 \sin^2 \theta + \sin^2 \theta = 1 \quad \{\text{using } (*)\}$$

$$\therefore 3 \sin^2 \theta = 1$$

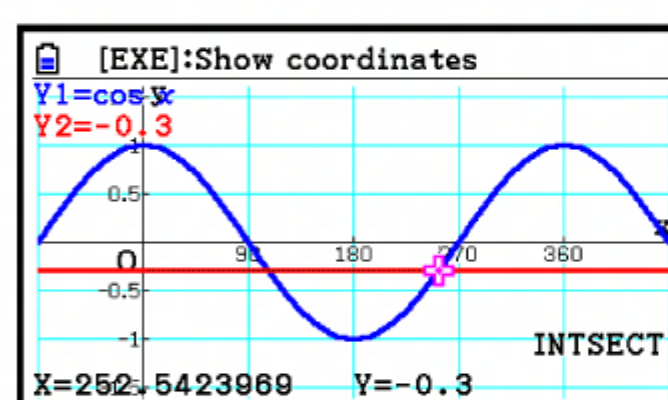
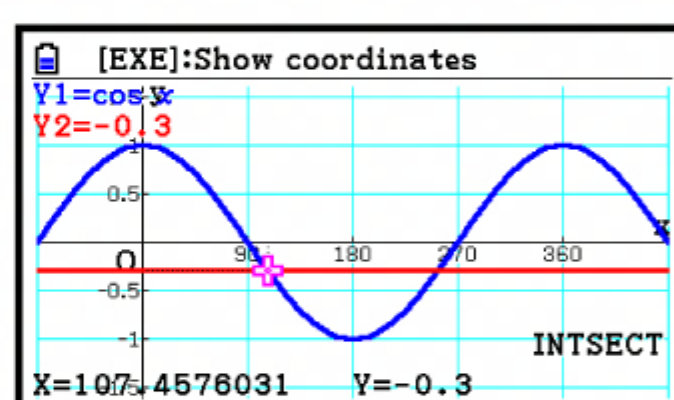
$$\therefore \sin^2 \theta = \frac{1}{3}$$

$$\therefore \sin \theta = -\frac{1}{\sqrt{3}} \quad \{\sin \theta < 0\}$$

$$\therefore \cos \theta = -\sqrt{\frac{2}{3}}$$



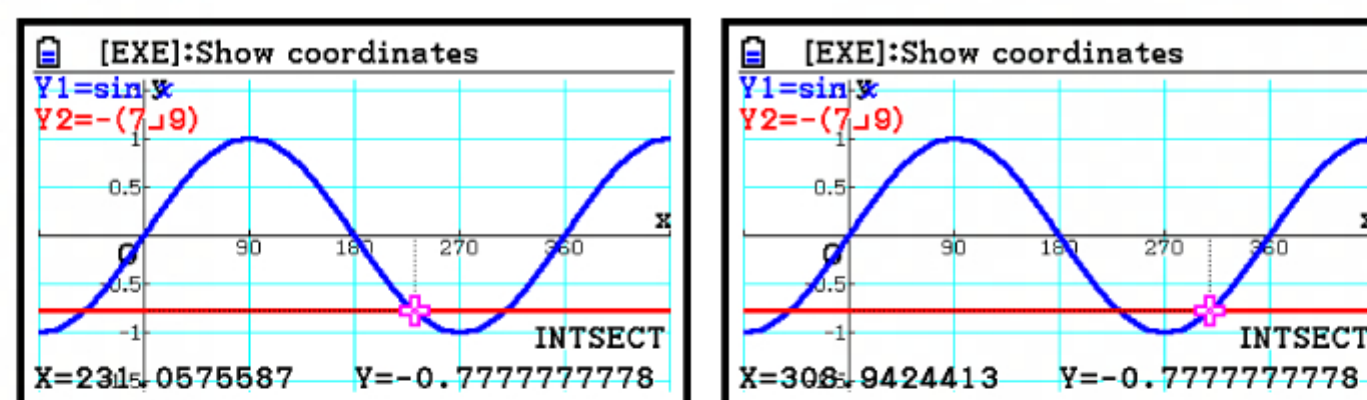
**51 a** We graph the functions  $Y_1 = \cos X$  and  $Y_2 = -0.3$  on the same set of axes.



The solutions are  $\theta \approx 107^\circ, 253^\circ$ .

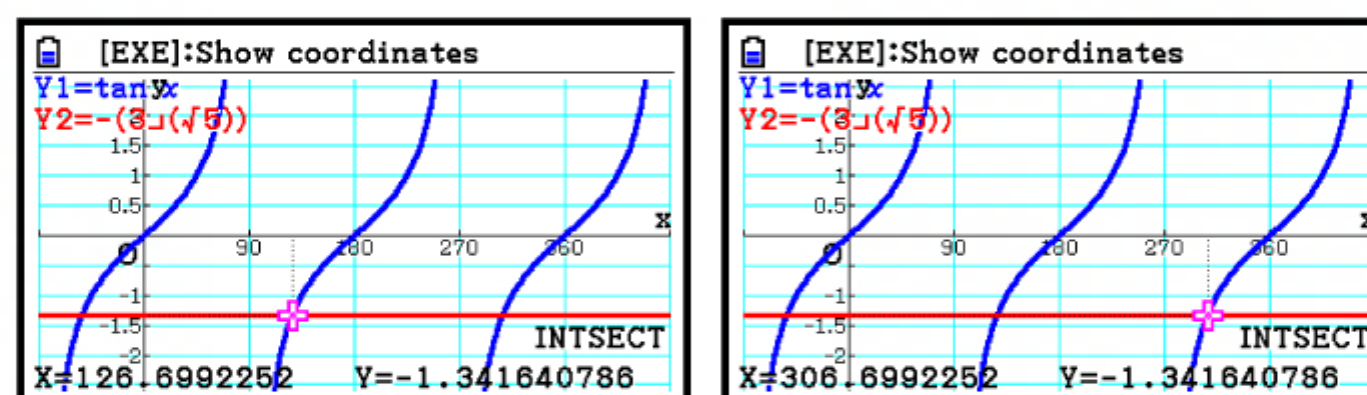


- b** We graph the functions  $Y_1 = \sin X$  and  $Y_2 = -\frac{7}{9}$  on the same set of axes.



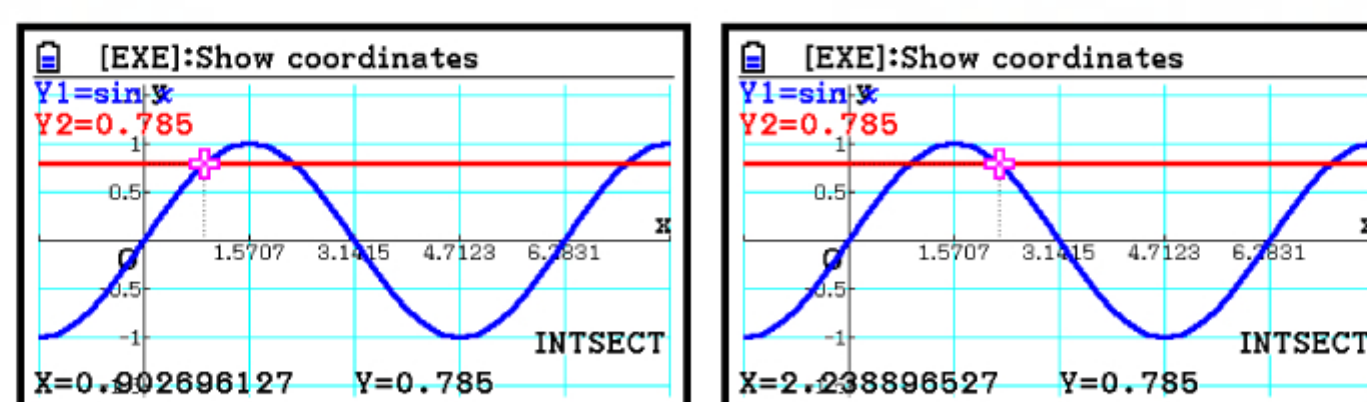
The solutions are  $\theta \approx 231^\circ, 309^\circ$ .

- c** We graph the functions  $Y_1 = \tan X$  and  $Y_2 = -\frac{3}{\sqrt{5}}$  on the same set of axes.



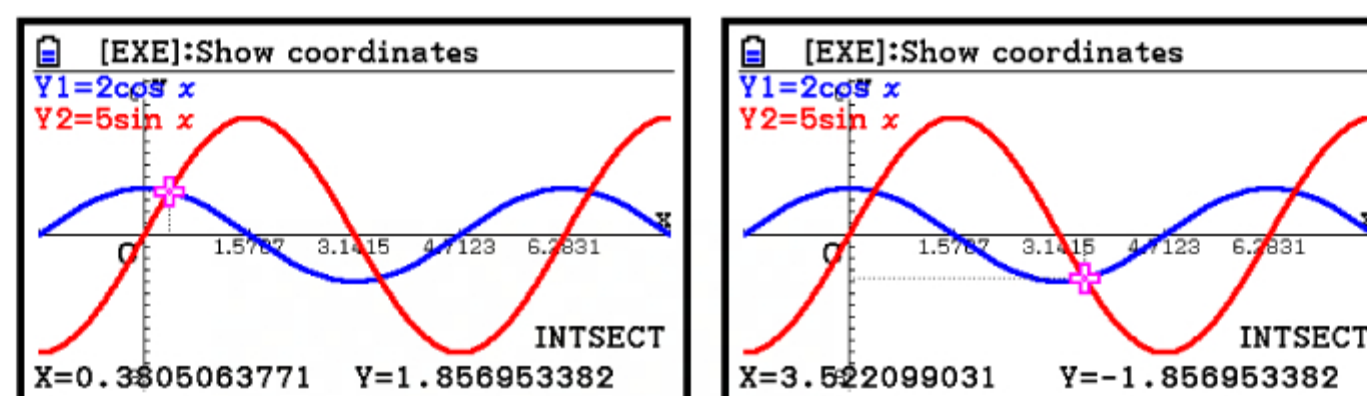
The solutions are  $\theta \approx 127^\circ, 307^\circ$ .

- 52 a** We graph the functions  $Y_1 = \sin X$  and  $Y_2 = 0.785$  on the same set of axes.



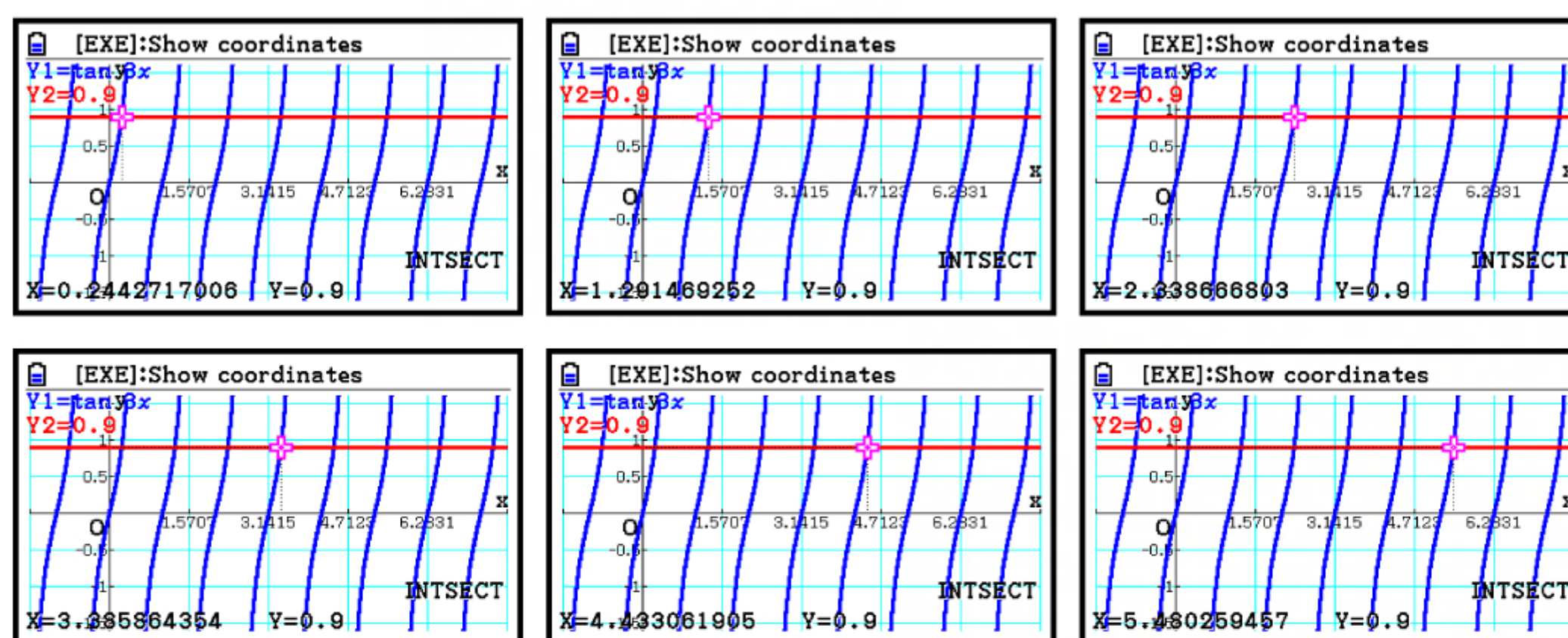
The solutions are  $x \approx 0.903, 2.24$ .

- b** We graph the functions  $Y_1 = 2 \cos X$  and  $Y_2 = 5 \sin X$  on the same set of axes.



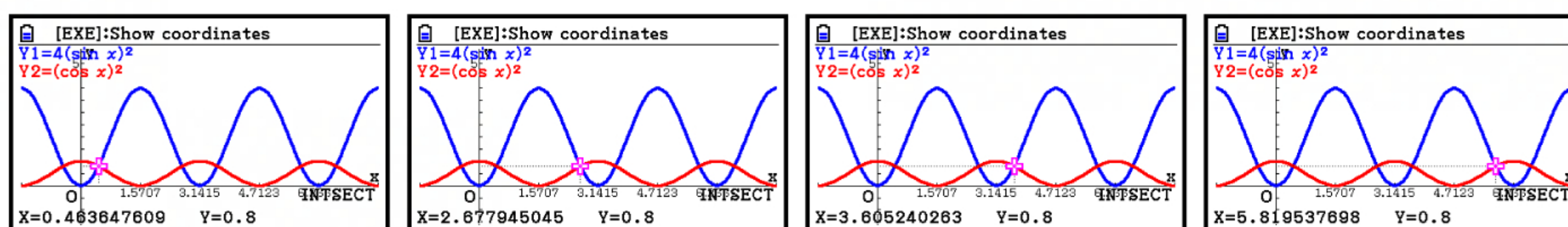
The solutions are  $x \approx 0.381, 3.52$ .

- c** We graph the functions  $Y_1 = \tan 3X$  and  $Y_2 = 0.9$  on the same set of axes.



The solutions are  $x \approx 0.244, 1.29, 2.34, 3.39, 4.43, 5.48$ .

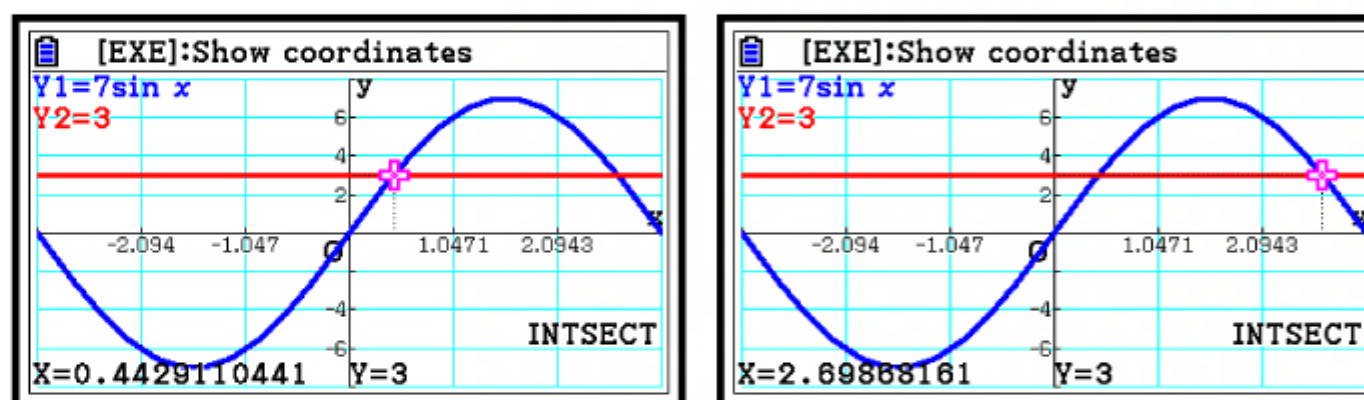
- d** We graph the functions  $Y_1 = 4 \sin^2 X$  and  $Y_2 = \cos^2 X$  on the same set of axes.



The solutions are  $x \approx 0.464, 2.68, 3.61, 5.82$ .

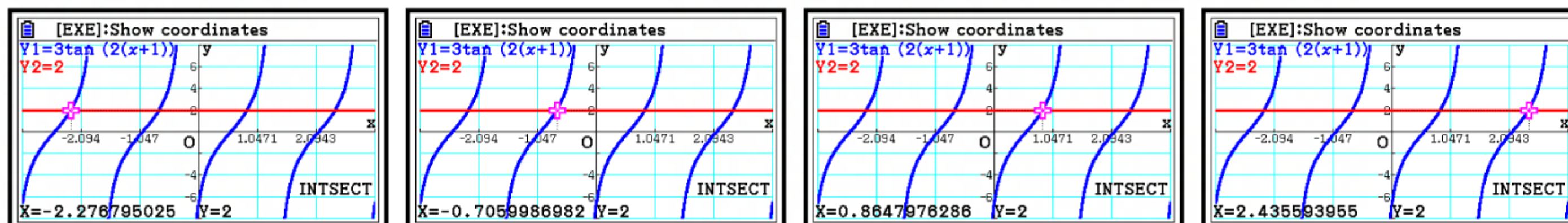


- 53 a** We graph the functions  $Y_1 = 7 \sin X$  and  $Y_2 = 3$  on the same set of axes.



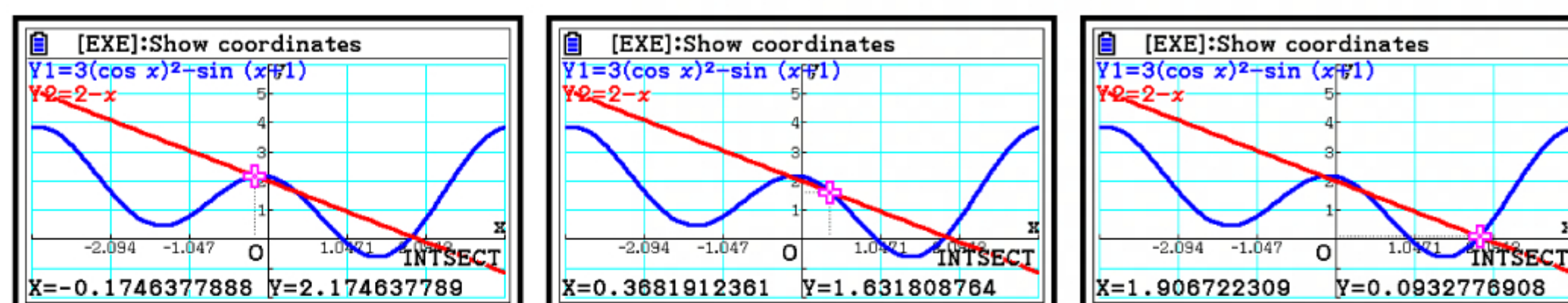
The solutions are  $x \approx 0.443, 2.70$ .

- b** We graph the functions  $Y_1 = 3 \tan(2(X + 1))$  and  $Y_2 = 2$  on the same set of axes.



The solutions are  $x \approx -2.28, -0.706, 0.865, 2.44$ .

- c** We graph the functions  $Y_1 = 3 \cos^2 X - \sin(X + 1)$  and  $Y_2 = 2 - X$  on the same set of axes.



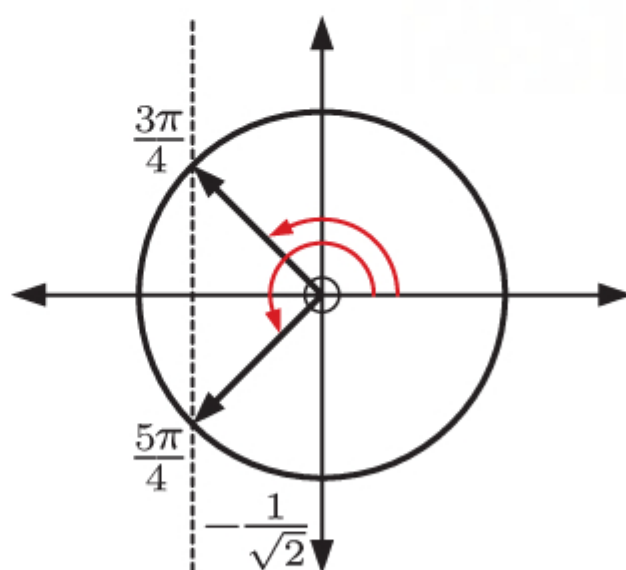
The solutions are  $x \approx -0.175, 0.368, 1.91$ .

- 54 a**  $\sqrt{2} \cos x + 1 = 0$

$$\therefore \sqrt{2} \cos x = -1$$

$$\therefore \cos x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

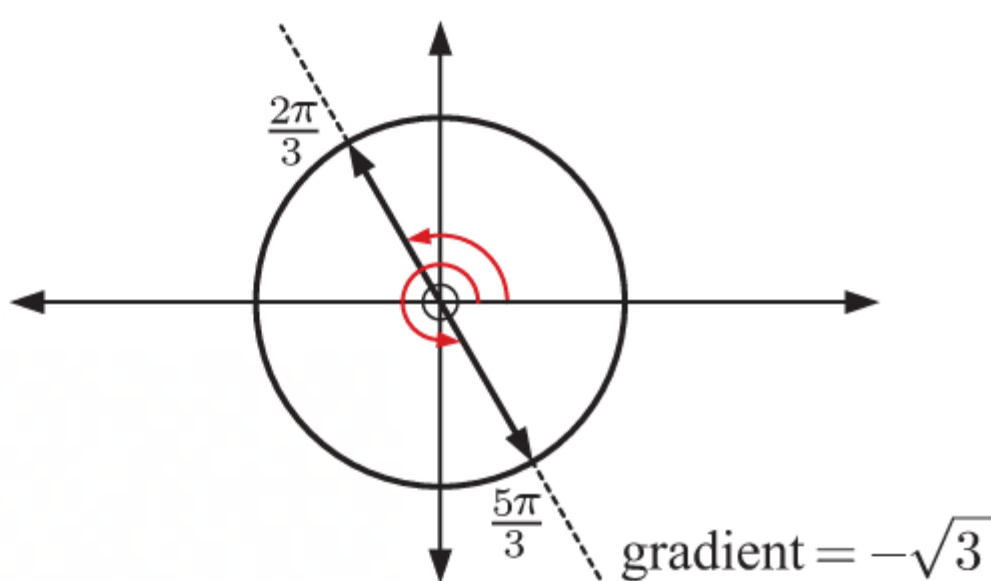


- b**  $\sin x = -\sqrt{3} \cos x$

$$\therefore \frac{\sin x}{\cos x} = -\sqrt{3}$$

$$\therefore \tan x = -\sqrt{3}$$

$$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3}$$



- 55**  $P(t) = 4500 + 500 \sin\left(\frac{2\pi}{7}(t - 3)\right)$ ,  $0 \leq t \leq 10$

- a i**  $P(0) = 4500 + 500 \sin\left(\frac{2\pi}{7}(0 - 3)\right)$   
 $= 4500 + 500 \sin\left(-\frac{6\pi}{7}\right)$   
 $\approx 4283.06$   
 $\approx 4280$  butterflies

- ii**  $P(3) = 4500 + 500 \sin\left(\frac{2\pi}{7}(3 - 3)\right)$   
 $= 4500 + 500 \sin 0$   
 $= 4500$  butterflies

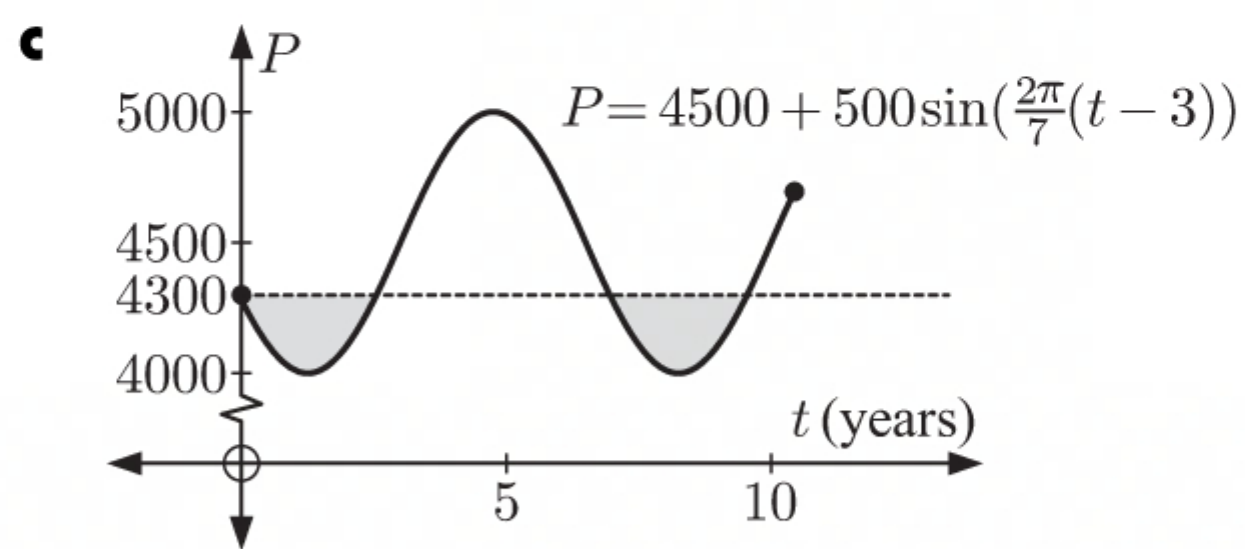
- b i** We need to solve  $P(t) = 4200$ , so  
 $4500 + 500 \sin\left(\frac{2\pi}{7}(t - 3)\right) = 4200$   
 Using technology,  $t \approx 0.217, 2.28, 7.22, 9.28$ .

So, the population is 4200 after about 0.217 years, 2.28 years, 7.22 years, and 9.28 years.

- ii** We need to solve  $P(t) = 4900$ , so  
 $4500 + 500 \sin\left(\frac{2\pi}{7}(t - 3)\right) = 4900$   
 Using technology,  $t \approx 4.03, 5.47$ .

So, the population is 4900 after about 4.03 years and 5.47 years.





We need to solve  $P(t) = 4300$ , so

$$4500 + 500 \sin\left(\frac{2\pi}{7}(t-3)\right) = 4300$$

Using technology,  $t \approx 2.54, 6.96, 9.54$ .

So, the population drops below 4300 between 0 years and  $\approx 2.54$  years, and between  $\approx 6.96$  years and  $\approx 9.54$  years.

- 56 a** The midpoint M of [AB] is  $\left(\frac{3+7}{2}, \frac{5+3}{2}\right)$  or (5, 4).

The gradient of [AB] is  $\frac{3-5}{7-3} = \frac{-2}{4} = -\frac{1}{2}$

$\therefore$  the gradient of the perpendicular bisector is 2.

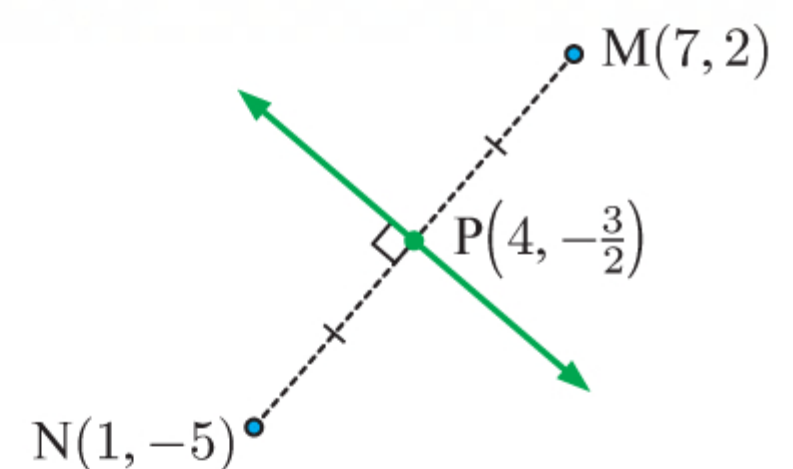
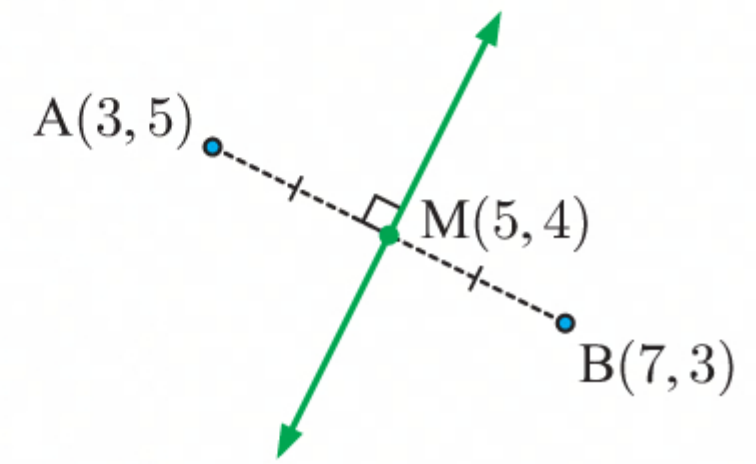
$\therefore$  the equation of the perpendicular bisector is  $2x - y = 2(5) - 4$   
which is  $2x - y = 6$ .

- b** The midpoint P of [MN] is  $\left(\frac{7+1}{2}, \frac{2+(-5)}{2}\right)$  or  $\left(4, -\frac{3}{2}\right)$ .

The gradient of [MN] is  $\frac{-5-2}{1-7} = \frac{-7}{-6} = \frac{7}{6}$ .

$\therefore$  the gradient of the perpendicular bisector is  $-\frac{6}{7}$ .

$\therefore$  the equation of the perpendicular bisector is  $6x + 7y = 6(4) + 7\left(-\frac{3}{2}\right)$   
which is  $6x + 7y = \frac{27}{2}$ .



- 57 a i**  $4x - 3y + 2 = 0$

$\therefore 3y = 4x + 2$

$\therefore y = \frac{4}{3}x + \frac{2}{3}$  has gradient  $\frac{4}{3}$

- ii** The perpendicular bisector has gradient  $-\frac{3}{4}$ .

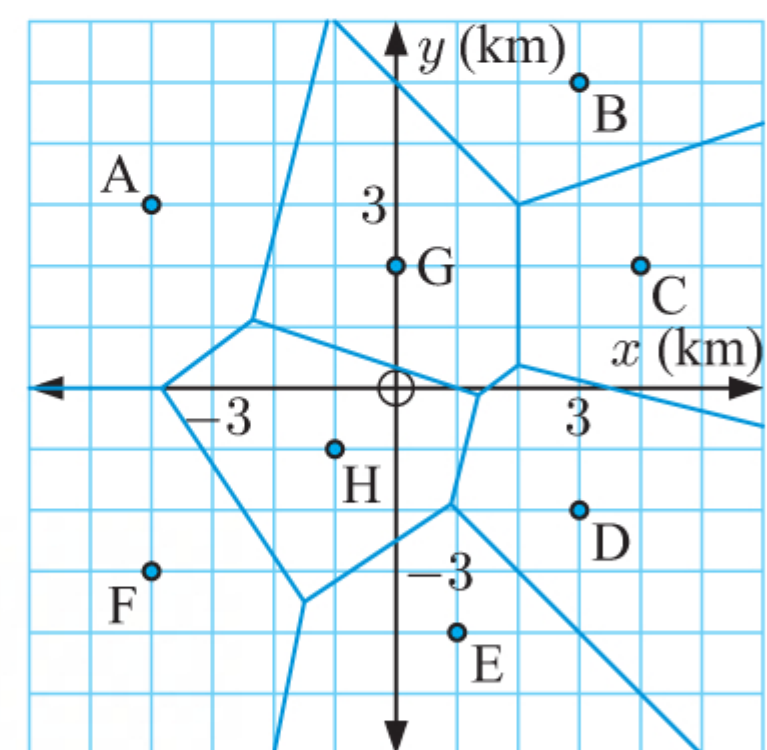
- b** The equation of the perpendicular bisector is  $3x + 4y = 3(4) + 4(6)$   
which is  $3x + 4y = 36$   
or  $3x + 4y - 36 = 0$

- 58 a i** (0, -2) lies in cell H, so the nearest bus stop is H.  
**ii** (3, 2) lies in cell C, so the nearest bus stop is C.  
**iii** (-5, 5) lies in cell A, so the nearest bus stop is A.  
**iv** (-1, -4) lies in cell E, so the nearest bus stop is E.

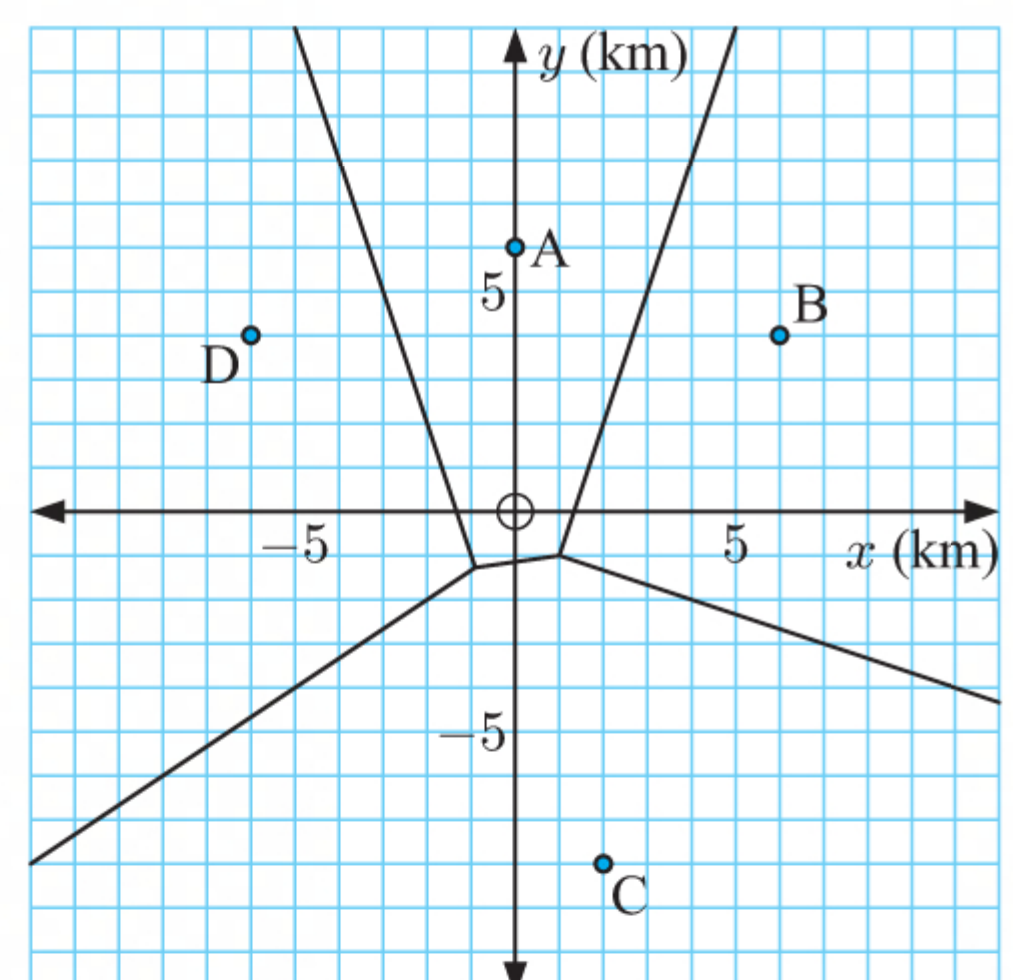
- b i** Jerome must live at the vertex adjacent to cells B, C, and G.  
 $\therefore$  Jerome lives at (2, 3).

- ii** Distance Jerome needs to walk = distance between vertex and B  
 $= \sqrt{(2-3)^2 + (3-5)^2}$   
 $= \sqrt{(-1)^2 + (-2)^2}$   
 $= \sqrt{5} \approx 2.24 \text{ km}$

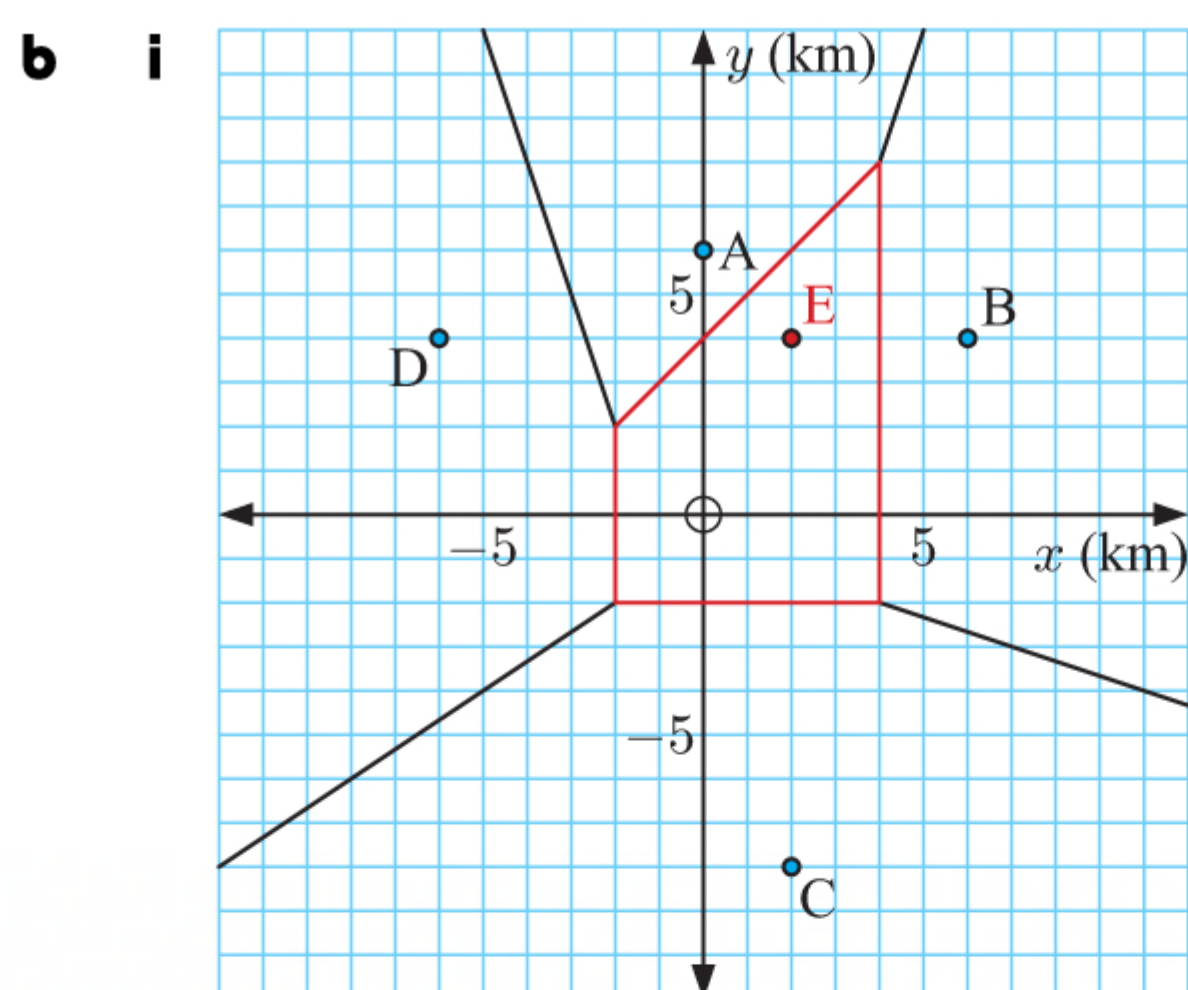
So, Jerome needs to walk about 2.24 km to get to any one of stops B, C, and G.



- 59 a i** (-2, 8) is in cell A, so the closest restaurant is A.  
**ii** (5, -5) is in cell C, so the closest restaurant is C.  
**iii** (-9, -6) is in cell D, so the closest restaurant is D.

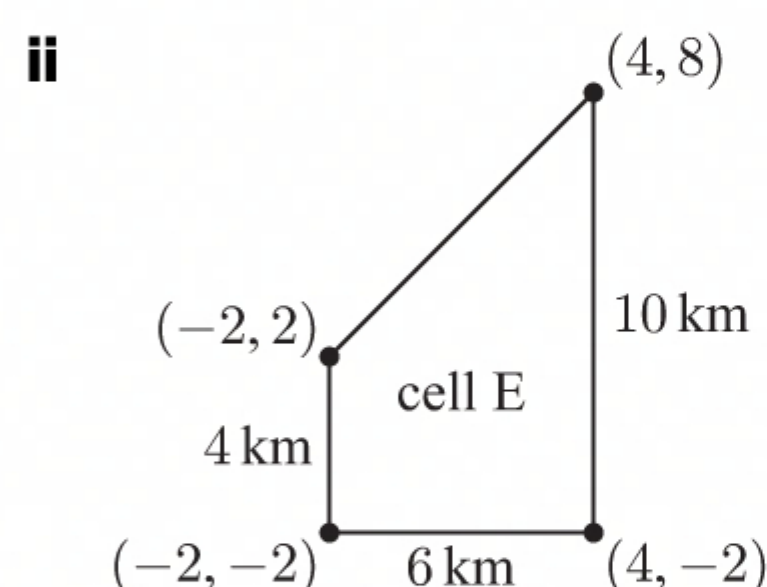




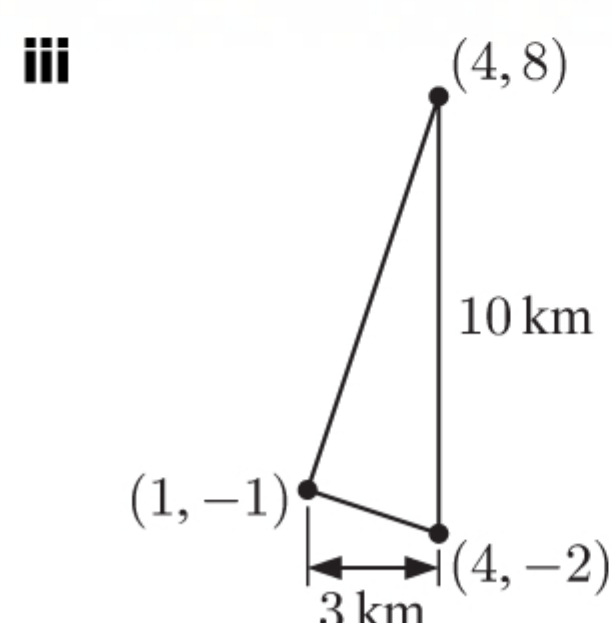


We construct  $PB(A, E)$ ,  $PB(B, E)$ ,  $PB(C, E)$ , and  $PB(D, E)$  within cells A, B, C, and D respectively.

We then remove the segments of edges which now lie within cell E, giving us the Voronoi diagram which includes site E.



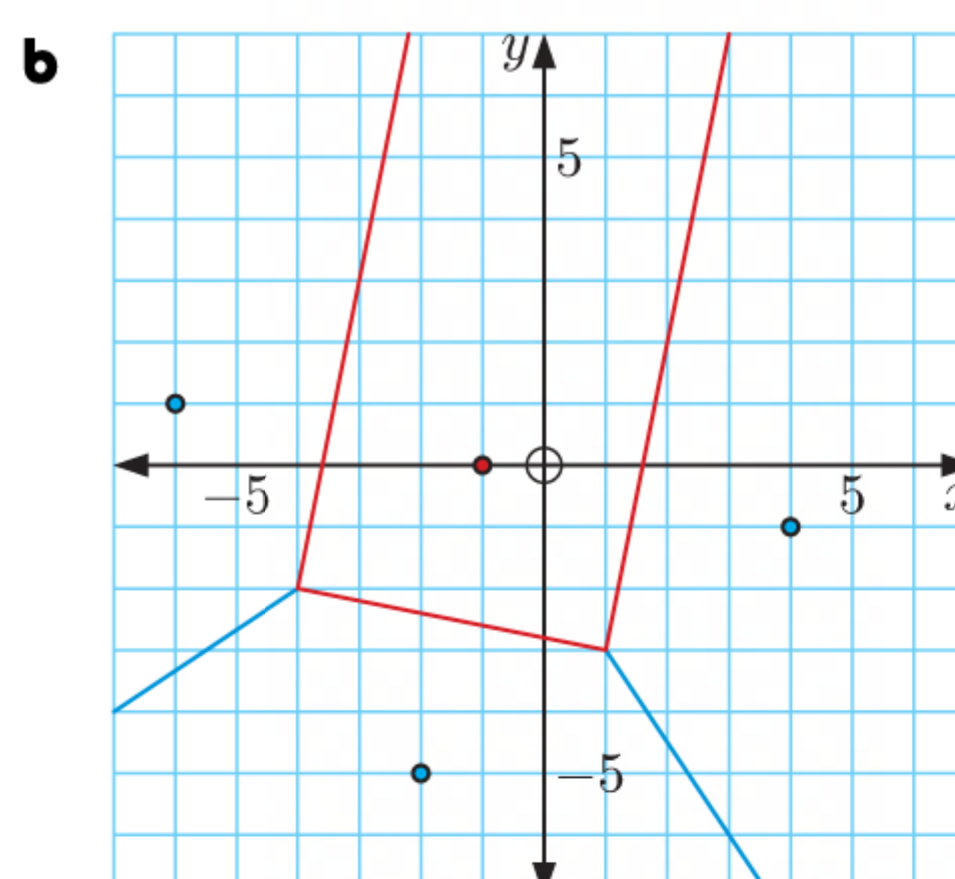
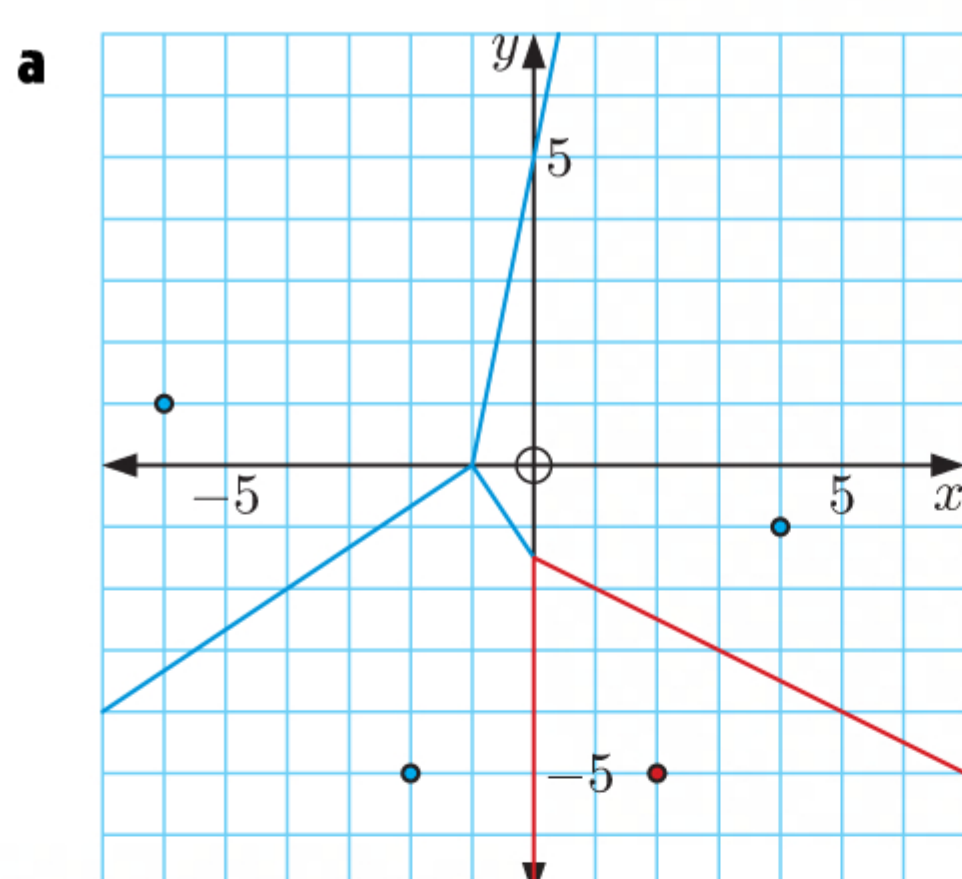
$$\begin{aligned} \text{Area of region} &= \text{area of cell E} \\ &= \left( \frac{10+4}{2} \right) \times 6 \\ &= 42 \text{ km}^2 \end{aligned}$$



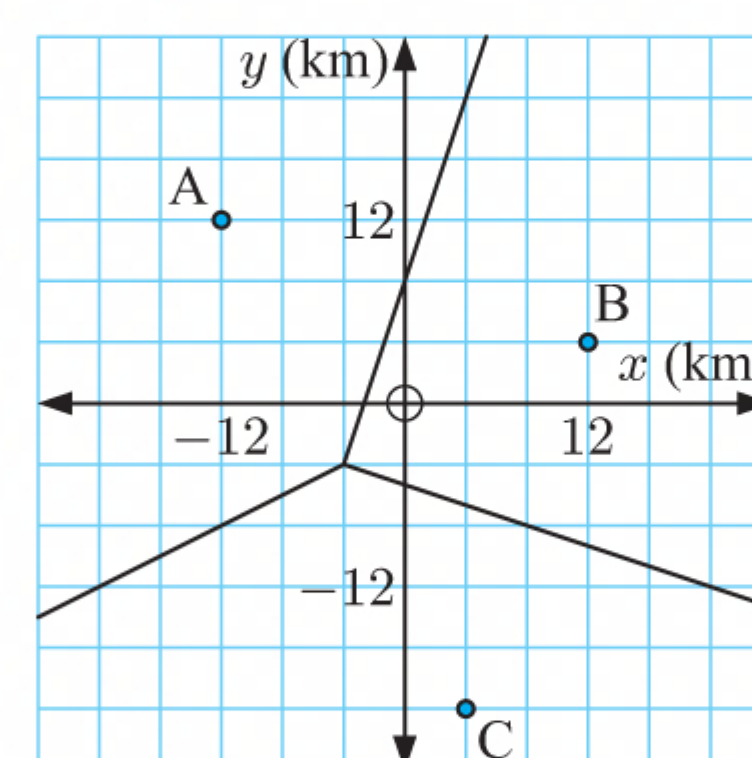
$$\begin{aligned} \text{Area of cell E that was cell B} &= \frac{1}{2} \times 3 \times 10 \\ &= 15 \text{ km}^2 \end{aligned}$$

Assuming that the density of residents is constant throughout the city, the proportion of residents now closest to E who were originally closest to B is  $\frac{15}{42} \approx 0.357$ .

- 60** We construct the perpendicular bisectors between the new site and each of the original ones, within their respective cells. We then remove the segments of edges which lie within the new cell, giving us the Voronoi diagram which includes the new site.



- 61**
- a** (12, 16) is closest to B, so we estimate the internet speed at 6 pm to be 42.7 Mbps at (12, 16).
- b** (-20, -12) is on the edge equidistant from A and C, so we estimate the internet speed at 6 pm to be  $\frac{44.3 + 45.9}{2} = 45.1$  Mbps at (-20, -12).
- c** (-4, 4) is the vertex equidistant from A, B, and C, so we estimate the internet speed at 6 pm to be  $\frac{44.3 + 42.7 + 45.9}{3} = 44.3$  Mbps at (-4, -4).





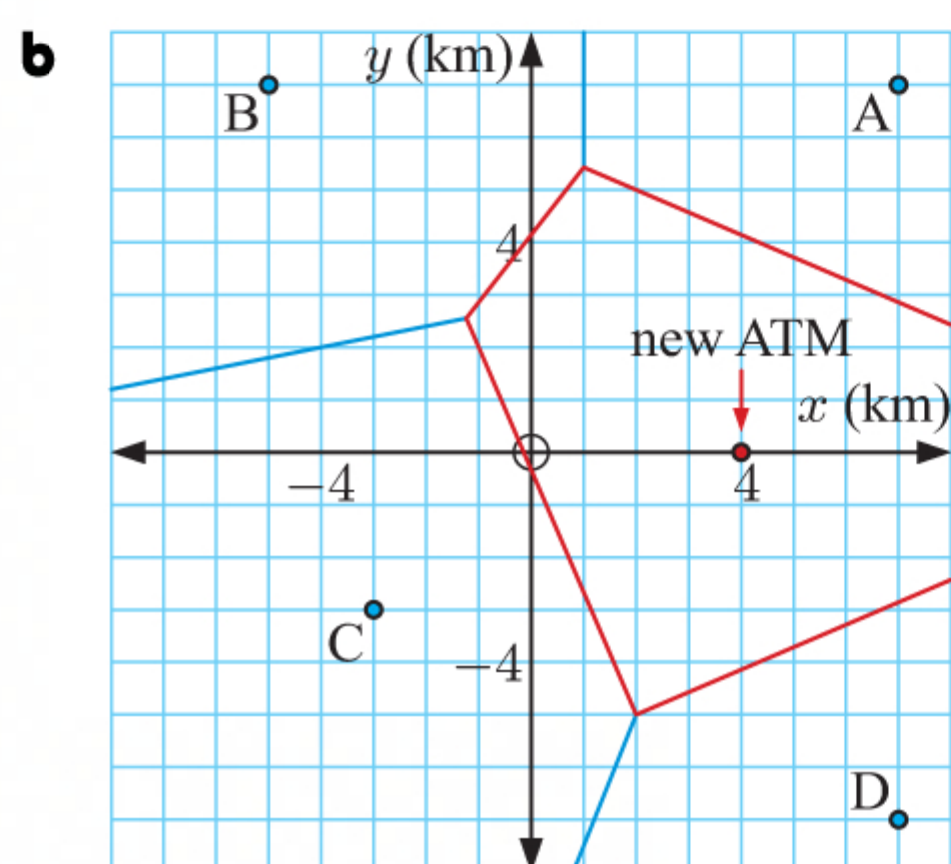
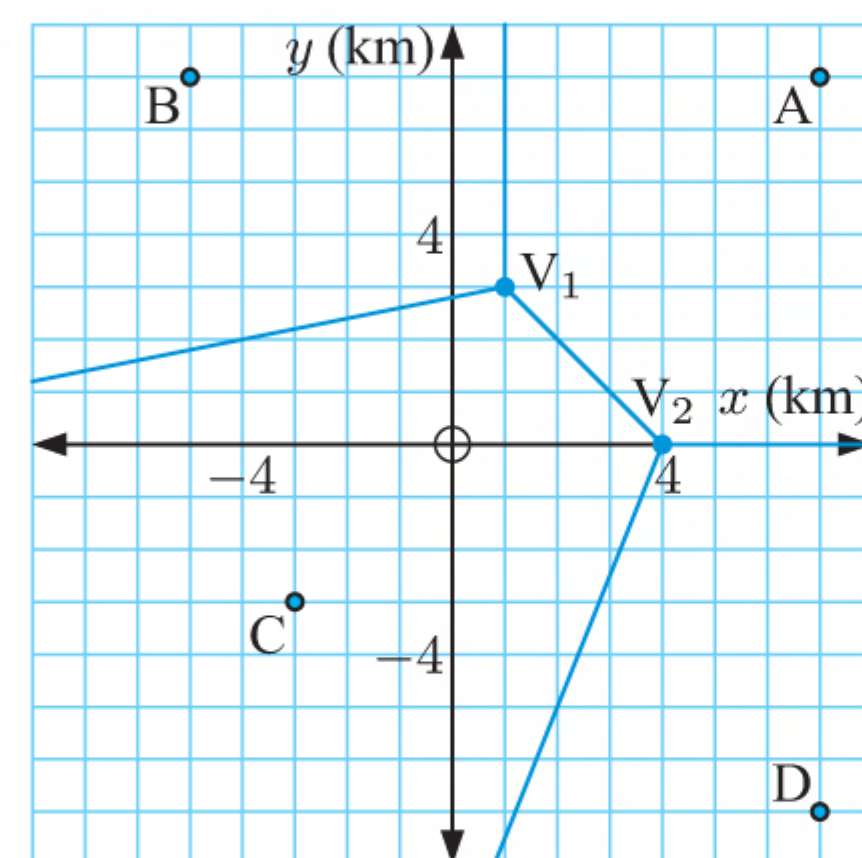
- 62 a** The new ATM needs to be placed at the centre of the largest empty circle, which will be centred at one of the vertices  $V_1(1, 3)$  or  $V_2(4, 0)$ .

$$\begin{aligned} V_1A &= \sqrt{(7-1)^2 + (7-3)^2} \\ &= \sqrt{6^2 + 4^2} \\ &= \sqrt{52} \text{ km} \end{aligned}$$

$$\begin{aligned} V_2A &= \sqrt{(7-4)^2 + (7-0)^2} \\ &= \sqrt{3^2 + 7^2} \\ &= \sqrt{58} \text{ km} \end{aligned}$$

So, the largest empty circle has centre  $V_2(4, 0)$ .

$\therefore$  the new ATM should be placed at  $(4, 0)$ .



We construct the perpendicular bisectors between the site of the new ATM and each of the original ones, within their respective cells.

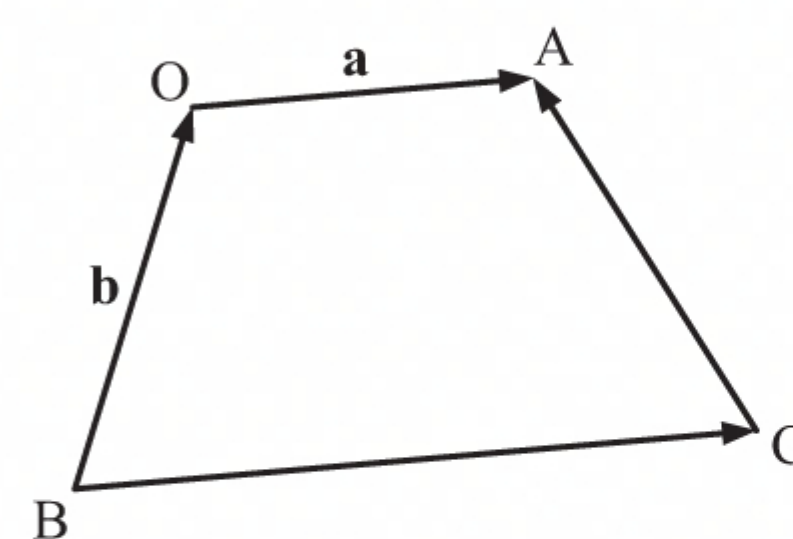
We then remove the segments of edges which lie within the new cell, giving us the Voronoi diagram which includes the new ATM at  $(4, 0)$ .

- c**  $(-1, 1)$  is in cell C, so Allan is closest to ATM C.

$$\begin{aligned} \therefore \text{distance Allan needs to walk} &= \sqrt{(-1 - (-3))^2 + (1 - (-3))^2} \\ &= \sqrt{2^2 + 4^2} \\ &= \sqrt{20} \\ &\approx 4.47 \text{ km} \end{aligned}$$

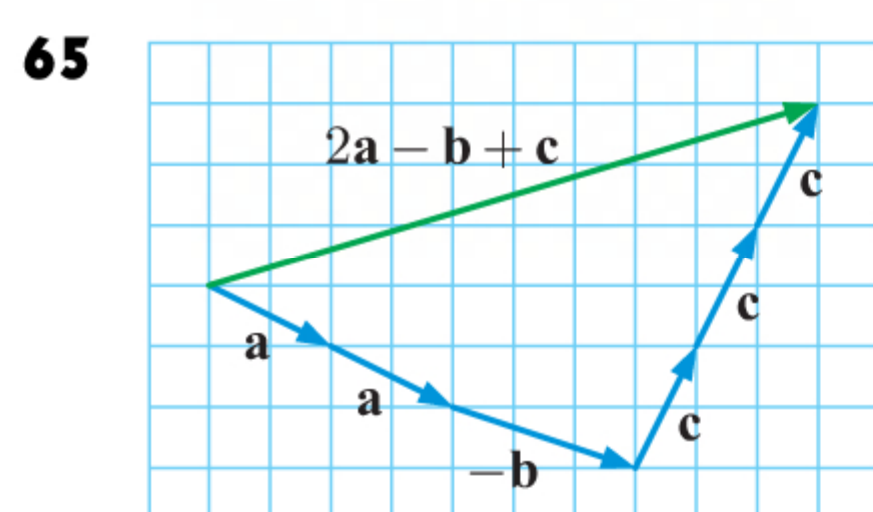
- d** In **c**, we assume that the path Allan has to walk to get to the nearest ATM is a straight line.

- 63 a**  $\vec{BC} = 2\mathbf{a}$       **b**  $\vec{CB} = -2\mathbf{a}$   
**c**  $\vec{BA} = \mathbf{b} + \mathbf{a}$       **d**  $\vec{OC} = -\mathbf{b} + 2\mathbf{a}$   
**e**  $\vec{AC} = -\mathbf{a} - \mathbf{b} + 2\mathbf{a}$       **f**  $\vec{CA} = -(\mathbf{a} - \mathbf{b})$   
 $= \mathbf{a} - \mathbf{b}$        $= \mathbf{b} - \mathbf{a}$



- 64 a**  $|\frac{1}{3}\mathbf{i} + k\mathbf{j}| = 1$  when  $\sqrt{(\frac{1}{3})^2 + k^2} = 1$   
 $\therefore \frac{1}{9} + k^2 = 1$   
 $\therefore k^2 = \frac{8}{9}$   
 $\therefore k = \pm\sqrt{\frac{8}{9}}$   
 $= \pm\frac{2\sqrt{2}}{3}$

- b**  $\left| \begin{pmatrix} 3 \\ k \\ k+2 \end{pmatrix} \right| = \sqrt{61}$  when  
 $\sqrt{3^2 + k^2 + (k+2)^2} = \sqrt{61}$   
 $\therefore 9 + k^2 + k^2 + 4k + 4 = 61$   
 $\therefore 2k^2 + 4k - 48 = 0$   
 $\therefore k^2 + 2k - 24 = 0$   
 $\therefore (k+6)(k-4) = 0$   
 $\therefore k = -6 \text{ or } 4$



$$\begin{aligned} 2\mathbf{a} - \mathbf{b} + 3\mathbf{c} &= 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 4 + 3 + 3 \\ -2 - 1 + 6 \end{pmatrix} \\ &= \begin{pmatrix} 10 \\ 3 \end{pmatrix} \text{ which agrees with the diagram.} \end{aligned}$$



$$66 \quad \mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{a} \quad \mathbf{c} - \mathbf{a} &= \begin{pmatrix} 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 4-1 \\ -1-2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{b} - 2\mathbf{c} - \mathbf{a} &= \begin{pmatrix} -2 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -2-2(4)-1 \\ 2-2(-1)-2 \end{pmatrix} \\ &= \begin{pmatrix} -11 \\ 2 \end{pmatrix} \end{aligned}$$

$$67 \quad \mathbf{a} = \mathbf{i} + \mathbf{j}, \quad \mathbf{b} = \mathbf{j} + \mathbf{k}, \quad \mathbf{c} = \mathbf{i} + \mathbf{k}$$

$$\begin{aligned} \mathbf{a} \quad \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c}) &= \frac{1}{2}(\mathbf{i} + \mathbf{j} + \mathbf{j} + \mathbf{k} + \mathbf{i} + \mathbf{k}) \\ &= \frac{1}{2}(2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{b} &= \mathbf{j} + \mathbf{k} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ \therefore |\mathbf{b}| &= \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \text{ units} \\ \text{So, } \frac{1}{|\mathbf{b}|}\mathbf{b} &= \frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k}) \\ &= \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \end{aligned}$$

$$68 \quad \overrightarrow{\text{OC}} = \begin{pmatrix} 5 \\ 22 \end{pmatrix}$$

$$\text{So, } \begin{pmatrix} 5 \\ 22 \end{pmatrix} = r \begin{pmatrix} 3 \\ -6 \end{pmatrix} + s \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 3r + 7s \\ -6r + 2s \end{pmatrix}$$

$$\therefore 3r + 7s = 5 \quad \dots (1)$$

$$-6r + 2s = 22 \quad \dots (2)$$

$$\therefore 6r + 14s = 10 \quad \{(1) \times 2\}$$

$$\begin{array}{r} -6r + 2s = 22 \quad \{(2)\} \\ \hline 16s = 32 \end{array}$$

$$\text{Adding, } 16s = 32$$

$$\therefore s = 2$$

$$\text{Substituting into (1), } 3r + 7(2) = 5$$

$$\therefore 3r = -9$$

$$\therefore r = -3$$

$$\text{So, } r = -3, s = 2.$$

$$\begin{aligned} \mathbf{b} \quad \frac{1}{2}\mathbf{c} + 3\mathbf{a} &= \frac{1}{2}\begin{pmatrix} 4 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(4) + 3(1) \\ \frac{1}{2}(-1) + 3(2) \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ \frac{11}{2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \mathbf{c} - 3\mathbf{a} + 2\mathbf{b} &= \begin{pmatrix} 4 \\ -1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 4-3(1)+2(-2) \\ -1-3(2)+2(2) \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ -3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore |\mathbf{c} - 3\mathbf{a} + 2\mathbf{b}| &= \sqrt{(-3)^2 + (-3)^2} \\ &= \sqrt{18} = 3\sqrt{2} \text{ units} \end{aligned}$$

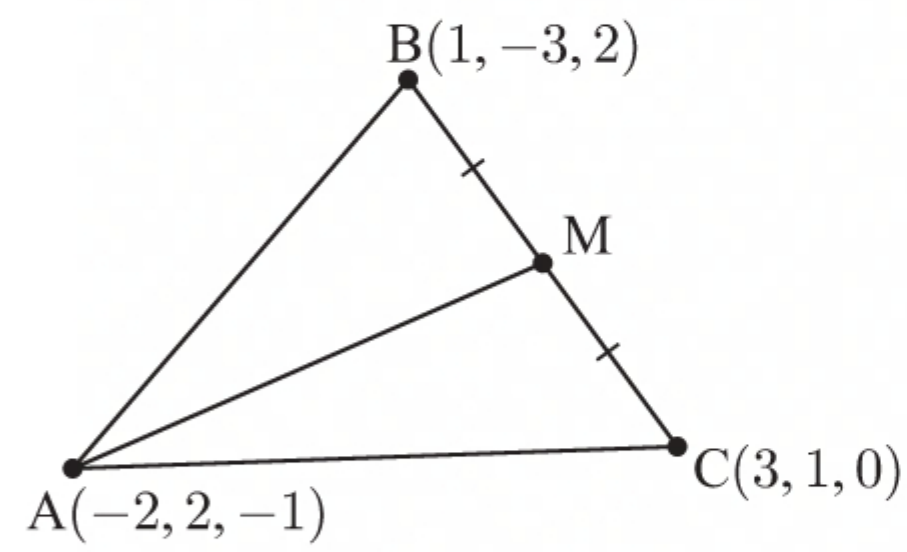
$$\mathbf{b} \quad -5\mathbf{c} = -5\mathbf{i} - 5\mathbf{k} = \begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}$$

$$\begin{aligned} \therefore |-5\mathbf{c}| &= \sqrt{(-5)^2 + 0^2 + (-5)^2} \\ &= \sqrt{50} = 5\sqrt{2} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 2\mathbf{a} - 3\mathbf{b} - \mathbf{c} &= 2(\mathbf{i} + \mathbf{j}) - 3(\mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{k}) \\ &= 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{j} - 3\mathbf{k} - \mathbf{i} - \mathbf{k} \\ &= \mathbf{i} - \mathbf{j} - 4\mathbf{k} \\ &= \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore |2\mathbf{a} - 3\mathbf{b} - \mathbf{c}| &= \sqrt{1^2 + (-1)^2 + (-4)^2} \\ &= \sqrt{18} = 3\sqrt{2} \text{ units} \end{aligned}$$





**69 a** M is  $\left(\frac{1+3}{2}, \frac{-3+1}{2}, \frac{2+0}{2}\right)$

$\therefore$  M is  $(2, -1, 1)$

**b**  $\vec{AB} = \begin{pmatrix} 1 - (-2) \\ -3 - 2 \\ 2 - (-1) \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 3 \end{pmatrix}$

$\vec{AM} = \begin{pmatrix} 2 - (-2) \\ -1 - 2 \\ 1 - (-1) \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$

$\vec{AC} = \begin{pmatrix} 3 - (-2) \\ 1 - 2 \\ 0 - (-1) \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$

**c**  $\frac{1}{2}\vec{AB} + \frac{1}{2}\vec{AC} = \frac{1}{2}\begin{pmatrix} 3 \\ -5 \\ 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} \quad \{\text{using b}\}$

$$= \begin{pmatrix} \frac{1}{2}(3) + \frac{1}{2}(5) \\ \frac{1}{2}(-5) + \frac{1}{2}(-1) \\ \frac{1}{2}(3) + \frac{1}{2}(1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

$$= \vec{AM} \quad \{\text{from b}\}$$

**70 a**  $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$  has length  $\sqrt{(-3)^2 + 2^2} = \sqrt{13}$  units

$\therefore$  the unit vector in the same direction is  $\frac{1}{\sqrt{13}}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$\therefore$  the vector of length 4 units in the same direction is  $\mathbf{v} = \frac{4}{\sqrt{13}}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{12}{\sqrt{13}} \\ \frac{8}{\sqrt{13}} \end{pmatrix}$

**b**  $\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$  has length  $\sqrt{(-1)^2 + 4^2 + 7^2} = \sqrt{66}$  units

$\therefore$  the unit vector in the opposite direction is  $-\frac{1}{\sqrt{66}}\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix} = \frac{1}{\sqrt{66}}\begin{pmatrix} 1 \\ -4 \\ -7 \end{pmatrix}$

$\therefore$  the vector of length 3 units in the opposite direction is  $\mathbf{v} = \frac{3}{\sqrt{66}}\begin{pmatrix} 1 \\ -4 \\ -7 \end{pmatrix} = \frac{\sqrt{66}}{22}\begin{pmatrix} 1 \\ -4 \\ -7 \end{pmatrix}$

$$= \begin{pmatrix} \frac{\sqrt{66}}{22} \\ -\frac{2\sqrt{66}}{11} \\ -\frac{7\sqrt{66}}{22} \end{pmatrix}$$

**71**  $\left|\begin{pmatrix} -3 \\ 4 \end{pmatrix}\right| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$

$\therefore$  the velocity vector of the object is  $\frac{3}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{9}{5} \\ \frac{12}{5} \end{pmatrix}$ .



**72**  $\mathbf{i} + 8\mathbf{j} - 4\mathbf{k} = \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix}$  has length  $\sqrt{1^2 + 8^2 + (-4)^2} = 9$  units

$\therefore$  the vector of length 6 units in the same direction is  $\frac{6}{9} \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix}$

$$= \begin{pmatrix} \frac{2}{3} \\ \frac{16}{3} \\ -\frac{8}{3} \end{pmatrix}$$

Now  $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{16}{3} \\ -\frac{8}{3} \end{pmatrix} = \begin{pmatrix} 2 + \frac{2}{3} \\ -1 + \frac{16}{3} \\ 3 + (-\frac{8}{3}) \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{13}{3} \\ \frac{1}{3} \end{pmatrix}$

$\therefore$  the point 6 units from  $(2, -1, 3)$  in the direction  $\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$  is  $(\frac{8}{3}, \frac{13}{3}, \frac{1}{3})$ .

**73**  $\mathbf{p} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ ,  $\mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}$ ,  $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

**a**  $\mathbf{p} \cdot \mathbf{r}$

$$= \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$= 3 \times 2 + 0 \times 1 + 1 \times (-1)$$

$$= 6 + 0 - 1$$

$$= 5$$

**b**  $\mathbf{q} \cdot (\mathbf{r} + \mathbf{p})$

$$= \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \cdot \left[ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2+3 \\ 1+0 \\ -1+1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$

$$= (-1) \times 5 + 0 \times 1 + 7 \times 0$$

$$= -5 + 0 + 0$$

$$= -5$$

**c**  $(2\mathbf{p} + \mathbf{q}) \cdot \mathbf{r}$

$$= \left[ 2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(3) + (-1) \\ 2(0) + 0 \\ 2(1) + 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 0 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$= 5 \times 2 + 0 \times 1 + 9 \times (-1)$$

$$= 10 + 0 - 9$$

$$= 1$$

**d**  $|\mathbf{q}|^2 = (-1)^2 + 0^2 + 7^2$

$$= 1 + 49$$

$$= 50$$

**e**  $k\mathbf{p} + \mathbf{q}$  is perpendicular to  $\mathbf{r}$  if  $(k\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} = 0$

$$\therefore \left[ k \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$\therefore \begin{pmatrix} 3k - 1 \\ 0 \\ k + 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$\therefore 2(3k - 1) + 0 - (k + 7) = 0$$

$$\therefore 6k - 2 - k - 7 = 0$$

$$\therefore 5k = 9$$

$$\therefore k = \frac{9}{5}$$

**74 a**  $\cos \theta = \frac{\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 1 \end{pmatrix}}{\left| \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right| \left| \begin{pmatrix} 8 \\ 1 \end{pmatrix} \right|}$

$$= \frac{2(8) + 3(1)}{\sqrt{2^2 + 3^2} \sqrt{8^2 + 1^2}}$$

$$= \frac{19}{\sqrt{13} \sqrt{65}}$$

$\therefore \theta = \cos^{-1} \left( \frac{19}{\sqrt{13} \sqrt{65}} \right) \approx 49.2^\circ$

**b**  $\cos \theta = \frac{\begin{pmatrix} -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -4 \end{pmatrix}}{\left| \begin{pmatrix} -1 \\ 5 \end{pmatrix} \right| \left| \begin{pmatrix} 6 \\ -4 \end{pmatrix} \right|}$

$$= \frac{-1(6) + 5(-4)}{\sqrt{(-1)^2 + 5^2} \sqrt{6^2 + (-4)^2}}$$

$$= \frac{-26}{\sqrt{26} \sqrt{52}}$$

$$= \frac{-26}{26\sqrt{2}}$$

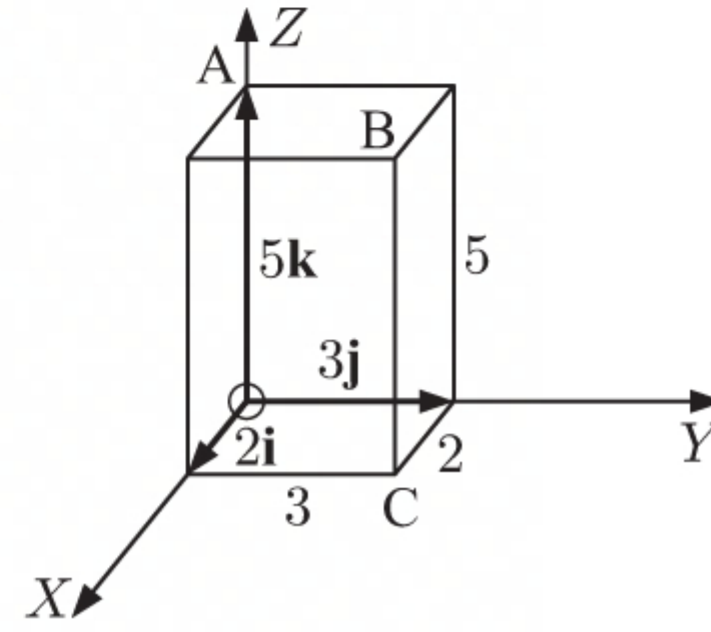
$$= -\frac{1}{\sqrt{2}}$$

$\therefore \theta = 135^\circ$



**75** The diagonal from O to B has direction  $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ .

The diagonal from A to C has direction  $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ .



If the acute angle between the diagonals is  $\theta$  then

$$\cos \theta = \frac{\left| \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \right|}{\left| \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \right|}$$

$$\therefore \cos \theta = \frac{|2(2) + 3(3) + 5(-5)|}{\sqrt{2^2 + 3^2 + 5^2} \sqrt{2^2 + 3^2 + (-5)^2}}$$

$$\therefore \cos \theta = \frac{12}{38}$$

$$\therefore \theta \approx 71.6^\circ.$$

**Note:** This angle may depend on the diagonals you select. The two other possible angles are about  $37.9^\circ$  and about  $58.2^\circ$ .

**76 a**  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$  where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . If  $\mathbf{a} \cdot \mathbf{b} < 0$ , then  $\cos \theta < 0$  and so  $90^\circ < \theta < 180^\circ$ .

**b i**  $\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = -6 - 1 + 3 = -4$

**ii**  $|\mathbf{a}| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14}$ ,  $|\mathbf{b}| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$ , and  $\cos \theta = \frac{-4}{\sqrt{14}\sqrt{11}} \approx -0.3223$   
 $\therefore \theta \approx 108.8^\circ$ .

**77**  $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ ,  $\mathbf{c} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$

**a**  $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 3 & 1 \end{vmatrix}$   
 $= \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} \mathbf{k}$   
 $= \begin{pmatrix} -5 \\ -3 \\ 4 \end{pmatrix}$

**b**  $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 0 & 4 & -1 \end{vmatrix}$   
 $= \begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 3 \\ 0 & 4 \end{vmatrix} \mathbf{k}$   
 $= \begin{pmatrix} -7 \\ -1 \\ -4 \end{pmatrix}$

So,  $(\mathbf{b} \times \mathbf{c}) \cdot 2\mathbf{a} = \begin{pmatrix} -7 \\ -1 \\ 4 \end{pmatrix} \cdot \left[ 2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right]$   
 $= \begin{pmatrix} -7 \\ -1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$   
 $= -14 - 2 - 16$   
 $= -32$



**78**  $\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$ ,  $\mathbf{b} = \mathbf{j} + 2\mathbf{k}$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ 0 & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

**b** Now  $\mathbf{a} \times \mathbf{b}$  has length  $\sqrt{5^2 + (-2)^2 + 1^2}$   
 $= \sqrt{30}$  units

$\therefore$  a vector of length 5 units perpendicular to both  $\mathbf{a}$  and

$$\begin{aligned} \mathbf{b} \text{ is } \frac{5}{\sqrt{30}} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} &= \frac{\sqrt{30}}{6} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5\sqrt{30}}{6} \\ -\frac{\sqrt{30}}{3} \\ \frac{\sqrt{30}}{6} \end{pmatrix} \end{aligned}$$

**79**  $\mathbf{a} = -4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$  and  $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$

$$\begin{aligned} m\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4m & 3m & -m \\ 2 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 3m & -m \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -4m & -m \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -4m & 3m \\ 2 & 0 \end{vmatrix} \mathbf{k} \\ &= 3m\mathbf{i} + 2m\mathbf{j} - 6m\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore |m\mathbf{a} \times \mathbf{b}| &= \sqrt{(3m)^2 + (2m)^2 + (-6m)^2} \\ &= \sqrt{9m^2 + 4m^2 + 36m^2} \\ &= \sqrt{49m^2} \end{aligned}$$

Now  $|m\mathbf{a} \times \mathbf{b}| = 35$

$$\therefore \sqrt{49m^2} = 35$$

$$\therefore 49m^2 = 1225$$

$$\therefore m^2 = \frac{1225}{49}$$

$$\therefore m = \pm \frac{35}{7}$$

$$\therefore m = \pm 5$$

**80**  $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{a} \times \mathbf{b} = \mathbf{j} - 2\mathbf{k}$

Now  $|\mathbf{a}| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$  units

and  $|\mathbf{a} \times \mathbf{b}| = \sqrt{0^2 + 1^2 + (-2)^2} = \sqrt{5}$  units

The angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $\frac{\pi}{4}$

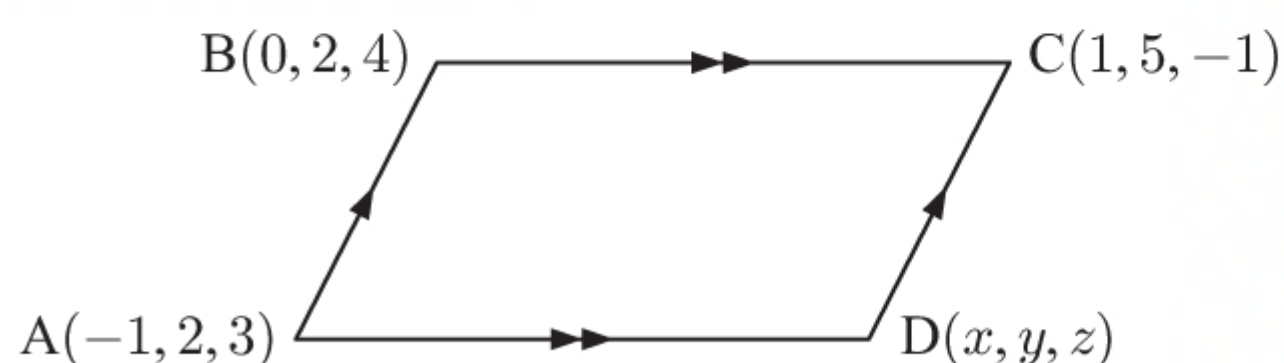
$$\therefore |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \frac{\pi}{4}$$

$$\therefore \sqrt{5} = \sqrt{11} \times |\mathbf{b}| \times \frac{1}{\sqrt{2}}$$

$$\begin{aligned} \therefore |\mathbf{b}| &= \frac{\sqrt{5} \times \sqrt{2}}{\sqrt{11}} \\ &= \sqrt{\frac{10}{11}} \text{ units} \end{aligned}$$



81



**a** Let D have coordinates  $(x, y, z)$ .

$$\overrightarrow{AD} = \overrightarrow{BC}$$

$$\therefore \begin{pmatrix} x+1 \\ y-2 \\ z-3 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 5-2 \\ -1-4 \end{pmatrix}$$

$$\therefore x+1=1, \quad y-2=3, \quad z-3=-5$$

$$\therefore x=0, \quad y=5, \quad z=-2$$

So, D is  $(0, 5, -2)$ .

**b**  $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{AD} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$

$$\begin{aligned} \therefore \overrightarrow{AB} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 3 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ 3 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} \mathbf{k} \\ &= -3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= |-3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}| \\ &= \sqrt{(-3)^2 + 6^2 + 3^2} \\ &= 3\sqrt{6} \text{ units}^2 \end{aligned}$$

**82** The defining vectors from R are  $\overrightarrow{RS} = \mathbf{s} - \mathbf{r}$

$$\begin{aligned} &= (3\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \\ &= \mathbf{i} + 3\mathbf{j} + \mathbf{k} \end{aligned}$$

and  $\overrightarrow{RT} = \mathbf{t} - \mathbf{r}$

$$\begin{aligned} &= (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \\ &= -\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \end{aligned}$$

$$\begin{aligned} \text{Now } \overrightarrow{RS} \times \overrightarrow{RT} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ -1 & 4 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} \mathbf{k} \\ &= -10\mathbf{i} + \mathbf{j} + 7\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= \frac{1}{2} |\overrightarrow{RS} \times \overrightarrow{RT}| \\ &= \frac{1}{2} \sqrt{(-10)^2 + 1^2 + 7^2} \\ &= \frac{1}{2} \sqrt{150} = \frac{5}{6} \sqrt{6} \text{ units}^2 \end{aligned}$$

**83 a**  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$  and  $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

**i** component of  $\mathbf{a}$  in the direction of  $\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$

$$\begin{aligned} &= \frac{2(1) + 3(-1) - 1(2)}{\sqrt{1^2 + (-1)^2 + 2^2}} \\ &= -\frac{3}{\sqrt{6}} \end{aligned}$$

**ii**  $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

$$\begin{aligned} &= \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\ &= 5\mathbf{i} - 5\mathbf{j} - 5\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{component of } \mathbf{a} \text{ perpendicular to } \mathbf{b} &= \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|} \\ &= \frac{\sqrt{5^2 + (-5)^2 + (-5)^2}}{\sqrt{1^2 + (-1)^2 + 2^2}} \\ &= \frac{\sqrt{75}}{\sqrt{6}} \\ &= \sqrt{\frac{25}{2}} \\ &= \frac{5}{\sqrt{2}} \end{aligned}$$



$$\mathbf{b} \quad \mathbf{a} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{b} = 5\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$$

$$\begin{aligned} \text{i} \quad \text{component of } \mathbf{a} \text{ in the direction of } \mathbf{b} &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \\ &= \frac{-2(5) - 1(-7) + 3(2)}{\sqrt{5^2 + (-7)^2 + 2^2}} \\ &= \frac{3}{\sqrt{78}} \end{aligned}$$

$$\begin{aligned} \text{ii} \quad \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 3 \\ 5 & -7 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ -7 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -1 \\ 5 & -7 \end{vmatrix} \mathbf{k} \\ &= 19\mathbf{i} + 19\mathbf{j} + 19\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{component of } \mathbf{a} \text{ perpendicular to } \mathbf{b} &= \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|} \\ &= \frac{\sqrt{19^2 + 19^2 + 19^2}}{\sqrt{5^2 + (-7)^2 + 2^2}} \\ &= \frac{19\sqrt{3}}{\sqrt{78}} \\ &= \frac{19}{\sqrt{26}} \end{aligned}$$

$$84 \quad \mathbf{a} \quad \text{i} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\text{ii} \quad x = 2 + 3\lambda, \quad y = -1 + 2\lambda, \quad z = 3 - \lambda, \quad \lambda \in \mathbb{R}$$

$$\mathbf{b} \quad \text{i} \quad \text{Since the line is perpendicular to the } YZ\text{-plane, } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ is a possible direction vector.}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\text{ii} \quad x = \lambda, \quad y = 1, \quad z = 2, \quad \lambda \in \mathbb{R}$$

$$85 \quad \mathbf{m} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{n} = -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$$

$$\mathbf{a} \quad \mathbf{m} + \mathbf{n} = \mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

$$\begin{aligned} \therefore |\mathbf{m} + \mathbf{n}| &= \sqrt{1 + 36 + 4} \\ &= \sqrt{41} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{m} \cdot \mathbf{n} &= 3(-2) + 1(5) + 2(-4) \\ &= -6 + 5 - 8 \\ &= -9 \end{aligned}$$

$$\mathbf{c} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$86 \quad \mathbf{a} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\mathbf{b} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\mathbf{c} \quad \text{Since the line is perpendicular to the } XZ\text{-plane, } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ is a possible direction vector.}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$87 \quad \mathbf{v}_A = \begin{pmatrix} 20 \\ -5 \\ 7 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_B = \begin{pmatrix} k \\ -11 \\ 15 \end{pmatrix}$$

$$\mathbf{a} \quad \text{The objects are moving in perpendicular directions.}$$

$$\begin{aligned} \mathbf{v}_A \cdot \mathbf{v}_B &= 0 \\ \therefore 20k + 55 + 105 &= 0 \\ \therefore 20k &= -160 \\ \therefore k &= -8 \end{aligned}$$



**b** Speed of object A =  $|\mathbf{v}_A|$ 

$$\begin{aligned}
&= \sqrt{20^2 + (-5)^2 + 7^2} \\
&= \sqrt{474} \\
&\approx 21.8 \text{ m s}^{-1}
\end{aligned}$$

Speed of object B =  $|\mathbf{v}_B|$ 

$$\begin{aligned}
&= \sqrt{(-8)^2 + (-11)^2 + 15^2} \\
&= \sqrt{410} \\
&\approx 20.2 \text{ m s}^{-1}
\end{aligned}$$

 $\therefore$  object A is moving faster.

$$88 \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{pmatrix} 0 - (-1) \\ 1 - 2 \\ 3 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\therefore \text{ line (AB) has equation } \mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$\mathbf{b} \quad \text{The line } L \text{ has direction vector } \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \text{ and (AB) has direction vector } \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

$$\begin{aligned}
\text{If } \theta \text{ is the angle between the lines, then } \cos \theta &= \frac{2 + 0 - 2}{\sqrt{2^2 + 0^2 + (-1)^2} \sqrt{1^2 + (-1)^2 + 2^2}} \\
&= 0 \\
\therefore \theta &= 90^\circ
\end{aligned}$$

So, the angle between (AB) and  $L$  is  $90^\circ$ .

$$89 \quad \text{The lines meet where } \begin{pmatrix} -3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

$$\therefore -3 + 2s = -1 - 3t \quad \text{and} \quad 2 - s = 6 + 4t$$

$$\therefore 2s + 3t = 2 \quad \dots (1) \quad \text{and} \quad s + 4t = -4 \quad \dots (2)$$

$$\begin{array}{ll}
2s + 3t = 2 & \{(1)\} \\
-2s - 8t = 8 & \{(2) \times (-2)\} \\
\hline
\therefore -5t = 10 & \\
\therefore t = -2 &
\end{array}$$

$$\begin{aligned}
\text{Using (2), } s + 4(-2) &= -4 \\
\therefore s &= 4
\end{aligned}$$

$$\text{Using line 1, } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + 4 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$\text{Checking in line 2, } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + (-2) \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad \checkmark$$

 $\therefore$  the lines meet at  $(5, -2)$ .

$$90 \quad \mathbf{a} \quad \text{For toy car A, the direction vector is } \begin{pmatrix} -2 \\ 3 \end{pmatrix} \text{ which has length } \sqrt{4 + 9} = \sqrt{13} \text{ units.}$$

$$\therefore \text{ the velocity vector is } \frac{\sqrt{13}}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\text{For toy car B, the direction vector is } \begin{pmatrix} 5 - (-1) \\ 7 - 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \text{ and takes 3 seconds to get there.}$$

$$\therefore \text{ the velocity vector is } \frac{1}{3} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad 0 \leq t \leq 3 \text{ s}$$



- b** Car  $A$  has coordinates  $(9 - 2t, -3 + 3t)$ .

$$\begin{aligned}\therefore x = 3 \text{ when } 9 - 2t &= 3 \\ \therefore 2t &= 6 \\ \therefore t &= 3\end{aligned}$$

At this time,  $y = -3 + 3 \times 3 = 6$

$\therefore$  car  $A$  passes through  $(3, 6)$  when  $t = 3$  seconds.

Car  $B$  has coordinates  $(-1 + 2t, 4 + t)$ .

$$\begin{aligned}\therefore x = 3 \text{ when } -1 + 2t &= 3 \\ \therefore 2t &= 4 \\ \therefore t &= 2\end{aligned}$$

At this time,  $y = 4 + 2 = 6$

$\therefore$  car  $B$  passes through  $(3, 6)$  when  $t = 2$  seconds.

So, both cars will pass through  $(3, 6)$ .

- c** The cars pass through  $(3, 6)$  at different times, so they will not collide.

**91 a** Ian:  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix} + t \begin{pmatrix} 0.6 \\ -0.5 \end{pmatrix}, \quad t \in \mathbb{R}$

Grant:  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}, \quad t \in \mathbb{R}$

**b i** When  $t = 2$ , Ian has position vector  $\begin{pmatrix} 9 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 0.6 \\ -0.5 \end{pmatrix} = \begin{pmatrix} 10.2 \\ 6 \end{pmatrix}$  and

Grant has position vector  $\begin{pmatrix} -3 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} = \begin{pmatrix} -1.6 \\ 2.6 \end{pmatrix}.$

$$\begin{aligned}\therefore \text{distance} &= \sqrt{(-1.6 - 10.2)^2 + (2.6 - 6)^2} \\ &= \sqrt{(-11.8)^2 + (-3.4)^2} \\ &= \sqrt{150.8} \\ &\approx 12.3 \text{ m}\end{aligned}$$

**ii** When  $t = 10$ , Ian has position vector  $\begin{pmatrix} 9 \\ 7 \end{pmatrix} + 10 \begin{pmatrix} 0.6 \\ -0.5 \end{pmatrix} = \begin{pmatrix} 15 \\ 2 \end{pmatrix}$  and

Grant has position vector  $\begin{pmatrix} -3 \\ 2 \end{pmatrix} + 10 \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$

$$\begin{aligned}\therefore \text{distance} &= \sqrt{(4 - 15)^2 + (5 - 2)^2} \\ &= \sqrt{(-11)^2 + 3^2} \\ &= \sqrt{130} \\ &\approx 11.4 \text{ m}\end{aligned}$$

- c** At time  $t$ , Ian is at  $(9 + 0.6t, 7 - 0.5t)$ , and Grant is at  $(-3 + 0.7t, 2 + 0.3t)$ .

$$\begin{aligned}\text{The distance between Ian and Grant at time } t \text{ is } D &= \sqrt{[(-3 + 0.7t) - (9 + 0.6t)]^2 + [(2 + 0.3t) - (7 - 0.5t)]^2} \\ &= \sqrt{(-12 + 0.1t)^2 + (-5 + 0.8t)^2} \\ &= \sqrt{144 - 2.4t + 0.01t^2 + 25 - 8t + 0.64t^2} \\ &= \sqrt{0.65t^2 - 10.4t + 169} \text{ m}\end{aligned}$$

Now  $D$  is minimised when  $D^2 = 0.65t^2 - 10.4t + 169$  is minimised.

This occurs when  $t = -\frac{-10.4}{2 \times 0.65} = 8$

When  $t = 8$ ,  $D = \sqrt{0.65(8)^2 - 10.4(8) + 169} = \sqrt{127.4} \approx 11.3$

The shortest distance between the swimmers is  $\approx 11.3$  m, and this occurs after 8 seconds.



- 92 a**  $L_1$  and  $L_2$  have direction vectors  $\begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$  respectively.

Since  $\begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ ,  $L_1$  is parallel to  $L_2$ .

- b**  $L_1$  and  $L_3$  meet if  $\begin{cases} 2 + 3t = 5 - 3s \\ 1 - 6t = 5 + 4s \\ -1 - 3t = 1 + 2s \end{cases}$  which is  $\begin{cases} 3s + 3t = 3 \\ 4s + 6t = -4 \\ 2s + 3t = -2 \end{cases}$ .

Solving this system using technology,  $s = 5$  and  $t = -4$ .

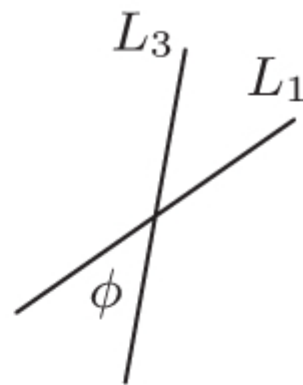
Substituting  $s = 5$  into  $L_3$ , the lines meet at  $(5 - 3(5), 5 + 4(5), 1 + 2(5))$  which is  $(-10, 25, 11)$ .

$L_3$  has direction vector  $\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \phi &= \frac{\left| \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|}{\sqrt{9 + 36 + 9} \sqrt{9 + 16 + 4}} \\ &= \frac{|-9 - 24 - 6|}{\sqrt{54} \times 5} \\ &= \frac{39}{\sqrt{1566}} \end{aligned}$$

$$\therefore \phi = \cos^{-1} \left( \frac{39}{\sqrt{1566}} \right) \approx 9.76^\circ$$

$\therefore$  the angle between  $L_1$  and  $L_3$  is about  $9.76^\circ$ .



- 93**  $x(t) = 1 - 2t$ ,  $y(t) = t^3 + t + 2$

**a**  $x(0) = 1$ ,  $y(0) = 2$

$\therefore$  the initial position of P is  $(1, 2)$ .

**c i** When  $t = 0$ , the velocity vector is  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ .

$$\therefore \text{speed} = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \approx 2.24 \text{ cm s}^{-1}.$$

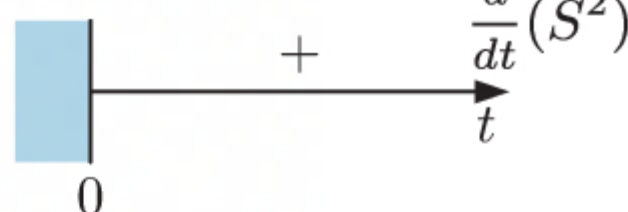
**ii** When  $t = 5$ , the velocity vector is  $\begin{pmatrix} -2 \\ 76 \end{pmatrix}$ .

$$\therefore \text{speed} = \sqrt{(-2)^2 + 76^2} = \sqrt{5780} \approx 76.0 \text{ cm s}^{-1}.$$

**d** At time  $t$ , P has speed  $S = \sqrt{(-2)^2 + (3t^2 + 1)^2}$   
 $= \sqrt{4 + 9t^4 + 6t^2 + 1}$   
 $= \sqrt{9t^4 + 6t^2 + 5}$

Now  $S$  is minimised when  $S^2 = 9t^4 + 6t^2 + 5$  is minimised.

$$\begin{aligned} \therefore \frac{d}{dt}(S^2) &= 36t^3 + 12t \\ &= 12t(3t^2 + 1) \end{aligned}$$



$$\therefore \frac{d}{dt}(S^2) \geq 0 \text{ for all } 0 \leq t \leq 5.$$

$\therefore S^2$  is increasing for all  $0 \leq t \leq 5$ .

$\therefore S$  is increasing for all  $0 \leq t \leq 5$ .

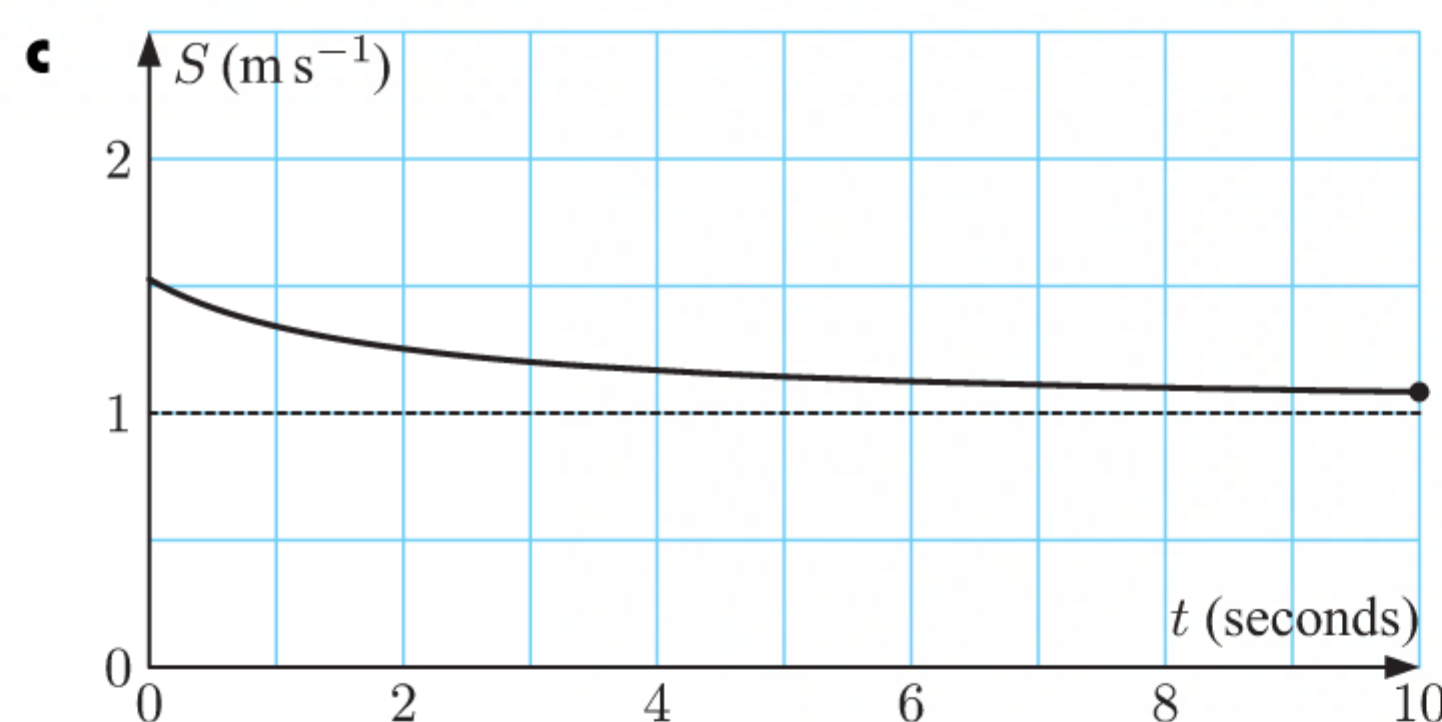
$\therefore$  minimum speed is  $\sqrt{5} \approx 2.24 \text{ cm s}^{-1}$  when  $t = 0$ , and maximum speed is  $\sqrt{5780} \approx 76.0 \text{ cm s}^{-1}$  when  $t = 5$ .

- 94 a** When  $t = 0$ , the velocity vector is  $\begin{pmatrix} \frac{2}{\sqrt{3}} \\ 1 \end{pmatrix}$ .

$$\therefore \text{initial speed} = \sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 + 1^2} = \sqrt{\frac{4}{3} + 1} = \sqrt{\frac{7}{3}} \text{ m s}^{-1}$$



$$\begin{aligned}
 \text{b } S &= \sqrt{\left(\frac{2}{\sqrt{2t+3}}\right)^2 + 1^2} \\
 &= \sqrt{\frac{4}{2t+3} + 1} \\
 &= \sqrt{\frac{4+2t+3}{2t+3}} \\
 &= \sqrt{\frac{2t+7}{2t+3}} \text{ m s}^{-1}
 \end{aligned}$$



**d** As  $t \rightarrow \infty$ ,  $S(t) \rightarrow 1 \text{ m s}^{-1}$ .

$$\begin{aligned}
 \text{e } x(t) &= \int \frac{2}{\sqrt{2t+3}} dt \\
 &= \int 2(2t+3)^{-\frac{1}{2}} dt \\
 &= 2(2t+3)^{\frac{1}{2}} + c \\
 &= 2\sqrt{2t+3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } x(3) &= 9 \quad \therefore 2\sqrt{2(3)+3} + c = 9 \\
 &\quad \therefore 2\sqrt{9} + c = 9 \\
 &\quad \therefore 6 + c = 9 \\
 &\quad \therefore c = 3
 \end{aligned}$$

$$\therefore x(t) = 2\sqrt{2t+3} + 3$$

$$\begin{aligned}
 y(t) &= \int 1 dt \\
 &= t + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } y(3) &= 4 \quad \therefore 3 + c = 4 \\
 &\quad \therefore c = 1
 \end{aligned}$$

$$\therefore y(t) = t + 1$$

- f i**  $x(0) = 2\sqrt{3} + 3 \approx 6.46$  and  $y(0) = 1$   
 $\therefore$  the initial position of the car was  $(2\sqrt{3} + 3, 1)$  or about  $(6.46, 1)$ .  
**ii**  $x(11) = 2\sqrt{2(11)+3} + 3 = 2\sqrt{25} + 3 = 13$  and  $y(11) = 12$   
 $\therefore$  after 11 seconds, the car is at  $(13, 12)$ .

**95 a**  $y(0) = 1.4$

$\therefore$  the ball was 1.4 m above the ground initially.

**b** When it reaches the ground,  $y = 0$

$$\therefore 1.4 + 14t - 4.9t^2 = 0$$

$$\therefore 4.9t^2 - 14t - 1.4 = 0$$

$$\therefore t = \frac{14 \pm \sqrt{(-14)^2 - 4(4.9)(-1.4)}}{9.8}$$

$$\therefore t = \frac{14 \pm \sqrt{223.44}}{9.8}$$

$$\therefore t \approx 2.954 \text{ or } -0.097$$

$\therefore$  the ball reached the ground when  $t \approx 2.954$   $\{t \geq 0\}$

$\therefore$  the ball was in the air for about 2.954 seconds.

**c**  $x(0) = 0$  and  $x(2.954) \approx 38.40$

$\therefore$  the ball travelled about 38.40 m horizontally.

**d**  $y'(t) = 14 - 9.8t$  which is 0 when  $t = \frac{14}{9.8} = \frac{10}{7}$  and  $y\left(\frac{10}{7}\right) = 1.4 + 14\left(\frac{10}{7}\right) - 4.9\left(\frac{10}{7}\right)^2 = 11.4$

$\therefore$  maximum height reached is 11.4 m.



$$\mathbf{e} \quad \mathbf{i} \quad \mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 13 \\ 14 - 9.8t \end{pmatrix}$$

$$\mathbf{ii} \quad \text{When } t \approx 2.954, \quad \mathbf{v} \approx \begin{pmatrix} 13 \\ -14.948 \end{pmatrix}$$

$$\tan \theta \approx \frac{14.948}{13}$$

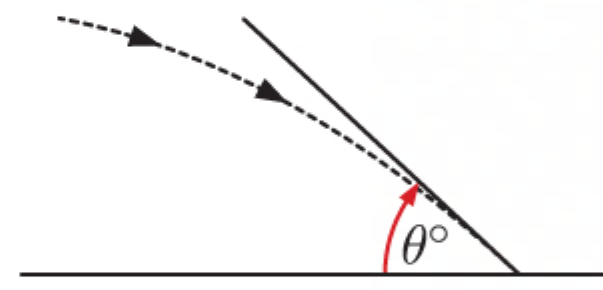
$$\therefore \theta \approx 49.0^\circ$$

$$\mathbf{f} \quad \mathbf{i} \quad \text{When } t = 0, \quad \mathbf{v} = \begin{pmatrix} 13 \\ 14 \end{pmatrix}$$

$$\therefore \text{initial speed} = \sqrt{13^2 + 14^2} \approx 19.1 \text{ m s}^{-1}$$

$$\mathbf{ii} \quad \text{When } t \approx 2.954, \quad \mathbf{v} \approx \begin{pmatrix} 13 \\ -14.948 \end{pmatrix}$$

$$\therefore \text{final speed} \approx \sqrt{13^2 + (-14.948)^2} \approx 19.8 \text{ m s}^{-1}$$



$$\mathbf{96} \quad \mathbf{a} \quad \mathbf{A} = \begin{pmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\mathbf{b} \quad \mathbf{i} \quad \mathbf{A} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} - \sqrt{3} \\ -\frac{\sqrt{3}}{2} - 1 \end{pmatrix}$$

$$\therefore \text{the image is } \left( \frac{1}{2} - \sqrt{3}, -\frac{\sqrt{3}}{2} - 1 \right)$$

$$\mathbf{ii} \quad \mathbf{A} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} + \frac{3}{2} \\ \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

$$\therefore \text{the image is } (1, \sqrt{3}).$$

$$\mathbf{97} \quad \mathbf{a} \quad \mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\mathbf{b} \quad \text{Since } \mathbf{A} \text{ has the form } \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \text{ and } |\mathbf{A}| = -\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = -1, \mathbf{A} \text{ is a reflection matrix where } \cos 2\alpha = \frac{1}{2}$$

$$\text{and } \sin 2\alpha = \frac{\sqrt{3}}{2}.$$

$$\therefore \tan 2\alpha = \sqrt{3}$$

$$\therefore 2\alpha = \frac{\pi}{3}$$

$$\therefore \alpha = \frac{\pi}{6}$$

$$\therefore \tan \alpha = \frac{1}{\sqrt{3}}$$

$$\therefore \text{the transformation is a reflection in the line } y = \frac{1}{\sqrt{3}}x.$$

$$\mathbf{c} \quad \mathbf{A} \begin{pmatrix} \sqrt{6} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \sqrt{6} \\ \sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{2} \\ \frac{\sqrt{18}}{2} - \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{6} \\ \sqrt{2} \end{pmatrix}$$

$$\therefore \text{the image is } (\sqrt{6}, \sqrt{2}), \text{ which is the same as the object point.}$$

Now since  $\frac{1}{\sqrt{3}}x = \frac{1}{\sqrt{3}} \times \sqrt{6} = \sqrt{2} = y$ ,  $(\sqrt{6}, \sqrt{2})$  lies on the line  $y = \frac{1}{\sqrt{3}}x$ , which is the mirror line of the reflection. Hence,  $(\sqrt{6}, \sqrt{2})$  is invariant under the transformation.



**98 a**  $\mathbf{A} = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{pmatrix}$

**b**  $\mathbf{A} \begin{pmatrix} 12 \\ -24 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 12 \\ -24 \end{pmatrix}$   
 $= \begin{pmatrix} 16 \\ -24 \end{pmatrix}$

$\therefore$  the image is  $(16, -24)$ .

**c**  $\mathbf{A}^2 = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{pmatrix}$   
 $= \begin{pmatrix} \frac{16}{9} & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore$  the transformation is a horizontal stretch with scale factor  $\frac{16}{9}$ .

**99 a**  $\mathbf{A} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{5}{2} \end{pmatrix}$

**b**  $\mathbf{A} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$   
 $= \begin{pmatrix} -5 \\ 10 \end{pmatrix}$

$\therefore$  the image is  $(-5, 10)$ .

**c**  $\mathbf{A}^{-1} = \frac{1}{(\frac{5}{2})^2 - 0} \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{5}{2} \end{pmatrix}$   
 $= \frac{4}{25} \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{5}{2} \end{pmatrix}$   
 $= \begin{pmatrix} \frac{2}{5} & 0 \\ 0 & \frac{2}{5} \end{pmatrix}$

$\therefore \mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$  is a reduction with scale factor  $\frac{2}{5}$ .

**100 a**  $\mathbf{A} = \begin{pmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

**b i**  $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  **ii**  $\mathbf{x}' = \mathbf{A}(\mathbf{x} + \mathbf{b})$   
 $\therefore \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{b}$

**c i** If the transformations in **b** are equivalent, then the matrix equations in **b i** and **b ii** must be the same.

$\therefore \cancel{\mathbf{A}\mathbf{x}} + \mathbf{A}\mathbf{b} = \cancel{\mathbf{A}\mathbf{x}} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$   
 $\therefore \mathbf{A}\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

**ii**  $\mathbf{A}\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  {from **i**}

$\therefore \mathbf{b} = \mathbf{A}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$   
 $= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$   $\{|\mathbf{A}| = 1\}$   
 $= \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} + \frac{1}{2} \end{pmatrix}$   
 $= \begin{pmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{pmatrix}$

**101 a i**  $m = \tan \alpha = -3$

$\therefore \alpha = \tan^{-1}(-3)$

$\therefore \cos 2\alpha = -0.8$  and  $\sin 2\alpha = -0.6$

$\therefore \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -0.8 & -0.6 \\ -0.6 & 0.8 \end{pmatrix}$

**ii**  $\mathbf{B} = \begin{pmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} 0 & -(-1) \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

**b i**  $\mathbf{BA} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -0.8 & -0.6 \\ -0.6 & 0.8 \end{pmatrix}$   
 $= \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$



ii  $\mathbf{BA}$  has the form  $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$  and  $|\mathbf{BA}| = -(0.6)^2 - (0.8)^2 = -1$ ,  $\mathbf{BA}$  is a reflection matrix where  $\cos 2\alpha = -0.6$  and  $\sin 2\alpha = 0.8$ .

$$\therefore \tan 2\alpha = -\frac{0.8}{0.6} \quad \left\{ \frac{\pi}{2} \leq 2\alpha \leq \pi \right\}$$

$$\therefore 2\alpha = \tan^{-1}\left(-\frac{0.8}{0.6}\right) + \pi$$

$$\therefore \alpha = \frac{1}{2} \tan^{-1}\left(-\frac{0.8}{0.6}\right) + \frac{\pi}{2}$$

$$\therefore \tan \alpha = 2$$

$\therefore$  the composite transformation is a reflection in the line  $y = 2x$ .

102 a  $\mathbf{x}' = \begin{pmatrix} \frac{5}{3} & 0 \\ 0 & \frac{5}{3} \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

b  $\mathbf{x}' = \begin{pmatrix} \frac{5}{3} & 0 \\ 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} -6 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}$   
 $= \begin{pmatrix} -10 \\ \frac{10}{3} \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}$   
 $= \begin{pmatrix} -11 \\ \frac{16}{3} \end{pmatrix}$

$\therefore$  the image is  $(-11, \frac{16}{3})$ .

103 a  $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} hk & 0 \\ 0 & k \end{pmatrix}$

b The transformation has matrix equation  $\mathbf{x}' = \begin{pmatrix} hk & 0 \\ 0 & k \end{pmatrix} \mathbf{x}$ .

The image of  $(4, 2)$  is  $(5, 10)$ .

$$\therefore \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} hk & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 4hk \\ 2k \end{pmatrix}$$

$$\therefore 4hk = 5 \quad \dots (1)$$

$$2k = 10 \quad \dots (2)$$

From (2),  $k = 5$  and substituting into (1) gives  $h = \frac{1}{4}$ .

104 a Out degree of S = 3

This is the number of parking areas that you can get to from the supermarket.

b  $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

c i Using technology,  $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}$

d i There are no 2-step routes from B to C.

c Area of image =  $|\det \mathbf{A}| \times \text{area of object}$

$$\therefore 60 = \left| \frac{25}{9} \right| \times \text{area of object}$$

$$\therefore \text{area of object} = 60 \times \frac{9}{25} = 21.6 \text{ units}^2$$

c Area of object = area of unit circle

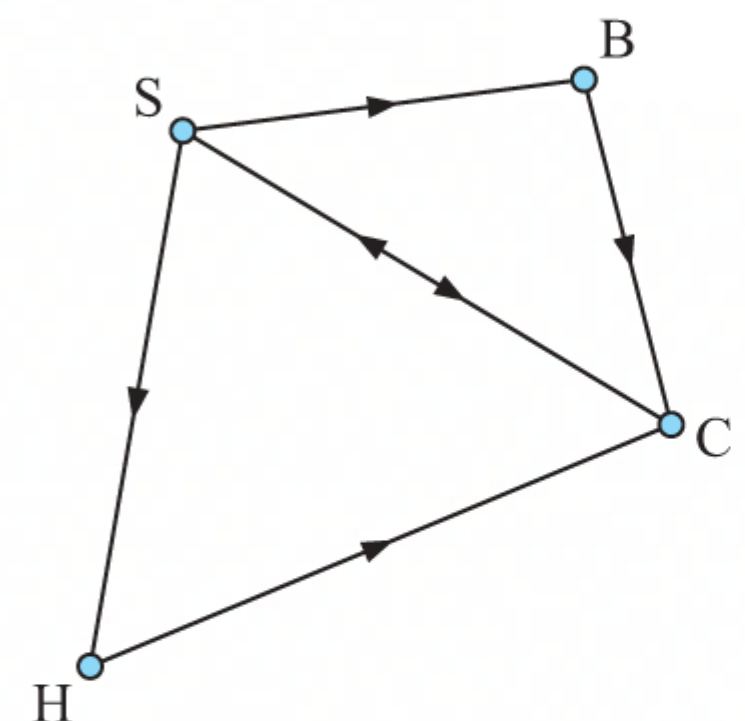
$$= \pi(1)^2$$

$$= \pi \text{ units}^2$$

Area of image =  $|\det \mathbf{A}| \times \text{area of object}$

$$= \left| \frac{5}{4} \times 5 - 0^2 \right| \times \pi$$

$$= \frac{25}{4} \pi \text{ units}^2$$



ii Using technology,  $\mathbf{A}^3 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

ii There are two 2-step routes from S to C.

e The least number of steps required to drive from the bakery to the hardware store is 3 because the element in row 1, column 3 of  $\mathbf{A}^n$  is non-zero for the first time when  $n = 3$ .



- 105 a** It is possible to travel from every vertex to every other vertex by following edges.

$\therefore$  the graph is connected.

- b** Degree of R = 4

**c**  $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

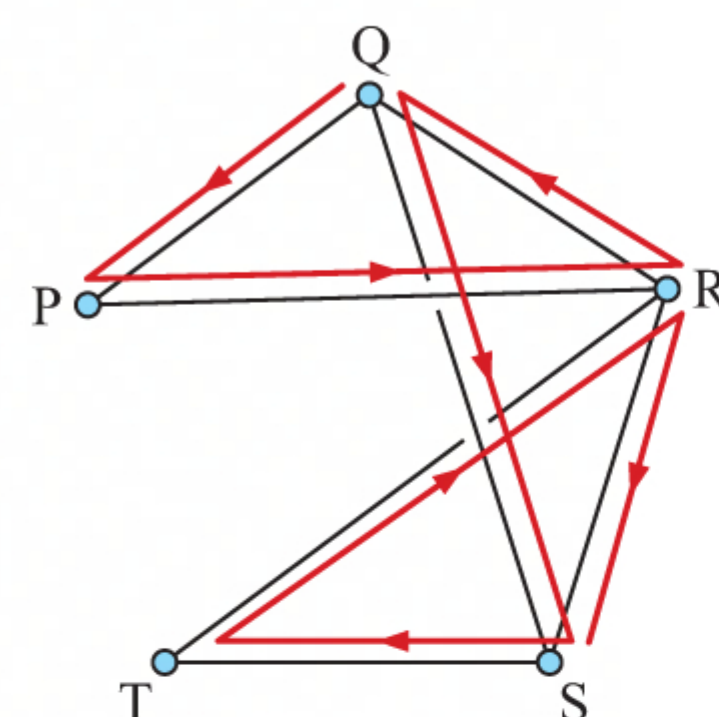
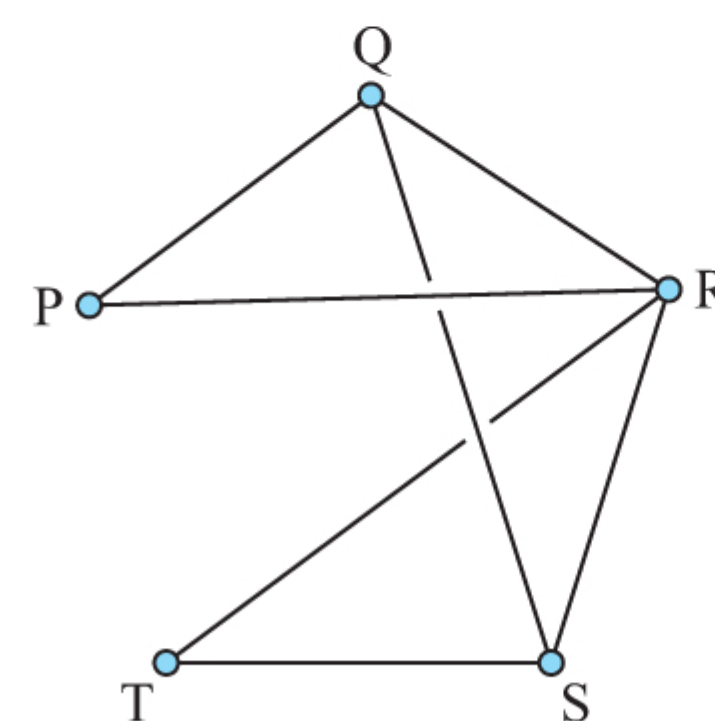
**d** Using technology,  $A^2 = \begin{pmatrix} 2 & 1 & 1 & 2 & 1 \\ 1 & 3 & 2 & 1 & 2 \\ 1 & 2 & 4 & 2 & 1 \\ 2 & 1 & 2 & 3 & 1 \\ 1 & 2 & 1 & 1 & 2 \end{pmatrix}$

The value in row 2, column 3 of  $A^2$  is 2.

$\therefore$  there are two 2-step routes from Q to R.

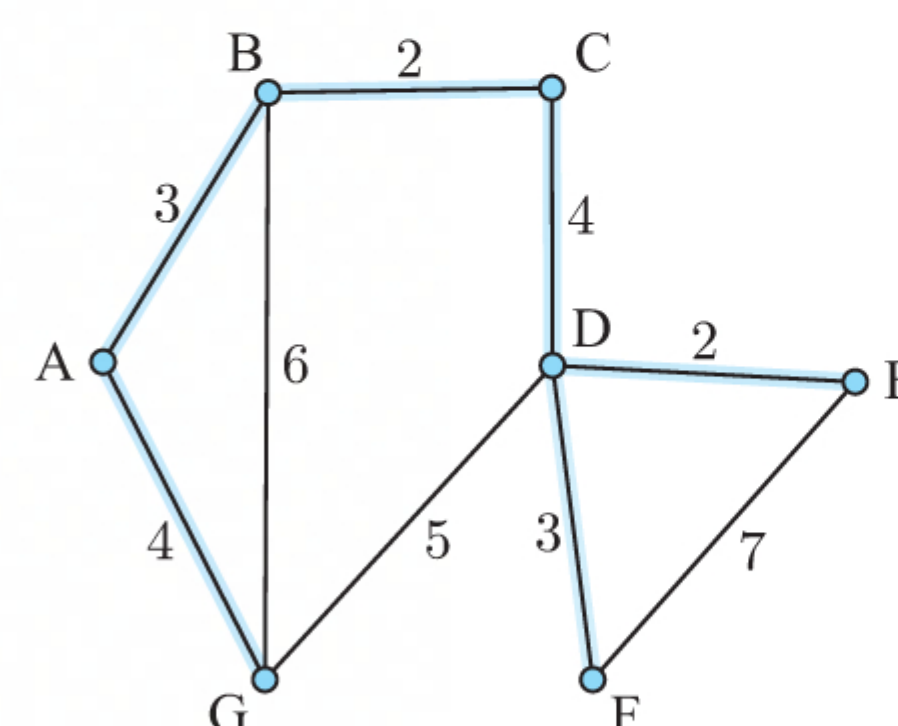
- e** The graph contains two vertices (Q and S) of odd degree, so the graph is semi-Eulerian.

- f** An Eulerian trail is  $Q \rightarrow P \rightarrow R \rightarrow Q \rightarrow S \rightarrow T \rightarrow R \rightarrow S$ .



- 106 a** There are 7 vertices, so we require 6 edges.

Edges BC and DE have the least weight. We then choose DF, BA, CD, and AG.



- b** Weight =  $2 + 2 + 3 + 3 + 4 + 4 = 18$

$\therefore$  the minimum total cost to connect all 7 cities is \$18 million.

- 107 a** The estimated cost of a bridge connecting islands B and E is \$750 000.

- b** We start with vertex A.

	①		④		②		③
	A	B	C	D	E		
A	—	650	800	300	340		
B	650	—	410	470	750		
C	800	410	—	450	500		
D	300	470	450	—	420		
E	340	750	500	420	—		

So, the edges of the minimum spanning tree are AD, AE, CD, and BC.

$\therefore$  bridges should be built between islands A and D, A and E, C and D, and B and C.

- c** Weight =  $300 + 340 + 450 + 410 = 1500$

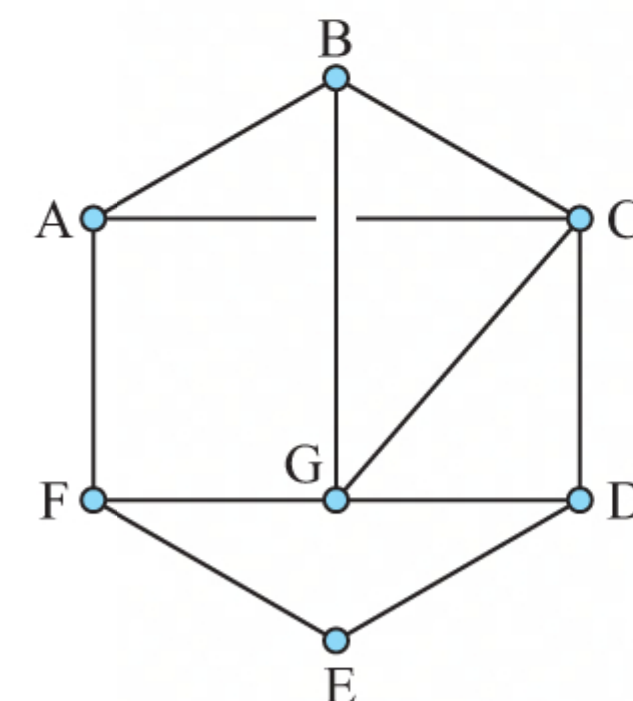
$\therefore$  the cost of constructing the bridges in **b** is \$1 500 000.



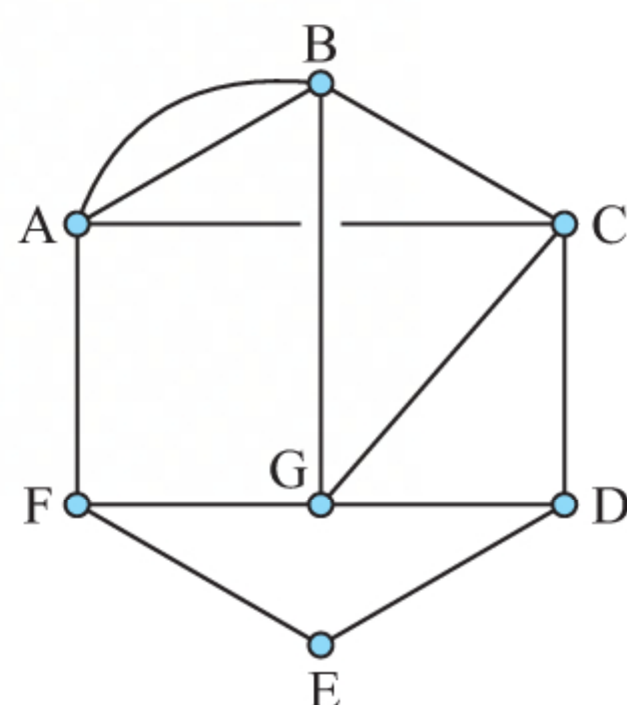
- d** Cost of constructing bridges between island B and every other island = \$650 000 + \$410 000 + \$470 000 + \$750 000  
= \$2 280 000

$\therefore$  it would cost \$2 280 000 – \$1 500 000 = \$780 000 more to construct bridges between island B and every other island.

- 108 a** The graph contains more than two vertices of odd degree (A, B, D, and F), so the graph is neither Eulerian nor semi-Eulerian.



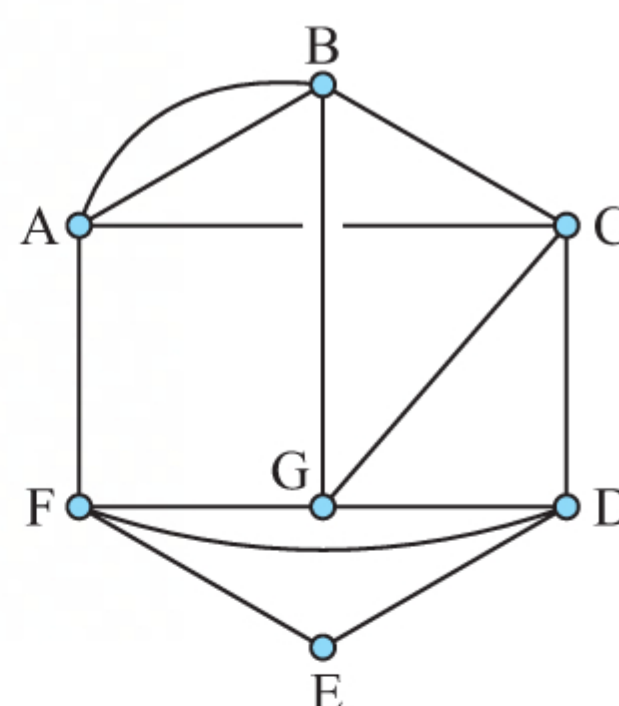
**b i**



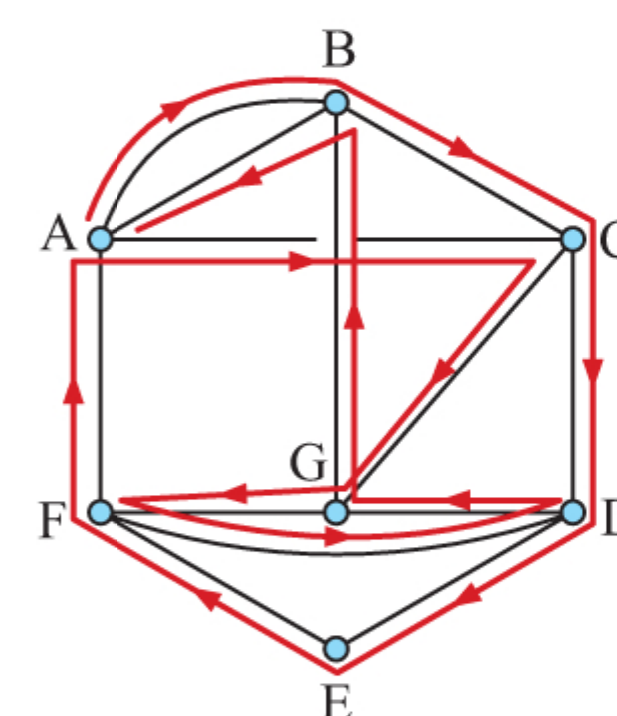
- ii** The graph contains two vertices (D and F) of odd degree, so the graph is semi-Eulerian.

- c i** The edge must be added between the vertices of odd degree: D and F.

**ii**



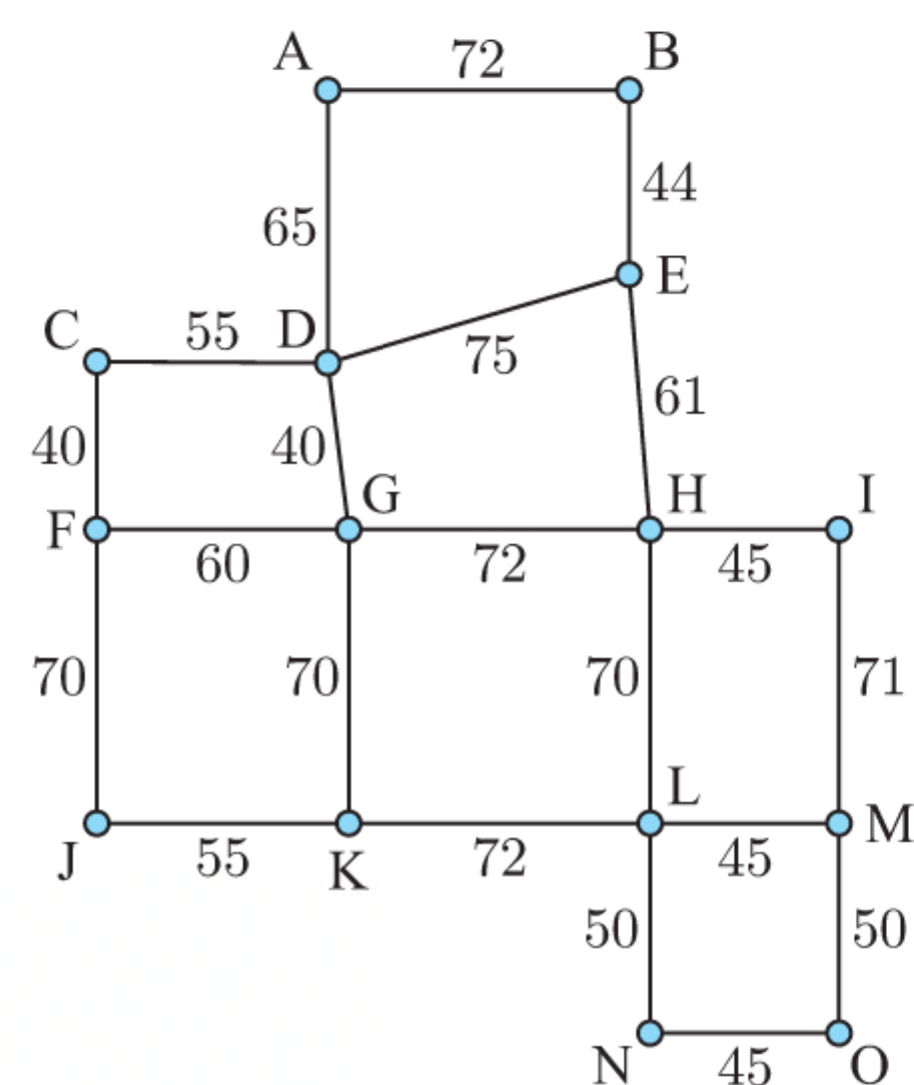
- iii** An Eulerian circuit is  $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow A \rightarrow C \rightarrow G \rightarrow F \rightarrow D \rightarrow G \rightarrow B \rightarrow A$ .



- 109 a** Not every vertex is connected to every other vertex by an edge.

$\therefore$  the graph is not complete.

- b** The vertices of odd degree are E, F, K, and M.





- c There are three possible pairings of the odd vertices: EF and KM, EK and FM, EM and FK.

For each case we find the shortest path between the vertices:

Pairing	Shortest path	Weight	Total weight
EF	$E \rightarrow D \rightarrow C \rightarrow F$	170	287
KM	$K \rightarrow L \rightarrow M$	117	
EK	$E \rightarrow D \rightarrow G \rightarrow K$	185	427
FM	$F \rightarrow J \rightarrow K \rightarrow L \rightarrow M$	242	
EM	$E \rightarrow H \rightarrow L \rightarrow M$	176	301
FK	$F \rightarrow J \rightarrow K$	125	

The combination of pairs with the lowest overall weight is EF and KM. The most efficient route traverses routes  $E \rightarrow D \rightarrow C \rightarrow F$  and  $K \rightarrow L \rightarrow M$  twice each.

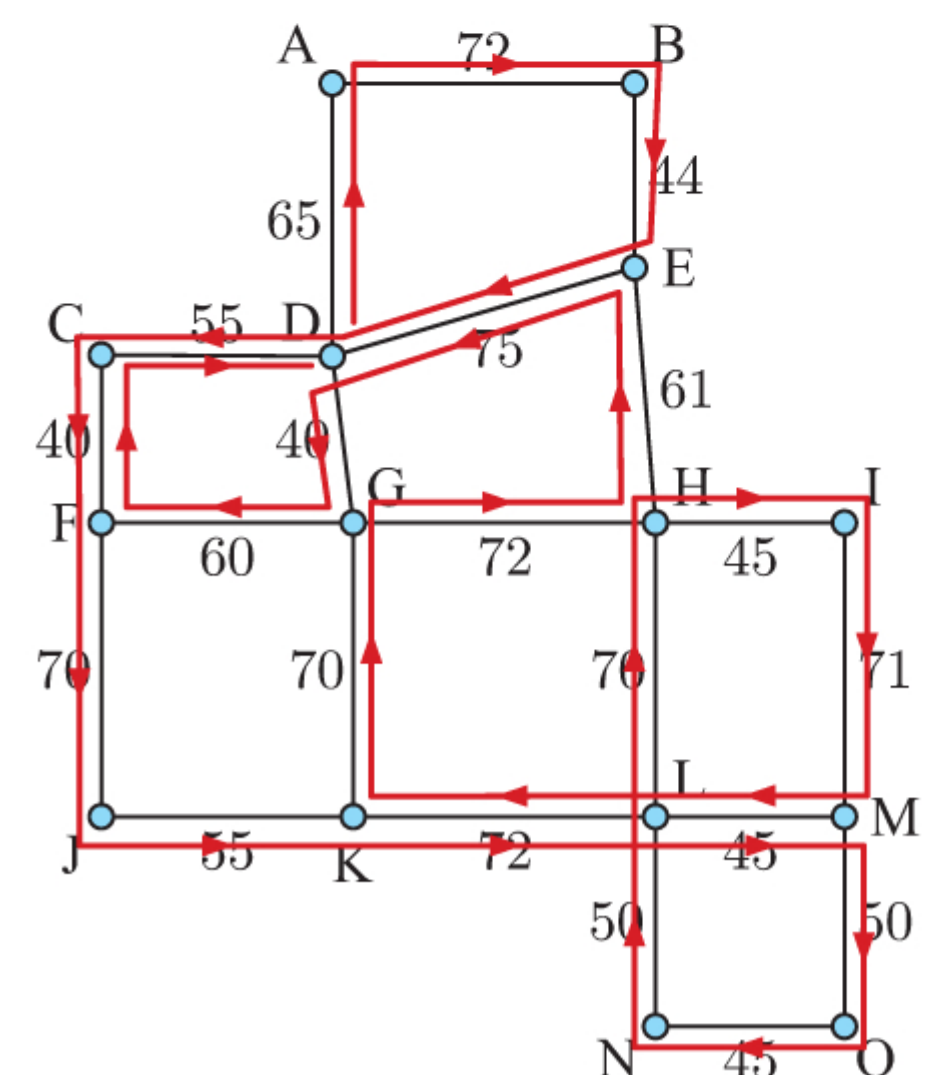
Minimum total weight

$$\begin{aligned}
 &= 72 + 44 + 61 + 45 + 71 + 50 + 45 + 50 + 72 + 55 + 70 + 40 + 55 + 65 + 75 + 72 + 70 \\
 &\quad + 45 + 70 + 60 + 40 + 287 \\
 &= 1514
 \end{aligned}$$

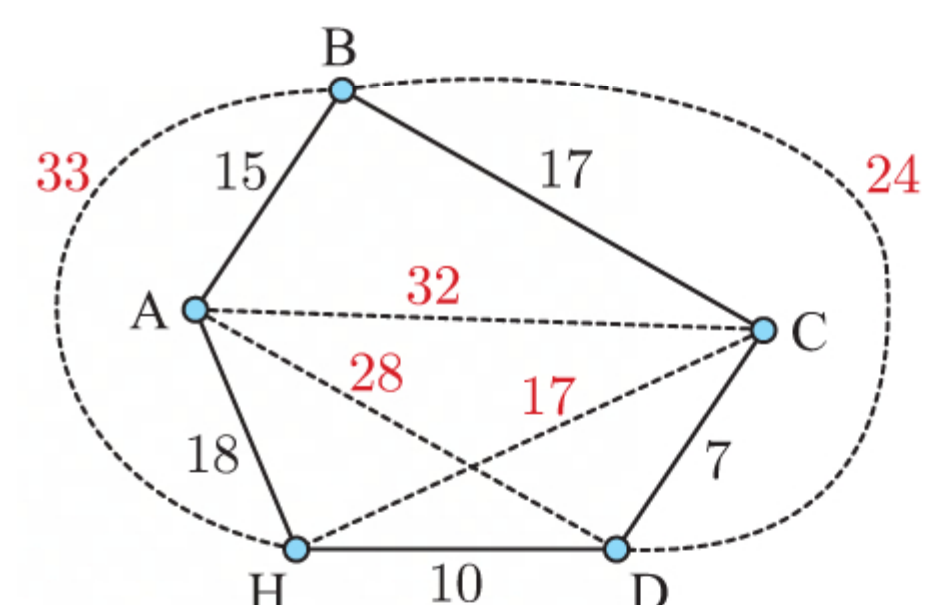
$\therefore$  the shortest distance the streetsweeper must travel is 1514 m or 1.514 km.

- d A possible route the streetsweeper can take is

$D \rightarrow A \rightarrow B \rightarrow E \rightarrow D \rightarrow C \rightarrow F \rightarrow J \rightarrow K \rightarrow L \rightarrow M \rightarrow O \rightarrow N \rightarrow L \rightarrow H \rightarrow I \rightarrow M \rightarrow L \rightarrow K \rightarrow G \rightarrow H \rightarrow E \rightarrow D \rightarrow G \rightarrow F \rightarrow C \rightarrow D$ .

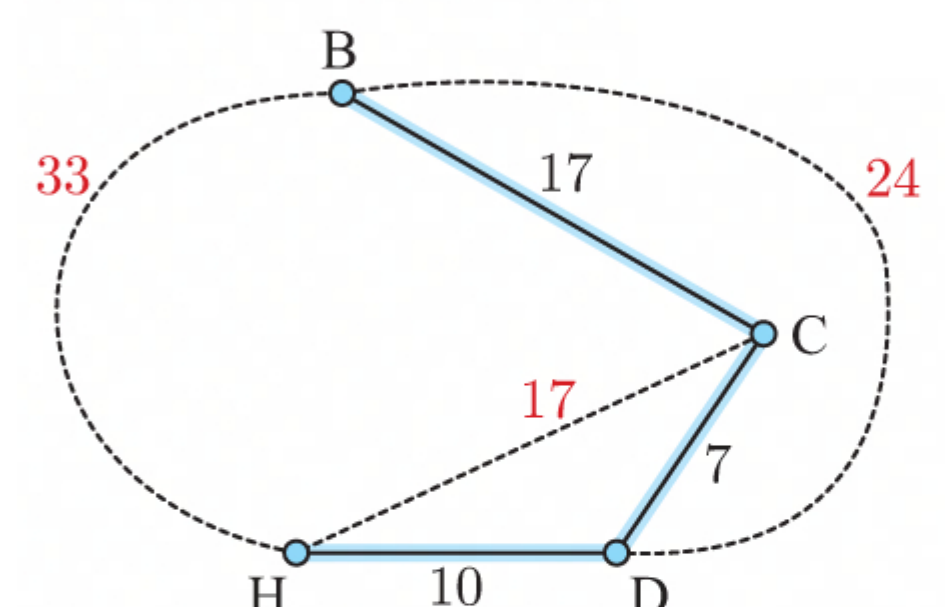


- 110 a The shortest time from A to C is  $15 + 17 = 32$  minutes.  
 The shortest time from A to D is  $18 + 10 = 28$  minutes.  
 The shortest time from B to D is  $17 + 7 = 24$  minutes.  
 The shortest time from B to H is  $15 + 18 = 33$  minutes.  
 The shortest time from C to H is  $7 + 10 = 17$  minutes.



- b Starting from vertex H, the nearest neighbour algorithm gives the Hamiltonian cycle  $H \rightarrow D \rightarrow C \rightarrow B \rightarrow A \rightarrow H$ .  
 So, an upper bound for the TSP and the time Carl will spend travelling is 67 minutes.

- c Deleting vertex A and the edges connected to it, we apply Kruskal's algorithm to find a minimum spanning tree. The minimum spanning tree has weight 34.  
 The two shortest deleted edges have weights 15 and 18.  
 So, a lower bound for the TSP and the time Carl will spend travelling is  $34 + 15 + 18 = 67$  minutes.



- d The lower bound in c is equal to the upper bound in b.  
 So, the Hamiltonian cycle in b is the quickest route.

Since the edges in that Hamiltonian cycle are from the original graph, the quickest route Carl can take to visit the universities is  $H \rightarrow D \rightarrow C \rightarrow B \rightarrow A \rightarrow H$ .



TOPIC 4 SKILL BUILDER QUESTIONS

- 1    **a** The sample size of only 10 students from a total of 500 is far too small, so this approach may produce a coverage error.
- b** The tape measure only allows Gerard to *estimate* the exact height of each student. With only 10 students being measured, any errors will significantly impact the results, so this approach may produce a measurement error.

- 2    **a**    **i** A survey of 50 students can be completed in a reasonable time frame. It is also large enough that the results can be representative of the whole student body.
- ii** Surveying all students is time-consuming and often impractical. A non-response error may be produced if students are absent.

	Boys	Girls
Year 8	135	140
Year 9	130	145
Year 10	125	130

- b** Total number of students = 135 + 140 + 130 + 145 + 125 + 130 = 805
- i** For the survey,  $\frac{135}{805} \times 50 \approx 8$  Year 8 boys will be selected.
- ii** For the survey,  $\frac{140 + 145 + 130}{805} \times 50 \approx 26$  girls will be selected.
- c** A stratified sample is better than a random sample in this case as a stratified sample will fairly represent each year level and gender. A random sample cannot guarantee such a fair representation.
- 3    **a** This is a convenience sample because it is more convenient for Marie to sample the first 10 people to visit her office than to sample 10 random people from the whole building for example.
- b** The preferences of the first 10 people to visit Marie’s office are likely to come from people who work with her. This may not be representative of the preferences of all people in the building, and so the sample may be biased.
- c** Marie could use a stratified sample where the subgroups may correspond to departments, floor number, and so on. In this way, a fair representation of preferences is more likely to be obtained.

- 4    **a** The ticket inspector selects passengers at regular intervals, so the sampling method used is systematic sampling.
- b** The first passenger to be checked is the 8th passenger.
- So, the next 6 passengers to be checked are the 28th, 48th, 68th, 88th, 108th, and 128th passengers.
- c** 5000 passengers left the terminal, and every 20th passenger was checked.
- $\therefore \frac{5000}{20} = 250$  passengers were checked.

- 5    **a** The question “What is your job?” could be interpreted as:
- “What is your occupation?”
  - “What do you do for volunteer work?”
  - “Where do you work?”

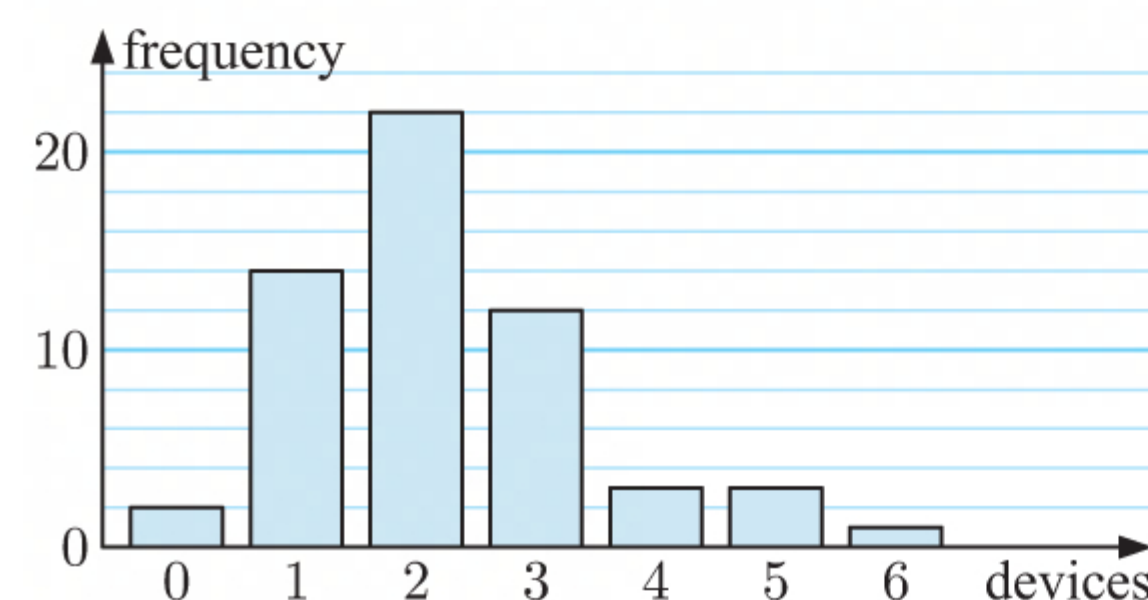
- “What is your role at home/your company/in society?”
  - “Are you unemployed?”
  - “Who do you work for?”
- b** The question could be rewritten as “What do you do for paid employment?”

- 6    **a**    **i** The question is likely to produce a measurement error because the respondents may not remember exactly what their annual income is and provide an estimate instead. In countries with income tax, some respondents may report before-tax income, while others may report after-tax income.
- ii** The question is likely to produce a non-response error because some people may not be comfortable saying how much they earn. For example, a person earning a low income may be embarrassed by their annual income and choose not to respond.
- b** The question can be improved by:
- Providing a list of income ranges from which the respondent can choose from.
  - Making the question optional.
  - Clearly specifying before or after-tax income.

- 7    **a** The number of houses on a particular street can be counted, so it is a discrete variable.
- b** The number of hours spent travelling on an airplane can be measured and can take any value between certain limits, so it is a continuous variable.
- c** The brand of laptop someone uses is a categorical variable.



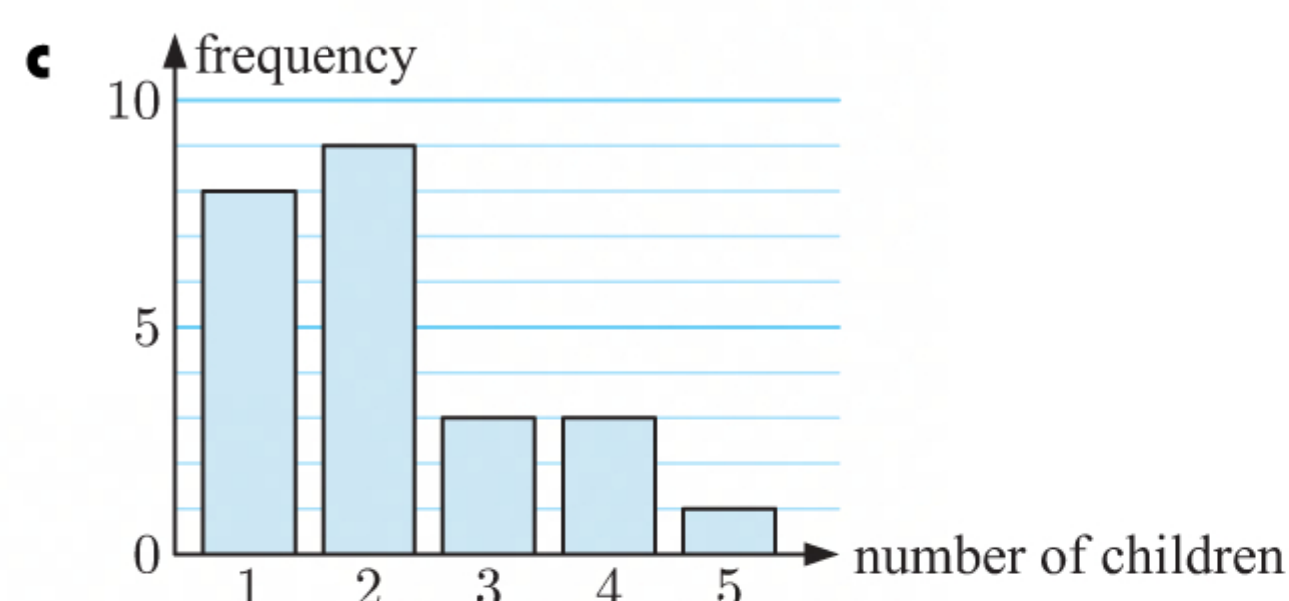
- 8 a**  $2 + 14 + 22 + 12 + 3 + 3 + 1 = 57$  people were surveyed.
- b** The mode of the data is 2 devices.
- c**  $\frac{14 + 22}{57} \times 100\% \approx 63.2\%$  of people browsed the internet using 1 or 2 devices.
- d** The data is positively skewed with no outliers.



- 9 a** The number of children in each family can be counted and takes exact number values.  
 $\therefore$  the data is discrete.

**b**

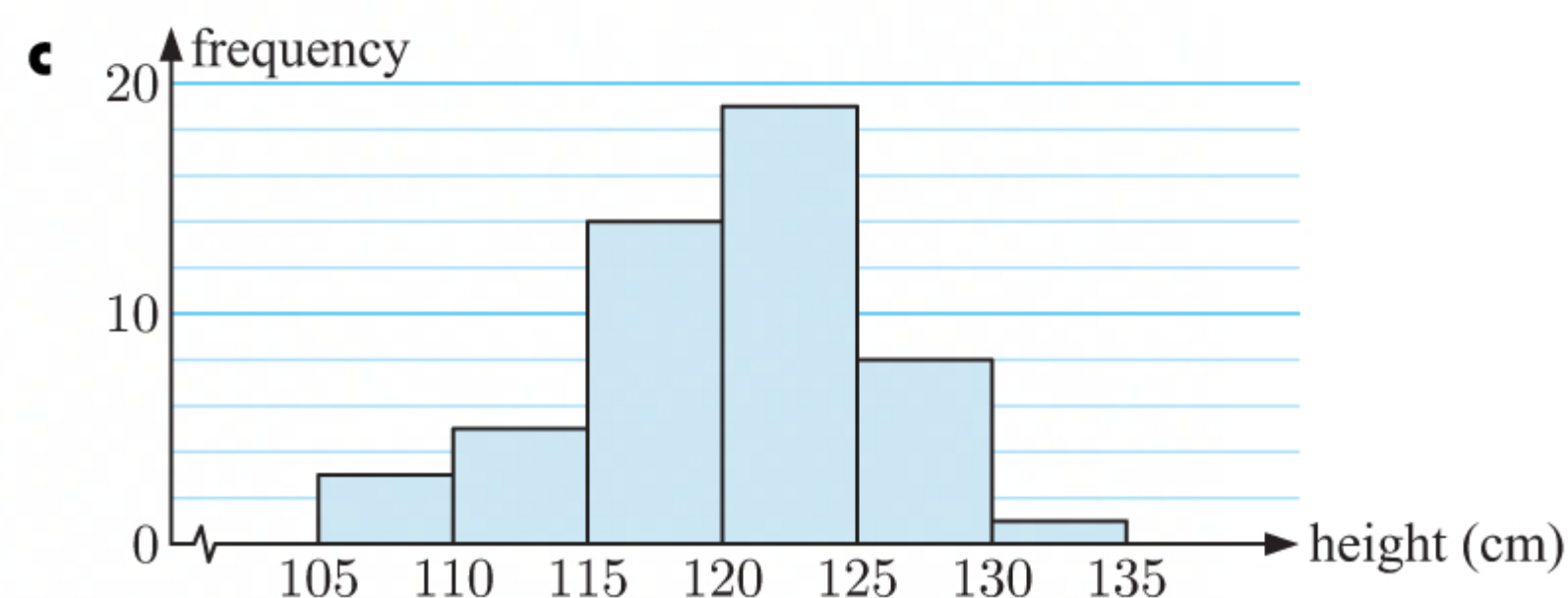
Number of children	Frequency
1	8
2	9
3	3
4	3
5	1
Total	24



- d** The data is positively skewed. There are no outliers.
- e**  $\frac{3 + 3 + 1}{24} \times 100\% \approx 29.2\%$  of families have 3 or more children.

- 10 a** The *height* of an emperor penguin is a numerical variable which can be measured.  
 $\therefore$  *height* is a continuous variable.

- b**  $3 + 5 + 14 + 19 + 8 + 1 = 50$  emperor penguins were measured.



Height ( $h$ cm)	Frequency
$105 \leq h < 110$	3
$110 \leq h < 115$	5
$115 \leq h < 120$	14
$120 \leq h < 125$	19
$125 \leq h < 130$	8
$130 \leq h < 135$	1

- d** The data appears to be approximately symmetrical.
- e** The modal class is  $120 \leq h < 125$ .  
 More emperor penguins have lengths in this interval than in any other interval.

- 11 a** 26 is the data value which occurs most often, so the mode is 26 customers.

- b** As  $n = 9$ ,  $\frac{n+1}{2} = 5$

The ordered data set is: ~~14~~ ~~16~~ ~~18~~ ~~23~~ **24** ~~25~~ ~~26~~ ~~26~~ ~~34~~

↑  
5th value

$\therefore$  median = 24 customers.

- c** mean =  $\frac{14 + 23 + \dots + 16 + 25}{9}$  ← sum of all the data values  
← 9 data values  
 $= \frac{206}{9}$   
 $\approx 22.9$  customers



**12 a i** mean number of visitors for exhibit A

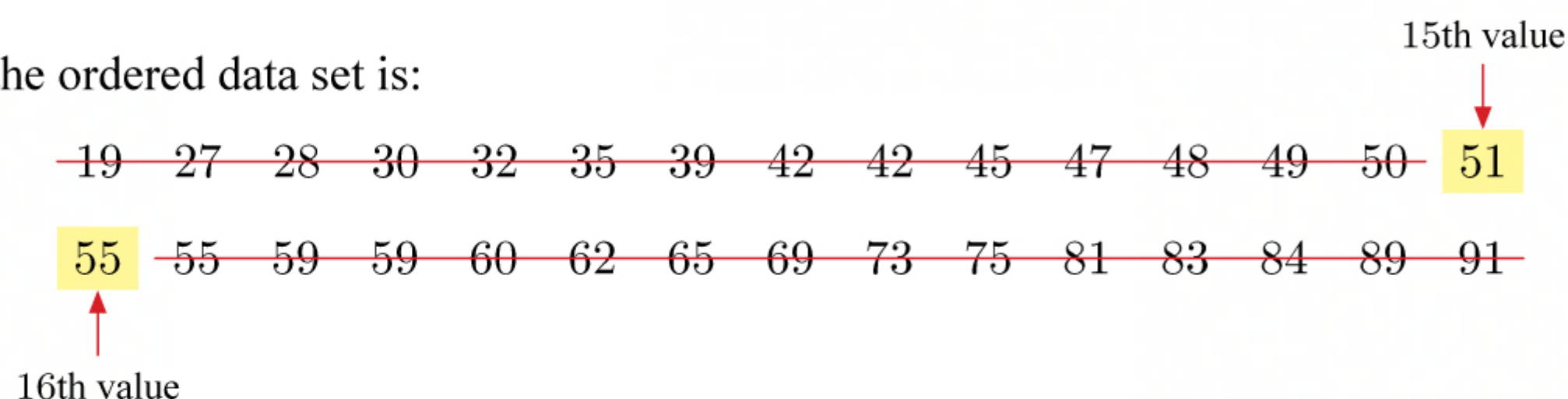
$$\begin{aligned}
 &= \frac{42 + 49 + 55 + 48 + \dots + 32}{30} \\
 &= \frac{1644}{30} \\
 &= 54.8 \text{ visitors}
 \end{aligned}$$

mean number of visitors for exhibit B

$$\begin{aligned}
 &= \frac{59 + 51 + 60 + 44 + \dots + 46}{30} \\
 &= \frac{1711}{30} \\
 &\approx 57.0 \text{ visitors}
 \end{aligned}$$

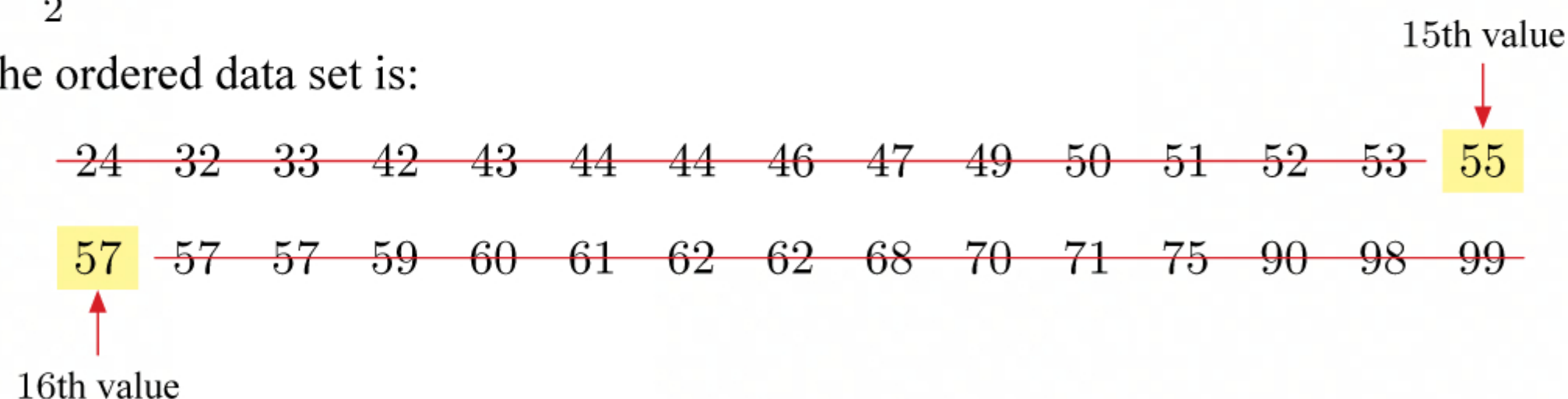
**ii** As  $n = 30$ , and  $\frac{n+1}{2} = 15.5$ , for both data sets, the median is the average of the 15th and 16th ordered data values.

For exhibit A, the ordered data set is:



$$\therefore \text{median} = \frac{51 + 55}{2} = 53 \text{ visitors}$$

For exhibit B, the ordered data set is:



$$\therefore \text{median} = \frac{55 + 57}{2} = 56 \text{ visitors}$$

**b** Exhibit B was more popular as the mean and median are both higher for exhibit B than for exhibit A.

**13 a**

$$\begin{aligned}
 \frac{9 + 10 + a + 13 + b + 16 + 21}{7} &= 14 \\
 \therefore \frac{69 + a + b}{7} &= 14 \\
 \therefore 69 + a + b &= 98 \\
 \therefore a + b &= 29
 \end{aligned}$$

Now,  $a$  and  $b$  are integers such that  $10 \leq a \leq 13$  and  $13 \leq b \leq 16$ .

$\therefore$  the only possible solution is  $a = 13$  and  $b = 16$ .

**b** Since  $n = 6$ ,  $\frac{n+1}{2} = 3.5$

So the median is the average of the 3rd and 4th ordered data values.

The ordered data set is: 1, 5, 9, 11, 16,  $p$   
two middle data values

$$\therefore \text{median} = \frac{9 + 11}{2} = 10$$

$$\text{Now, } \frac{1 + 5 + 9 + 11 + 16 + p}{6} = 10$$

$$\therefore \frac{42 + p}{6} = 10$$

$$\therefore 42 + p = 60$$

$$\therefore p = 18$$

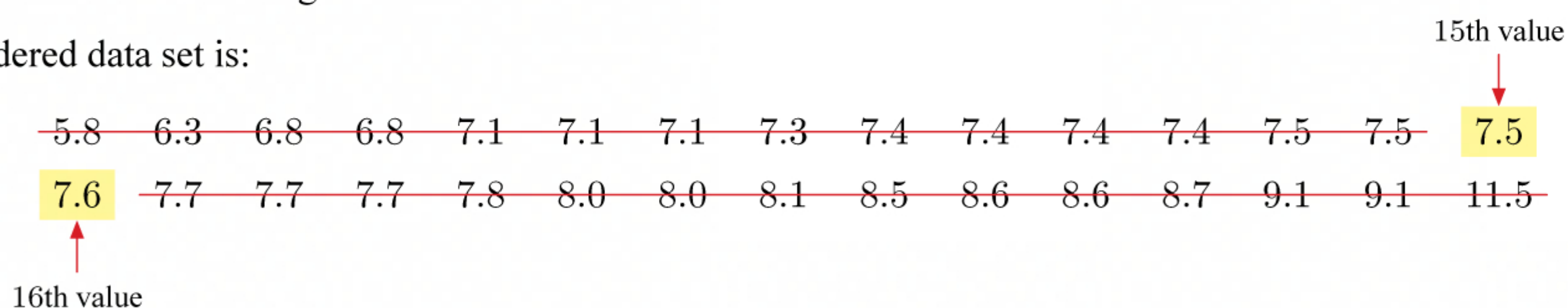
**14 a**

$$\begin{aligned}
 \text{mean} &= \frac{7.5 + 6.8 + \dots + 8.5}{30} \\
 &= \frac{233.1}{30} \\
 &= 7.77 \text{ hours}
 \end{aligned}$$

$$\text{As } n = 30, \frac{n+1}{2} = 15.5$$

So the median is the average of the 15th and 16th data values.

The ordered data set is:



$$\therefore \text{median} = \frac{7.5 + 7.6}{2} = 7.55 \text{ hours}$$

**b** The outlier is 11.5 hours.



$$\begin{aligned}\text{c i mean} &= \frac{7.5 + 6.8 + \dots + 8.5}{29} \\ &= \frac{221.6}{29} \\ &\approx 7.64 \text{ hours}\end{aligned}$$

As  $n = 29$  with the outlier removed,  $\frac{n+1}{2} = 15$ .

The ordered data set is:

15th value  
↓

<del>5.8</del>	<del>6.3</del>	<del>6.8</del>	<del>6.8</del>	<del>7.1</del>	<del>7.1</del>	<del>7.1</del>	<del>7.3</del>	<del>7.4</del>	<del>7.4</del>	<del>7.4</del>	<del>7.4</del>	<del>7.5</del>	<del>7.5</del>	7.5
<del>7.6</del>	<del>7.7</del>	<del>7.7</del>	<del>7.7</del>	<del>7.8</del>	<del>8.0</del>	<del>8.0</del>	<del>8.1</del>	<del>8.5</del>	<del>8.6</del>	<del>8.6</del>	<del>8.7</del>	<del>9.1</del>	<del>9.1</del>	

$\therefore$  median = 7.5 hours

- ii The measure of centre which is most affected by extreme values is the mean. So, the mean is most affected if the outlier is removed.

**15**

Number of touchdowns ( $x$ )	Frequency ( $f$ )	Product ( $xf$ )	Cumulative frequency
0	2	0	2
1	10	10	12
2	7	14	19
3	6	18	25
4	4	16	29
5	2	10	31
6	1	6	32
Total	$\sum f = 32$	$\sum xf = 74$	

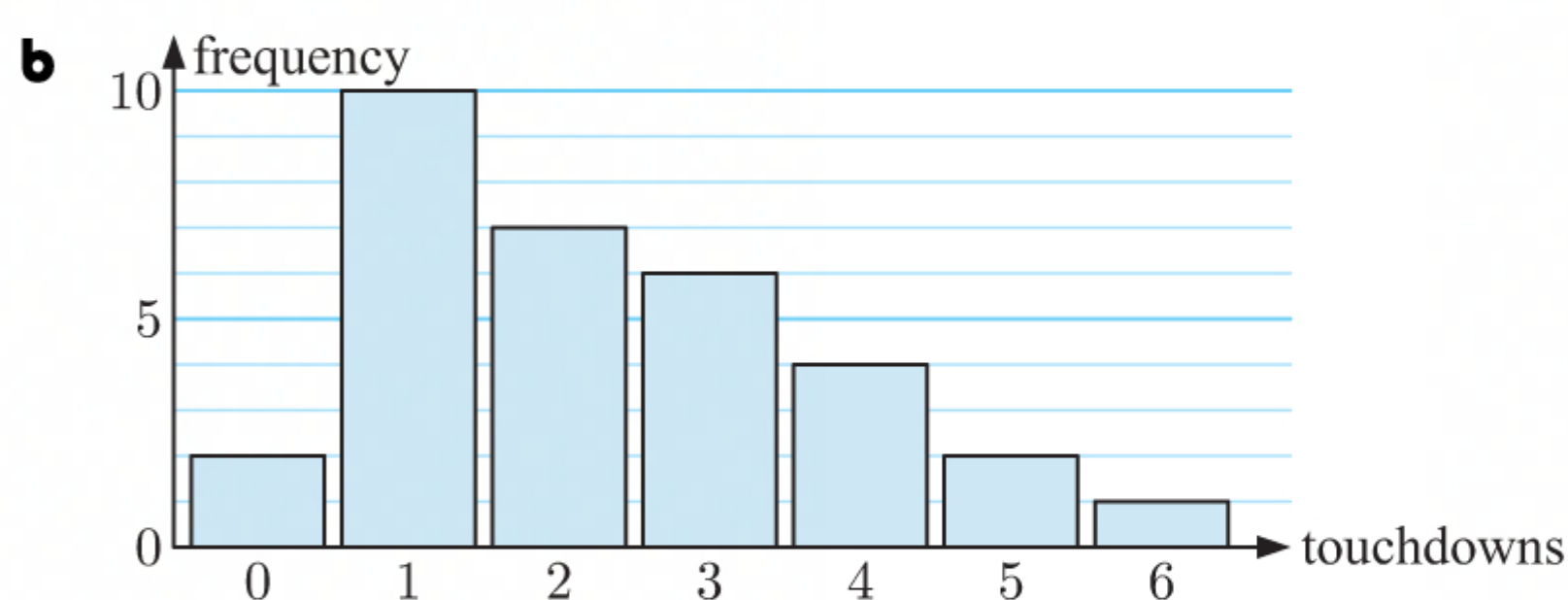
$$\begin{aligned}\text{a i } \bar{x} &= \frac{\sum xf}{\sum f} \\ &= \frac{74}{32} \\ &= 2.3125 \text{ touchdowns}\end{aligned}$$

- ii There are 32 data values, so  $n = 32$ .  $\frac{n+1}{2} = 16.5$ , so the median is the average of the 16th and 17th ordered data values. From the cumulative frequency column, the 13th to 19th ordered data values are 2 touchdowns.

$\therefore$  the 16th and 17th ordered data values are 2 touchdowns.

$\therefore$  the median is 2 touchdowns.

- iii Looking down the frequency column, the highest frequency is 10. This corresponds to 1 touchdown, so the mode is 1 touchdown.



- c** The data appears to be positively skewed.

- d** The data appears to be positively skewed, so the median is not a suitable measure of centre for this data.

The mean is affected by extreme values, so the mean is not a suitable measure of centre for this data.

The mode gives the most frequent number of touchdowns scored, and so is the most suitable measure of centre for this data.



**16 a**

Number of cars	Frequency	Cumulative frequency
0	78	78
1	117	195
2	69	264
3	18	282
4	2	284
Total	284	

**b i**  $\text{mean} = \frac{0 \times 78 + 1 \times 117 + 2 \times 69 + 3 \times 18 + 4 \times 2}{284}$

$$= \frac{317}{284}$$

$$\approx 1.12 \text{ cars}$$

- ii** There are 284 data values, so  $n = 284$ .  $\frac{n+1}{2} = 142.5$ , so the median is the average of the 142nd and 143rd ordered data values.

From the cumulative frequency column, the 79th to 195th ordered data values are 1 car.

$\therefore$  the 142nd and 143rd data values are 1 car.

$$\therefore \text{median} = \frac{1+1}{2} = 1 \text{ car}$$

- iii** Looking down the frequency column, the highest frequency is 117. This corresponds to 1 car, so the mode is 1 car.

**17**

Score	7	9	$a$	13	16
Frequency	1	2	1	2	1

**a**  $\bar{x} = \frac{\sum xf}{\sum f}$

$$\therefore 11 = \frac{7 \times 1 + 9 \times 2 + a \times 1 + 13 \times 2 + 16 \times 1}{1 + 2 + 1 + 2 + 1}$$

$$\therefore 11 = \frac{a + 67}{7}$$

$$\therefore a + 67 = 77$$

$$\therefore a = 10$$

- b** Let  $k$  be the number of goals that Kai will need to score in the next game.

Since she averaged 11 goals in her first 7 games, her average after the next game  $= \frac{7 \times 11 + k}{8} = \frac{k + 77}{8}$ .

For her overall average to improve to 12, we require  $\frac{k + 77}{8} = 12$

$$\therefore k + 77 = 96$$

$$\therefore k = 19$$

So, Kai will need to score 19 goals in the next game to improve her overall average to 12.

**18**

Weekly rent (€ $r$ )	Frequency ( $f$ )	Midpoint ( $x$ )	Product ( $xf$ )
$80 \leq r < 100$	3	90	270
$100 \leq r < 120$	15	110	1650
$120 \leq r < 140$	26	130	3380
$140 \leq r < 160$	30	150	4500
$160 \leq r < 180$	14	170	2380
$180 \leq r < 200$	1	190	190
Total	$\sum f = 89$		$\sum xf = 12\,370$

**a**  $\bar{x} = \frac{\sum xf}{\sum f}$

$$= \frac{12\,370}{89}$$

$$\approx 139$$

$\therefore$  the mean weekly rent was about €139.

**b**  $P(r \geq 140) = \frac{30 + 14 + 1}{89}$

$$\approx 0.506$$



**19** The ordered data sets are:

*Cailan:* 79 80 81 83 84 85 86 87 90 92 (10 data values)

*Miles:* 82 82 83 84 84 85 87 88 90 91 (10 data values)

**a** *Cailan:*

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum} \\ &= 92 - 79 \\ &= 13\end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set into two:

lower half                      upper half  
79 80 81 83 84    85 86 87 90 92

$$Q_1 = \text{median of lower half} = 81$$

$$Q_3 = \text{median of upper half} = 87$$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &= 87 - 81 \\ &= 6\end{aligned}$$

*Miles:*

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum} \\ &= 91 - 82 \\ &= 9\end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set into two:

lower half                      upper half  
82 82 83 84 84    85 87 88 90 91

$$Q_1 = \text{median of lower half} = 83$$

$$Q_3 = \text{median of upper half} = 88$$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &= 88 - 83 \\ &= 5\end{aligned}$$

**b** **i** Miles' data has the lower range.

**ii** Miles' data has the lower interquartile range.

**c** The interquartile range is more appropriate than the range for determining who is generally the more consistent golfer as it is less affected by outliers.

**20** **a** The ordered data set is:

4 9 10 12 12 14 14 15 16 16 16 17 18 18 18 20 23 26 31 {19 data values}

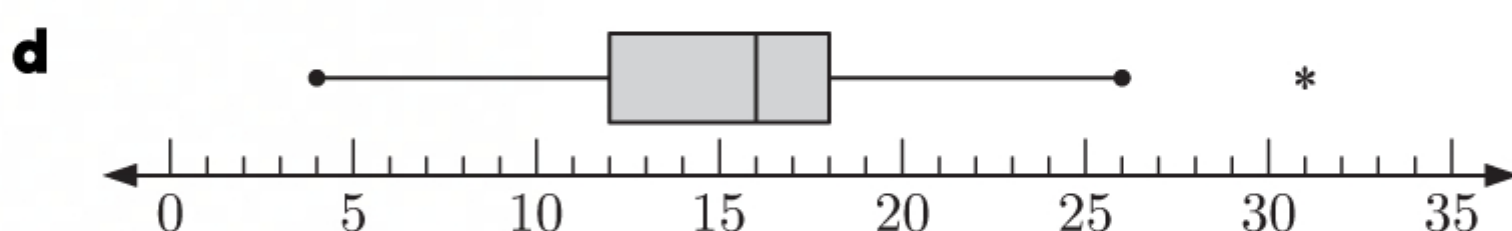
$\downarrow$                        $\downarrow$                        $\downarrow$   
 $Q_1 = 12$                       median = 16                       $Q_3 = 18$

So the five-number summary is:  $\begin{cases} \text{minimum} = 4 & Q_1 = 12 \\ \text{median} = 16 & Q_3 = 18 \\ \text{maximum} = 31 \end{cases}$

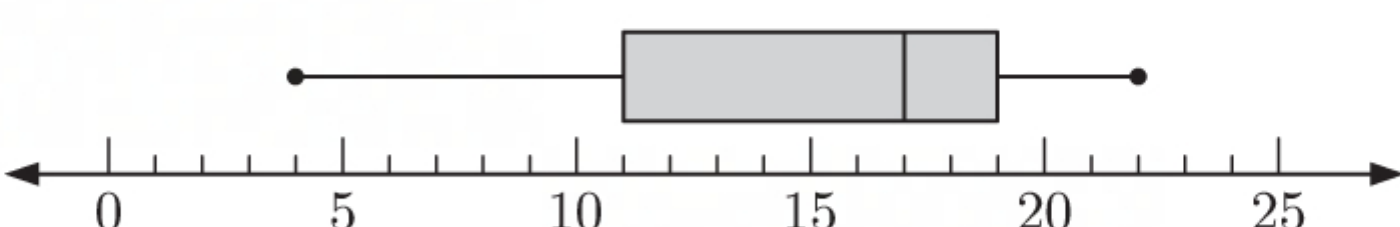
**b**  $\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &= 18 - 12 \\ &= 6\end{aligned}$

**c**    upper boundary                      lower boundary  
 $\begin{aligned} &= \text{upper quartile} + 1.5 \times \text{IQR} & &= \text{lower quartile} - 1.5 \times \text{IQR} \\ &= 18 + 1.5 \times 6 & &= 12 - 1.5 \times 6 \\ &= 27 & &= 3 \end{aligned}$

31 is above the upper boundary, so it is an outlier.



**21**



**a** minimum value = 4 cm

**c** median = 17 cm

**e** lower quartile = 11 cm

**f**  $\begin{aligned}\text{range} &= \text{maximum} - \text{minimum} \\ &= 22 - 4 \\ &= 18 \text{ cm}\end{aligned}$

**b** maximum value = 22 cm

**d** upper quartile = 19 cm

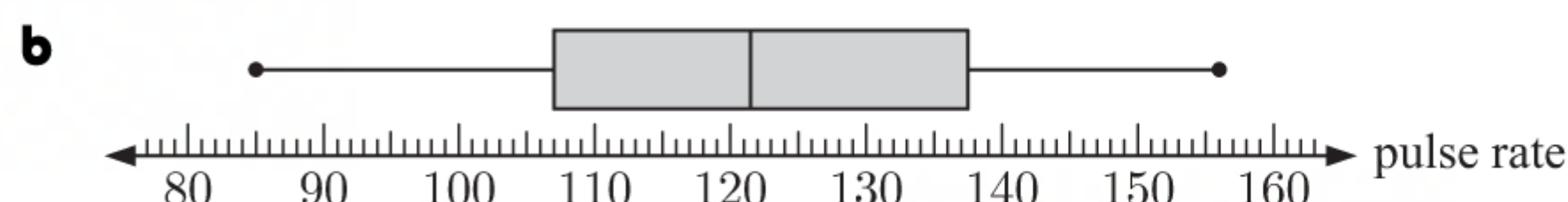
**g**  $\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &= 19 - 11 \\ &= 8 \text{ cm}\end{aligned}$



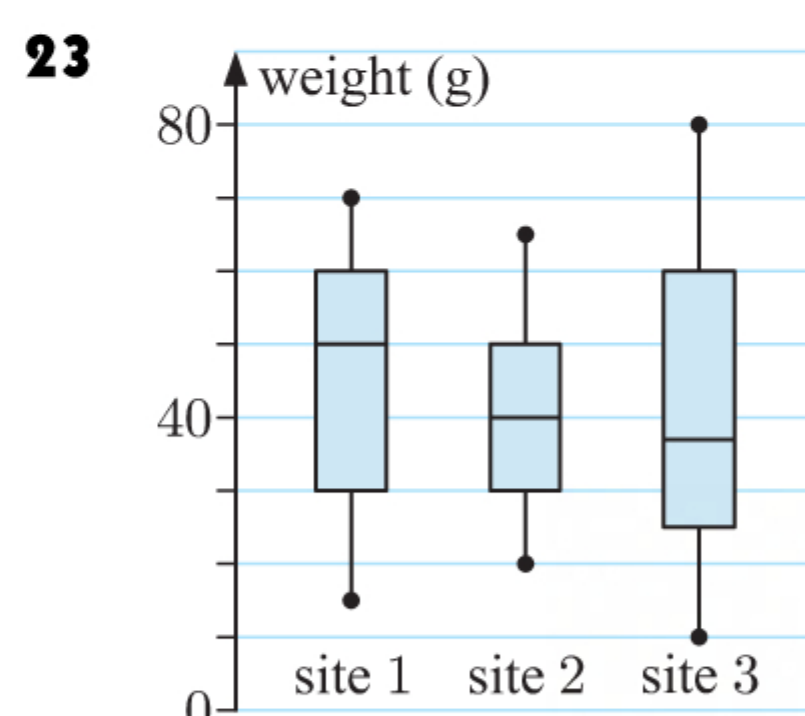
**22 a** The ordered data set is:

85 96 98 100 105 106 108 108 112 112 118 120 123 125 126 128 133 135 140 144 144 148 148 156  
 $Q_1 = 107$  median = 121.5  $Q_3 = 137.5$  {24 data values}

So the five-number summary is:  $\begin{cases} \text{minimum} = 85 & Q_1 = 107 \\ \text{median} = 121.5 & Q_3 = 137.5 \\ \text{maximum} = 156 \end{cases}$



**c** Range = maximum – minimum      IQR =  $Q_3 - Q_1$   
 $= 156 - 85$        $= 137.5 - 107$   
 $= 71$        $= 30.5$



**a** The five-number summary for site 1 is:

$\begin{cases} \text{minimum} = 15 & Q_1 = 30 \\ \text{median} = 50 & Q_3 = 60 \\ \text{maximum} = 70 \end{cases}$

**b** Site 3 has the greatest range of weights.

**c** The weights of fungi have the least variation at site 2.

**d** Site 1 has the highest median weight of fungi.

**e** Site 1 has the highest proportion of weights above 40 grams.

**24 a** Old recipe:

1-Variable	
n	=12
minX	=6
Q1	=7
Med	=7.75
Q3	=8
maxX	=9

New recipe:

1-Variable	
n	=12
minX	=4
Q1	=6.25
Med	=7.25
Q3	=8
maxX	=9

The five number summaries are:

Old recipe: minimum = 6

$Q_1 = 7$

median = 7.75

$Q_3 = 8$

maximum = 9

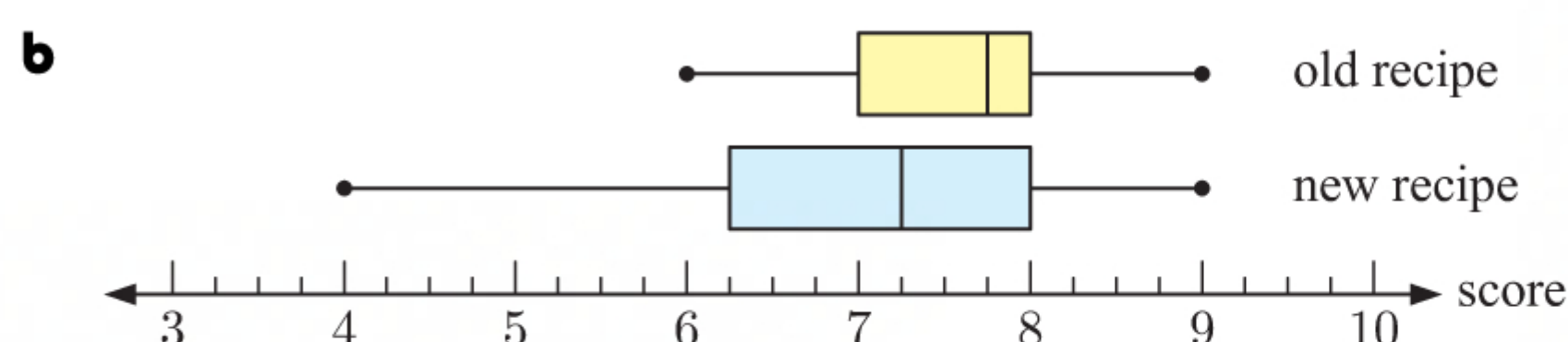
New recipe: minimum = 4

$Q_1 = 6.25$

median = 7.25

$Q_3 = 8$

maximum = 9



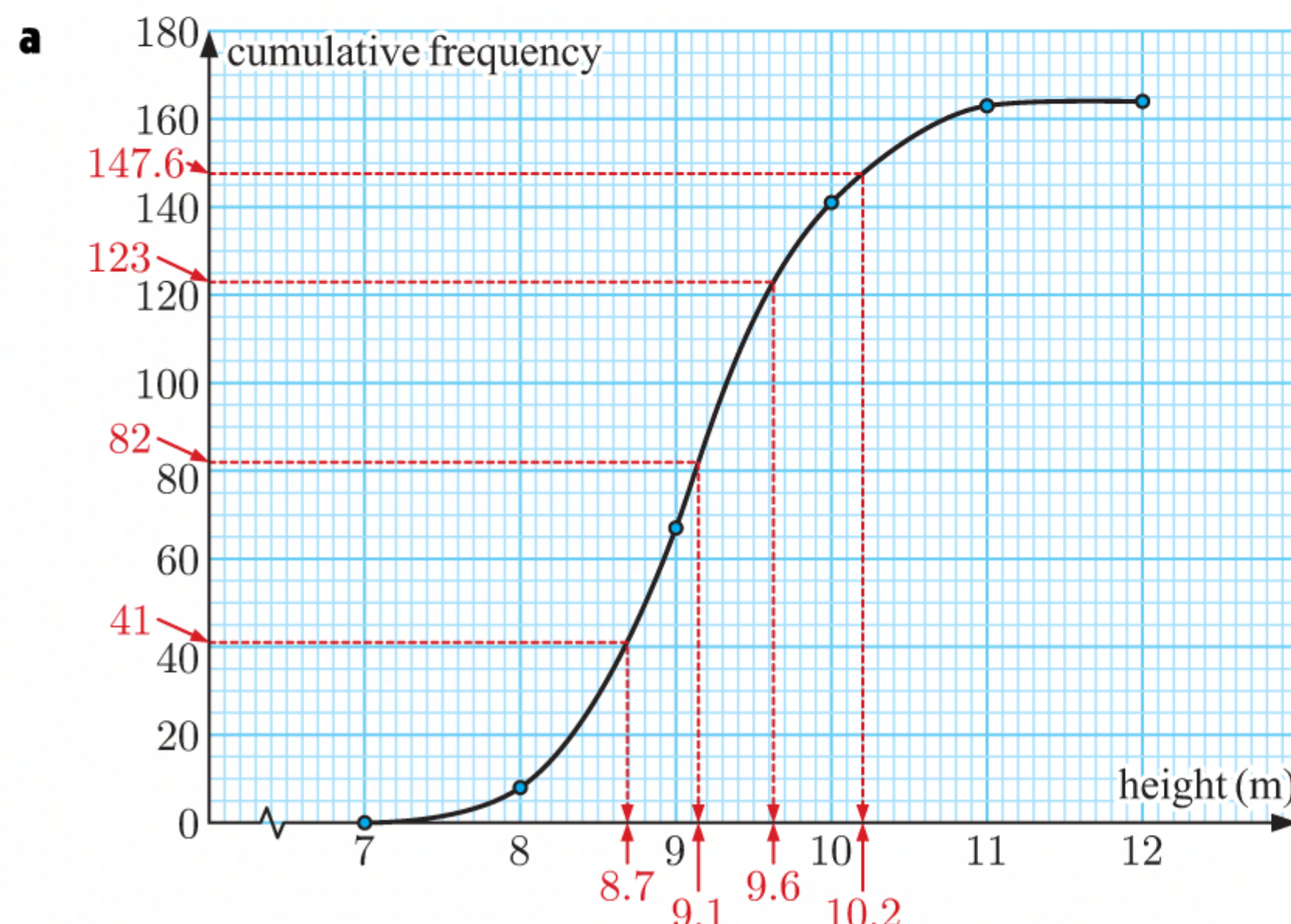
**c** The minimum,  $Q_1$ , and the median are higher for the old recipe, and  $Q_3$  and the maximum are the same for both recipes.

The group generally preferred the old recipe over the new recipe, so the distributor should not adopt this new recipe for their drink.



25

Height ( $h$ m)	Frequency	Cumulative frequency
$7 \leq h < 8$	8	8
$8 \leq h < 9$	59	67
$9 \leq h < 10$	74	141
$10 \leq h < 11$	22	163
$11 \leq h < 12$	1	164



- b The median is the 50th percentile. As 50% of 164 is 82, we start with the cumulative frequency 82 and find the corresponding height.

From the graph, the median  $\approx 9.1$  m.

- c  $Q_1$  is the 25th percentile. As 25% of 164 is 41, we start with the cumulative frequency 41 and find the corresponding height.

From the graph,  $Q_1 \approx 8.7$  m

$Q_3$  is the 75th percentile. As 75% of 164 is 123, we start with the cumulative frequency 123 and find the corresponding height.

From the graph,  $Q_3 \approx 9.6$  m

$$\text{IQR} = Q_3 - Q_1$$

$$\approx 9.6 - 8.7$$

$$\approx 0.9 \text{ m}$$

- d As 90% of 164 is 147.6, we start with the cumulative frequency 147.6 and find the corresponding height.

The 90th percentile  $\approx 10.2$  m which means that 90% of trees are shorter than about 10.2 m.

- 26 Anthony:  $1\frac{1}{2}$ , 2,  $2\frac{1}{2}$ , 4,  $4\frac{1}{2}$ , 3,  $3\frac{1}{2}$ , 5, 6, 6

Katherine: 3,  $3\frac{1}{2}$ , 4, 3, 3,  $3\frac{1}{2}$ , 4, 4,  $4\frac{1}{2}$ , 4

- a Using technology:

Anthony:

	Des(Norm1)	d/c(Real)
1-Variable		
$\bar{x}$	=3.8	
$\Sigma x$	=38	
$\Sigma x^2$	=167	
$\sigma x$	=1.50332963	
$sx$	=1.58464857	
$n$	=10	

The mean  $\mu = 3.8$  hours and the standard deviation  $\sigma \approx 1.50$  hours.

Katherine:

	Des(Norm1)	d/c(Real)
1-Variable		
$\bar{x}$	=3.65	
$\Sigma x$	=36.5	
$\Sigma x^2$	=135.75	
$\sigma x$	=0.50249378	
$sx$	=0.52967495	
$n$	=10	

The mean  $\mu = 3.65$  hours and the standard deviation  $\sigma \approx 0.502$  hours.

- b Anthony's mean is higher than Katherine's, so Anthony generally practised for longer.
- c Katherine's standard deviation is lower than Anthony's, so there is less deviation from the mean for her data set. Katherine therefore practised more consistently than Anthony.

27

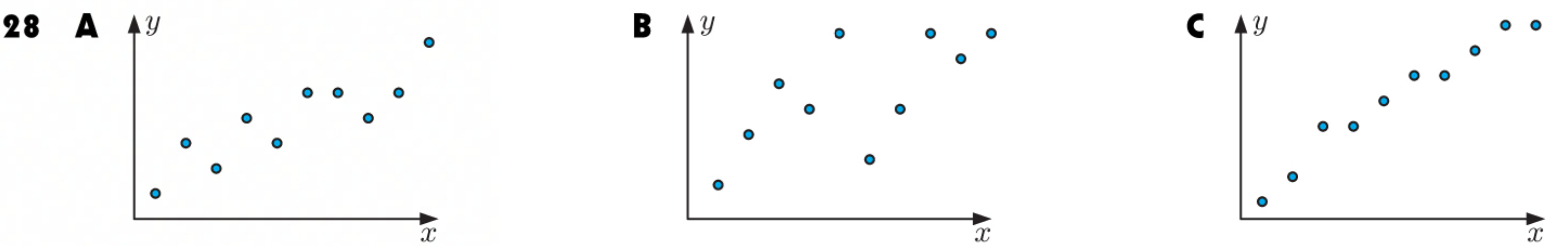
Mark	3	4	5	6	7	8	9	10
Frequency	1	3	5	8	4	2	0	1

Using technology:

	Des(Norm1)	d/c(Real)
1-Variable		
$\bar{x}$	=5.91666666	
$\Sigma x$	=142	
$\Sigma x^2$	=894	
$\sigma x$	=1.49768339	
$sx$	=1.52989532	
$n$	=24	

The mean test score  $\mu \approx 5.92$ , and the population standard deviation  $\sigma \approx 1.50$ .





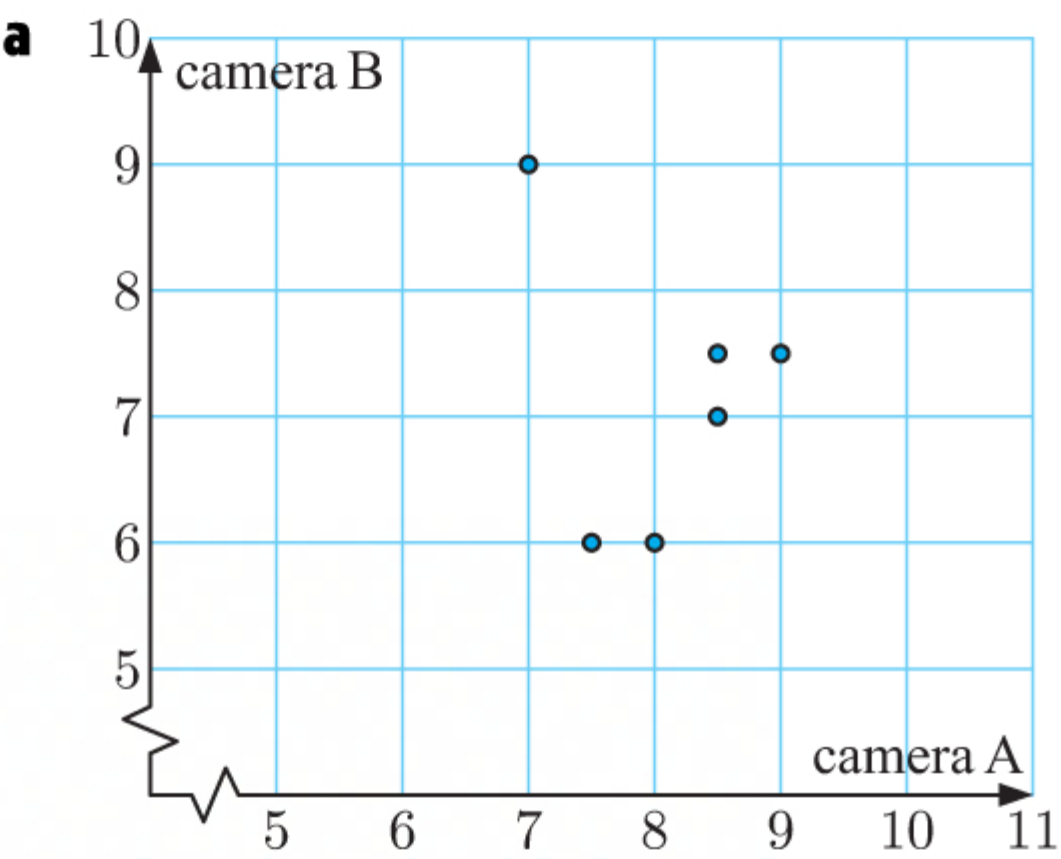
- a** In each scatter diagram, as  $x$  increases  $y$  generally increases.  
 $\therefore$  each scatter diagram shows a positive association between  $x$  and  $y$ .

**b**

Strength of correlation	Scatter diagram
Weak	<b>B</b>
Moderate	<b>A</b>
Strong	<b>C</b>

**29**

Camera A	8.5	8	9	7	8.5	7.5
Camera B	7	6	7.5	9	7.5	6

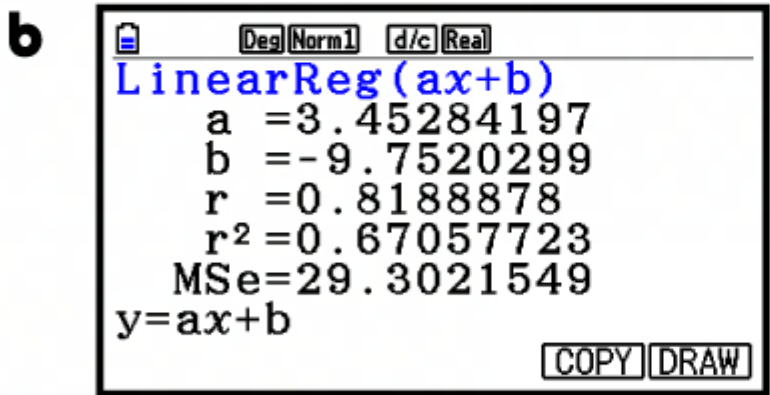
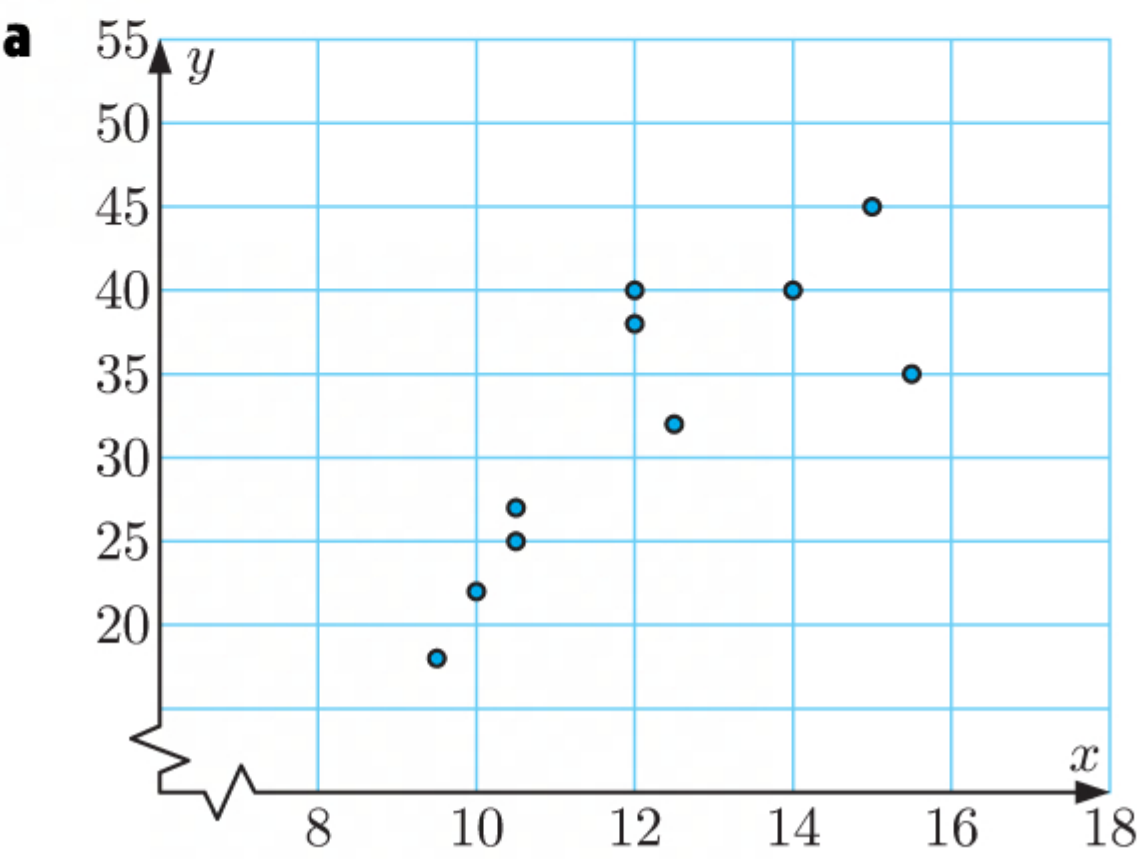
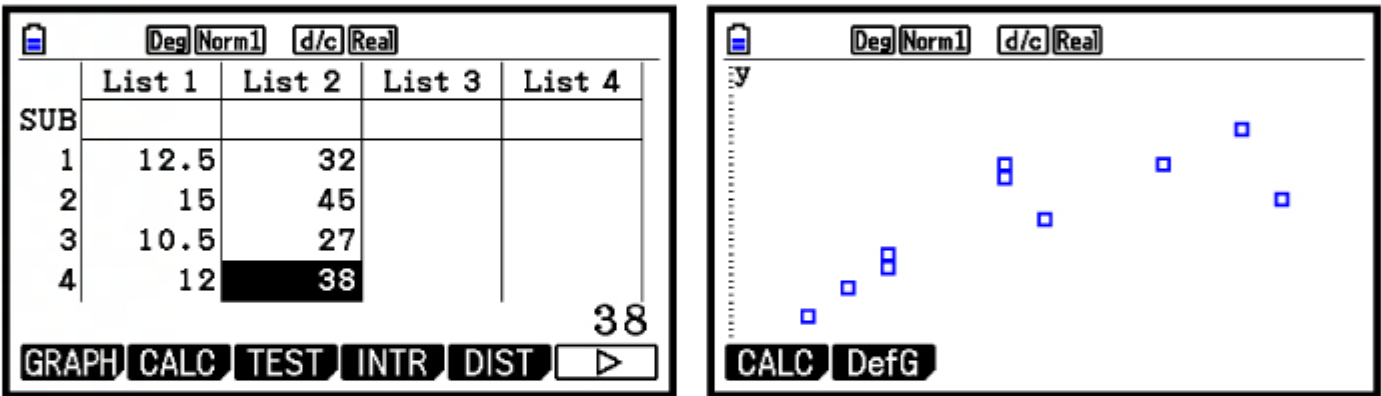


- b** The point  $(7, 9)$  appears to be an outlier. This corresponds to the review scoring camera A a 7, and camera B a 9.

- c**
- i** With the outlier removed, there appears to be a strong, positive, linear correlation between camera A's scores and camera B's scores.
- ii** No, an increase in camera A's scores is not likely to cause an increase in camera B's scores. It is more likely that both scores are related to the preferences of each reviewer.

**30**

Language ( $x$ )	12.5	15.0	10.5	12.0	9.5	10.5	15.5	10.0	14.0	12.0
Mathematics ( $y$ )	32	45	27	38	18	25	35	22	40	40



So,  $r \approx 0.819$ .

- c** The data suggests that there is a moderate, positive, linear correlation between the students' language scores and their mathematics scores. So, "Those who do well in languages also do well in mathematics." is a moderately reasonable statement.



**31**

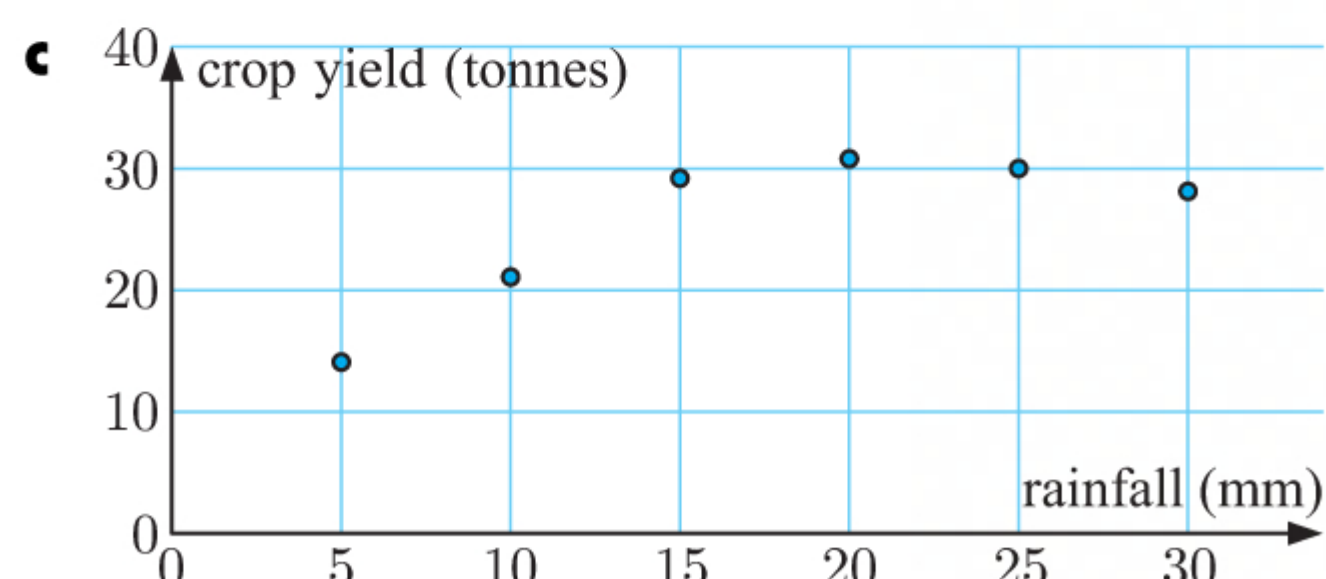
Monthly rainfall (mm)	5	10	15	20	25	30
Crop yield (tonnes)	14	21	29	31	30	28

**a**

```

LinearReg(ax+b)
a = 0.56571428
b = 15.6
r = 0.79505891
r² = 0.63211867
MSe = 20.3714285
y = ax + b
    
```

So,  $r \approx 0.795$ .



**b** A moderate, positive relationship may exist between crop yield and rainfall.

**d** The relationship between rainfall and crop yield does not appear to be linear and so Pearson's correlation coefficient may not be appropriate for this data.

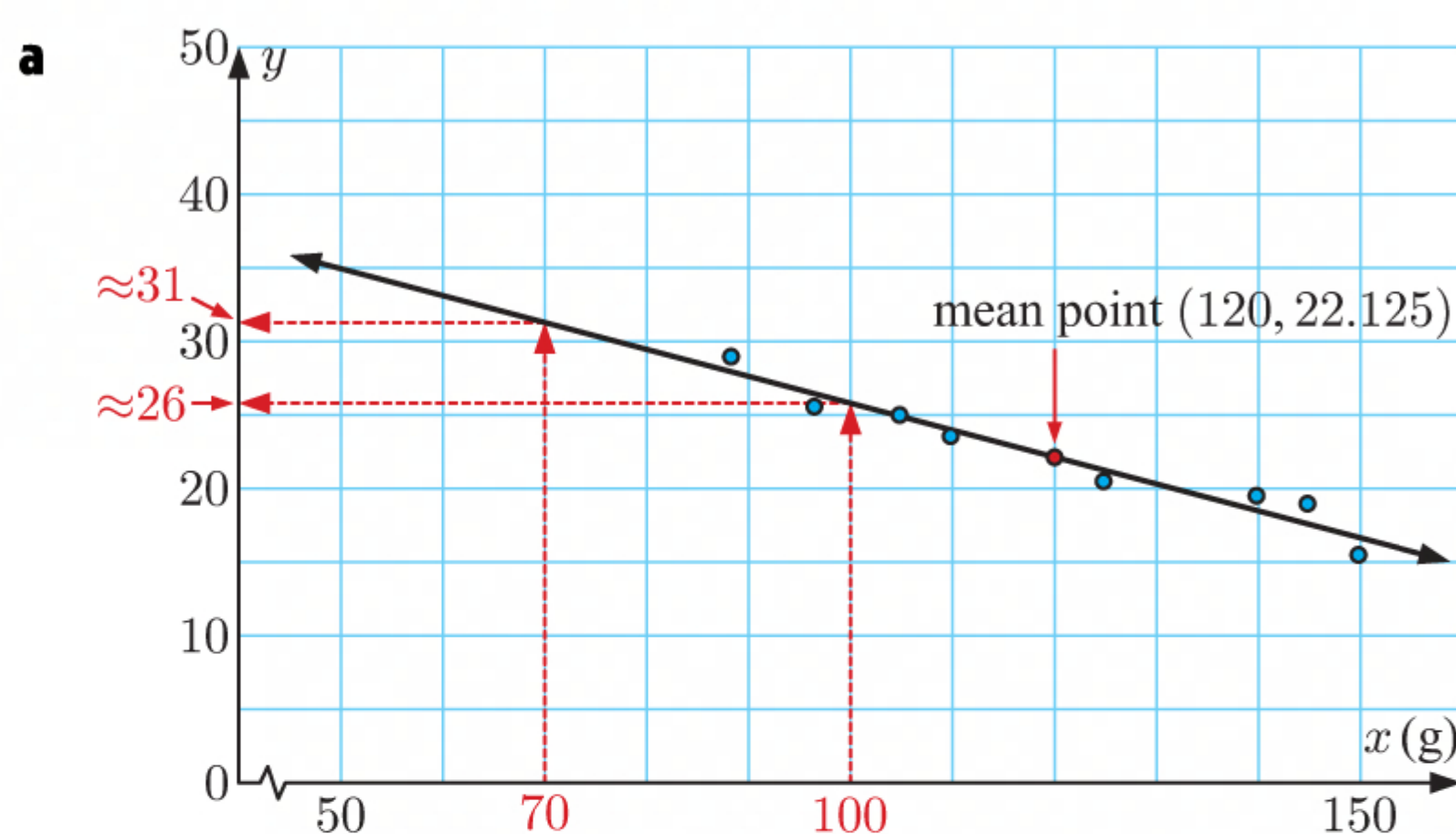
**32**  $r \approx -0.6321$

$\therefore r^2 \approx (-0.6321)^2 \approx 0.3996$

$\therefore$  about 39.96% of the variation in the *average speed* is explained by the variation in the *distance travelled*.

**33**

Median weight ( $x$ g)	88	97	105	110	125	140	145	150
Number in bag ( $y$ )	28	26	26	23	21	19	18	16



**b i** When  $x = 100$ ,  $y \approx 26$ .

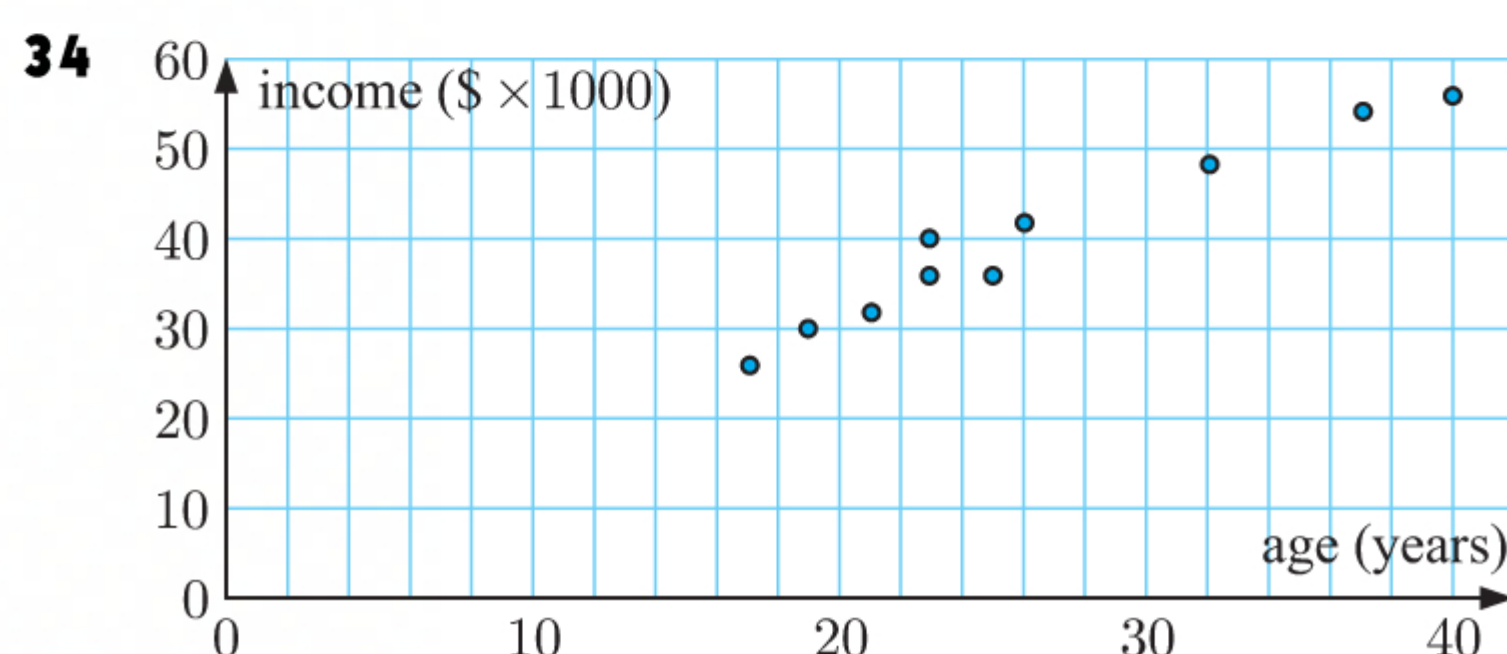
If the median weight is 100 grams, there are about 26 potatoes in a bag.

**ii** When  $x = 70$ ,  $y \approx 31$ .

If the median weight is 70 grams, there are about 31 potatoes in a bag.

**c** The estimate in **b i** is an interpolation, and the estimate in **b ii** is an extrapolation.

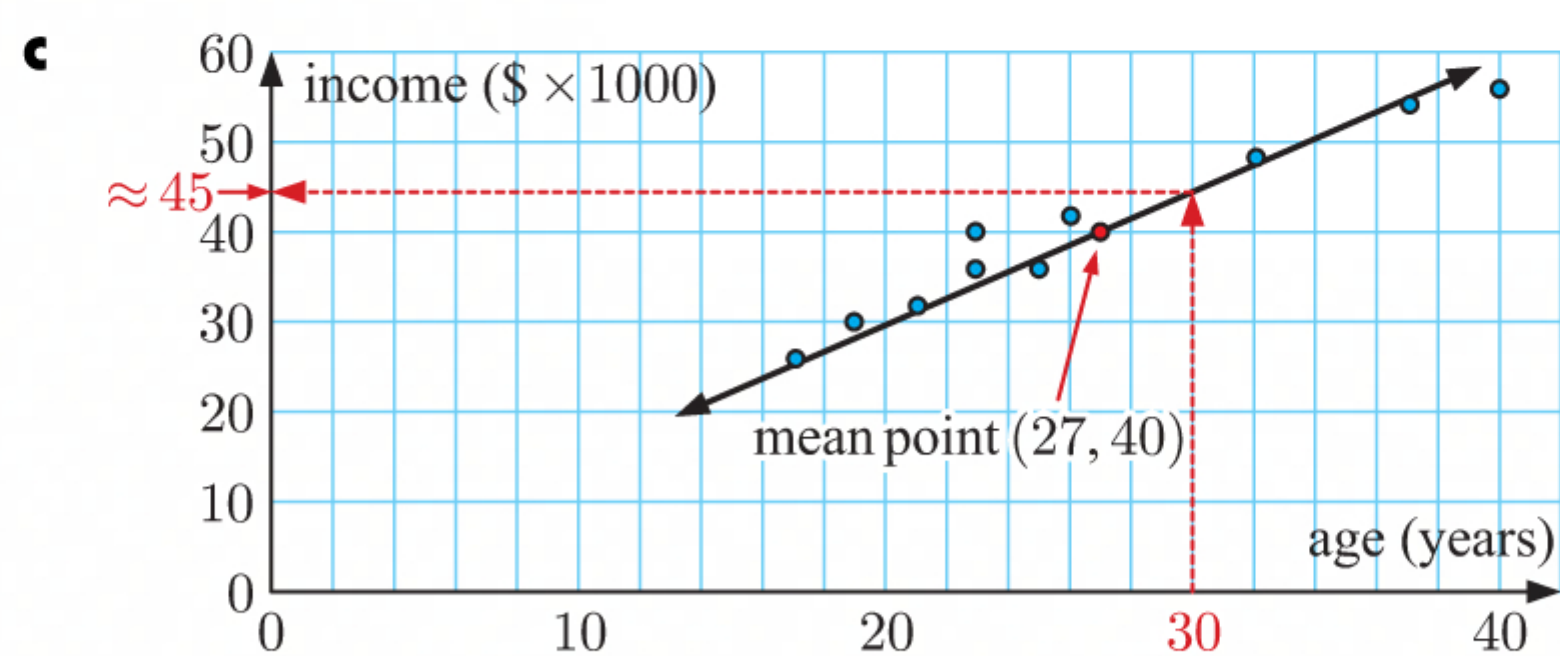
So, the estimate in **b i** is likely to be more reliable.



**a** There is a strong, positive correlation between the age of an individual and their annual income.

**b** No, the relationship is more likely dependent on the amount of professional experience or qualifications an individual has.





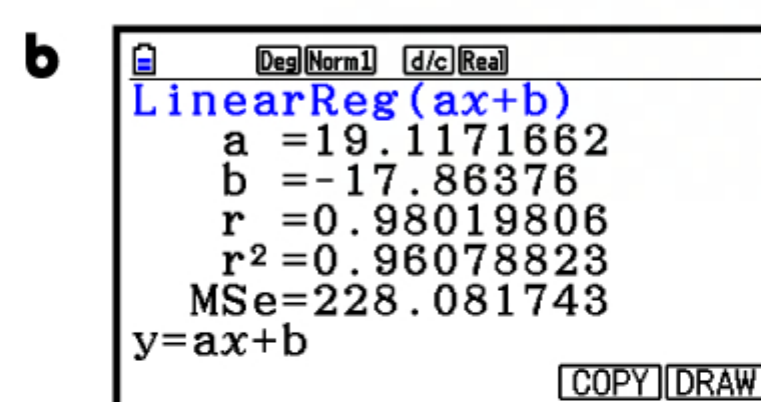
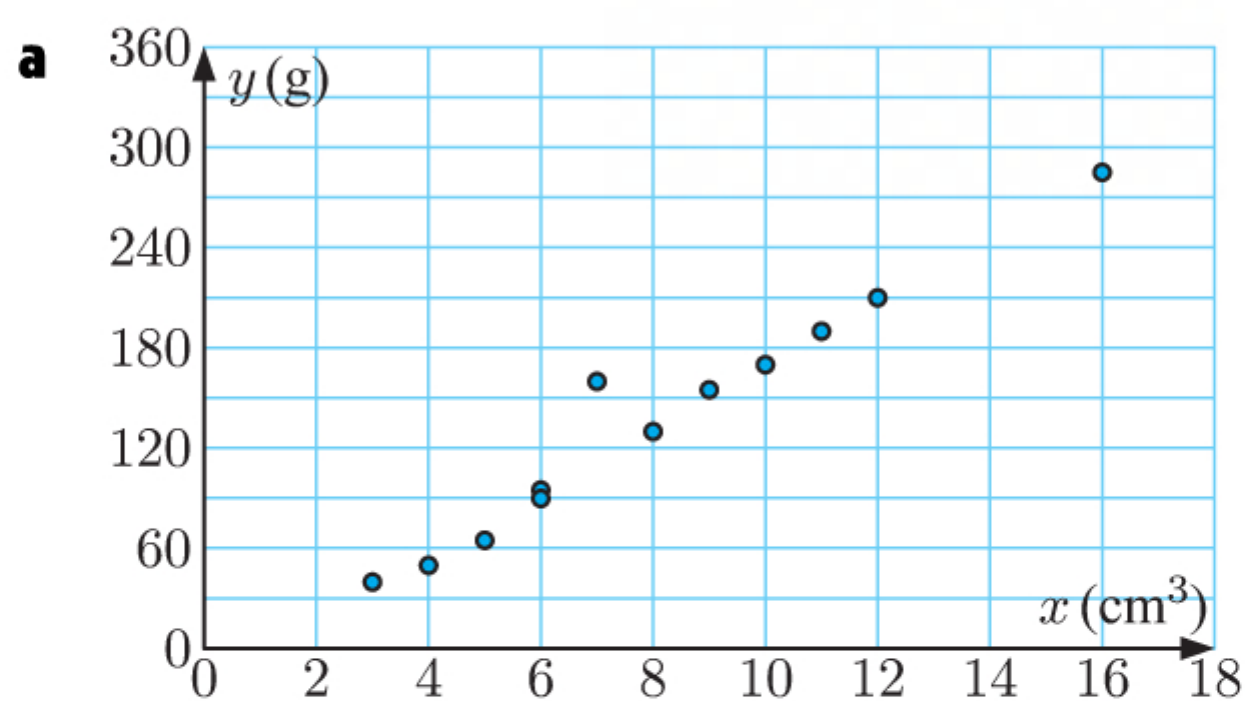
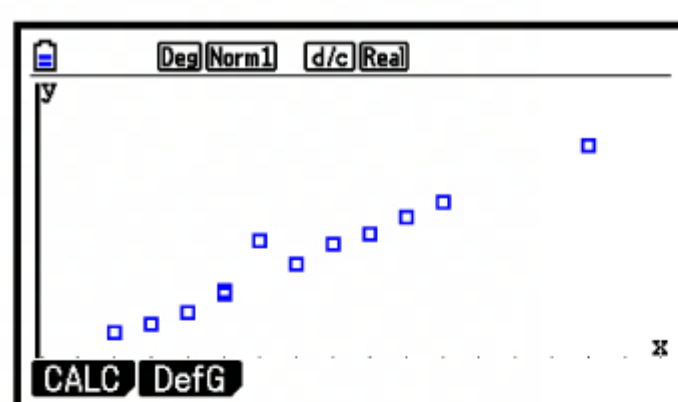
**d** When  $x = 30$ ,  $y \approx 45$ .

The annual income of someone who is 30 years old is approximately \$45 000. This is an interpolation, so the estimate is reliable.

**35**

Sample	A	B	C	D	E	F	G	H	I	J	K	L
Volume ( $x \text{ cm}^3$ )	3	6	4	7	16	8	5	12	9	6	10	11
Mass ( $y \text{ g}$ )	40	95	50	160	285	130	65	210	155	90	170	190

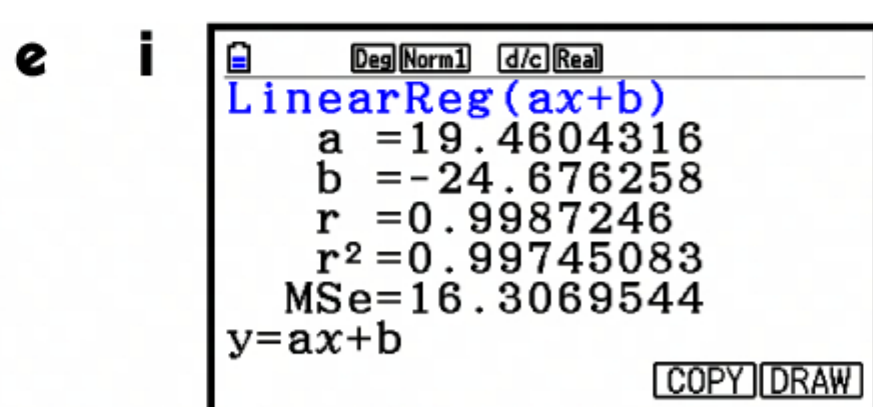
	List 1	List 2	List 3	List 4
SUB				
1	3	40		
2	6	95		
3	4	50		
4	7	160		



So,  $r \approx 0.980$ .

**c** There appears to be a strong, positive correlation between the *volume* of a sample of silver and its *mass*.

**d** The data point (7, 160) which corresponds to sample D appears to be an outlier. We therefore agree with the jeweller that there is a fake sample.



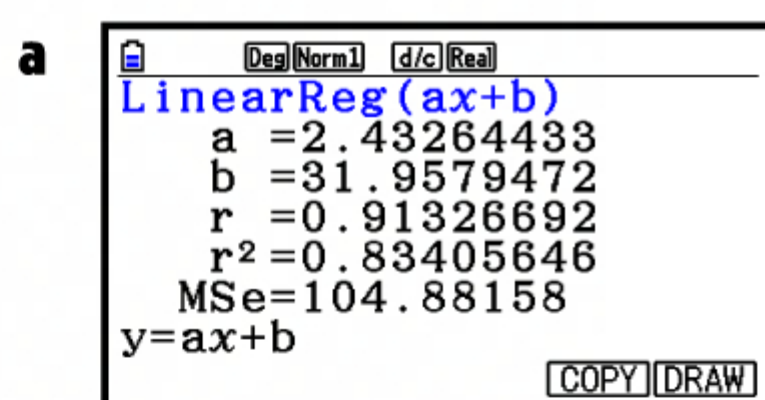
**ii** When  $x = 7$ ,  $y \approx 19.5(7) - 24.7$   
 $\approx 112$

So, a sample of silver with volume  $7 \text{ cm}^3$  would weigh approximately 112 g.

Using technology, the regression line is  
 $y \approx 19.5x - 24.7$ .

**36**

Study time ( $x \text{ h}$ )	7	6	3	16	15	11	18	32	20
Result ( $y \%$ )	56	42	25	80	65	60	85	96	90



Using technology, the least squares regression line is  $y \approx 2.43x + 32.0$ .

**b** From **a**,  $r \approx 0.913$  and  $r^2 \approx 0.834$ .

**c** There is a strong, positive correlation between the number of hours that a student studies and their examination result.

**d** Yes, this is a causal relationship as spending more time studying for the examination is likely to cause a better result.



- e When  $y = 70$ ,  $70 \approx 2.43x + 32.0$   
 $\therefore 38 \approx 2.43x$   
 $\therefore x \approx 15.6$

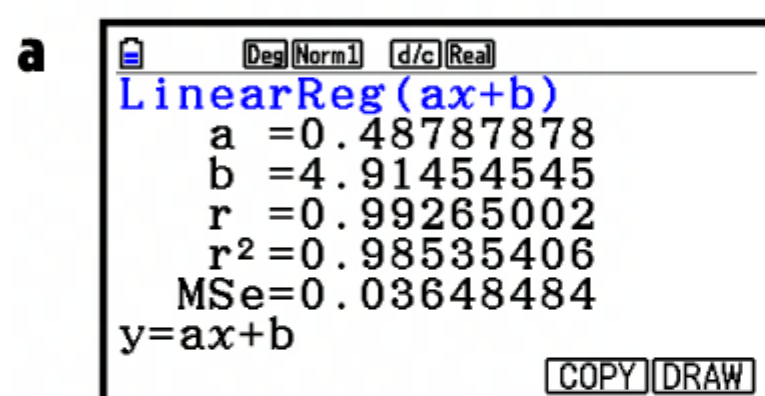
So, Tony studied for approximately 15.6 hours.

- f The  $y$ -intercept of the line of best fit  $\approx 32.0$ . This indicates that if a student did not spend any time studying, they would obtain a result of 32% on average.

The gradient of the line of best fit  $\approx 2.43$ . This indicates that for every additional hour of study, the result obtained increases by an average of 2.43%.

**37**

Time ( $t$ days)	0	1	2	3	4	5	6	7	8	9
Height ( $h$ mm)	5	5.7	5.7	6.2	6.8	7.1	8	8.3	9	9.3



So,  $r \approx 0.993$ .

- c  $h \approx 0.4879t + 4.9145$

- i When  $t = 14$ ,  $h \approx 0.4879(14) + 4.9145$   
 $\approx 11.7$

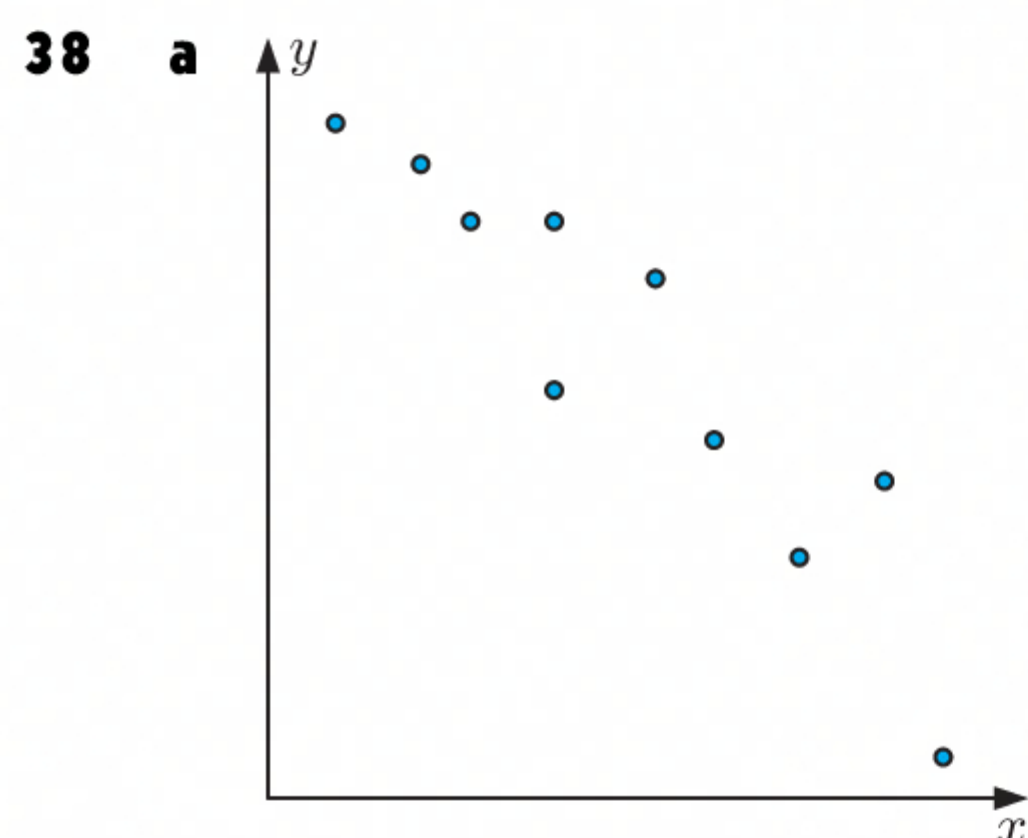
$\therefore$  after 14 days, the grass is about 11.7 mm high.

- ii When  $h = 20$ ,  $20 \approx 0.4879t + 4.9145$

$$\therefore 15.0855 \approx 0.4879t$$

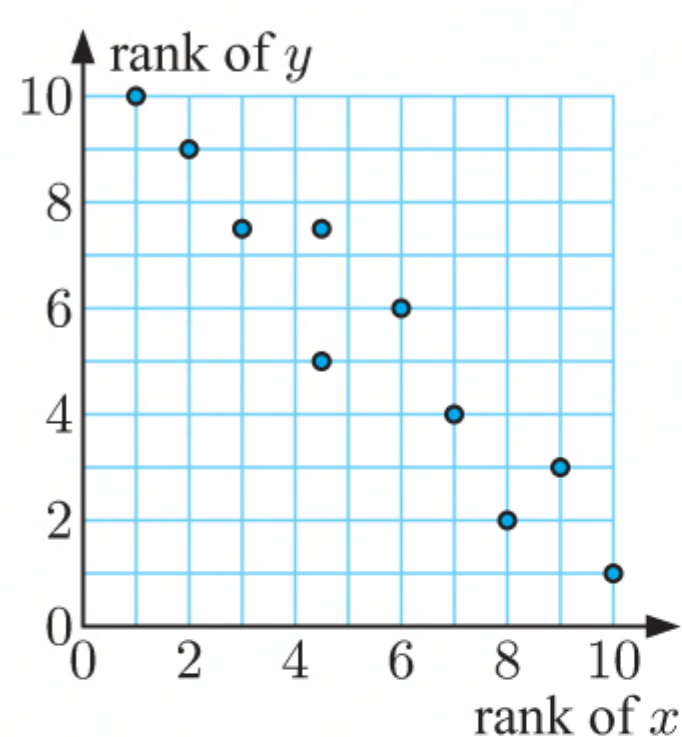
$$\therefore t \approx 30.9$$

$\therefore$  the grass reaches a height of 20 mm after about 30.9 days.



As  $x$  increases,  $y$  generally decreases.

So as the rank of  $x$  increases, the rank of  $y$  generally decreases.



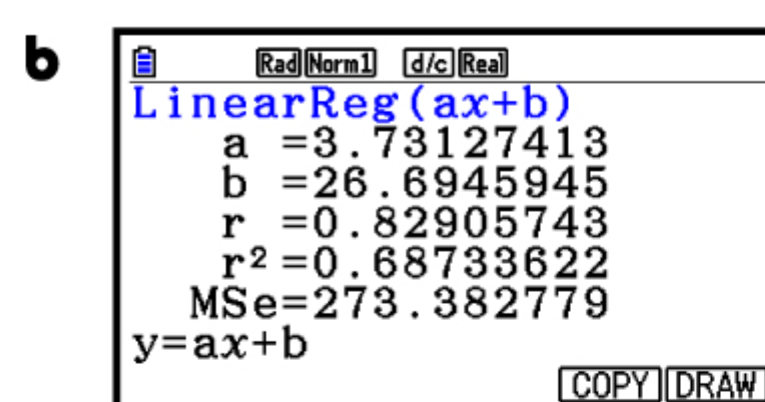
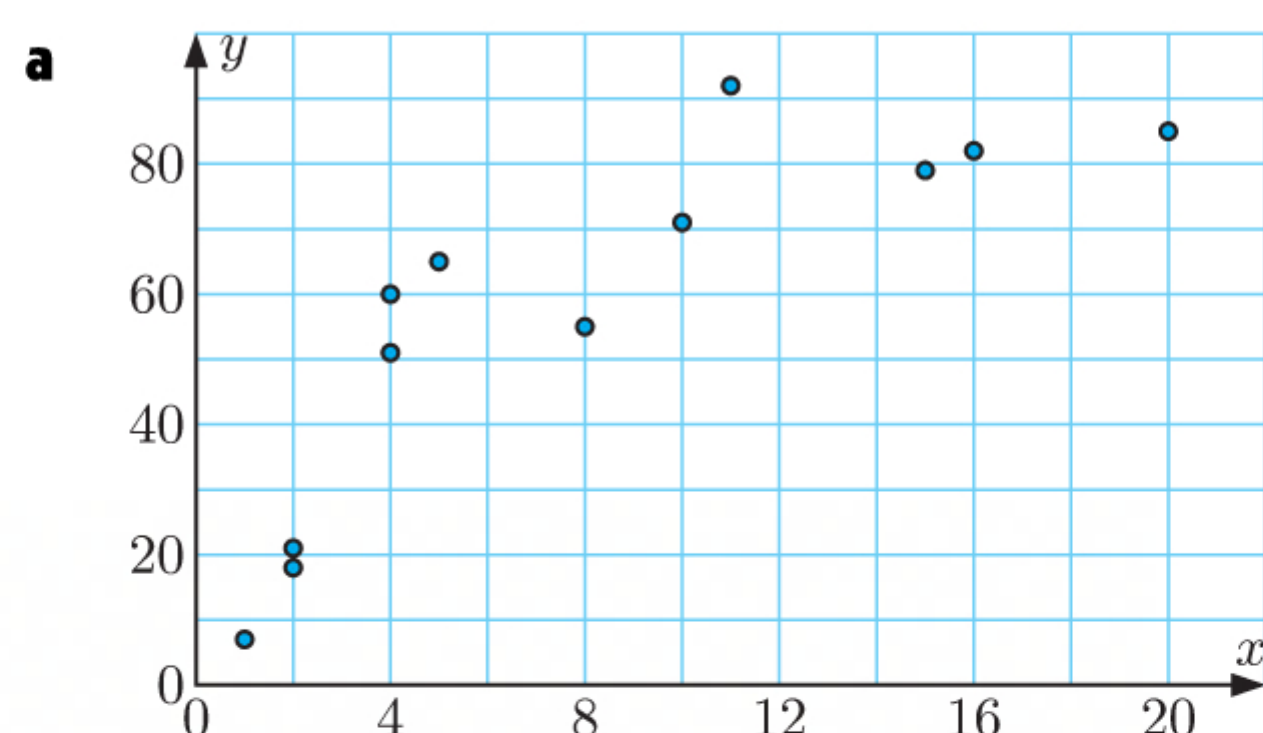
Looking at the first few points, we see that **B** is the correct rank scatter diagram.

- b The rank scatter diagram has a strong, negative linear correlation, so the correct value of Spearman's rank correlation coefficient is  $r_s \approx -0.960$  (C).



**39**

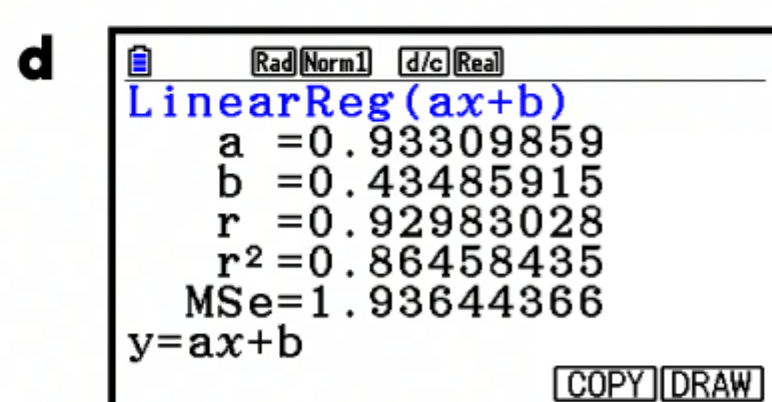
Number of matches played ( $x$ )	11	5	10	16	2	1	8	20	15	2	4	4
Highest score ( $y$ )	92	65	71	82	21	7	55	85	79	18	60	51



So,  $r_p \approx 0.829$ .

**c**

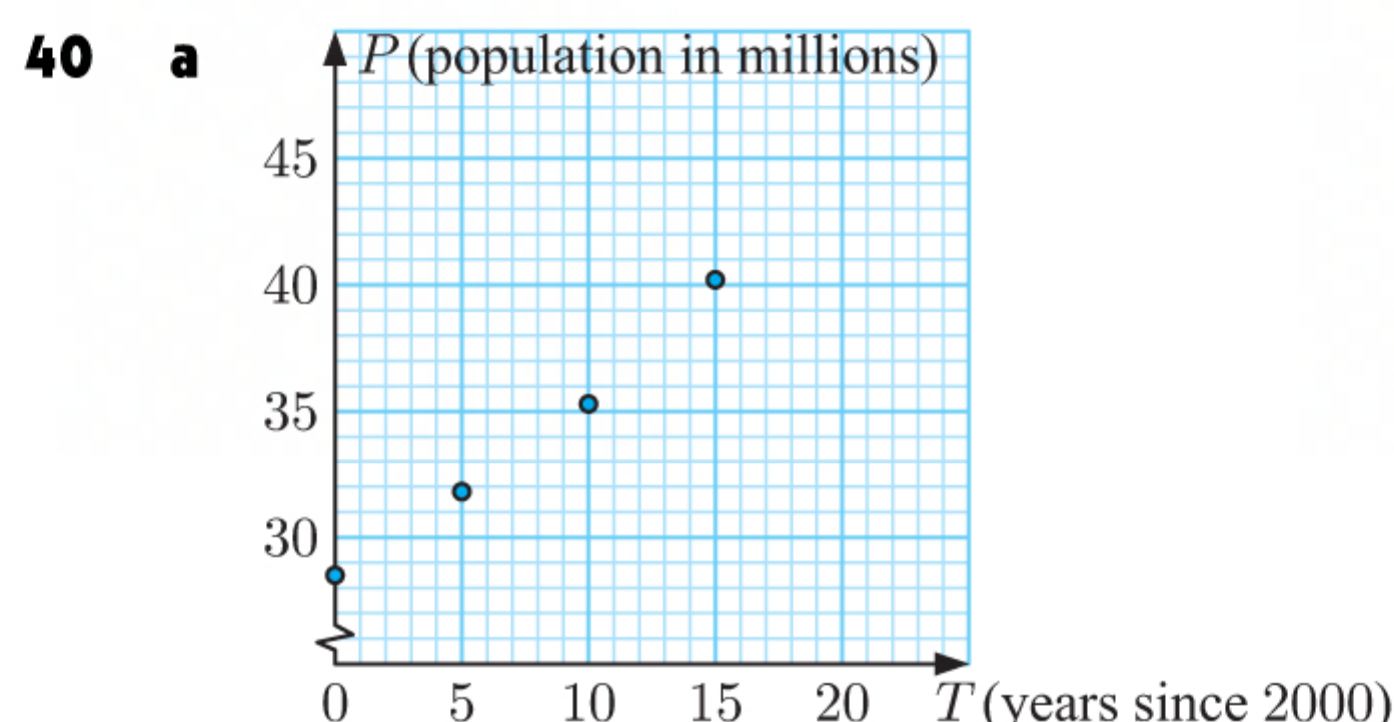
Number of matches played ( $x$ )	11	5	10	16	2	1	8	20	15	2	4	4
rank of $x$	9	6	8	11	2.5	1	7	12	10	2.5	4.5	4.5
Highest score ( $y$ )	92	65	71	82	21	7	55	85	79	18	60	51
rank of $y$	12	7	8	10	3	1	5	11	9	2	6	4



So,  $r_s \approx 0.930$ .

**e** The scatter diagram in **a** shows a non-linear trend.

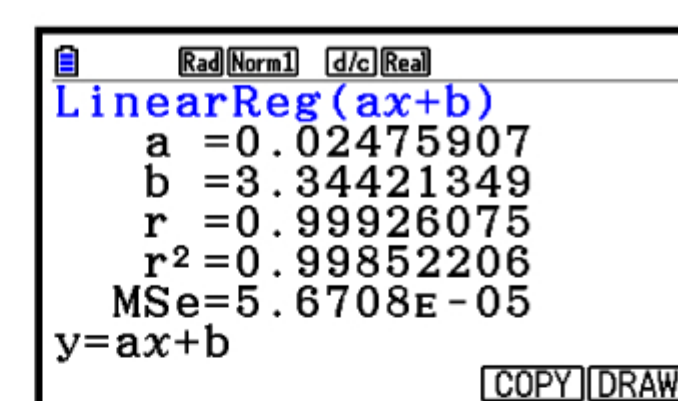
Using **d**, there appears to be a strong, positive, non-linear correlation between number of matches played and highest score.



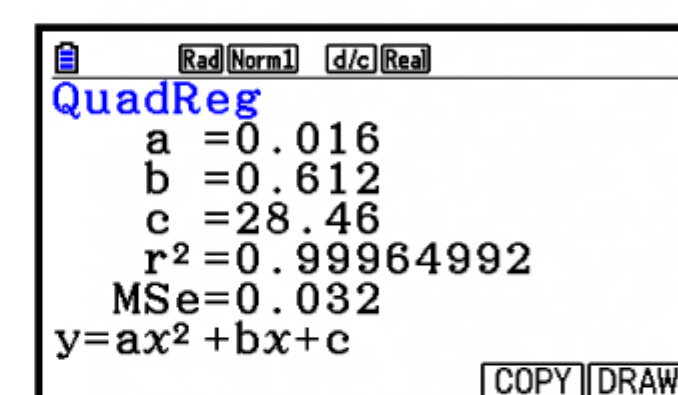
$T$	0	5	10	15
$P$	28.5	31.8	36.3	41.2
$\ln P$	3.35	3.46	3.59	3.72

**b i** Using technology, the linear model connecting  $\ln P$  and  $T$  is  
 $\ln P \approx 0.0248T + 3.34$

$$\begin{aligned} \therefore \text{the exponential model connecting } P \text{ and } T \text{ is } P &\approx e^{0.0248T+3.34} \\ &\therefore P \approx e^{0.0248T} \times e^{3.34} \\ &\therefore P \approx e^{3.34} \times (e^{0.0248})^T \\ &\therefore P \approx 28.3 \times (1.03)^T \end{aligned}$$



**ii** Using technology, the quadratic model connecting  $P$  and  $T$  is  
 $P \approx 0.016T^2 + 0.612T + 28.46$ .



**c i**

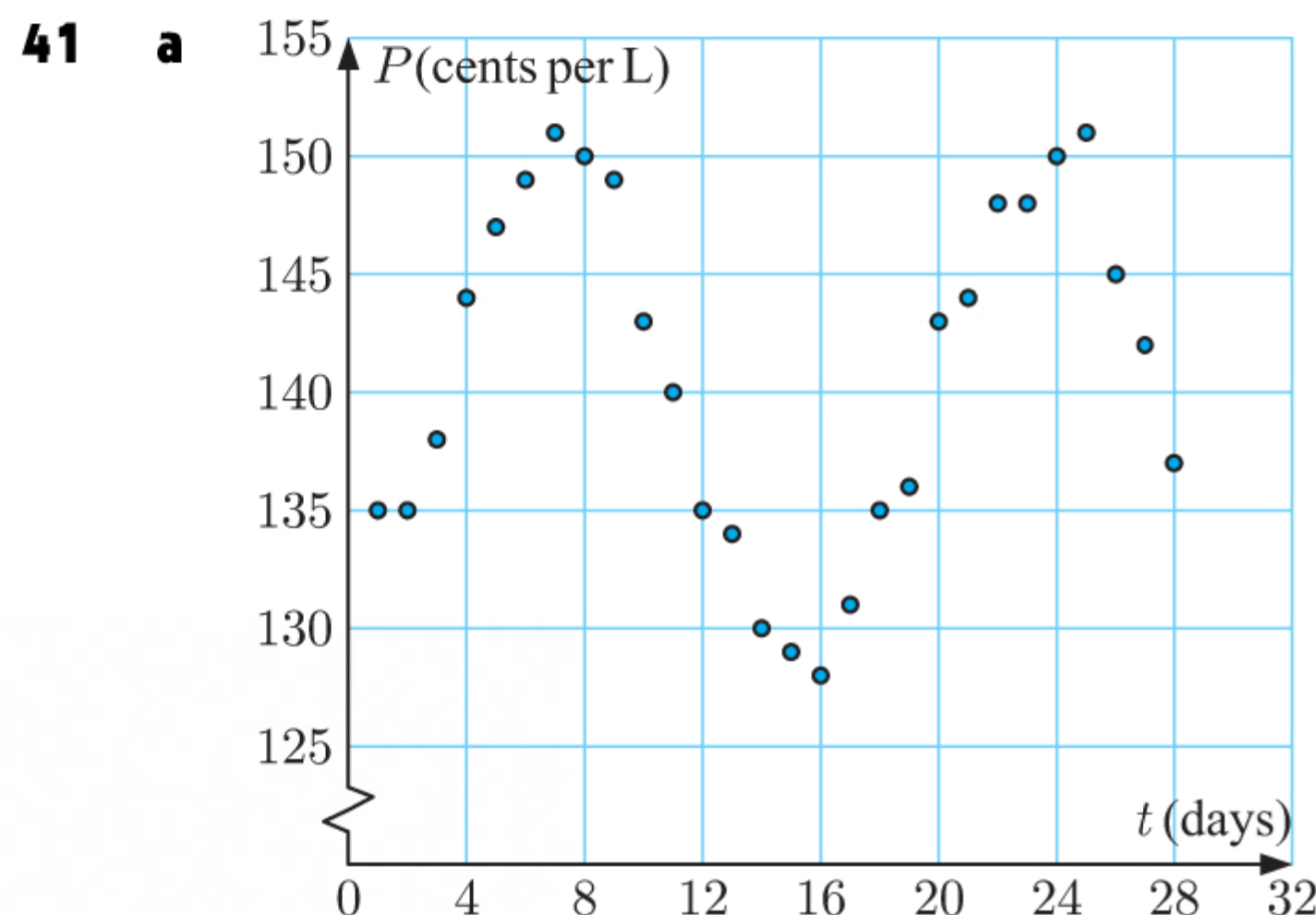
$T$ (years since 2000)	0	5	10	15
$P$ (population in millions)	28.5	31.8	36.3	41.2
Residual for exponential model	$\approx 0.162$	$\approx -0.273$	$\approx 0.000488$	$\approx 0.117$
Residual for quadratic model	0.04	-0.12	0.12	-0.04

$$\begin{aligned} \text{For the exponential model, } SS_{\text{res}} &\approx (0.162)^2 + (-0.273)^2 + (0.000488)^2 + (0.117)^2 \\ &\approx 0.114 \end{aligned}$$

$$\begin{aligned} \text{For the quadratic model, } SS_{\text{res}} &\approx (0.04)^2 + (-0.12)^2 + (0.12)^2 + (-0.04)^2 \\ &\approx 0.032 \end{aligned}$$



ii The quadratic model has the lower  $SS_{\text{res}}$  value, so the quadratic model fits the data better.



**b i** From the scatter diagram, the time between peaks (period) is about 16 days.

So,  $\frac{2\pi}{b} \approx 16$  and  $\therefore b \approx \frac{\pi}{8}$ .

ii The amplitude  $= \frac{\text{max} - \text{min}}{2} \approx \frac{151 - 128}{2} \approx 11.5$ , so  $a \approx 11.5$ .

iii The principal axis is midway between the maximum and minimum, so  $d \approx \frac{151 + 128}{2} \approx 139.5$ .

iv The model is  $P \approx 11.5 \sin\left(\frac{\pi}{8}(t - c)\right) + 139.5$  for some constant  $c$ .



From the scatter diagram, the first period starts somewhere between  $t = 3$  and  $t = 4$ . So we estimate  $c \approx 3.5$ .

So, the model is  $P \approx 11.5 \sin\left(\frac{\pi}{8}(t - 3.5)\right) + 139.5$

**c** Using technology,  $P \approx 10.7 \sin(0.381t - 1.12) + 139.4$



Rad(Norm)	d/c(Real)
SinReg	
a	=10.6561657
b	=0.38074639
c	=-1.1221973
d	=139.405144
MSe	=1.54194482
y=a·sin(bx+c)+d	
COPY	

**d** Using technology,  $SS_{\text{res}} \approx 80.2$  for the model in **b**.

		Rad(Norm)		d/c(Real)	
SUB		List 1	List 2	List 3	List 4
1	1	135	129.93	5.0619	
2	2	135	133.11	1.889	
3	3	138	137.25	0.7435	
4	4	144	141.74	2.2564	
5.061900541					
GRAPH CALC TEST INTR DIST 					

Math(Rad(Norm))	d/c(Real)
Sum (List 4) <sup>2</sup>	
80.21941831	
JUMP DELETE MAT MATH	

Using technology,  $SS_{\text{res}} \approx 40.1$  for the model in **c**.

	Rad(Norm)		d/c(Real)	
	List 1	List 2	List 3	List 4
SUB				
1	1	135	132.2	2.7915
2	2	135	135.64	-0.644
3	3	138	139.61	-1.618
4	4	144	143.56	0.4374
			2.791588956	
GRAPH CALC TEST INTR DIST 				

Math(Rad(Norm))	d/c(Real)
Sum (List 4) <sup>2</sup>	
40.09056535	
JUMP DELETE MAT MATH	

The model in **c** has a much lower  $SS_{\text{res}}$  value, so it is a better fit for the data.

**42**

$VO_2 \text{ max}$ (mL/kg/min)	52.1	63.2	47.7	58.8	59.1	50.8	62.9	55.6	66.2	54.1
Time taken (minutes)	62.8	37.3	65.2	50.7	43.5	58.1	39.2	50.4	34.1	61.1

**a**

Rad(Norm)	d/c(Real)
LinearReg(ax+b)	
a	=-1.811129
b	=153.564913
r	=-0.9597359
r <sup>2</sup>	=0.921093
MSe	=11.409596
y=ax+b	
COPY DRAW	

So,  $r \approx -0.960$ .

**b** Criterion validity is being considered here as the  $VO_2 \text{ max}$  of each athlete is being used to predict the time taken to run 10 km, the latter of which is the criterion variable.

**c** There is a very strong negative correlation between the variables. So, the predictor variable  $VO_2 \text{ max}$  has a very high criterion validity.



- d** The correlation between  $VO_2 \text{ max}$  and *time taken* is negative, so as  $VO_2 \text{ max}$  increases, the *time taken* decreases. Shorter times are better, because that means higher running speeds.  
 $\therefore$  a high  $VO_2 \text{ max}$  value will indicate that the athlete will perform *better* in the 10 km run.

- 43 a** Each test is identical and is done under the same conditions by Xavier every morning.

Test-retest reliability is being considered here.

- b** Each test has the same number of times tables questions to answer, and is done under the same conditions by Abigail's students every Friday. However, the tests are not identical as the 20 times tables are randomly chosen each week.

Parallel forms reliability is being considered here.

- c** Each match would be held under similar conditions throughout the season. However, the matches would not be identical as the tennis player would have different opponents each time.

Parallel forms reliability is being considered here.

**44**

Time (min)	Frequency
35 - 39	10
40 - 44	46
45 - 49	43
50+	15
Total	114

**a**  $P(40 \text{ to } 44 \text{ minutes}) \approx \frac{46}{114}$   
 $\approx 0.404$

**b**  $P(\text{at least } 50 \text{ minutes}) \approx \frac{15}{114}$   
 $\approx 0.132$

**c**  $P(\text{between } 35 \text{ and } 49 \text{ minutes inclusive}) \approx \frac{10 + 46 + 43}{114}$   
 $\approx 0.868$

**45**

Division	2017	2018	2019
1	4	5	5
2	6	7	8
3	13	12	14
4	18	10	14
5	20	17	16
Total	61	51	57

**a**  $P(\text{player in the 2017 tournament played in division 1})$   
 $\approx \frac{4}{61}$   $\leftarrow$  number of division 1 players in 2017 tournament  
 $\leftarrow$  total number of players in 2017 tournament  
 $\approx 0.0656$

**b** There were  $13 + 12 + 14 = 39$  division 3 players in total,  
and  $61 + 51 + 57 = 169$  players in total.

$\therefore P(\text{player in any of the past tournaments played in division 3}) \approx \frac{39}{169}$   
 $\approx 0.231$

- c** In the 2019 tournament, 8 players played in division 2 and 14 players played in division 4. So,  $57 - 8 - 14 = 35$  players in the 2019 tournament did *not* play in division 2 or 4.

$\therefore P(\text{player in the 2019 tournament did not play in division 2 or 4}) \approx \frac{35}{57}$   
 $\approx 0.614$

**46 a**

	< 40	40 - 59	$\geq 60$	Total
Male	56	127	419	602
Female	75	113	230	418
Total	131	240	649	1020

- b i** 602 out of the 1020 patients were male.

$\therefore P(\text{male}) \approx \frac{602}{1020} \approx 0.590$

- ii** 75 out of the 1020 patients were female and younger than 40.

$\therefore P(\text{female and younger than 40}) \approx \frac{75}{1020} \approx 0.0735$

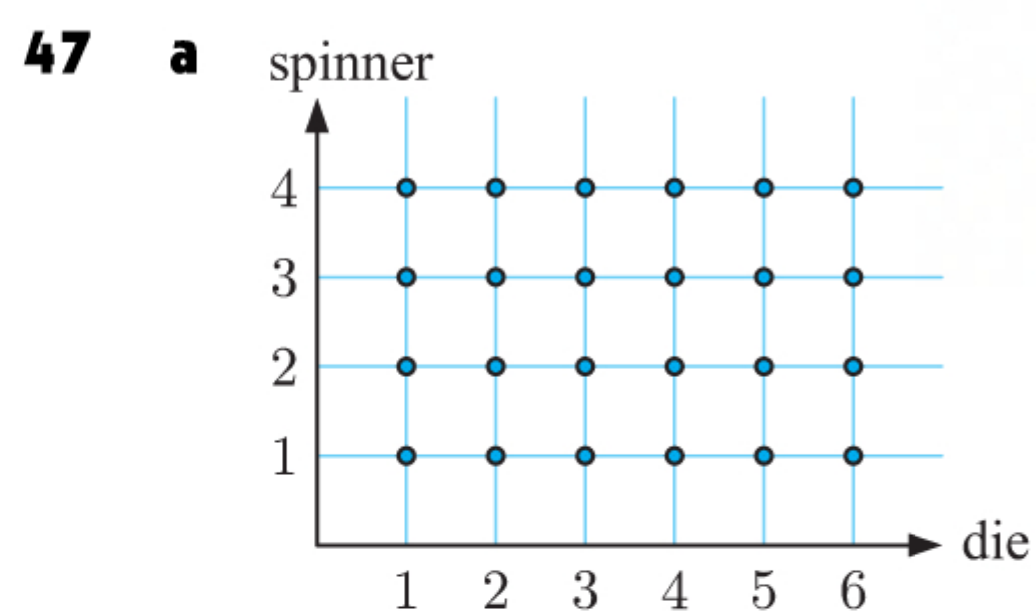
- iii** 230 out of the 418 female patients were 60 or older.

$\therefore P(60 \text{ or older, given they were female}) \approx \frac{230}{418} \approx 0.550$

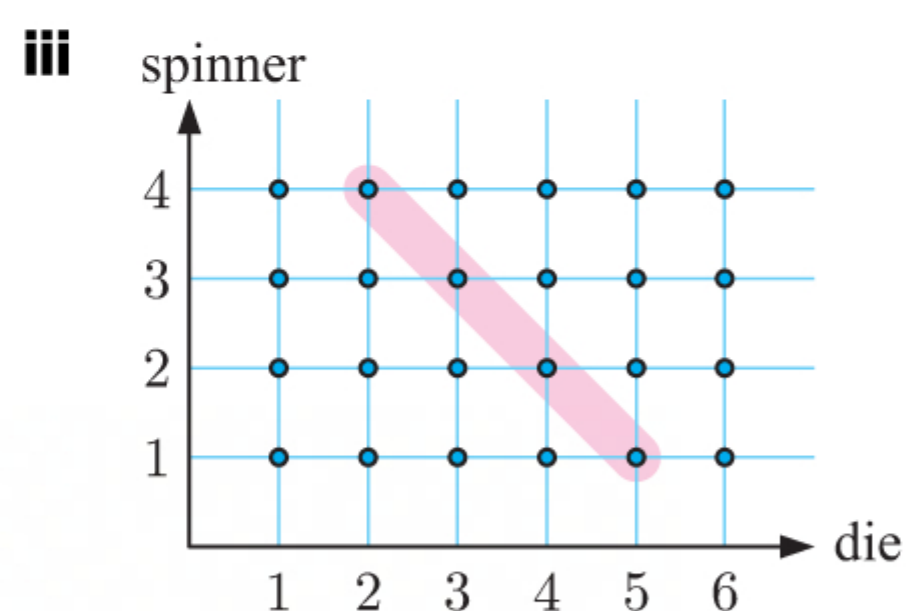
- iv**  $127 + 419 = 546$  out of the  $240 + 649 = 889$  patients who were 40 or older were male.

$\therefore P(\text{male, given they were 40 or older}) \approx \frac{546}{889} \approx 0.614$

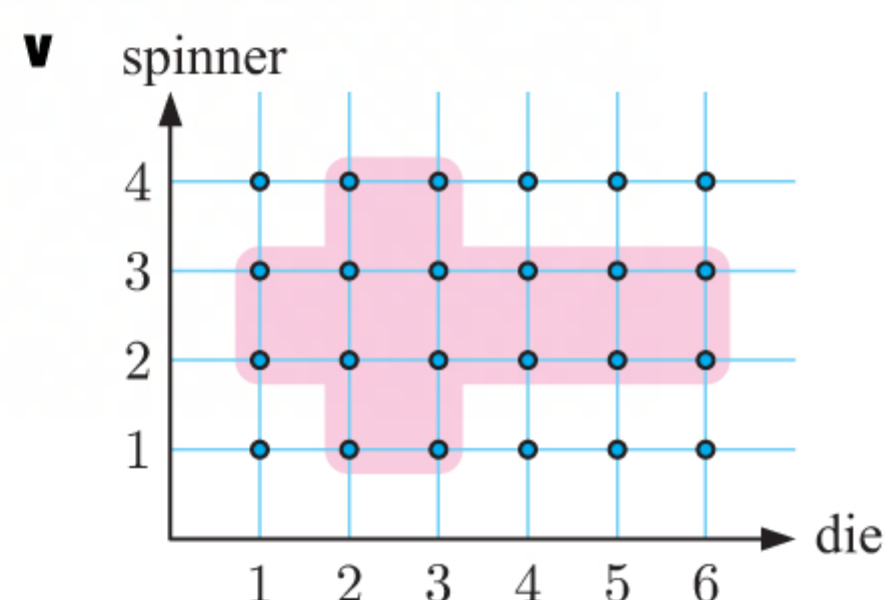




**b i**  $P(\text{two 1s}) = \frac{1}{24}$

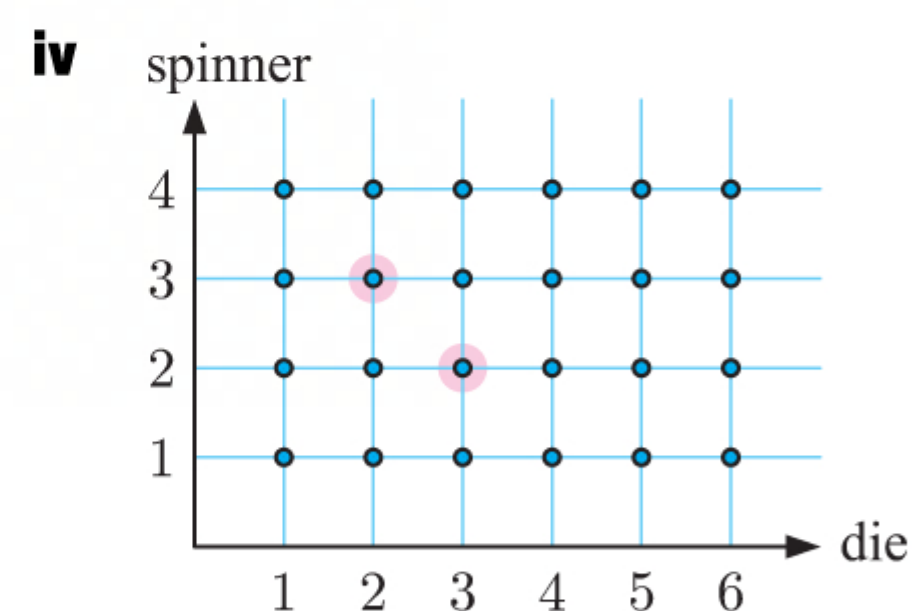


$$P(\text{a sum of 6}) = \frac{4}{24} \\ = \frac{1}{6}$$

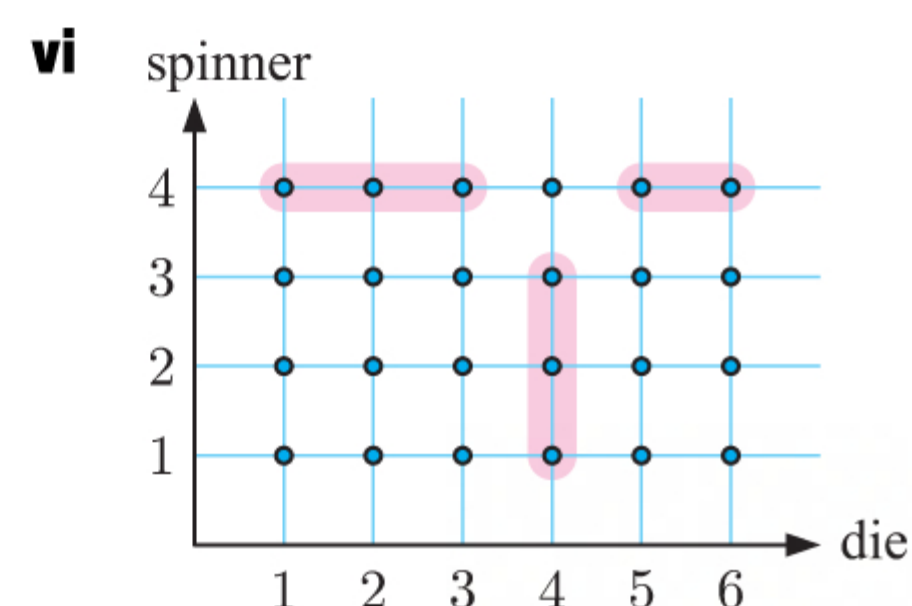


$$P(\text{a 2 or a 3 (or both)}) = \frac{16}{24} \\ = \frac{2}{3}$$

**ii**  $P(\text{two 5s}) = 0$  {the spinner does not have a 5}



$$P(\text{a 2 and a 3}) = \frac{2}{24} \\ = \frac{1}{12}$$



$$P(\text{exactly one 4}) = \frac{8}{24} \\ = \frac{1}{3}$$

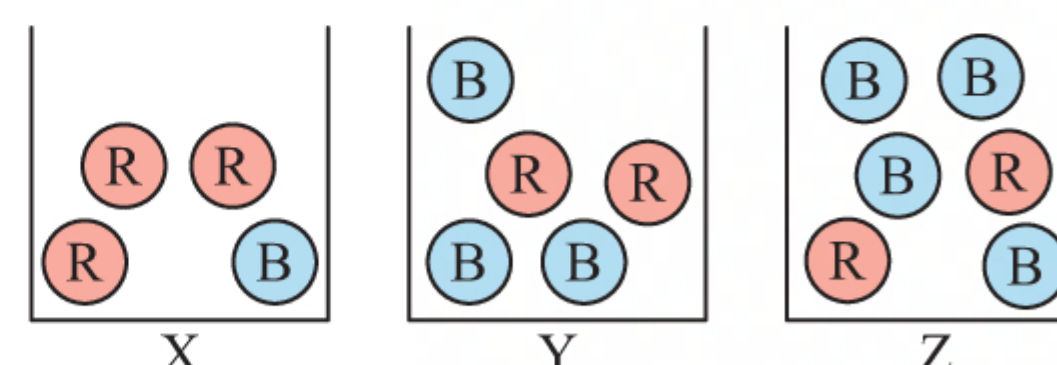
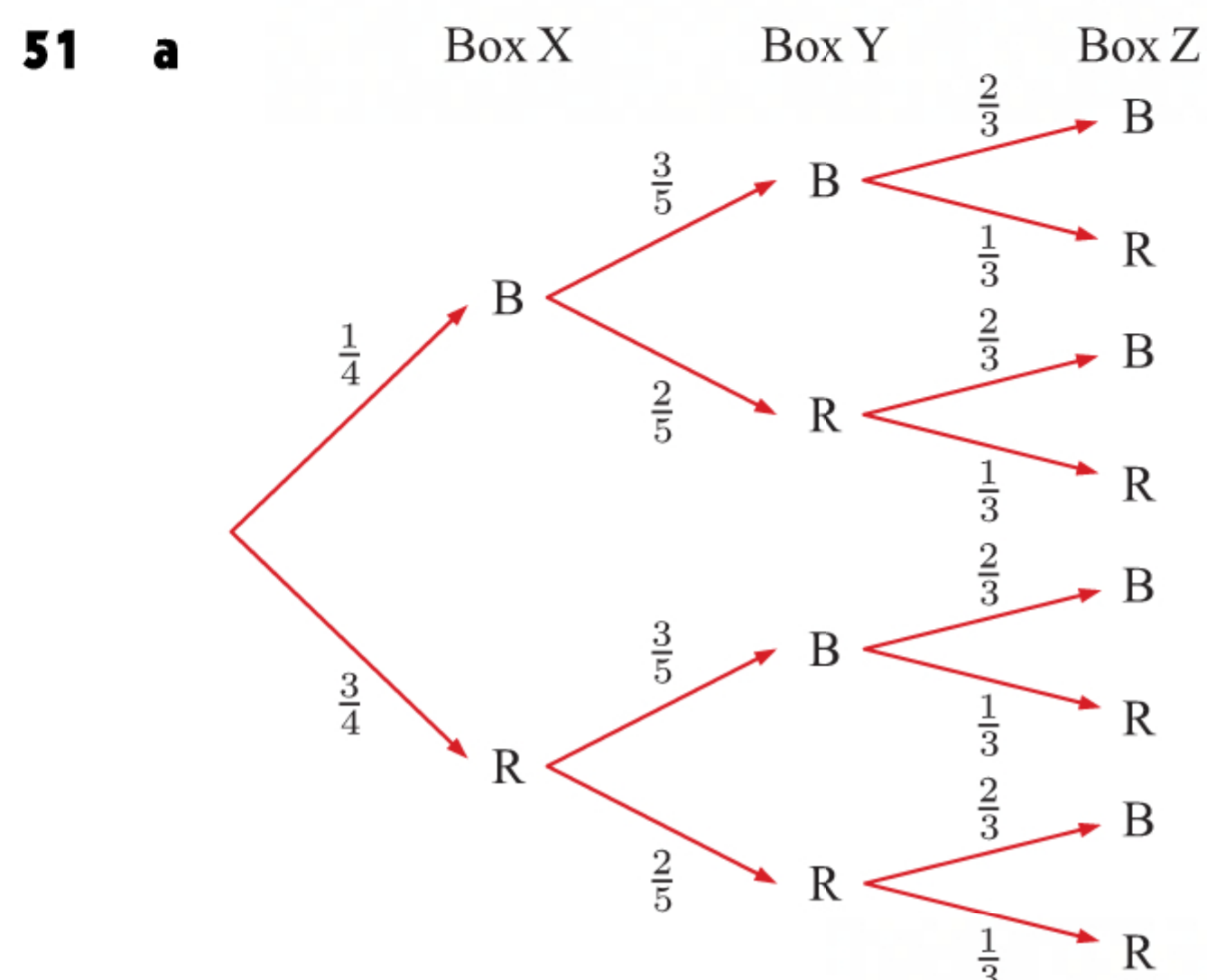
**48 a**  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$   
 $\therefore 0.78 = 0.37 + 0.41 - P(A \cap B)$   
 $\therefore P(A \cap B) = 0$

**b** Since  $P(A \cap B) = 0$ ,  $A$  and  $B$  are mutually exclusive events.

**49**  $A$  and  $B$  are mutually exclusive  $\therefore P(A \cap B) = 0$

Now  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$   
 $= P(A) + P(B)$   
 $\therefore 0.55 = P(A) + 0.3$   
 $\therefore P(A) = 0.25$

**50**  $P(A \cup B) = 1 - P((A \cup B)')$  Now  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$   
 $= 1 - \frac{1}{12}$   $\therefore \frac{11}{12} = \frac{23}{50} + \frac{5}{7} - P(A \cap B)$   
 $= \frac{11}{12}$   $\therefore P(A \cap B) = \frac{541}{2100}$





**b i**  $P(\text{exactly 2 red balls are drawn})$ 

$$\begin{aligned}
 &= P(RRB) + P(RBR) + P(BRR) \\
 &= \frac{3}{4} \times \frac{2}{5} \times \frac{2}{3} + \frac{3}{4} \times \frac{3}{5} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{5} \times \frac{1}{3} \\
 &= \frac{1}{5} + \frac{3}{20} + \frac{1}{30} \\
 &= \frac{23}{60}
 \end{aligned}$$

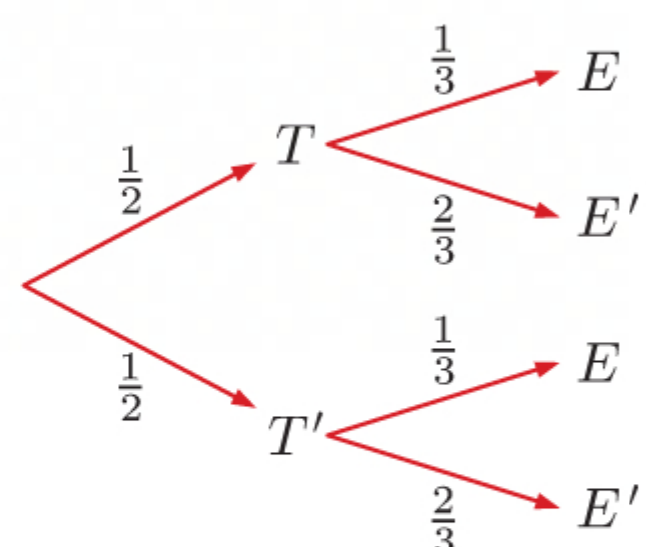
**ii**  $P(\text{blue balls are drawn from boxes X and Z})$ 

$$\begin{aligned}
 &= P(BBB) + P(BRB) \\
 &= \frac{1}{4} \times \frac{3}{5} \times \frac{2}{3} + \frac{1}{4} \times \frac{2}{5} \times \frac{2}{3} \\
 &= \frac{1}{10} + \frac{1}{15} \\
 &= \frac{1}{6}
 \end{aligned}$$

**iii**  $P(\text{at most one blue ball is drawn}) = P(\text{no blue balls are drawn}) + P(\text{exactly one blue ball is drawn})$ 

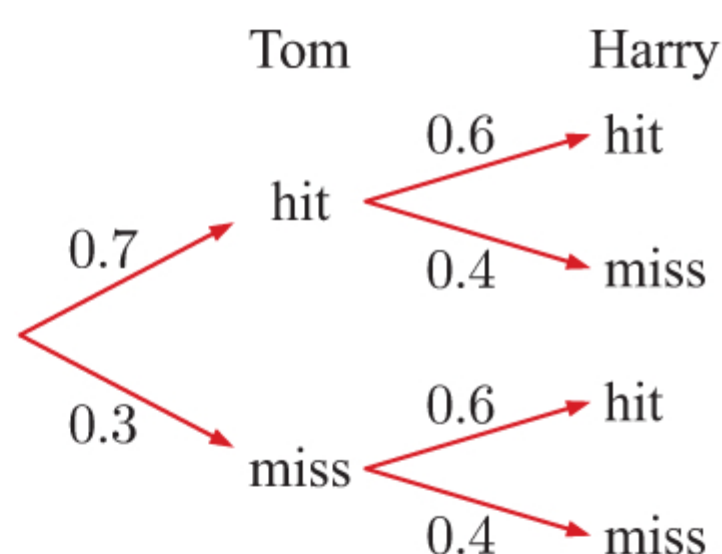
$$\begin{aligned}
 &= P(RRR) + [P(BRR) + P(RBR) + P(RRB)] \\
 &= \frac{3}{4} \times \frac{2}{5} \times \frac{1}{3} + \left[ \frac{1}{4} \times \frac{2}{5} \times \frac{1}{3} + \frac{3}{4} \times \frac{3}{5} \times \frac{1}{3} + \frac{3}{4} \times \frac{2}{5} \times \frac{2}{3} \right] \\
 &= \frac{1}{10} + \frac{1}{30} + \frac{3}{20} + \frac{1}{5} \\
 &= \frac{29}{60}
 \end{aligned}$$

**c** If an extra red ball is added to box Y, the probabilities in **b i** and **b iii** will be affected.**52 a**

	outcome	probability
	$T \text{ and } E$	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
	$T \text{ and } E'$	$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
	$T' \text{ and } E$	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
	$T' \text{ and } E'$	$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
	total	$\frac{6}{6} = 1$

**b i**  $P(T \cap E') = \frac{1}{3}$  {from **a**}

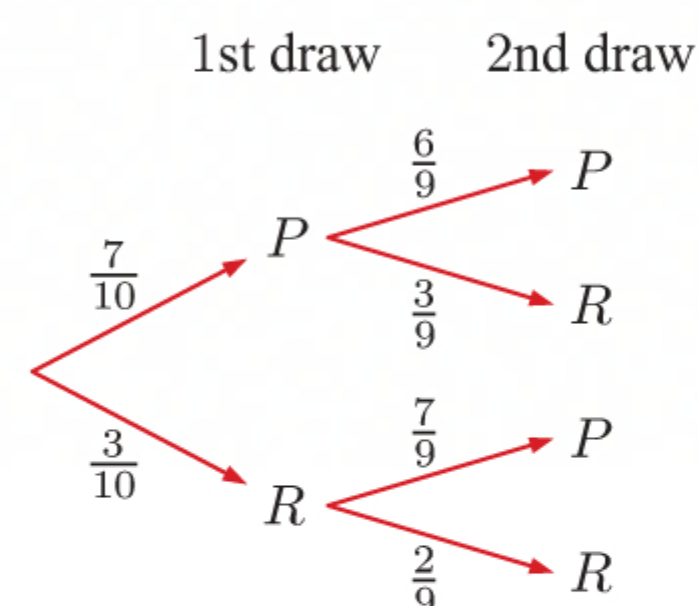
$$\begin{aligned}
 \text{ii } P(T \cup E') &= P(T) + P(E') - P(T \cap E') \\
 &= \frac{1}{2} + \frac{2}{3} - \frac{1}{3} \quad \{\text{from **a**}\} \\
 &= \frac{5}{6}
 \end{aligned}$$

**53****a**  $P(\text{only one of them is successful})$ 

$$\begin{aligned}
 &= 0.7 \times 0.4 + 0.3 \times 0.6 \\
 &= 0.46
 \end{aligned}$$

**b**  $P(\text{at least one is successful})$ 

$$\begin{aligned}
 &= 1 - P(\text{both miss}) \\
 &= 1 - 0.3 \times 0.4 \\
 &= 1 - 0.12 \\
 &= 0.88
 \end{aligned}$$

**54 a** Let  $P$  be the event that a purple ticket is drawn, and  $R$  be the event that a red ticket is drawn.**b i**  $P(\text{at least one red ticket})$ 

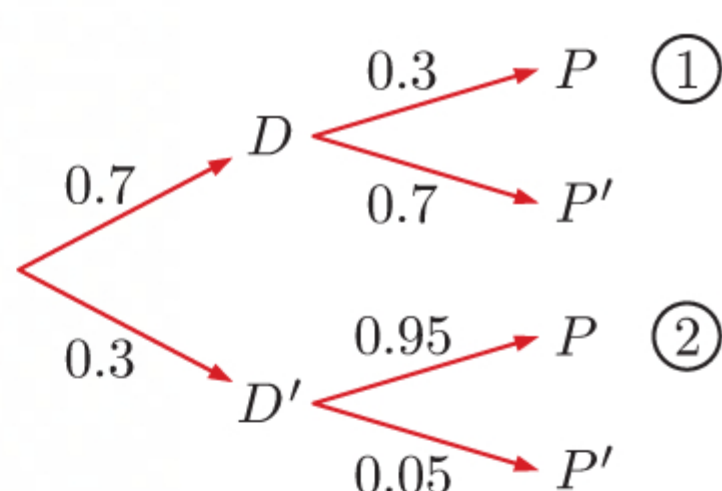
$$\begin{aligned}
 &= 1 - P(\text{no red tickets}) \\
 &= 1 - P(P) \\
 &= 1 - \frac{7}{10} \times \frac{6}{9} \\
 &= 1 - \frac{42}{90} \\
 &= \frac{48}{90} = \frac{8}{15}
 \end{aligned}$$

**ii**  $P(\text{one ticket of each colour})$ 

$$\begin{aligned}
 &= P(PR \text{ or } RP) \\
 &= \frac{7}{10} \times \frac{3}{9} + \frac{3}{10} \times \frac{7}{9} \\
 &= \frac{21}{90} + \frac{21}{90} \\
 &= \frac{42}{90} = \frac{7}{15}
 \end{aligned}$$

**iii**  $P(\text{purple ticket second})$ 

$$\begin{aligned}
 &= P(P \text{ or } RP) \\
 &= \frac{7}{10} \times \frac{6}{9} + \frac{3}{10} \times \frac{7}{9} \\
 &= \frac{42}{90} + \frac{21}{90} \\
 &= \frac{63}{90} = \frac{7}{10}
 \end{aligned}$$

**55** Let  $D$  be the event that Donna goes shopping with Nick, and  $P$  be the event that Nick purchases a packet of potato chips.

$$\begin{aligned}
 \text{a } P(P) &= \underbrace{P(D \cap P)}_{\text{①}} + \underbrace{P(D' \cap P)}_{\text{②}} \\
 &= 0.7 \times 0.3 + 0.3 \times 0.95 \\
 &= 0.495
 \end{aligned}$$

$$\begin{aligned}
 \text{b } P(D | P) &= \frac{P(D \cap P)}{P(P)} \\
 &= \frac{0.7 \times 0.3}{0.495} \quad \leftarrow \text{① from **a**} \\
 &\approx 0.424
 \end{aligned}$$



$$56 \quad \mathbf{a} \quad P(2 \text{ white truffles}) = P(\text{first is white} \cap \text{second is white})$$

$$= P(\text{first is white}) \times P(\text{second is white given first is white})$$

$$= \frac{2}{12} \times \frac{1}{11}$$

$$= \frac{2}{132}$$

$$= \frac{1}{66}$$

$$\mathbf{b} \quad P(2 \text{ white truffles}) = \frac{1}{66} \quad \{\text{from a}\}$$

$$P(2 \text{ dark brown truffles}) = P(\text{first is dark brown} \cap \text{second is dark brown})$$

$$= P(\text{first is dark brown}) \times P(\text{second is dark brown given first is dark brown})$$

$$= \frac{6}{12} \times \frac{5}{11}$$

$$= \frac{30}{132}$$

$$= \frac{5}{22}$$

$$P(2 \text{ light brown truffles}) = P(\text{first is light brown} \cap \text{second is light brown})$$

$$= P(\text{first is light brown}) \times P(\text{second is light brown given first is light brown})$$

$$= \frac{4}{12} \times \frac{3}{11}$$

$$= \frac{12}{132}$$

$$= \frac{1}{11}$$

$$P(\text{different coloured truffles}) = 1 - P(\text{same coloured truffles})$$

$$= 1 - \left( \frac{1}{66} + \frac{5}{22} + \frac{1}{11} \right)$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

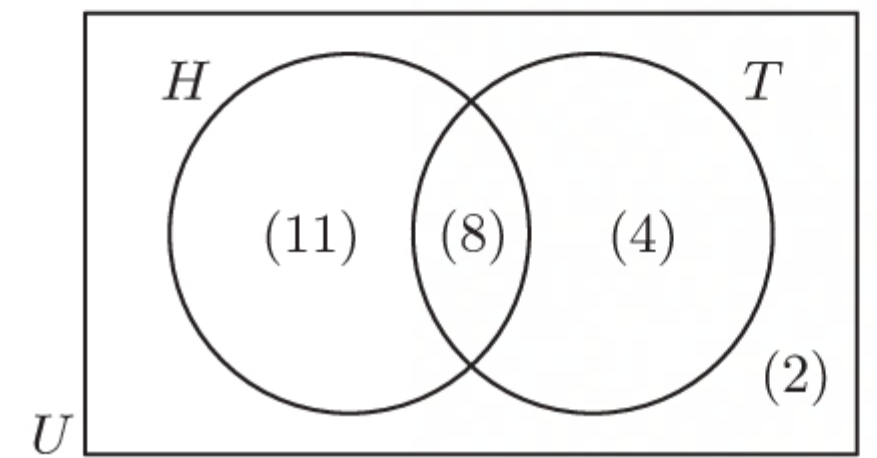
$$57 \quad \text{There are } 11 + 8 + 4 + 2 = 25 \text{ students.}$$

$$\mathbf{a} \quad P(H) = \frac{11+8}{25} \\ = \frac{19}{25}$$

$$\mathbf{b} \quad P(T') = \frac{11+2}{25} \\ = \frac{13}{25}$$

$$\mathbf{c} \quad P(\text{plays at least one sport}) \\ = P(H \cup T) \\ = \frac{11+8+4}{25} \\ = \frac{23}{25}$$

$$\mathbf{d} \quad P(T | H) = \frac{8}{11+8} \\ = \frac{8}{19}$$



$$58 \quad \mathbf{a} \quad \text{Let } O \text{ represent a student who owns an orange highlighter, and} \\ B \text{ represent a student who owns a blue highlighter.}$$

Let the proportion of students in  $O \cap B$  be  $x$ .

$\therefore$  the proportion in  $O \cap B'$  is  $0.4 - x$  and

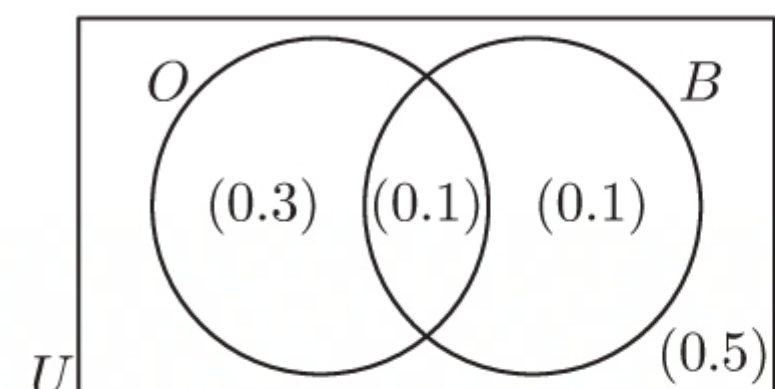
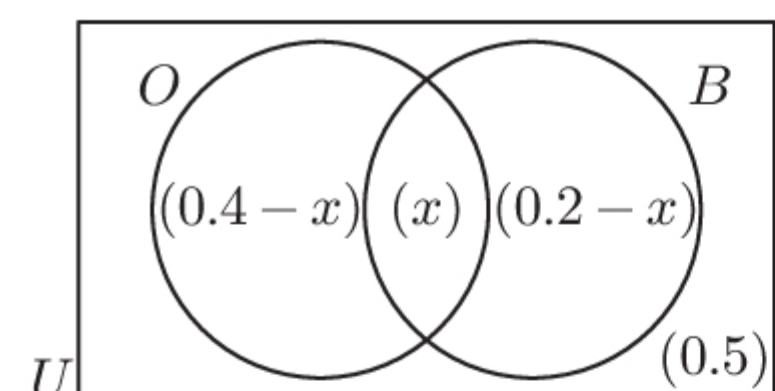
the proportion in  $O' \cap B$  is  $0.2 - x$ .

The proportion in  $O' \cap B'$  is 0.5.

$$\therefore (0.4 - x) + x + (0.2 - x) = 1 - 0.5$$

$$\therefore 0.6 - x = 0.5$$

$$\therefore x = 0.1$$

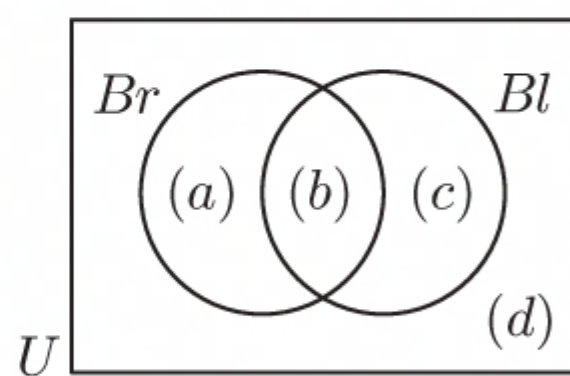


$$\mathbf{b} \quad \mathbf{i} \quad P(B | O) = \frac{P(B \cap O)}{P(O)} \\ = \frac{0.1}{0.4} \\ = \frac{1}{4}$$

$$\mathbf{ii} \quad P(O | B') = \frac{P(O \cap B')}{P(B')} \\ = \frac{0.3}{0.8} \\ = \frac{3}{8}$$



- 59 a** Let  $Br$  represent a student with brown hair, and  $Bl$  represent a student with blue eyes.



$$a + b + c + d = 30$$

$$a + b = 17$$

$$b + c = 12$$

$$d = 4$$

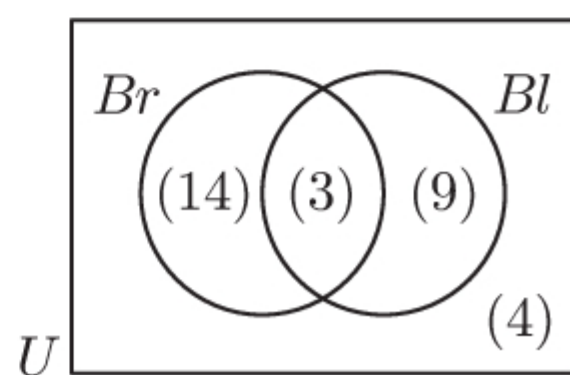
$$\therefore a + b + c = 26$$

$$\text{Since } a + b = 17, \quad 17 + c = 26$$

$$\therefore c = 9$$

$$\text{Since } b + c = 12, \quad b + 9 = 12$$

$$\therefore b = 3$$



$$a + b = 17$$

$$\therefore a + 3 = 17$$

$$\therefore a = 14$$

**b i**  $P(Bl \text{ but not } Br) = \frac{9}{30} = 0.3$

**ii**  $P(Br | Bl) = \frac{3}{12} = 0.25$

**60**  $n = 180$  attempts

$$p = P(\text{Roger getting a "first serve" in}) = \frac{7}{9}$$

Roger would be expected to get  $np = 180 \times \frac{7}{9} = 140$  "first serves" in.

- 61** Let  $H$  represent a head, and  $T$  represent a tail.

**a**  $P(2 \text{ heads and } 1 \text{ tail})$

$$= P(\text{HHT, HTH, or THH})$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$= \frac{3}{8}$$

**b**  $n = 400$  times

$$p = P(\text{exactly } 1 \text{ tail}) = \frac{3}{8} \quad \{\text{from a}\}$$

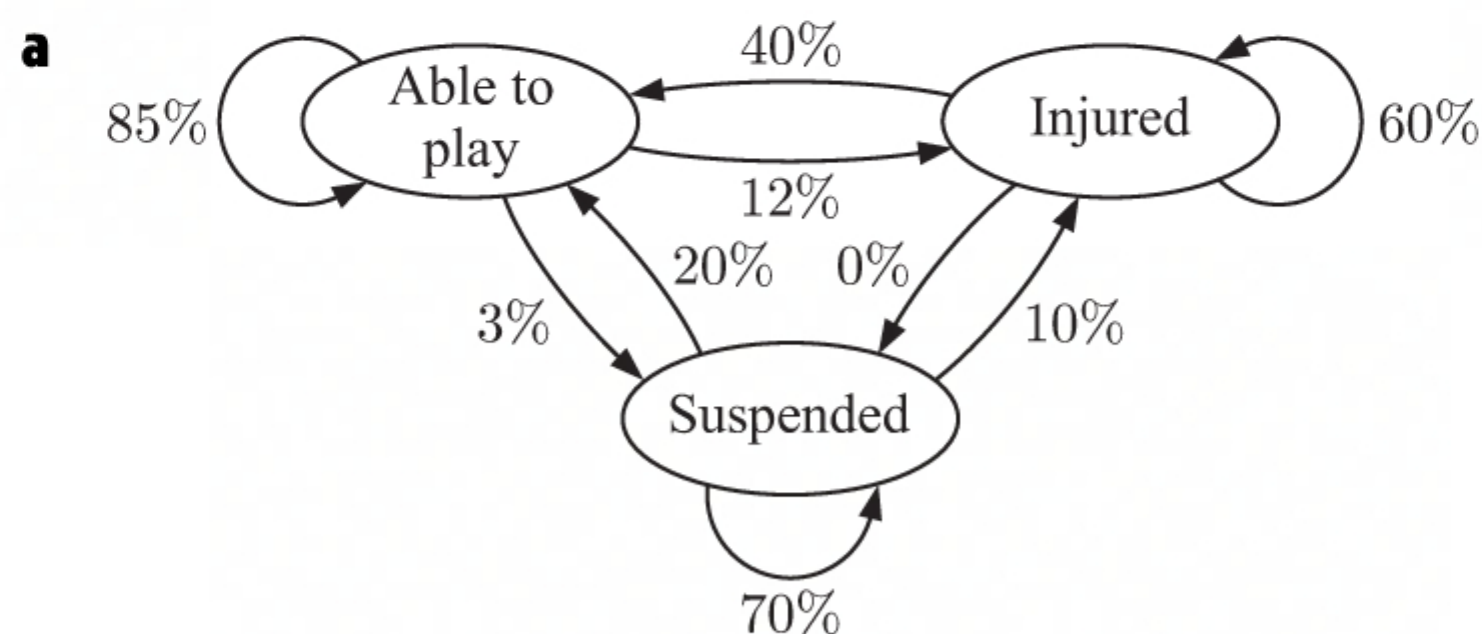
You would expect to see exactly one tail

$$np = 400 \times \frac{3}{8} = 150 \text{ times.}$$

**62**

*This week*

		<i>This week</i>		
		Able to play	Injured	Suspended
<i>Next week</i>	Able to play	85%	40%	20%
	Injured	12%	60%	10%
	Suspended	3%	0%	70%



**b**  $\mathbf{T} = \begin{pmatrix} 0.85 & 0.4 & 0.2 \\ 0.12 & 0.6 & 0.1 \\ 0.03 & 0 & 0.7 \end{pmatrix}$

- c** The numbers in the first column of  $\mathbf{T}$  represent the proportions of players currently able to play who will be in each category next week.

**d**  $\mathbf{T}^2 = \begin{pmatrix} 0.85 & 0.4 & 0.2 \\ 0.12 & 0.6 & 0.1 \\ 0.03 & 0 & 0.7 \end{pmatrix}^2 = \begin{pmatrix} 0.7765 & 0.58 & 0.35 \\ 0.177 & 0.408 & 0.154 \\ 0.0465 & 0.012 & 0.496 \end{pmatrix}$

The numbers in the second column of  $\mathbf{T}^2$  represent the proportions of players currently injured who will be in each category in 2 weeks' time.



- e Percentage of players currently able to play, who will not be able to play in two weeks' time due to injury or suspension =  $17.7\% + 4.65\%$   
 $= 22.35\%$

f  $s_0 = \begin{pmatrix} 40 \\ 0 \\ 0 \end{pmatrix}$

i  $s_1 = Ts_0 = \begin{pmatrix} 0.85 & 0.4 & 0.2 \\ 0.12 & 0.6 & 0.1 \\ 0.03 & 0 & 0.7 \end{pmatrix} \begin{pmatrix} 40 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 34 \\ 4.8 \\ 1.2 \end{pmatrix}$

Rounding to the nearest player, the manager would expect about 34 players to be able to play, 5 players to be injured, and 1 player to be suspended after one week.

ii  $s_2 = T^2s_0 = \begin{pmatrix} 0.7765 & 0.58 & 0.35 \\ 0.177 & 0.408 & 0.154 \\ 0.0465 & 0.012 & 0.496 \end{pmatrix} \begin{pmatrix} 40 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 31.06 \\ 7.08 \\ 1.86 \end{pmatrix}$

Rounding to the nearest player, the manager would expect about 31 players to be able to play, 7 players to be injured, and 2 players to be suspended after two weeks.

63

		This week	
		Alfred	Mae
Next week	Alfred	80%	15%
	Mae	20%	85%

a  $T = \begin{pmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{pmatrix}$

b  $T^3 = \begin{pmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{pmatrix}^3 = \begin{pmatrix} 0.5855 & 0.310875 \\ 0.4145 & 0.689125 \end{pmatrix}$

Column 2 of  $T^3$  tells us how the people who voted for Mae will vote in 3 weeks' time. About 31.1% will vote for Alfred, and about 68.9% will vote for Mae again.

c  $s_0 = \begin{pmatrix} 1 - 0.7 \\ 0.7 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$

$s_4 = T^4s_0 = \begin{pmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{pmatrix}^4 \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \approx \begin{pmatrix} 0.4056 \\ 0.5944 \end{pmatrix}$

$\therefore$  in 4 weeks' time, Mae will have about 59.4% of the vote.

d If  $s = \begin{pmatrix} a \\ b \end{pmatrix}$  is the steady state matrix then  $Ts = s$   
 $\therefore \begin{pmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$   
 $\therefore \begin{pmatrix} 0.8a + 0.15b \\ 0.2a + 0.85b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

Using either row,  $0.8a + 0.15b = a$

$\therefore 0.15b = 0.2a$

$\therefore b = \frac{4}{3}a$

Now  $s$  is a matrix of probabilities, so  $a + b = 1$

$\therefore a + \frac{4}{3}a = 1$

$\therefore \frac{7}{3}a = 1$

$\therefore a = \frac{3}{7}$  and  $b = \frac{4}{7}$

So,  $s = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}$ .

In the long term,  $\frac{3}{7} \times 100\% \approx 42.9\%$  of people living in the area will vote for Alfred, and  $\frac{4}{7} \times 100\% \approx 57.1\%$  will vote for Mae.



**64 a**

$x$	0	1	2	3
$P(X = x)$	$k$	0.2	0.5	0.1

$$\begin{aligned}\sum P(X = x) &= 1 \\ \therefore k + 0.2 + 0.5 + 0.1 &= 1 \\ \therefore k + 0.8 &= 1 \\ \therefore k &= 0.2\end{aligned}$$

**b**

$x$	2	4	6
$P(X = x)$	$2k$	0.1	$0.6 - k$

$$\begin{aligned}\sum P(X = x) &= 1 \\ \therefore 2k + 0.1 + (0.6 - k) &= 1 \\ \therefore k + 0.7 &= 1 \\ \therefore k &= 0.3\end{aligned}$$

**65 a**  $P(x) = k(x + 3), \quad x = 0, 1, 2, 3, 4$

$$\therefore P(0) = 3k, \quad P(1) = 4k, \quad P(2) = 5k, \\ P(3) = 6k, \quad P(4) = 7k$$

Since  $P(x)$  is a probability mass function,

$$\begin{aligned}\sum_{i=1}^n P(x_i) &= 1 \\ \therefore 3k + 4k + 5k + 6k + 7k &= 1 \\ \therefore 25k &= 1 \\ \therefore k &= \frac{1}{25}\end{aligned}$$

**b**  $P(x) = \frac{k^{x-3}}{x-1}, \quad x = 3, 4, 5$

$$\therefore P(3) = \frac{k^0}{3-1}, \quad P(4) = \frac{k^1}{4-1}, \quad P(5) = \frac{k^2}{5-1}$$

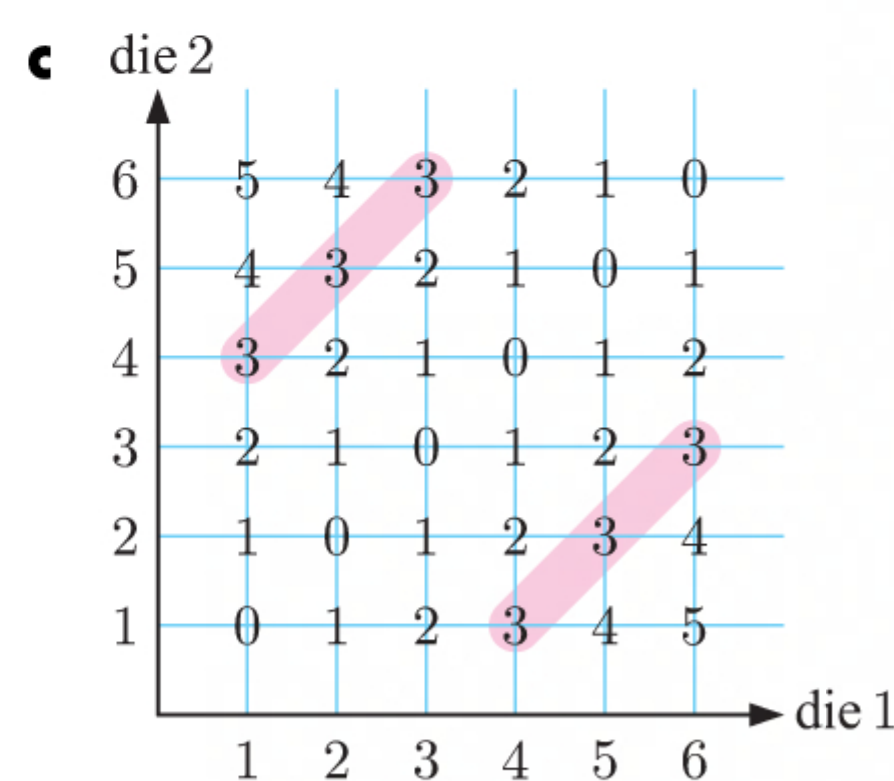
$$= \frac{1}{2} \qquad = \frac{k}{3} \qquad = \frac{k^2}{4}$$

Since  $P(x)$  is a probability mass function,

$$\begin{aligned}\sum_{i=1}^n P(x_i) &= 1 \\ \therefore \frac{1}{2} + \frac{k}{3} + \frac{k^2}{4} &= 1 \\ \therefore \frac{k^2}{4} + \frac{k}{3} - \frac{1}{2} &= 0 \\ \therefore 3k^2 + 4k - 6 &= 0 \\ \therefore k &= \frac{-4 \pm \sqrt{16 + 72}}{2 \times 3} \\ &= \frac{-4 \pm \sqrt{88}}{6} \\ &= \frac{-2 \pm \sqrt{22}}{3} \quad \{k > 0\}\end{aligned}$$

**66 a**  $X$  is the difference of a number from one die and a number from the other die. So  $X$  is a discrete random variable because  $X$  has a set of distinct possible values.

**b**  $X = 0, 1, 2, 3, 4, 5$



$$\begin{aligned}P(X = 3) &= \frac{6}{36} \\ &= \frac{1}{6}\end{aligned}$$

**67 a**  $P(x) = \frac{a}{(x-3)^2}, \quad x = 0, 1, 2$

$$\therefore P(0) = \frac{a}{9}, \quad P(1) = \frac{a}{4}, \quad P(2) = a$$

Since  $P(x)$  is a probability mass function,  $\sum_{i=1}^n P(x_i) = 1$

$$\begin{aligned}\therefore \frac{a}{9} + \frac{a}{4} + a &= 1 \\ \therefore \frac{49}{36}a &= 1 \\ \therefore a &= \frac{36}{49}\end{aligned}$$

**b**  $P(X = 2) = P(2)$

$$\begin{aligned}&= a \\ &= \frac{36}{49} \quad \{\text{from a}\}\end{aligned}$$

**c** Since  $P(X = 2) = \frac{36}{49}$  is the greatest probability, the mode of the distribution is 2.

$$p_1 = \frac{a}{9} = \frac{\frac{36}{49}}{9} = \frac{4}{49} \approx 0.0816$$

$$p_1 + p_2 = \frac{a}{9} + \frac{a}{4} = \frac{4}{49} + \frac{\frac{36}{49}}{4} = \frac{4}{49} + \frac{9}{49} = \frac{13}{49} \approx 0.265$$

Since  $p_1 + p_2 + p_3 = 1 \geq 0.5$ , the median is 2.

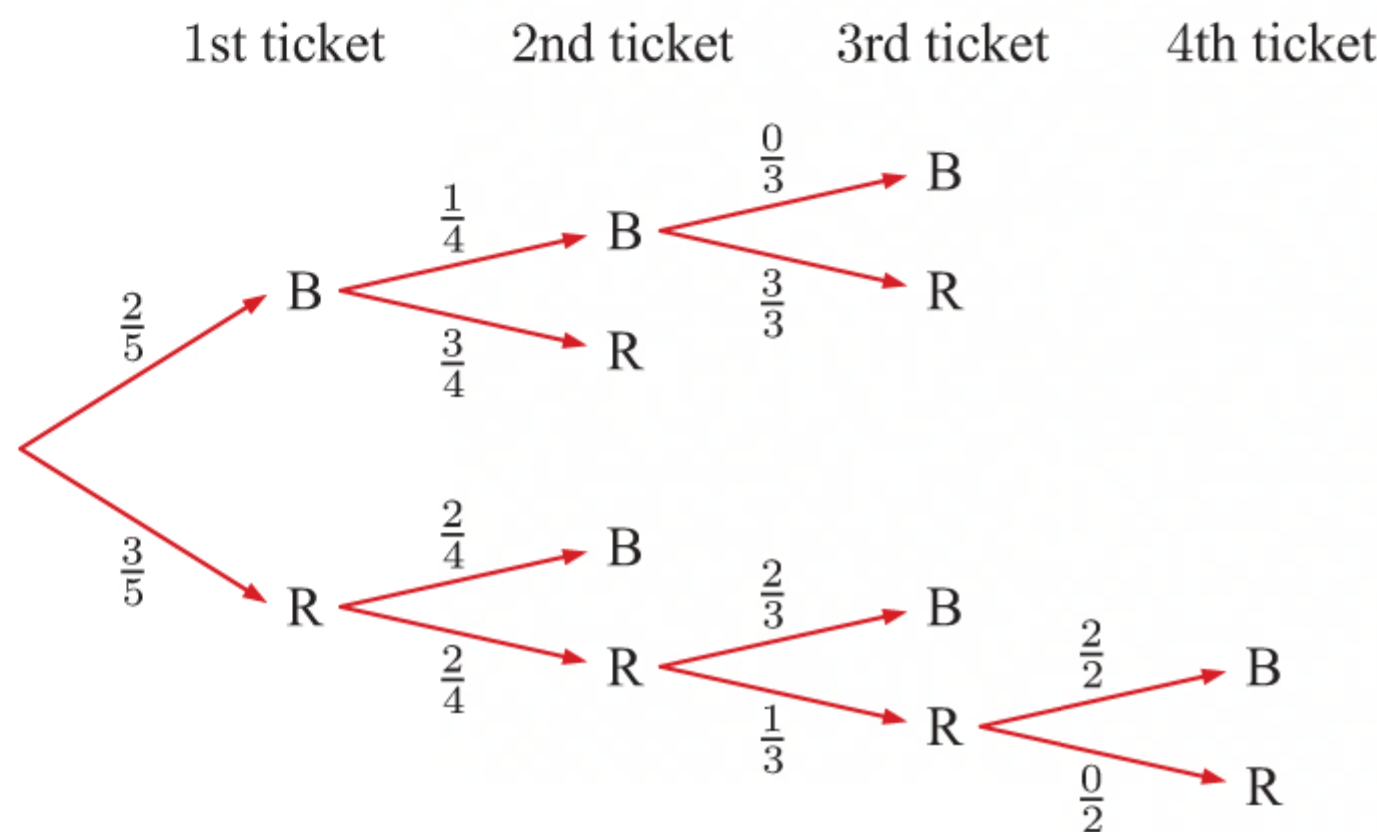


**68 a**  $X = 2, 3, 4$

**b** Let B represent a blue ticket, and R represent a red ticket.

The possible selections that can be made are:

BR      BBR      RRRB  
 RB      RRB  
 ( $X = 2$ )   ( $X = 3$ )   ( $X = 4$ )



$$\begin{aligned}\text{So, } P(X = 2) &= \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} = \frac{3}{5} \\ P(X = 3) &= \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} + \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} = \frac{3}{10} \\ P(X = 4) &= \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2} = \frac{1}{10}\end{aligned}$$

$\therefore$  the probability distribution of  $X$  is

$x$	2	3	4
$P(X = x)$	$\frac{3}{5}$	$\frac{3}{10}$	$\frac{1}{10}$

**c** It is most likely that 2 tickets are drawn, so the mode is 2.

$$\begin{aligned}\mathbf{d} \quad E(X) &= \sum_{i=1}^n x_i p_i \\ &= 2\left(\frac{3}{5}\right) + 3\left(\frac{3}{10}\right) + 4\left(\frac{1}{10}\right) \\ &= \frac{5}{2} = 2.5\end{aligned}$$

**69**  $P(x) = P(X = x) = \frac{1}{24}(x + 6), \quad x = 1, 2, 3$

$$\begin{aligned}\mathbf{a} \quad P(1) &= \frac{1}{24}(1 + 6) = \frac{7}{24} \\ P(2) &= \frac{1}{24}(2 + 6) = \frac{8}{24} = \frac{1}{3} \\ P(3) &= \frac{1}{24}(3 + 6) = \frac{9}{24} = \frac{3}{8}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad E(X) &= \sum_{i=1}^3 x_i P(x_i) \\ &= 1\left(\frac{7}{24}\right) + 2\left(\frac{8}{24}\right) + 3\left(\frac{9}{24}\right) \quad \{\text{from a}\} \\ &= \frac{50}{24} = 2\frac{1}{12}\end{aligned}$$

<b>70</b>	Cups of tea	0	1	2	3	4	5
	Probability	0.1	0.07	0.16	0.37	0.21	0.09

**a** Russell is most likely to drink 3 cups of tea in one day, so the mode is 3 cups.

$$\begin{aligned}\mathbf{b} \quad \text{Expected number of cups of tea} &= 0 \times 0.1 + 1 \times 0.07 + 2 \times 0.16 + 3 \times 0.37 + 4 \times 0.21 + 5 \times 0.09 \\ &= 2.79 \text{ cups}\end{aligned}$$

$\therefore$  on average, Russell drinks 2.79 cups of tea per day.

**71 a**  $P(x) = \frac{x^2 + kx}{50}, \quad x = 1, 2, 3, 4$

$$\therefore P(1) = \frac{k+1}{50}, \quad P(2) = \frac{2k+4}{50}, \quad P(3) = \frac{3k+9}{50}, \quad P(4) = \frac{4k+16}{50}$$

Since  $P(x)$  is a probability mass function,  $\sum_{i=1}^n P(x_i) = 1$

$$\begin{aligned}\therefore \frac{k+1}{50} + \frac{2k+4}{50} + \frac{3k+9}{50} + \frac{4k+16}{50} &= 1 \\ \therefore \frac{10k+30}{50} &= 1 \\ \therefore 10k+30 &= 50 \\ \therefore 10k &= 20 \\ \therefore k &= 2\end{aligned}$$

**b**  $\mu = E(X)$

$$\begin{aligned}&= \sum_{i=1}^n x_i p_i \\ &= 1\left(\frac{3}{50}\right) + 2\left(\frac{8}{50}\right) + 3\left(\frac{15}{50}\right) + 4\left(\frac{24}{50}\right) \\ &= 3.2\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad P(X \geq 2) &= 1 - P(X = 1) \\ &= 1 - P(1) \\ &= 1 - \frac{3}{50} \\ &= \frac{47}{50}\end{aligned}$$



**72**

$x$	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.4	0.2	0.1

$$\begin{aligned}\mathbf{a} \quad \mu &= \sum x_i p_i \\ &= 1(0.1) + 2(0.2) + 3(0.4) + 4(0.2) + 5(0.1) \\ &= 3\end{aligned}$$

**b** Since  $P(X = 3) = 0.4$  is the greatest probability, the mode of the distribution is 3.

$$\begin{aligned}\mathbf{c} \quad \sigma^2 &= \sum (x_i - \mu)^2 p_i \\ &= (1 - 3)^2(0.1) + (2 - 3)^2(0.2) + (3 - 3)^2(0.4) + (4 - 3)^2(0.2) + (5 - 3)^2(0.1) \\ &= 1.2\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad \sigma &= \sqrt{\sigma^2} \\ &= \sqrt{1.2} \\ &\approx 1.10\end{aligned}$$

$$\mathbf{73} \quad E(X) = 7, \sigma(X) = 2 \quad \therefore \text{Var}(X) = 2^2 = 4$$

$$\begin{aligned}\mathbf{a} \quad Y &= 4X + 3 \\ \therefore E(Y) &= E(4X + 3) & \text{Var}(Y) &= \text{Var}(4X + 3) \\ &= 4E(X) + 3 & &= 4^2 \text{Var}(X) \\ &= 4(7) + 3 & &= 16 \text{Var}(X) \\ &= 28 + 3 & &= 16(4) \\ &= 31 & &= 64\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad Y &= \frac{1}{2}(5 - X) = \frac{5}{2} - \frac{1}{2}X \\ \therefore E(Y) &= E\left(\frac{5}{2} - \frac{1}{2}X\right) & \text{Var}(Y) &= \text{Var}\left(\frac{5}{2} - \frac{1}{2}X\right) \\ &= \frac{5}{2} - \frac{1}{2}E(X) & &= \left(-\frac{1}{2}\right)^2 \text{Var}(X) \\ &= \frac{5}{2} - \frac{1}{2}(7) & &= \frac{1}{4} \text{Var}(X) \\ &= \frac{5}{2} - \frac{7}{2} & &= \frac{1}{4}(4) \\ &= -\frac{2}{2} & &= 1 \\ &= -1\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad Y &= \frac{2X - 1}{3} = \frac{2}{3}X - \frac{1}{3} \\ \therefore E(Y) &= E\left(\frac{2}{3}X - \frac{1}{3}\right) & \text{Var}(Y) &= \text{Var}\left(\frac{2}{3}X - \frac{1}{3}\right) \\ &= \frac{2}{3}E(X) - \frac{1}{3} & &= \left(\frac{2}{3}\right)^2 \text{Var}(X) \\ &= \frac{2}{3}(7) - \frac{1}{3} & &= \frac{4}{9} \text{Var}(X) \\ &= \frac{14}{3} - \frac{1}{3} & &= \frac{4}{9}(4) \\ &= \frac{13}{3} & &= \frac{16}{9}\end{aligned}$$

$$\mathbf{74} \quad E(X) = 3.5, \text{Var}(X) = 1.19, \text{ and } Y = 20 + 3X$$

$$\begin{aligned}\mathbf{a} \quad E(Y) &= E(20 + 3X) \\ &= 20 + 3E(X) \\ &= 20 + 3(3.5) \\ &= 30.5 \text{ points}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \text{Var}(Y) &= \text{Var}(20 + 3X) \\ &= 3^2 \text{Var}(X) \\ &= 9 \text{Var}(X) \\ &= 9(1.19) \\ &= 10.71\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \sigma(Y) &= \sqrt{\text{Var}(Y)} \\ &= \sqrt{10.71} \\ &\approx 3.27 \text{ points}\end{aligned}$$



<b>75</b>	$x$	1	2	3	4
	$P(X = x)$	0.25	0.38	0.17	0.2

$$\begin{aligned}\mathbf{a} \quad E(X) &= \sum x_i p_i \\ &= 1(0.25) + 2(0.38) + 3(0.17) + 4(0.2) \\ &= 2.32\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad E(X^2) &= \sum x_i^2 p_i \\ &= 1^2(0.25) + 2^2(0.38) + 3^2(0.17) + 4^2(0.2) \\ &= 6.5\end{aligned}$$

$$\begin{aligned}\therefore \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 6.5 - (2.32)^2 \\ &= 1.1176\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \sigma(X) &= \sqrt{\text{Var}(X)} \\ &= \sqrt{1.1176} \\ &\approx 1.06\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad E(X + 2) &= E(X) + 2 \\ &= 2.32 + 2 \\ &= 4.32\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad \text{Var}(2 - 3X) &= (-3)^2 \text{Var}(X) \\ &= 9 \text{Var}(X) \\ &= 9(1.1176) \\ &= 10.0584\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad \sigma(2X - 10) &= |2| \sigma(X) \\ &= 2\sqrt{1.1176} \\ &\approx 2.11\end{aligned}$$

**76 a** Let  $X$  be the return from each game.

$$\begin{aligned}E(X) &= 40\left(\frac{1}{12}\right) + 20\left(\frac{3}{12}\right) + 5\left(\frac{8}{12}\right) \\ &= \frac{40 + 60 + 40}{12} \\ &= \frac{140}{12} \approx \$11.67\end{aligned}$$

<i>Ticket colour</i>	Blue	Red	Yellow
<i>Winnings</i>	\$40	\$20	\$5
<i>Probability</i>	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{8}{12}$

**b** The expected return per game is \$11.67. It costs \$15 to play.

So, the expected gain  $\approx \$11.67 - \$15 \approx -\$3.33$

It is not advisable to play this game many times as the player can expect to lose \$3.33 on average per game.

**c** Let  $k$  be the number of extra red tickets added to the bag.

<i>Ticket colour</i>	Blue	Red	Yellow
<i>Winnings</i>	\$40	\$20	\$5
<i>Probability</i>	$\frac{1}{12+k}$	$\frac{k+3}{12+k}$	$\frac{8}{12+k}$

For the game to be fair, the expected return must equal the cost of each game.

$$\therefore E(X) = 40\left(\frac{1}{12+k}\right) + 20\left(\frac{k+3}{12+k}\right) + 5\left(\frac{8}{12+k}\right) = 15 \quad \{\text{the cost of the game is \$15}\}$$

$$\therefore \frac{40}{12+k} + \frac{20k+60}{12+k} + \frac{40}{12+k} = 15$$

$$\therefore \frac{20k+140}{12+k} = 15$$

$$\therefore 20k + 140 = 15(12+k)$$

$$\therefore 20k + 140 = 180 + 15k$$

$$\therefore 5k = 40$$

$$\therefore k = 8$$

So, 8 extra red tickets should be added to the bag to make the game fair.

<b>77 a</b>	1	1				$n = 1$
	1	2	1			$n = 2$
	1	3	3	1		$n = 3$
	1	4	6	4	1	$n = 4$



- b i** Let  $X$  be the number of times we get a number greater than 3.

$$\therefore X \sim B(4, \frac{5}{8}) \text{ and } P(X = x) = \binom{4}{x} \left(\frac{5}{8}\right)^x \left(\frac{3}{8}\right)^{4-x}$$

$$\begin{aligned} \text{So, } P(X = 3) &= \binom{4}{3} \left(\frac{5}{8}\right)^3 \left(\frac{3}{8}\right)^1 \\ &= 4 \left(\frac{5}{8}\right)^3 \left(\frac{3}{8}\right) \\ &\approx 0.366 \end{aligned}$$

- ii** Let  $X$  be the number of times we get a number greater than 5.

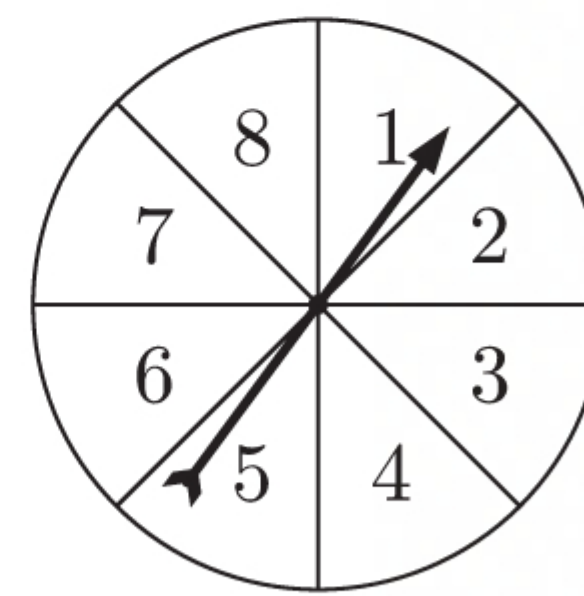
$$\therefore X \sim B(4, \frac{3}{8}) \text{ and } P(X = x) = \binom{4}{x} \left(\frac{3}{8}\right)^x \left(\frac{5}{8}\right)^{4-x}$$

$$\begin{aligned} \text{So, } P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{4}{0} \left(\frac{3}{8}\right)^0 \left(\frac{5}{8}\right)^4 - \binom{4}{1} \left(\frac{3}{8}\right)^1 \left(\frac{5}{8}\right)^3 \\ &= 1 - \left(\frac{5}{8}\right)^4 - 4 \left(\frac{3}{8}\right) \left(\frac{5}{8}\right)^3 \\ &\approx 0.481 \end{aligned}$$

- iii** Let  $X$  be the number of times we get a number that is spelt with 3 letters.

$$\therefore X \sim B(4, \frac{3}{8}) \text{ and } P(X = x) = \binom{4}{x} \left(\frac{3}{8}\right)^x \left(\frac{5}{8}\right)^{4-x}$$

$$\begin{aligned} \text{So, } P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \binom{4}{0} \left(\frac{3}{8}\right)^0 \left(\frac{5}{8}\right)^4 + \binom{4}{1} \left(\frac{3}{8}\right)^1 \left(\frac{5}{8}\right)^3 \\ &= \left(\frac{5}{8}\right)^4 + 4 \left(\frac{3}{8}\right) \left(\frac{5}{8}\right)^3 \\ &\approx 0.519 \end{aligned}$$

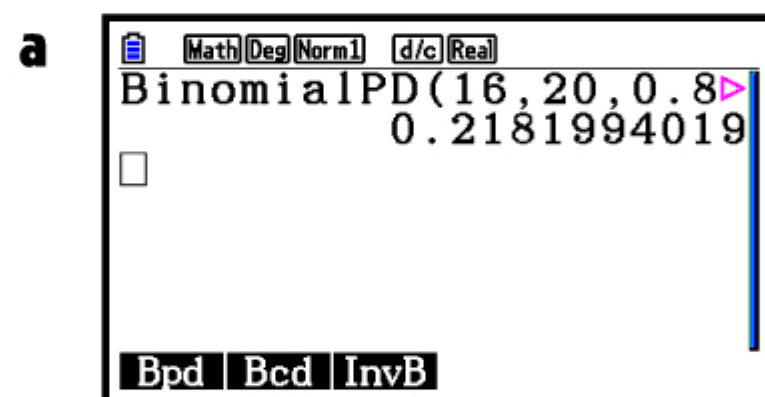


one two three four  
five six seven eight

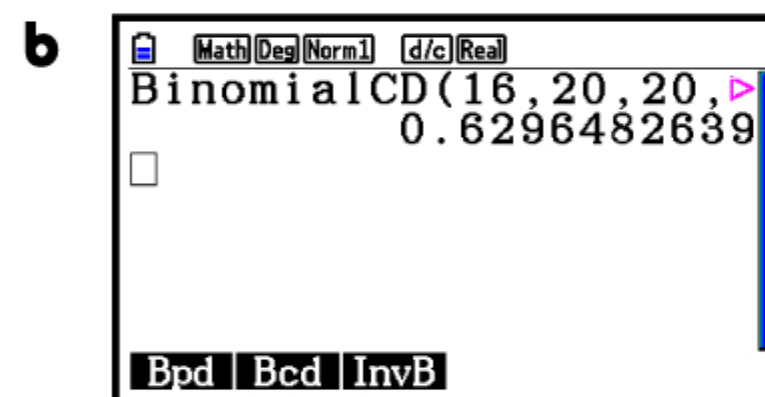
- 78** Let  $X$  be the number of residents who oppose the construction.

$n = 20$ , so  $X = 0, 1, 2, 3, \dots$ , or 20, and  $p = 80\% = 0.8$

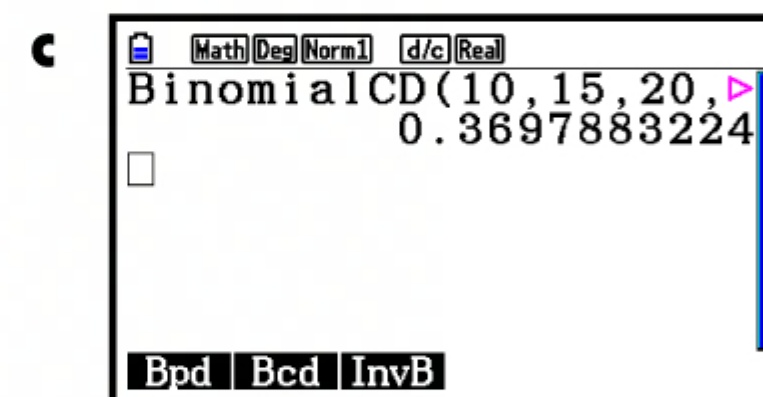
$$\therefore X \sim B(20, 0.8)$$



$$P(X = 16) \approx 0.218$$



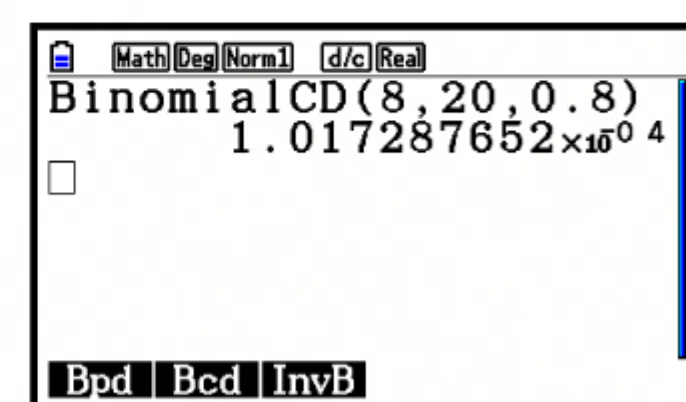
$$P(X \geq 16) \approx 0.630$$



$$P(10 \leq X \leq 15) \approx 0.370$$

- d** If more than 8 residents support the construction, then 8 or fewer residents oppose the construction.

$$P(X \leq 8) \approx 0.000102$$



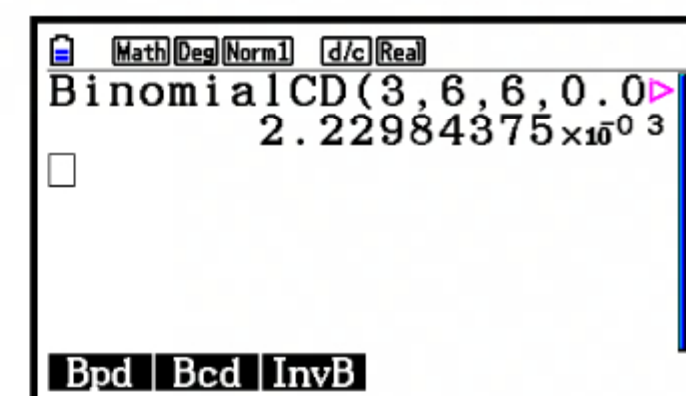
- 79 a** Let  $X$  be the number of defective items.

$n = 6$ , so  $X = 0, 1, 2, 3, 4, 5$ , or 6, and  $p = 5\% = 0.05$

$$\therefore X \sim B(6, 0.05)$$

$$\begin{aligned} \text{Using technology, } P(X > 2) &= P(X \geq 3) \\ &\approx 0.00223 \end{aligned}$$

$\therefore$  the manufacturer will have to pay a refund on about  $0.00223 \times 100\% \approx 0.223\%$  of boxes.



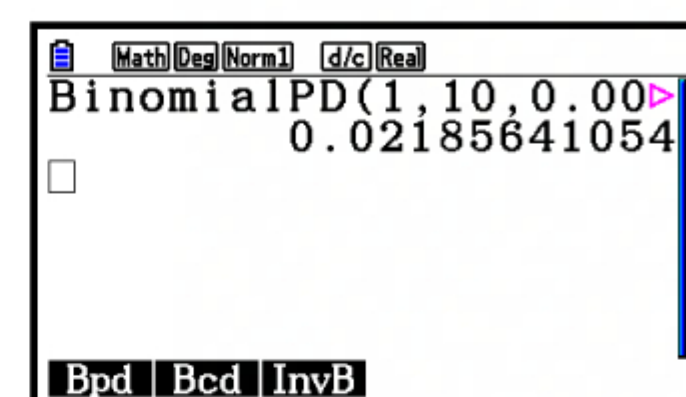
- b** Let  $Y$  be the number of boxes refunded.

$n = 10$ , so  $Y = 0, 1, 2, 3, \dots$ , or 10, and  $p \approx 0.00223$  {from **a**}

$$\therefore Y \sim B(10, 0.00223)$$

$$\text{Using technology, } P(Y = 1) \approx 0.0219$$

So, the probability that Patrick will get a refund for exactly 1 box is about 0.0219.





**80** Let  $X$  be the number of germinations in one row.

$n = 10$ , so  $X = 0, 1, 2, 3, \dots$ , or  $10$ , and  $p = \frac{1}{2}$

$$\therefore X \sim B(10, \frac{1}{2})$$

$$\begin{aligned} P(X \geq 8) &= P(X = 8) + P(X = 9) + P(X = 10) \\ &= \binom{10}{8} \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + \binom{10}{9} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 + \binom{10}{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0 \\ &= \frac{7}{128} \end{aligned}$$

So, the probability that at least 8 seeds germinate in one row is  $\frac{7}{128}$ .

Let  $Y$  be the number of rows with at least 8 seeds germinating.

$n = 10$ , so  $Y = 0, 1, 2, 3, \dots$ , or  $10$ , and  $p = \frac{7}{128}$

$$\therefore Y \sim B(10, \frac{7}{128})$$

$$\begin{aligned} P(Y \geq 1) &= 1 - P(Y = 0) \\ &= 1 - \binom{10}{0} \left(\frac{7}{128}\right)^0 \left(\frac{121}{128}\right)^{10} \\ &\approx 0.430 \end{aligned}$$

So, the probability that the row with the maximum number of germinations contains at least 8 seedlings is about 0.430.

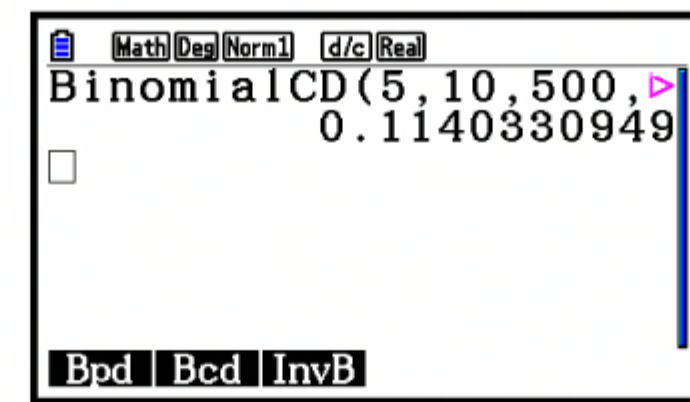
**81** Let  $F$  be the event of a faulty chip.

$$\therefore P(F) = 0.03 \text{ and } P(F') = 0.97$$

If  $X$  is the number which are faulty then  $X \sim B(500, 0.03)$ .

1% is 5 chips and 2% is 10 chips.

Using technology,  $P(5 \leq X \leq 10) \approx 0.114$



<b>82</b>	Score	1	2	3	4
	Probability	$\frac{1}{12}$	$k$	$\frac{1}{4}$	$\frac{1}{3}$

**a** Since this is a probability distribution,

$$\begin{aligned} \sum_{i=1}^n P(x_i) &= 1 \\ \therefore \frac{1}{12} + k + \frac{1}{4} + \frac{1}{3} &= 1 \\ \therefore k &= \frac{1}{3} \end{aligned}$$

**b** Let  $X$  be the number of 2s rolled.

$n = 2400$ , so  $X = 0, 1, 2, 3, \dots$ , or  $2400$ , and

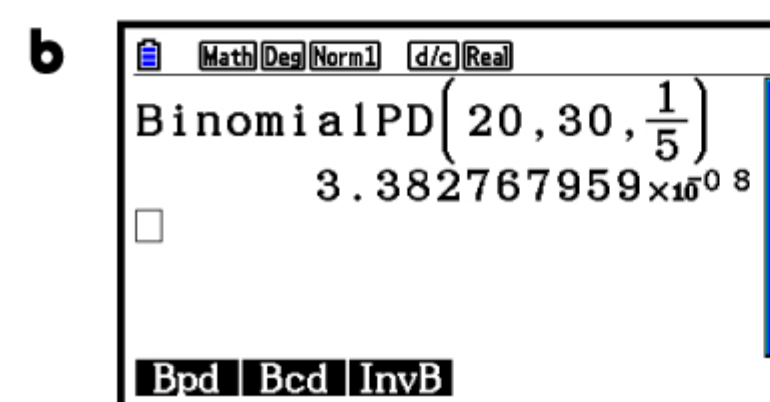
$$p = \frac{1}{3} \quad \{\text{from a}\}$$

$$\therefore X \sim B(2400, \frac{1}{3})$$

$$\begin{aligned} \text{So, } \mu &= np & \text{and } \sigma &= \sqrt{np(1-p)} \\ &= 2400 \times \frac{1}{3} & &= \sqrt{2400 \times \frac{1}{3} \times \frac{2}{3}} \\ &= 800 & &= \sqrt{\frac{1600}{3}} \\ & & &= \frac{40}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ & & &= \frac{40\sqrt{3}}{3} \end{aligned}$$

**83**  $Y \sim B(30, \frac{1}{5})$

$$\begin{aligned} \text{a } \mu &= np & \text{and } \sigma &= \sqrt{np(1-p)} \\ &= 30 \times \frac{1}{5} & &= \sqrt{30 \times \frac{1}{5} \times \frac{4}{5}} \\ &= 6 & &= \sqrt{\frac{24}{5}} \\ & & &\approx 2.19 \end{aligned}$$



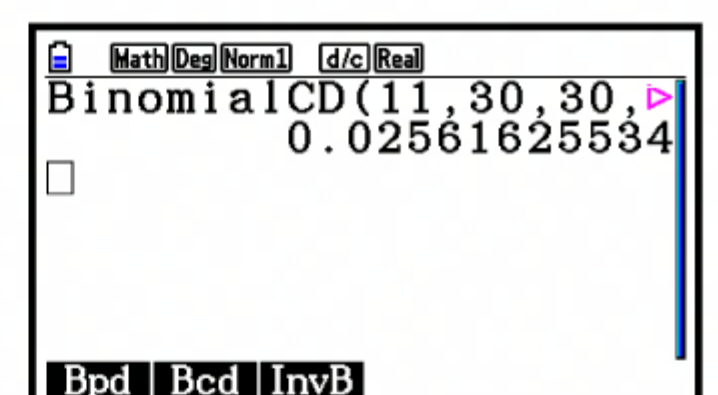
$$P(Y = 20) \approx 3.38 \times 10^{-8}$$

$$\text{c } P(Y \geq \mu + 2\sigma) = P\left(Y \geq 6 + 2\sqrt{\frac{24}{5}}\right) \quad \{\text{from a}\}$$

$$\text{But } 6 + 2\sqrt{\frac{24}{5}} \approx 10.38$$

$$\begin{aligned} \therefore P(Y \geq \mu + 2\sigma) &= P(Y \geq 11) \\ &\approx 0.0256 \end{aligned}$$

{using technology}

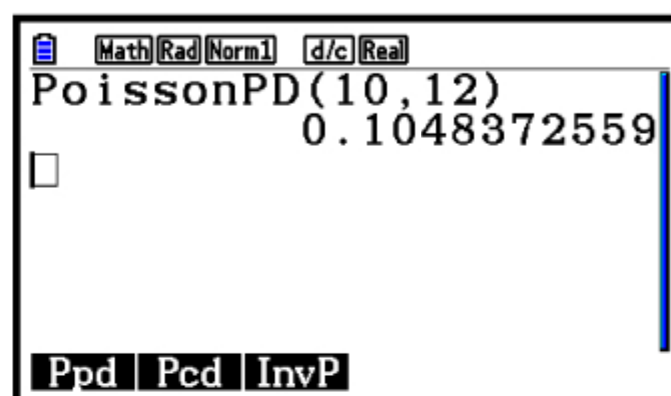




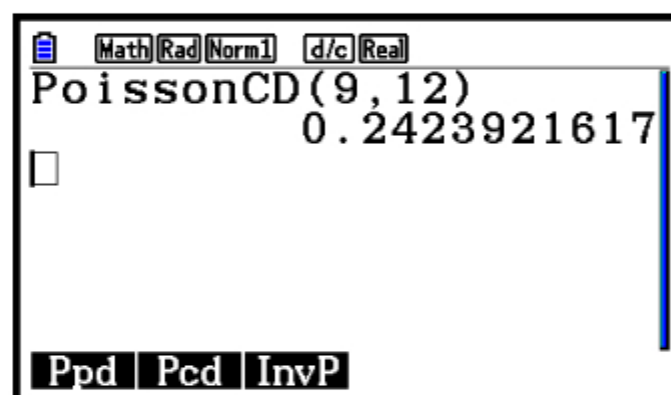
**84 a**  $X$  is a Poisson random variable.  $X \sim \text{Po}(12)$

**b** mean =  $E(X) = 12$  errors       $\text{Var}(X) = 12$       standard deviation =  $\sqrt{\text{Var}(X)}$   
 $= \sqrt{12}$  errors

**c**  $P(X = 10) \approx 0.105$

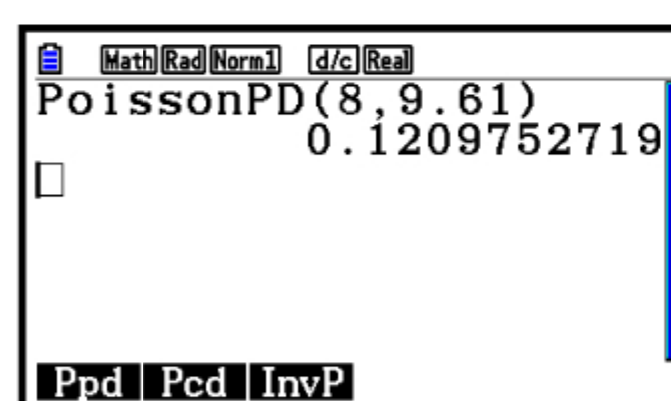


**d**  $P(X \geq 10) = 1 - P(X \leq 9)$   
 $\approx 1 - 0.242$   
 $\approx 0.758$

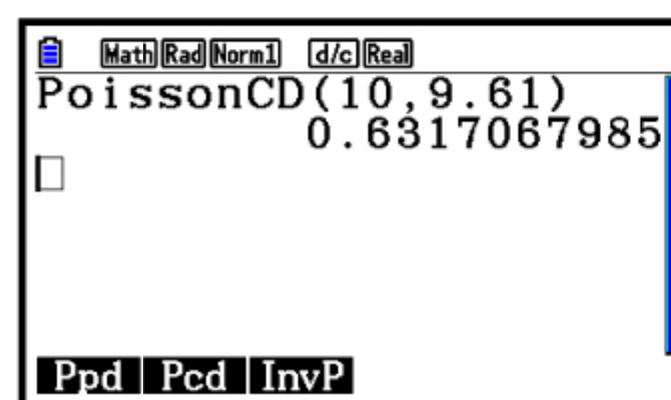


**85 a** mean = variance =  $\sigma^2 = 9.61$

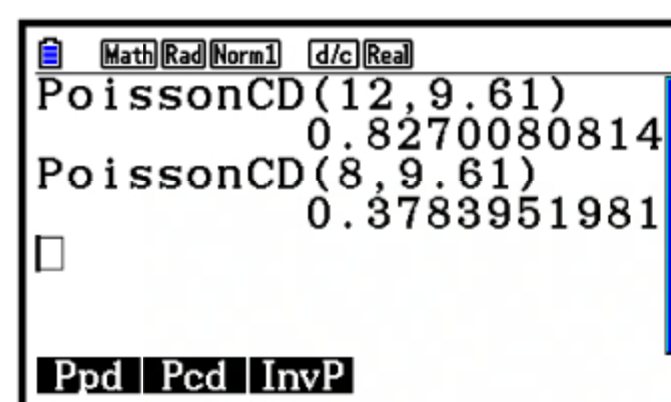
**b i**  $P(X = 8) \approx 0.121$



**ii**  $P(X \geq 11) = 1 - P(X \leq 10)$   
 $\approx 1 - 0.632$   
 $\approx 0.368$

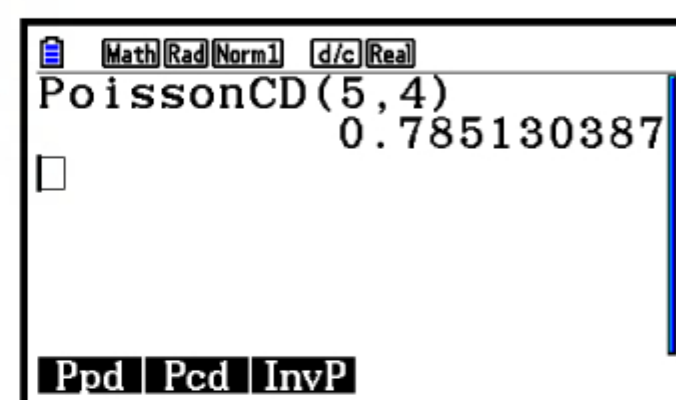


**iii**  $P(X \geq 13 | X \geq 9) = \frac{P(X \geq 13 \cap X \geq 9)}{P(X \geq 9)}$   
 $= \frac{P(X \geq 13)}{P(X \geq 9)}$   
 $= \frac{1 - P(X \leq 12)}{1 - P(X \leq 8)}$   
 $\approx \frac{1 - 0.827}{1 - 0.378}$   
 $\approx 0.278$



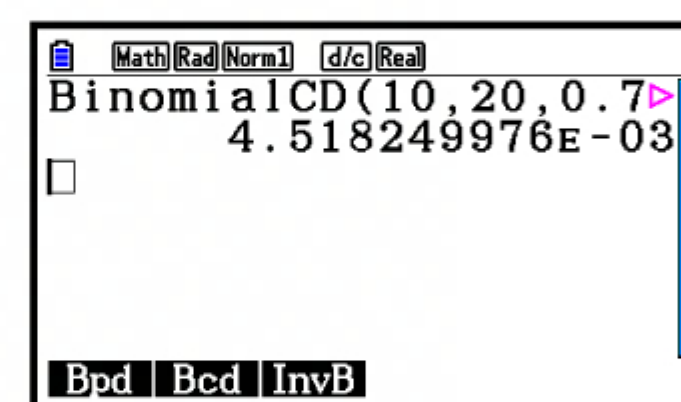
**86** The average number of amoebas in 10 mL of water is 4.  $\therefore X \sim \text{Po}(4)$

**a**  $P(X < 6) = P(X \leq 5)$   
 $\approx 0.785$

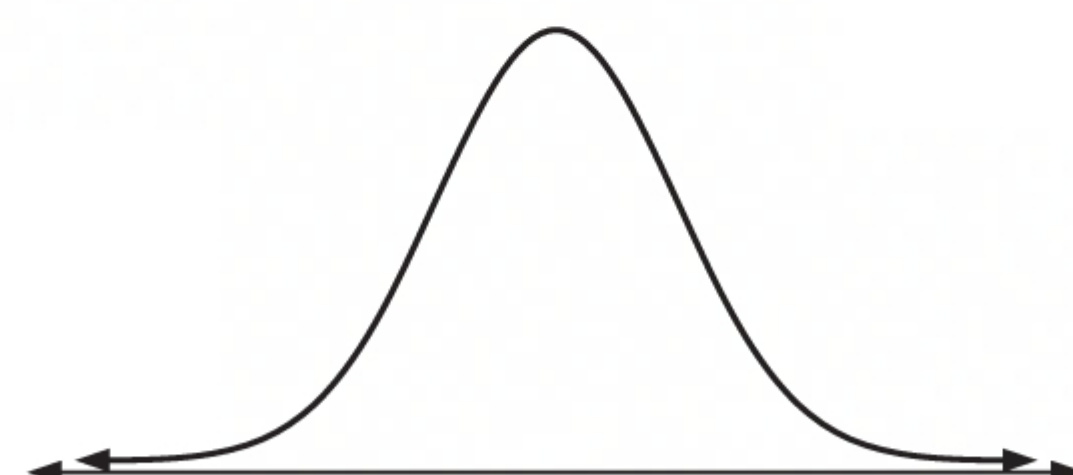


**b** If  $Y$  is the number of days where less than 6 amoebas are collected then  $Y \sim B(20, 0.78513)$ .

$P(Y \geq 11) = 1 - P(Y \leq 10)$   
 $\approx 1 - 0.00452$   
 $\approx 0.995$

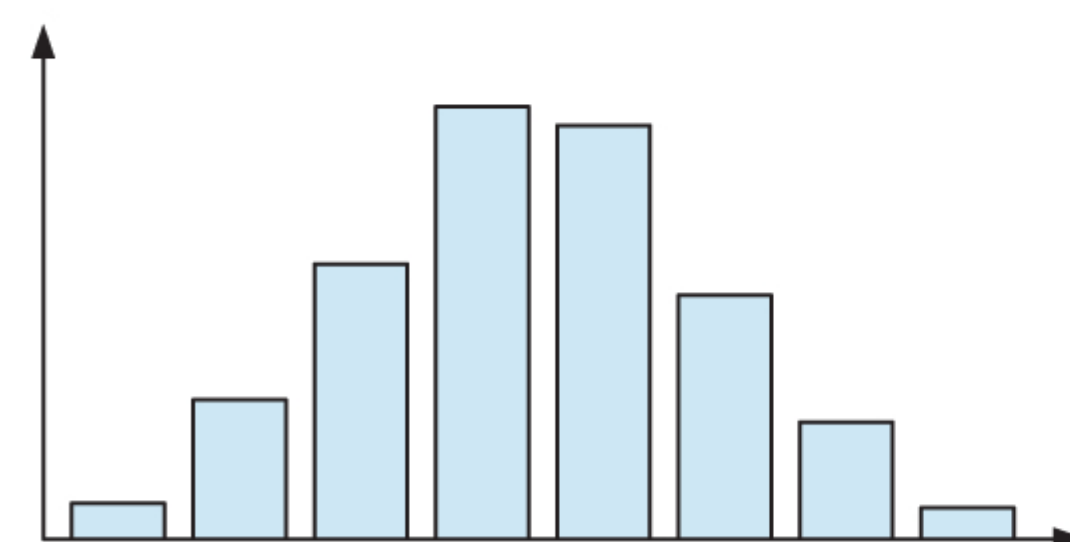


**87 a** The variable is likely to be normally distributed as the amount of sleep will be generally centred around the mean, but will vary due to factors such as diet, exercise, and so on.

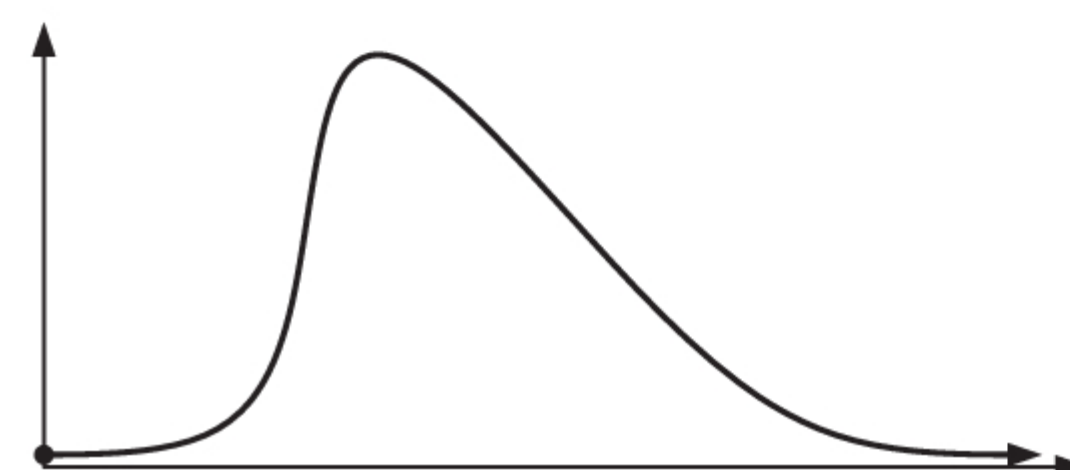




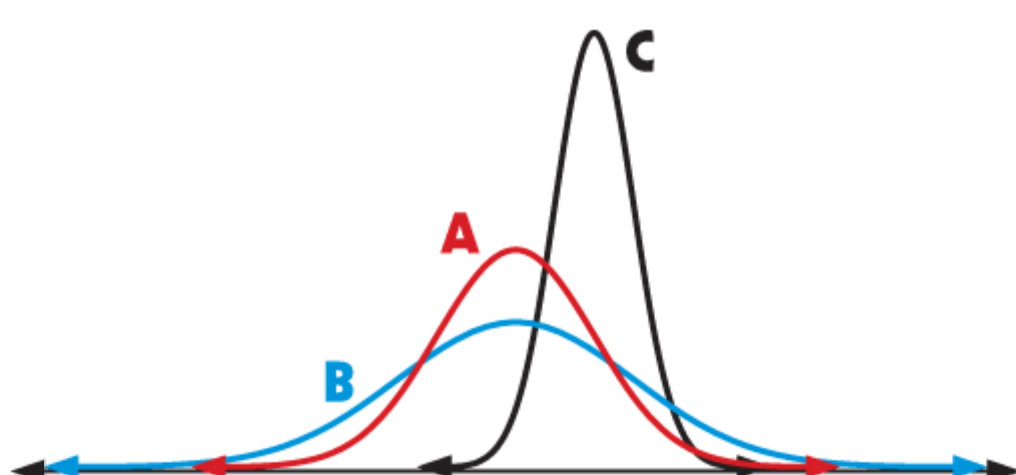
- b** The variable is not likely to be normally distributed as it is a discrete variable. The number of lollies may vary from bag to bag, so the distribution may appear symmetric.



- c** The variable is not likely to be normally distributed as it is more likely that there would be more people younger than the mean age than there are older. The distribution may be positively skewed.



88



**A** and **B** both have the same mean, and **C** has a greater mean.

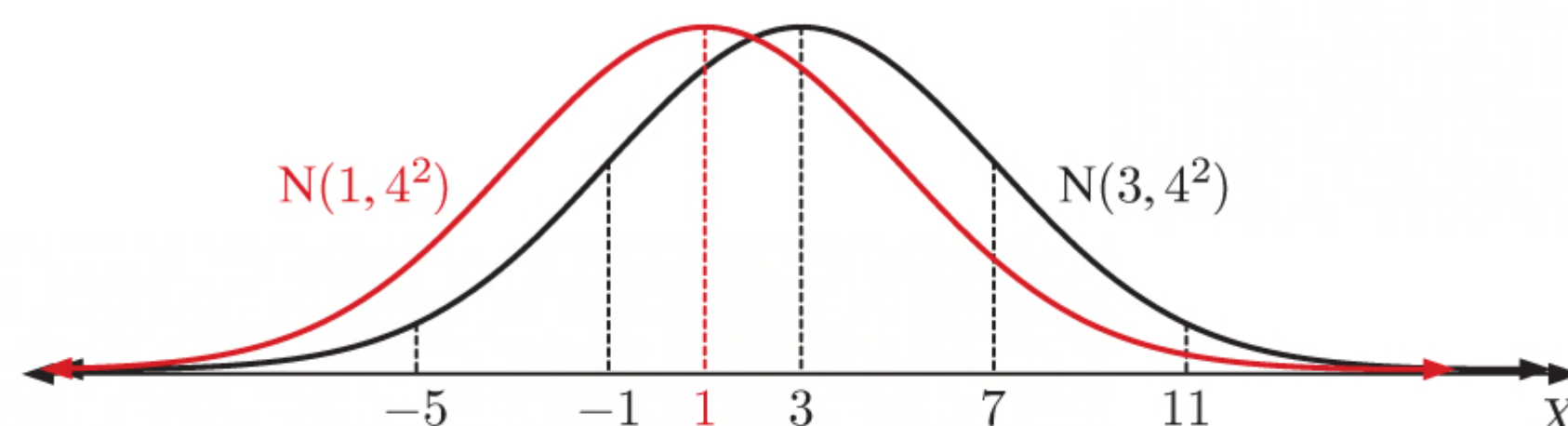
**B** has a greater spread, and hence a larger standard deviation than **A**.

**a**  $\mu = 4, \sigma = 1$  corresponds to **C**

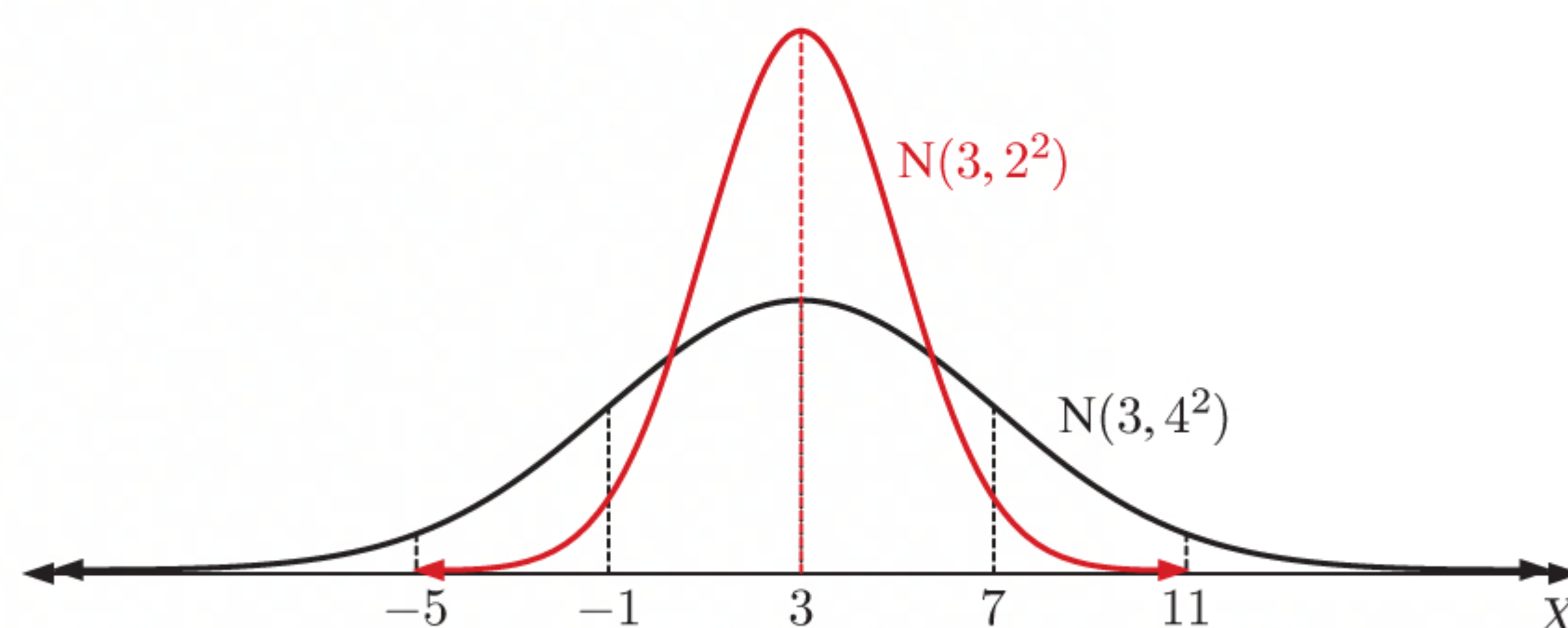
**b**  $\mu = 2, \sigma = 2$  corresponds to **A**

**c**  $\mu = 2, \sigma = 3$  corresponds to **B**

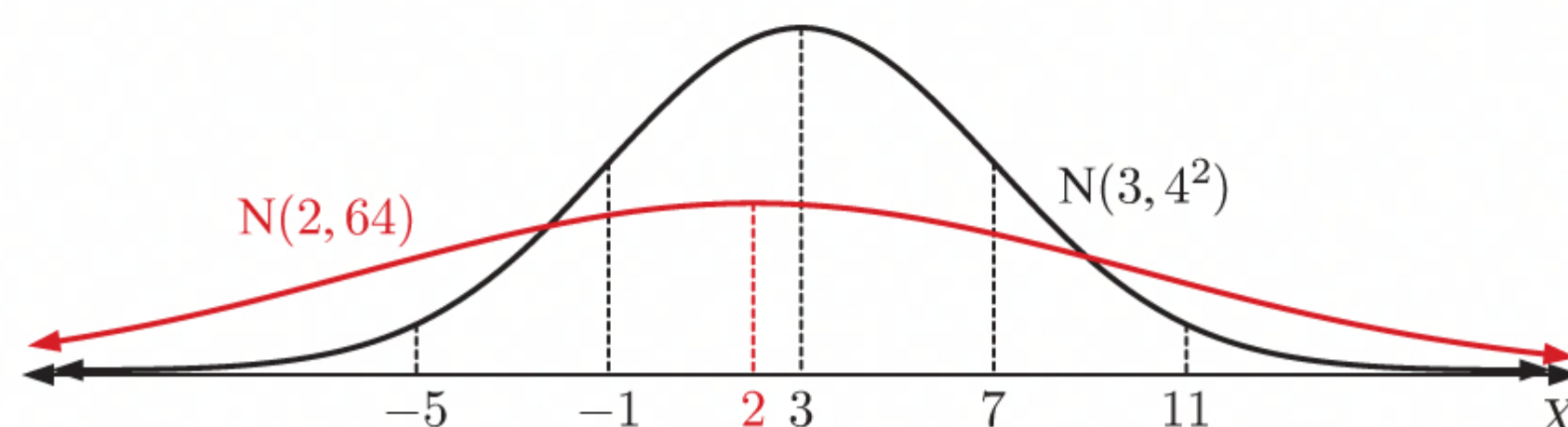
89 a



b



c



- 90 a** Approximately 68% of the population lies between 25 and 35.

- b** Approximately 95% of the population lies between 20 and 40.

- c** Approximately 99.7% of the population lies between 15 and 45.

- 91** Let the capacity of a randomly selected container be  $X$  mL.

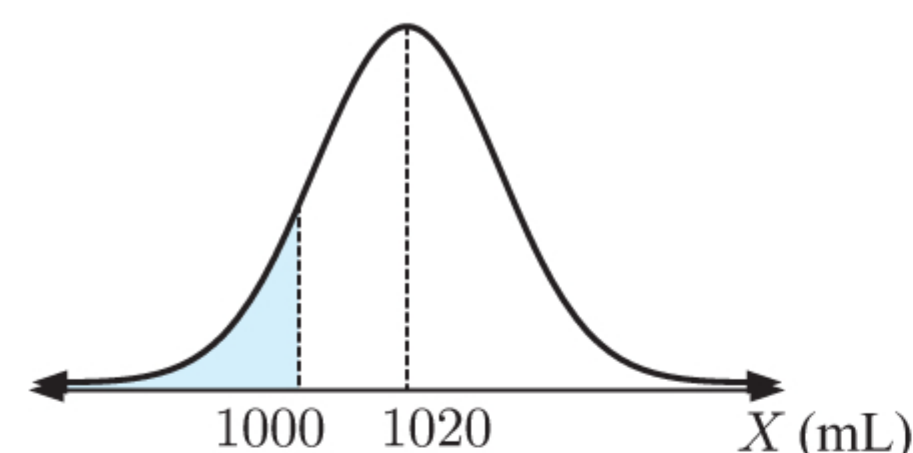
So,  $X \sim N(1020, 17^2)$ .

a

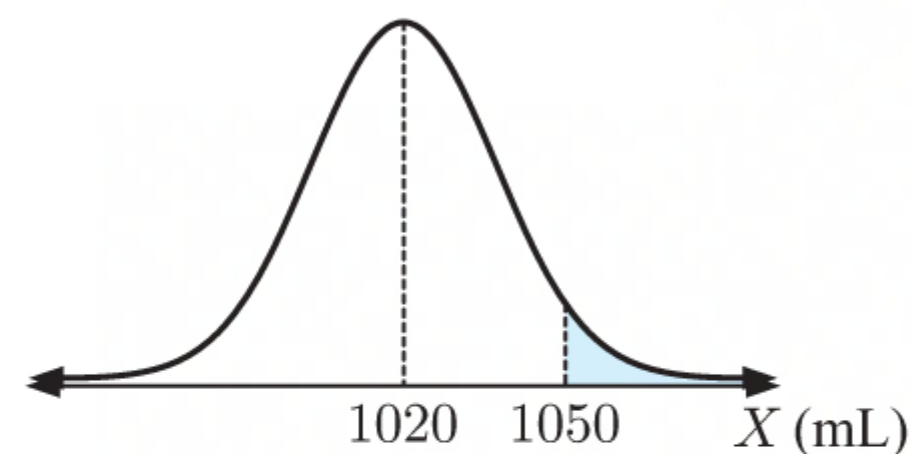
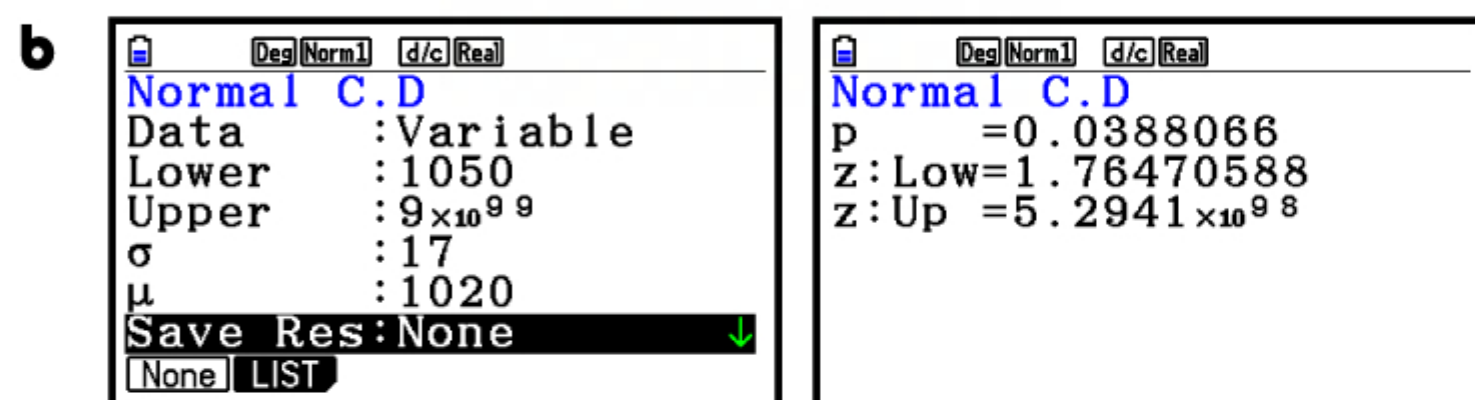
	Des	Norm1	d/c	Real
Normal C.D				
Data	:	Variable		
Lower	:	$-9 \times 10^9$		
Upper	:	1000		
$\sigma$	:	17		
$\mu$	:	1020		
Save Res:	:	None		
		[None] LIST		

	Des	Norm1	d/c	Real
Normal C.D				
p	:	0.11970343		
z: Low	:	$-5.294 \times 10^9$		
z: Up	:	-1.1764706		

$$P(X \leq 1000) \approx 0.120$$







$$P(X \geq 1050) \approx 0.0388$$

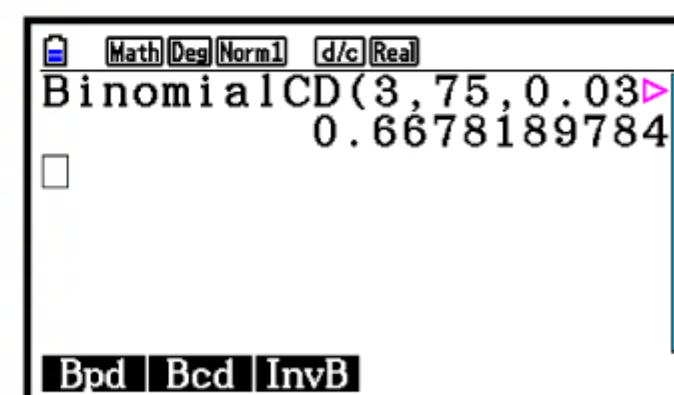
$\therefore$  about 3.88% of containers overflow.

**c** Let  $Y$  be the number of containers which overflow.

$n = 75$ , so  $Y = 0, 1, 2, 3, \dots$ , or 75 and  $p \approx 0.0388$  {from **b**}

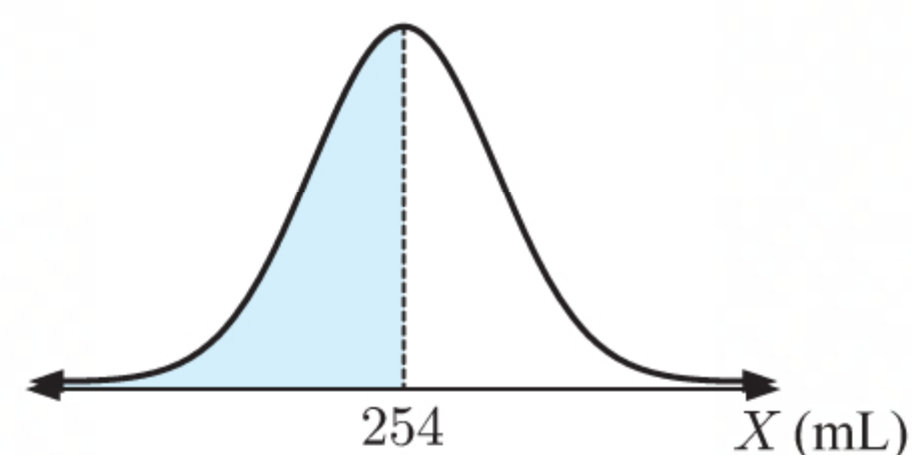
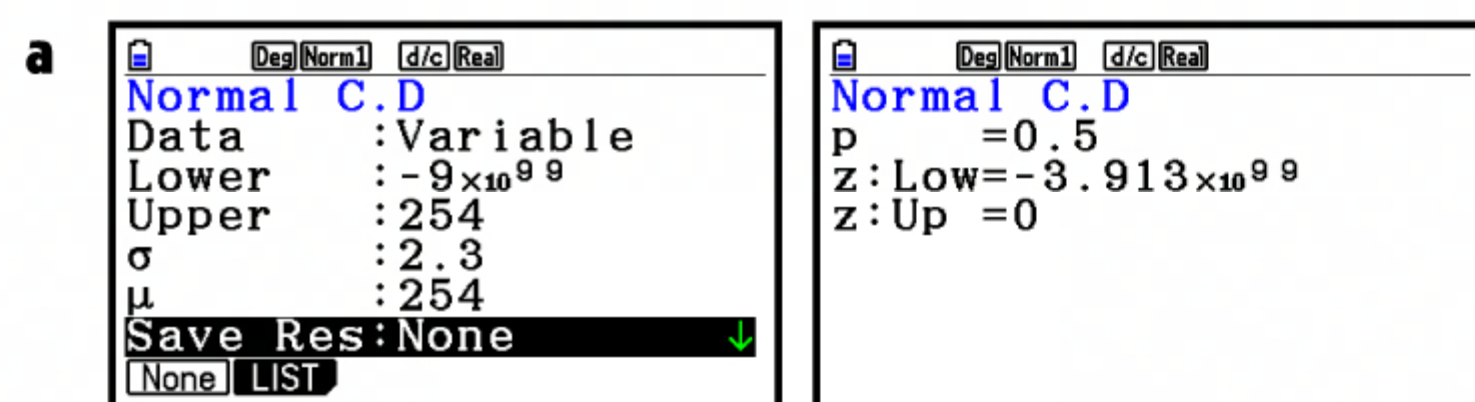
$$\therefore Y \sim B(75, 0.0388)$$

Using technology,  $P(Y \leq 3) \approx 0.668$

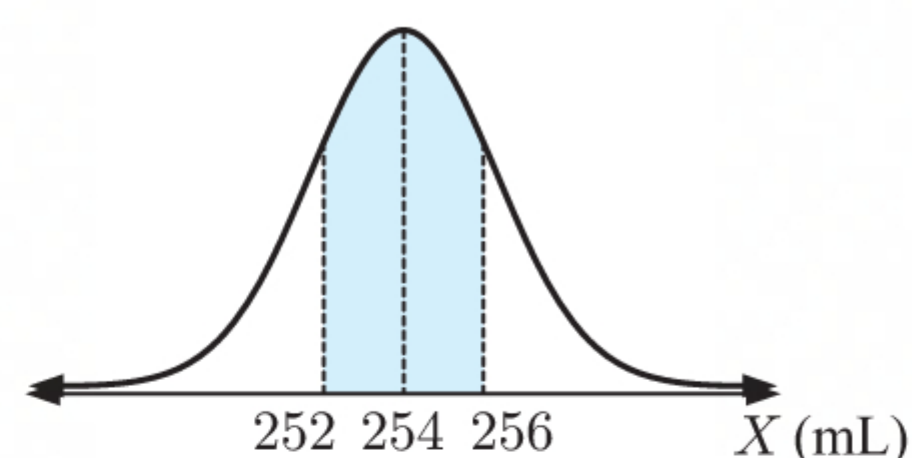
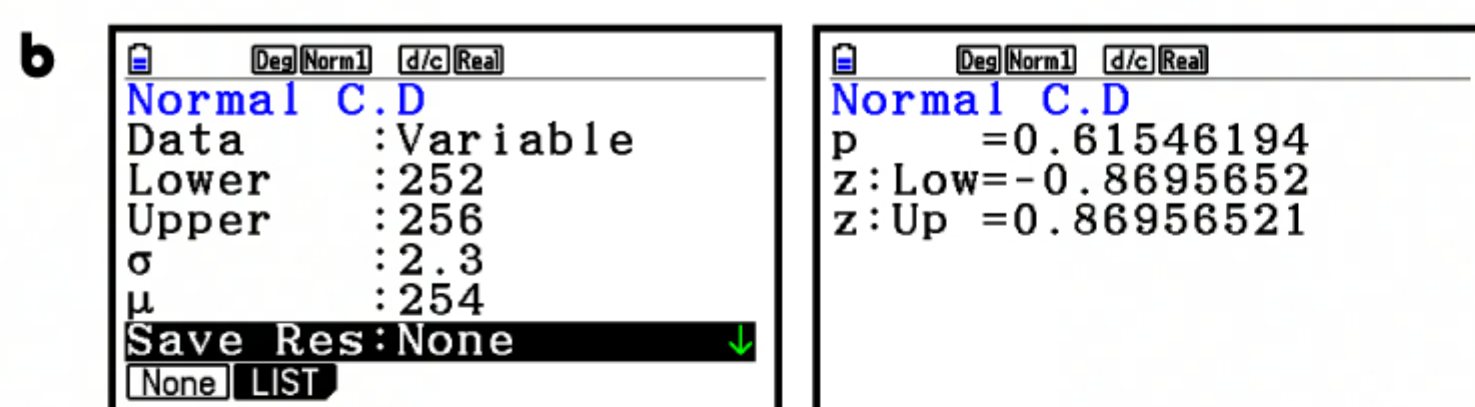


**92** Let the volume of a randomly selected drink be  $X$  mL.

So,  $X \sim N(254, (2.3)^2)$ .

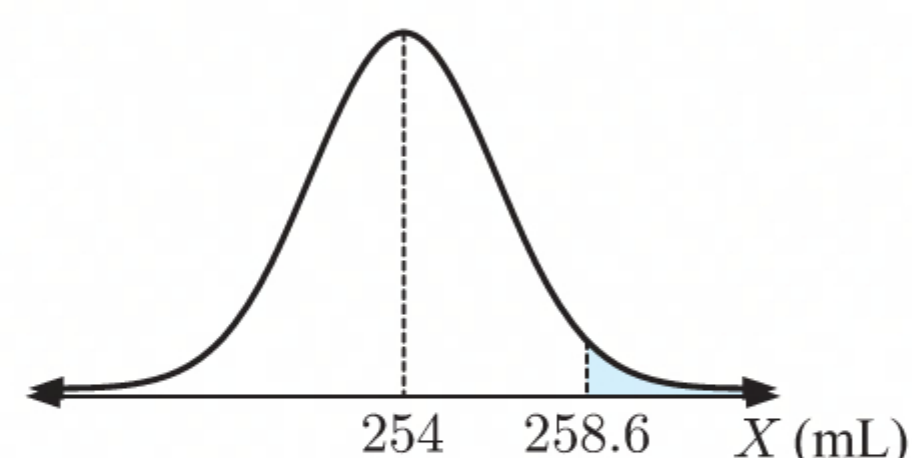
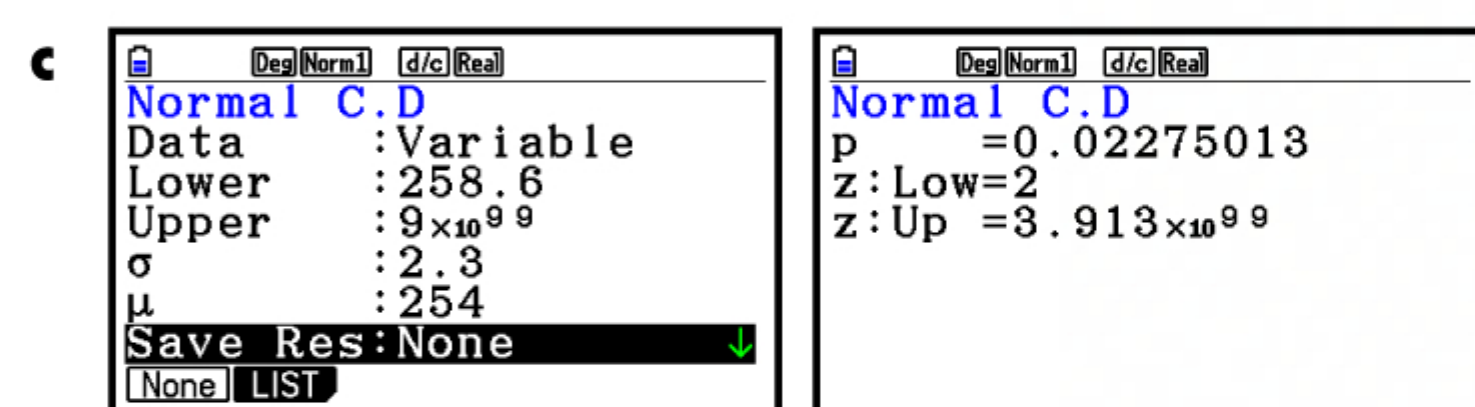


$$P(X < 254) = 0.5$$



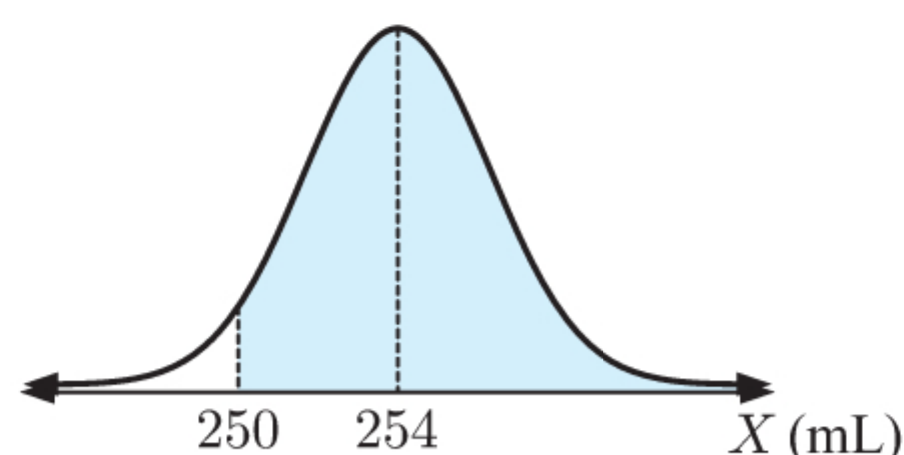
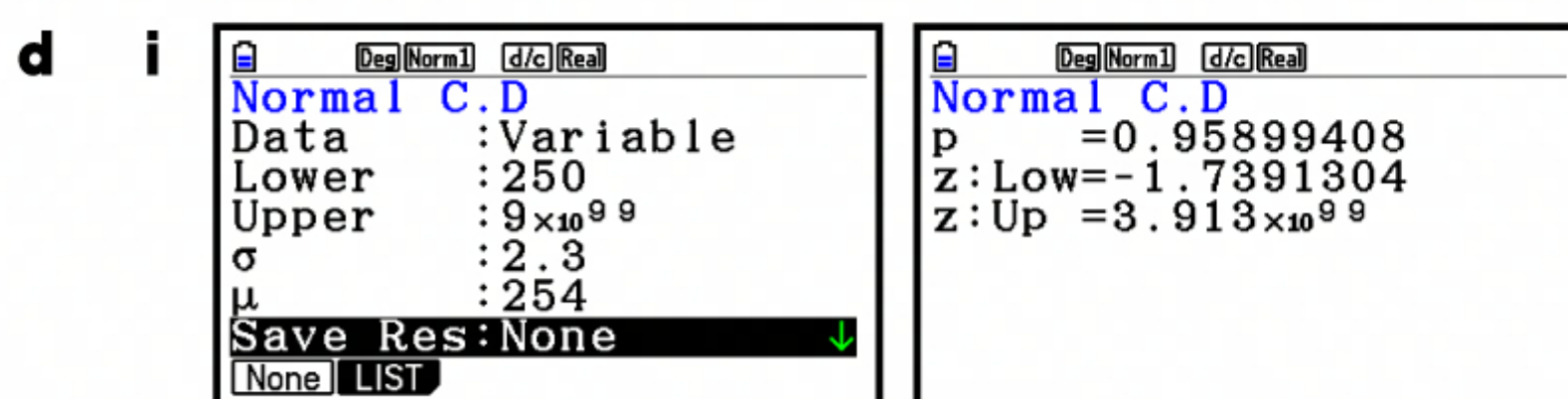
$$P(252 \leq X \leq 256) \approx 0.615$$

$\therefore$  about 61.5% of drinks dispensed by the machine have volume between 252 mL and 256 mL.



$$P(X \geq 254 + 2 \times 2.3) = P(X \geq 258.6) \approx 0.0228$$

$\therefore$  we expect about  $0.0228 \times 80 \approx 2$  drinks to have volume at least two standard deviations above the mean.



$$P(X \geq 250) \approx 0.959$$

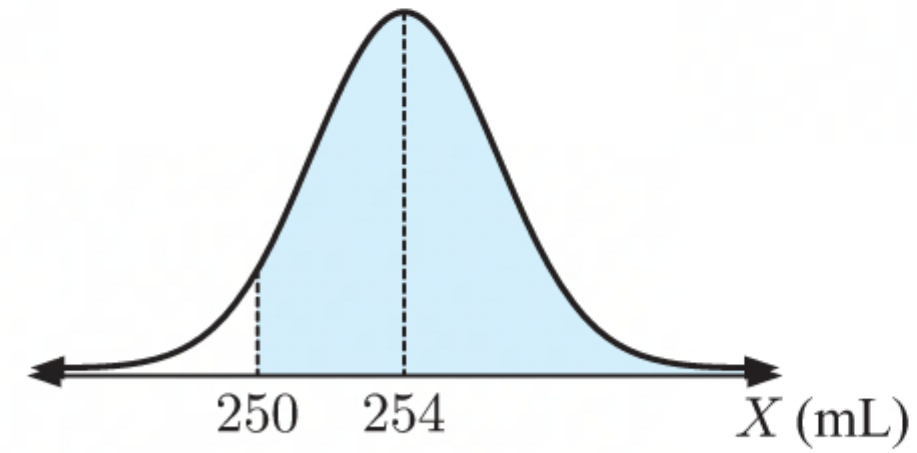
$\therefore$  the operator's guarantee that at least 95% of drinks will have volume at least 250 mL is valid.



- ii Suppose  $X \sim N(254, (2.5)^2)$ .

Normal C.D	
Data	: Variable
Lower	: 250
Upper	: $9 \times 10^9$
$\sigma$	: 2.5
$\mu$	: 254
Save Res	: None
None	LIST

Normal C.D	
p	= 0.9452007
z: Low	= -1.6
z: Up	= $3.6 \times 10^9$



$$P(X \geq 250) \approx 0.945$$

$\therefore$  about 94.5% of drinks will have volume at least 250 mL.

So, the operator's guarantee is no longer valid.

- 93 a Let the volume of sauce in a randomly selected bottle be  $X$  mL.

$$X \sim N(500, (2.5)^2)$$

$$\therefore P(X < 495) \approx 0.0228$$

Math	Des	Norm1	d/c	Real
NormCD(-9×10 <sup>99</sup> , 495, 2.5)				
0.02275013195				
Npd Ncd InvN				

- b Let  $Y$  be the number of bottles which require extra sauce.

$$Y \sim B(200, 0.0228) \quad \{\text{from a}\}$$

$$\therefore P(Y \geq 8) \approx 0.0892$$

Math	Des	Norm1	d/c	Real
BinomialCD(8, 200, 200)				
0.08921204063				
Bpd Bcd InvB				

- 94 a Let the completion time of a randomly selected run be  $X$  seconds.

$$X \sim N(45, 4^2)$$

i  $P(X < 40) \approx 0.106$

ii  $P(\text{two consecutive runs under 40 seconds})$

$$= [P(X < 40)]^2$$

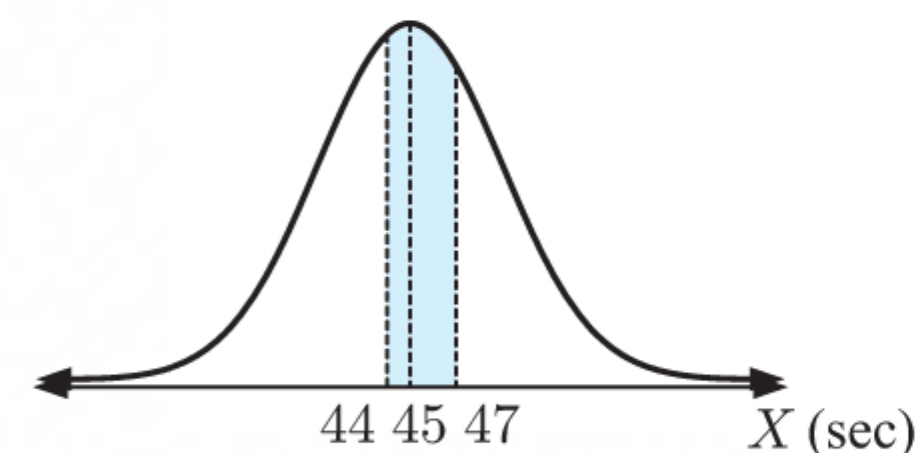
$$\approx 0.0112$$

Math	Des	Norm1	d/c	Real
NormCD(-9×10 <sup>99</sup> , 40, 4, 4)				
0.1056497737				
Ans <sup>2</sup>				
0.01116187468				
Npd Ncd InvN				

b

Normal C.D	
Data	: Variable
Lower	: 44
Upper	: 47
$\sigma$	: 4
$\mu$	: 45
Save Res	: None
None	LIST

Normal C.D	
p	= 0.29016878
z: Low	= -0.25
z: Up	= 0.5



$$P(44 \leq X \leq 47) \approx 0.290$$

$\therefore$  we expect about  $0.290 \times 60 \approx 17$  runs to take between 44 seconds and 47 seconds.

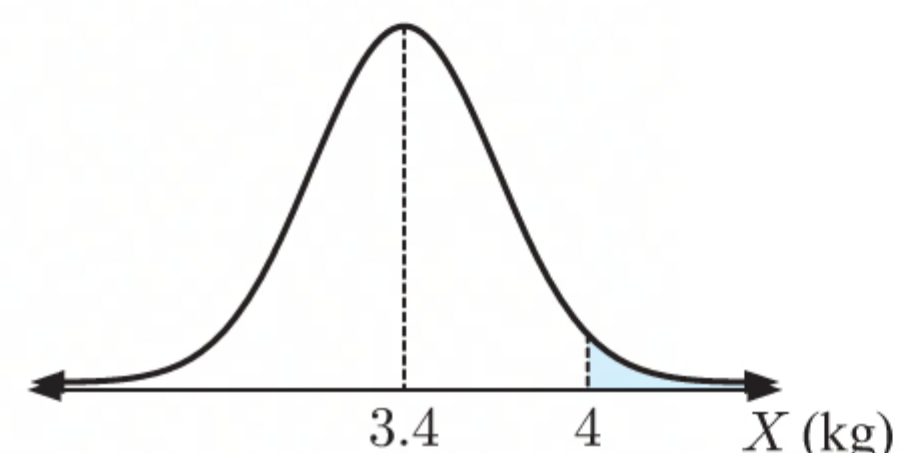
- 95 Let the birth weight of a randomly selected baby be  $X$  kg.

$$\text{So, } X \sim N(3.4, (0.3)^2).$$

a i

Normal C.D	
Data	: Variable
Lower	: 4
Upper	: $9 \times 10^9$
$\sigma$	: 0.3
$\mu$	: 3.4
Save Res	: None
None	LIST

Normal C.D	
p	= 0.02275013
z: Low	= 2
z: Up	= $9.99 \times 10^9$



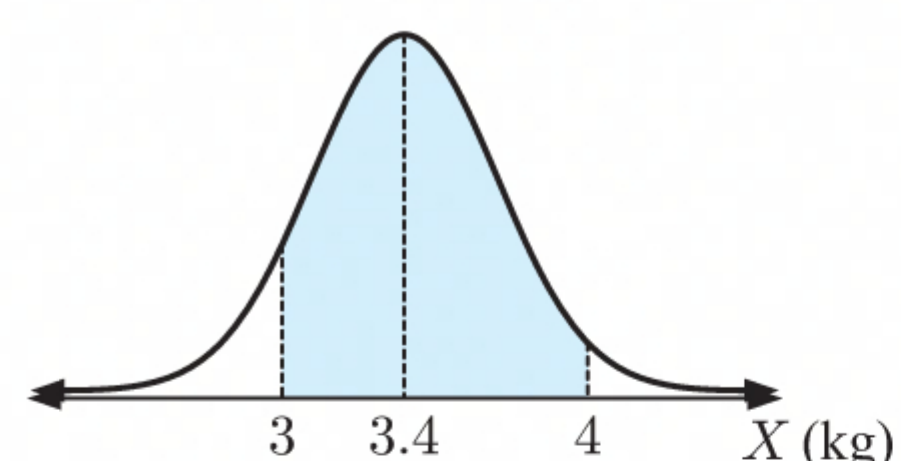
$$P(X > 4) \approx 0.0228$$

$\therefore$  about 2.28% of babies have birth weights in excess of 4 kg.

ii

Normal C.D	
Data	: Variable
Lower	: 3
Upper	: 4
$\sigma$	: 0.3
$\mu$	: 3.4
Save Res	: None
None	LIST

Normal C.D	
p	= 0.88603864
z: Low	= -1.3333333
z: Up	= 2



$$P(3 \leq X \leq 4) \approx 0.886$$

$\therefore$  about 88.6% of babies have birth weights between 3 kg and 4 kg.



b

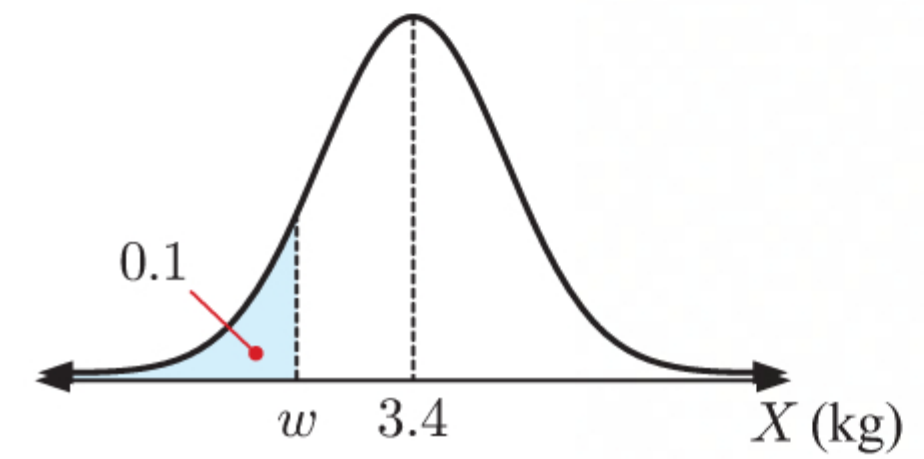
Inverse Normal
Data : Variable
Tail : Left
Area : 0.1
$\sigma$ : 0.3
$\mu$ : 3.4
Save Res: None
None LIST

Inverse Normal
xInv=3.01553453

$$P(X \leq w) = 0.1$$

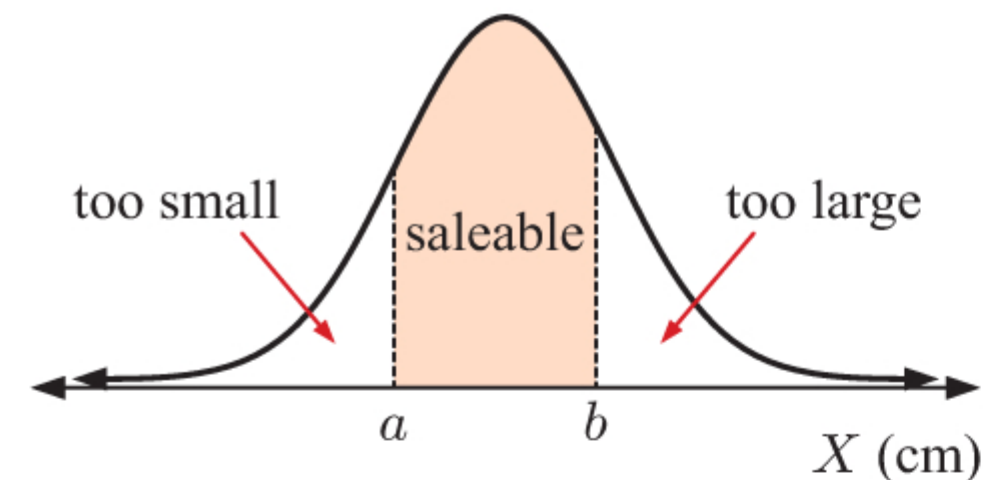
$$\therefore w \approx 3.02$$

$\therefore$  the weight below which a baby is classified as having *low birth weight* is about 3.02 kg.



**96** Let the length of a randomly selected zucchini be  $X$  cm.

$$\text{So, } X \sim N(24.3, (6.83)^2).$$



**a** 15% of zucchinis are too small.

$$P(X < a) = 0.15$$

$$\therefore a \approx 17.2$$

Inverse Normal
Data : Variable
Tail : Left
Area : 0.15
$\sigma$ : 6.83
$\mu$ : 24.3
Save Res: None
None LIST

Inverse Normal
xInv=17.2211599

20% of zucchinis are too large.

$$P(X > b) = 0.2$$

$$\therefore P(X \leq b) = 0.8$$

$$\therefore b \approx 30.0$$

Inverse Normal
Data : Variable
Tail : Left
Area : 0.8
$\sigma$ : 6.83
$\mu$ : 24.3
Save Res: None
None LIST

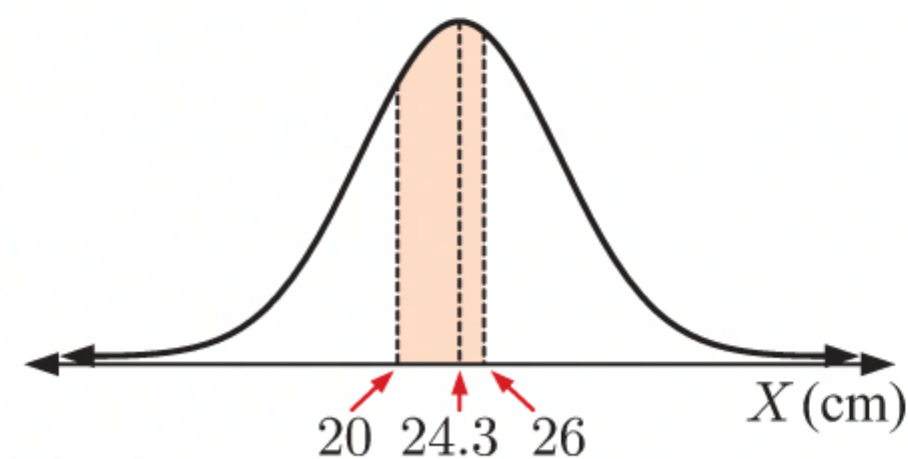
Inverse Normal
xInv=30.048273

**b i**  $P(\text{saleable length}) = 1 - 0.2 - 0.15$   
 $= 0.65$

ii

Normal C.D
Data : Variable
Lower : 20
Upper : 26
$\sigma$ : 6.83
$\mu$ : 24.3
Save Res: None
None LIST

Normal C.D
p = 0.33379545
z: Low = -0.6295754
z: Up = 0.2489019

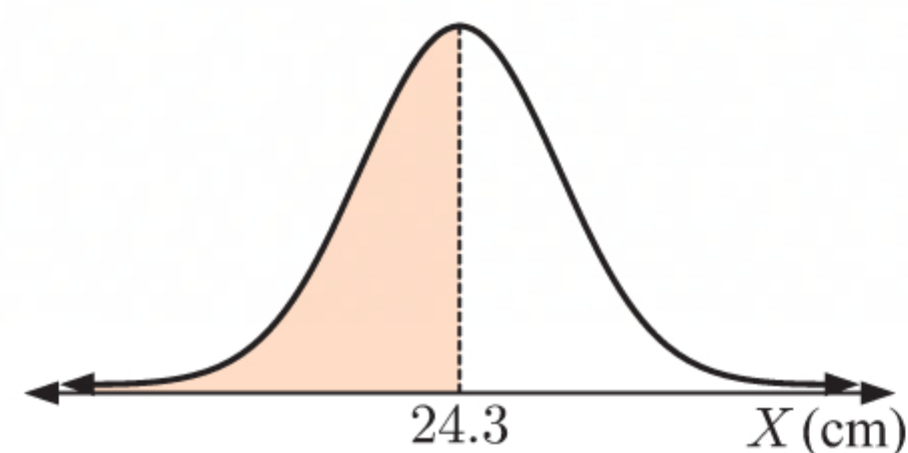


$$P(20 \leq X \leq 26) \approx 0.334$$

iii

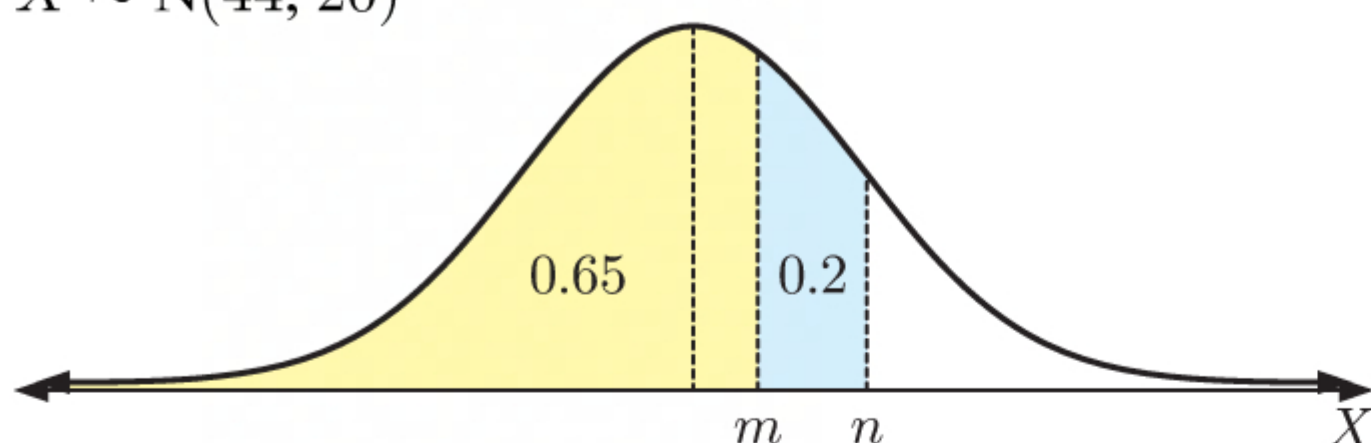
Normal C.D
Data : Variable
Lower : $-9 \times 10^9$
Upper : 24.3
$\sigma$ : 6.83
$\mu$ : 24.3
Save Res: None
None LIST

Normal C.D
p = 0.5
z: Low = $-1.318 \times 10^9$
z: Up = 0



$$P(X < 24.3) = 0.5$$

**97**  $X \sim N(44, 20)$



$$P(X \leq m) = 0.65$$

$$\therefore m \approx 45.72 \quad \{\text{using technology}\}$$

$$P(X < n) = 0.65 + 0.2 = 0.85$$

$$\therefore n \approx 48.64 \quad \{\text{using technology}\}$$

$$\therefore n - m \approx 48.64 - 45.72$$

$$\approx 2.92$$

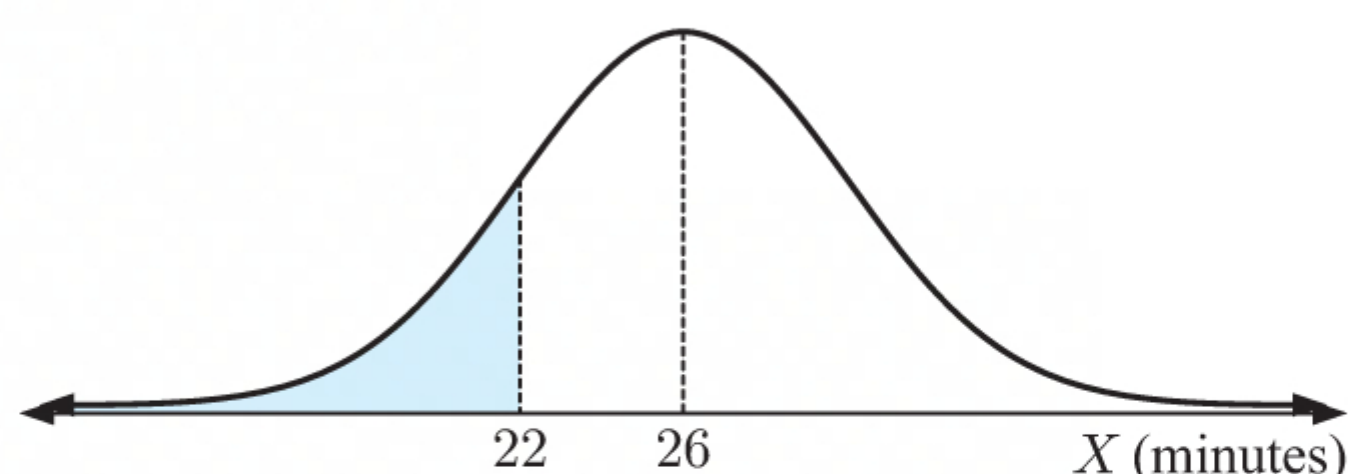
InvNormCD(0.65, $\sqrt{20}$ , 4)
45.72320551
InvNormCD(0.85, $\sqrt{20}$ , 4)
48.63507103
□
Npd Ncd InvN



**98** Let the finishing time of a randomly selected runner be  $X$  minutes.

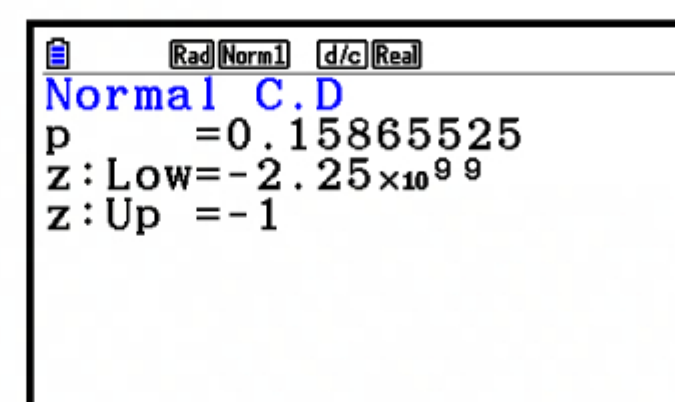
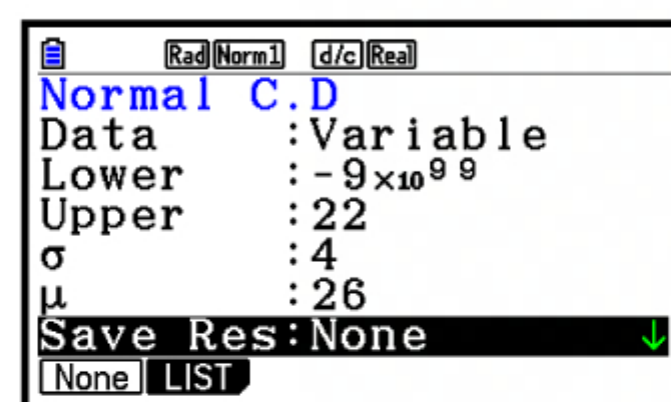
$$\therefore X \sim N(26, 4^2)$$

**a i**

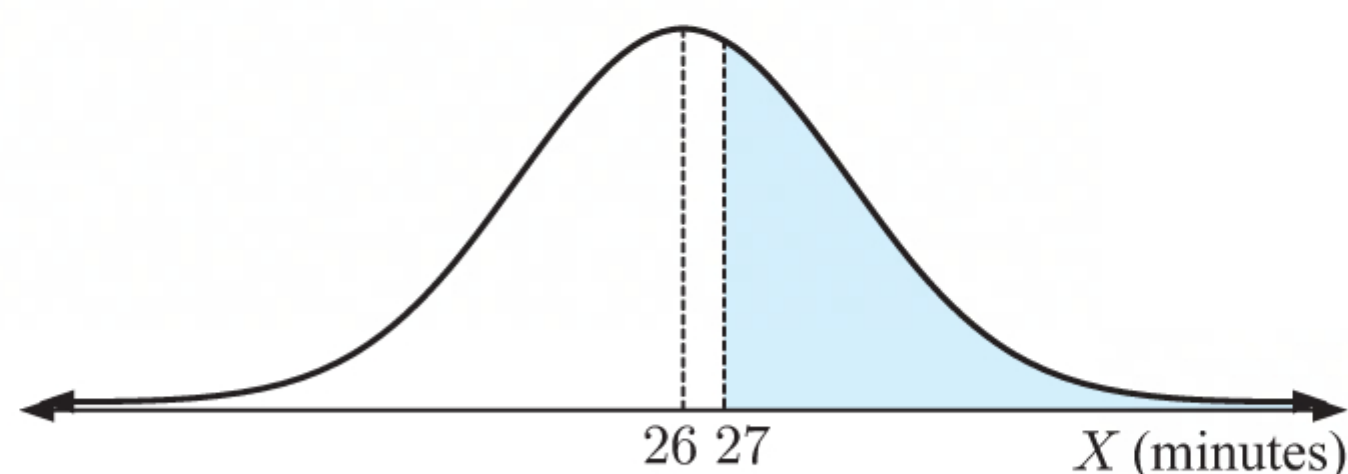


$$P(X < 22) \approx 0.159$$

$\therefore$  we expect about  $200 \times 0.159 \approx 32$  runners to have completed the course in less than 22 minutes.

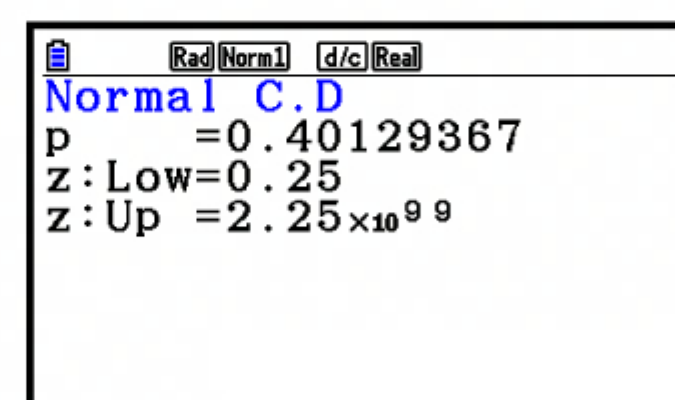
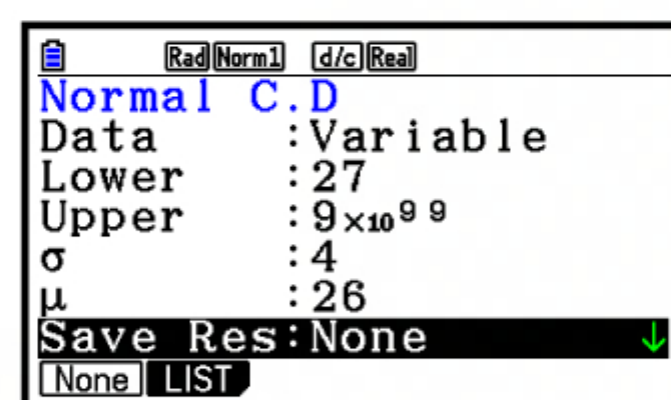


**ii**

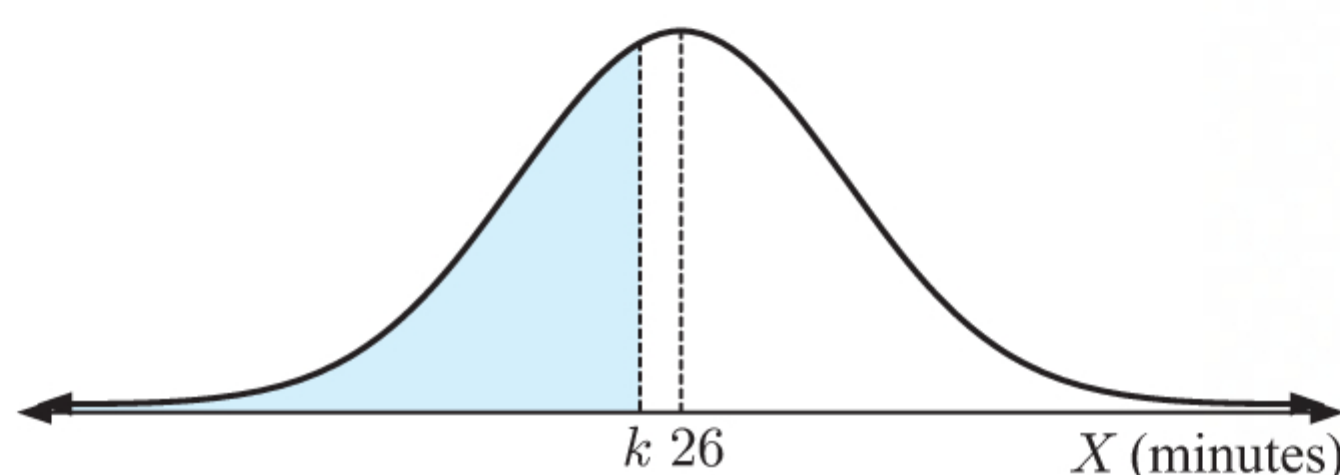


$$P(X > 27) \approx 0.401$$

$\therefore$  we expect about  $200 \times 0.401 \approx 80$  runners to have completed the course in more than 27 minutes.



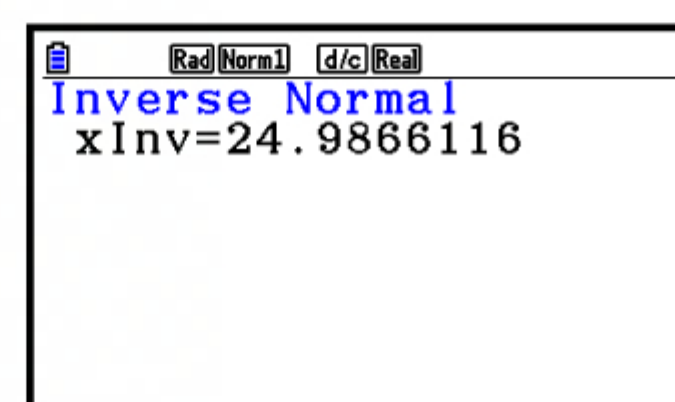
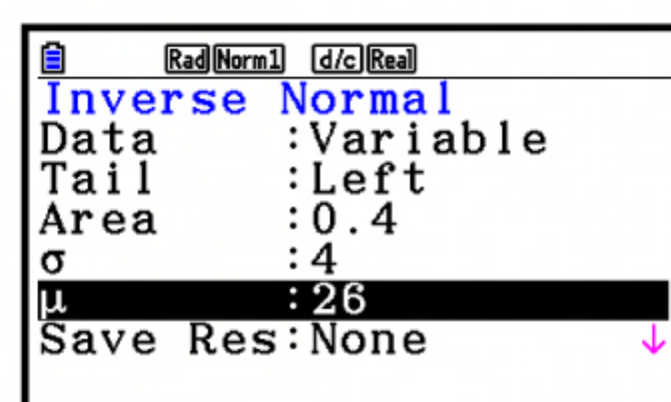
**b**



$$P(X < k) = 0.4$$

Using technology,  $k \approx 25.0$ .

The fastest 40% of runners finished quicker than about 25.0 minutes.



**99**

$x$	0	1	2	3	4	5
$P(X = x)$	0.1	0.05	0.25	0.3	0.15	0.15

**a**  $Y = 2X + 5$

**b i**  $E(X)$

$$= 0(0.1) + 1(0.05) + 2(0.25) + 3(0.3) + 4(0.15) + 5(0.15)$$

$$= 2.8 \text{ chores}$$

**ii**  $E(Y) = E(2X + 5)$

$$= 2E(X) + 5$$

$$= 2(2.8) + 5 \quad \{\text{using i}\}$$

$$= \$10.60$$

**c** Let  $W$  be the number of times Alisdair buys a chocolate bar each week.

$$\therefore W \sim B(5, 0.4)$$

$$\text{Expected amount of money left over each week} = E(Y - W)$$

$$= E(Y) - E(W)$$

$$= 10.6 - 5 \times 0.4$$

$$= 10.6 - 2$$

$$= \$8.60$$

**100**

**a i**

$$E(3X - 2Y) = 4$$

$$\text{and } E(2X + Y) = 5$$

$$\therefore 3E(X) - 2E(Y) = 4 \quad \dots (1) \quad \text{and} \quad 2E(X) + E(Y) = 5 \quad \dots (2)$$

$$3E(X) - 2E(Y) = 4 \quad \{(1)\}$$

$$4E(X) + 2E(Y) = 10 \quad \{(2) \times 2\}$$

$$\text{Adding, } 7E(X) = 14$$

$$\therefore E(X) = 2$$

$$\text{Substituting } E(X) = 2 \text{ into (2) gives } 2(2) + E(Y) = 5$$

$$\therefore E(Y) = 1$$



$$\begin{aligned}
 \text{ii } E(2Y - X) &= 2E(Y) - E(X) \\
 &= 2(1) - 2 \quad \{\text{using i}\} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{b i } \quad \text{Var}(X - Y) &= \frac{3}{4} & \text{and} & \quad \text{Var}(3Y - X) = \frac{11}{4} \\
 \therefore \text{Var}(X) + (-1)^2 \text{Var}(Y) &= \frac{3}{4} & \text{and} & \quad 3^2 \text{Var}(Y) + (-1)^2 \text{Var}(X) = \frac{11}{4} \\
 \therefore \text{Var}(X) + \text{Var}(Y) &= \frac{3}{4} \quad \dots (3) & \text{and} & \quad 9 \text{Var}(Y) + \text{Var}(X) = \frac{11}{4} \quad \dots (4)
 \end{aligned}$$

$$\begin{aligned}
 &\text{Var}(X) + \text{Var}(Y) = \frac{3}{4} \quad \{(3)\} \\
 &-\text{Var}(X) - 9 \text{Var}(Y) = -\frac{11}{4} \quad \{(4) \times -1\} \\
 \text{Adding, } &\underline{-8 \text{Var}(Y) = -2} \\
 &\therefore \text{Var}(Y) = \frac{1}{4}
 \end{aligned}$$

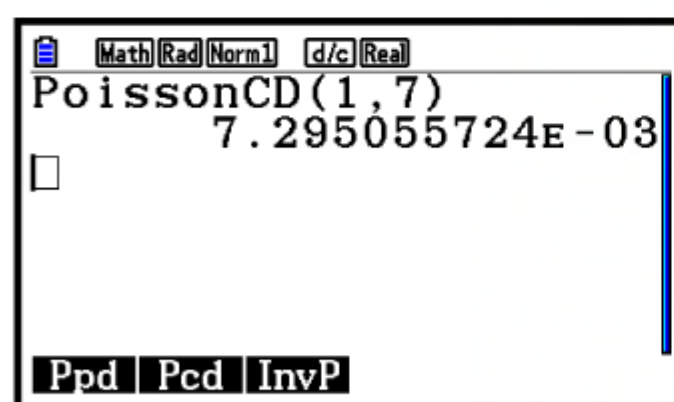
$$\begin{aligned}
 \text{Substituting } \text{Var}(Y) = \frac{1}{4} \text{ into (3) gives } \text{Var}(X) + \frac{1}{4} &= \frac{3}{4} \\
 \therefore \text{Var}(X) &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } \text{Var}(7X + 5Y) &= 7^2 \text{Var}(X) + 5^2 \text{Var}(Y) \\
 &= 49 \text{Var}(X) + 25 \text{Var}(Y) \\
 &= 49\left(\frac{1}{2}\right) + 25\left(\frac{1}{4}\right) \quad \{\text{using i}\} \\
 &= \frac{123}{4}
 \end{aligned}$$

**101**  $X \sim \text{Po}(5)$  and  $Y \sim \text{Po}(2)$  independently.

**a**  $(X + Y) \sim \text{Po}(7)$

**b**  $P(X + Y \leq 1) \approx 0.00730$



Math Rad Norm1 d/c Real  
PoissonCD(1, 7)  
7.295055724E-03  
□  
Ppd Pcd InvP

$$\begin{aligned}
 \text{c } P(Y = 1 \mid X + Y \leq 1) &= \frac{P(Y = 1 \cap X + Y \leq 1)}{P(X + Y \leq 1)} \\
 &= \frac{P(Y = 1 \cap X = 0)}{P(X + Y \leq 1)} \\
 &= \frac{P(Y = 1)P(X = 0)}{P(X + Y \leq 1)} \quad \{\text{independence}\} \\
 &\approx \frac{0.271 \times 0.00674}{0.00730} \quad \{\text{using b}\} \\
 &\approx 0.250
 \end{aligned}$$

**102 a** Let the heights of the basketball players be the independent random variables  $X_1, X_2, \dots, X_5$ .

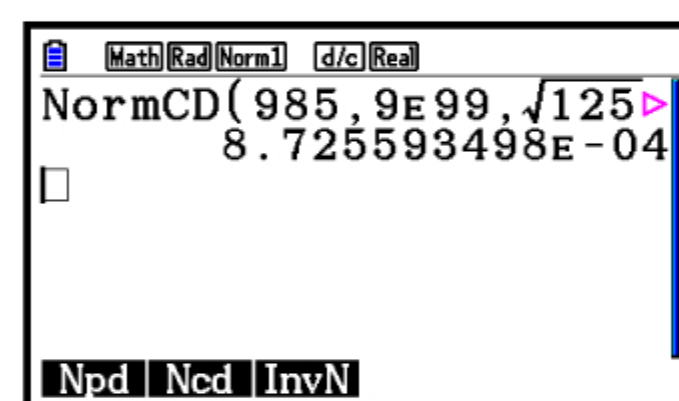
Let the sum of their heights be  $Y_5 = \sum_{k=1}^5 X_k$ , where  $X_k \sim N(190, 5^2)$ ,  $k = 1, 2, \dots, 5$ .

$$\begin{aligned}
 \text{Now } E(Y_5) &= \sum_{k=1}^5 E(X_k) & \text{and } \text{Var}(Y_5) &= \sum_{k=1}^5 \text{Var}(X_k) \\
 &= 5 \times 190 & &= 5 \times 5^2 \\
 &= 950 \text{ cm} & &= 125 \text{ cm}^2
 \end{aligned}$$

$$\therefore Y_5 \sim N(950, 125)$$

$$\text{Now } P(Y_5 \geq 985) \approx 0.000873$$

$\therefore$  the probability that the combined height of the 5 players is at least 985 cm is about 0.000873.



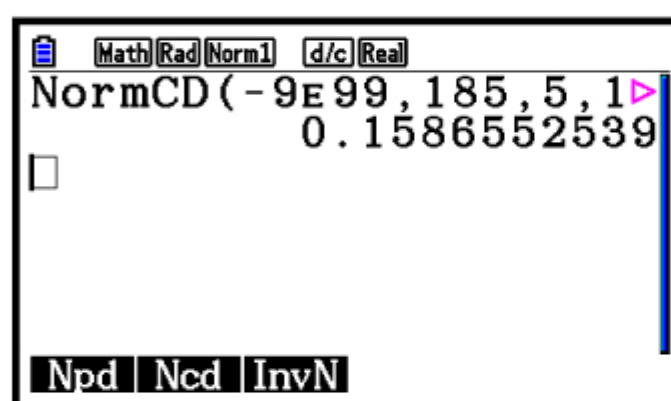
Math Rad Norm1 d/c Real  
NormCD(985, 950, sqrt(125))  
8.725593498E-04  
□  
Npd Ncd InvN



- b** Let  $X$  be the height of a *single* basketball player.

$$\therefore X \sim N(190, 5^2)$$

$$\therefore P(X < 185) \approx 0.1587$$

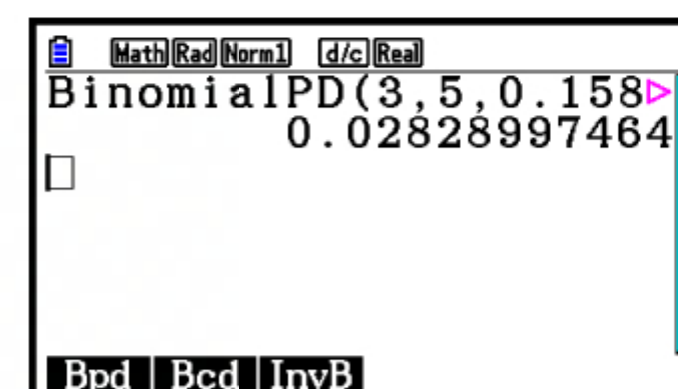


Let  $Y$  be the number of basketball players in the sample that are shorter than 185 cm.

$$\therefore Y \sim B(5, 0.1587)$$

$$\therefore P(Y = 3) \approx 0.0283$$

$\therefore$  the probability that exactly 3 players are shorter than 185 cm is about 0.0283.



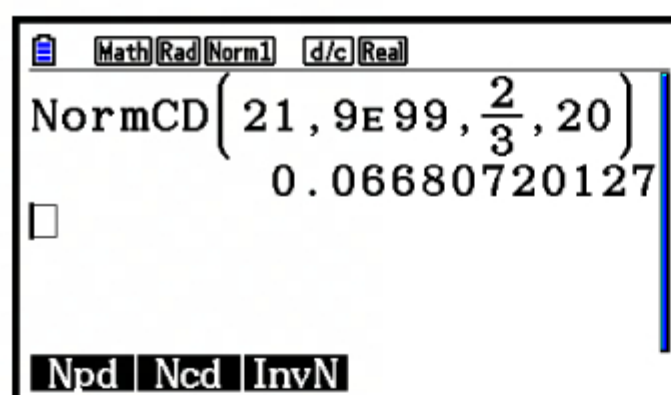
- 103**  $X$  is normally distributed with mean  $\mu = 20$  and standard deviation  $\sigma = 2$ .

**a i** mean of  $\bar{X}_9$  = mean of  $X$   
 $= \mu$   
 $= 20$

**ii** standard deviation of  $\bar{X}_9 = \frac{\sigma}{\sqrt{9}}$   
 $= \frac{2}{3}$

**b** Since  $X$  is normally distributed,  $\bar{X}_9$  is also normally distributed.

**c**  $P(\bar{X}_9 \geq 21) \approx 0.0668$



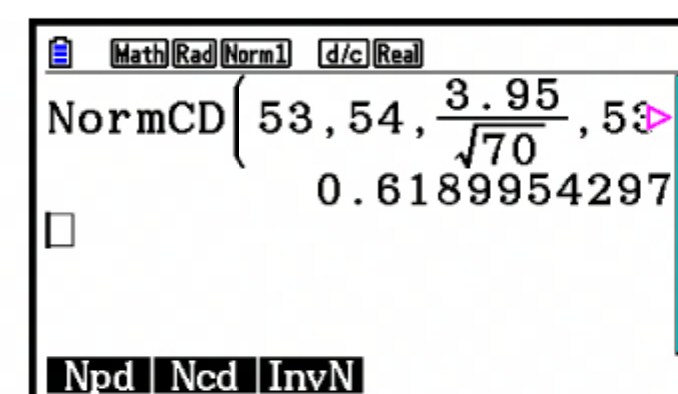
- 104** Let  $X$  be the amount of vitamin C in one orange.

Since the sample has size  $n = 70$ , we apply the Central Limit Theorem.

$\therefore \bar{X}_{70}$  is approximately normally distributed with mean 53.2 mg and standard deviation  $\frac{3.95}{\sqrt{70}}$  mg.

$$\therefore P(53 < \bar{X}_{70} < 54) \approx 0.619$$

$\therefore$  the probability that the average amount of vitamin C per orange lies between 53 mg and 54 mg is about 0.619.



- 105** We are given that  $\bar{x} = 0.4$  seconds and  $\sigma = 0.1$  seconds.

Since the sample has size  $n = 100$ , we apply the Central Limit Theorem.

$$\therefore \text{the 95\% confidence interval is } \bar{x} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}}$$

$$\therefore 0.4 - 1.96 \times \frac{0.1}{\sqrt{100}} \leq \mu \leq 0.4 + 1.96 \times \frac{0.1}{\sqrt{100}}$$

$$\therefore 0.3804 \leq \mu \leq 0.4196$$

So, we are 95% confident that the population mean reaction time lies between 0.3804 seconds and 0.4196 seconds.

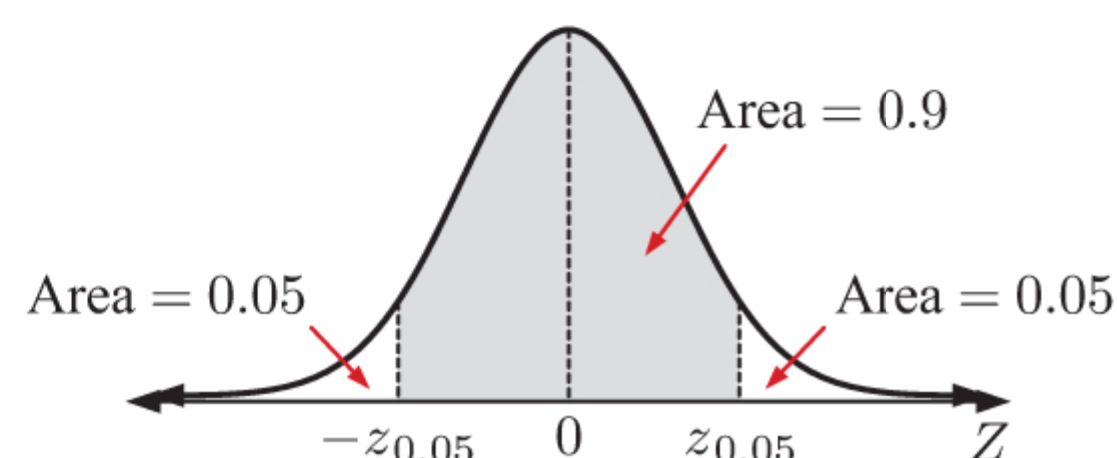
- 106 a** For a 90% confidence interval,  $\alpha = 0.1$

$$\therefore \frac{\alpha}{2} = 0.05$$

$$\therefore P(z \geq z_{0.05}) = 0.05$$

$$\therefore z_{0.05} \approx 1.645 \quad \{\text{using technology}\}$$

We are given that  $\bar{x} = 850$  g and  $\sigma = 9.7$  g.





Since the sample has size  $n = 40$ , we apply the Central Limit Theorem.

$\therefore$  the 90% confidence interval is  $\bar{x} - 1.645 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 1.645 \frac{\sigma}{\sqrt{n}}$

$$\therefore 850 - 1.645 \times \frac{9.7}{\sqrt{40}} \leq \mu \leq 850 + 1.645 \times \frac{9.7}{\sqrt{40}}$$

$$\therefore 847.48 \leq \mu \leq 852.52$$

So, we are 90% confident that the population mean weight of cabbages lies between 847.48 g and 852.52 g.

**b** Width  $\approx 852.52 - 847.48$

$$\approx 5.04 \text{ g}$$

**c i** If a higher confidence level is used, the corresponding value of  $z_{\frac{\alpha}{2}}$  is larger.

$\therefore$  the margin of error is larger.

$\therefore$  the width of the confidence interval will increase.

**ii** If a larger sample is taken,  $n$  is larger.

$\therefore$  the margin of error is smaller.

$\therefore$  the width of the confidence interval will decrease.

**107 a** Using technology,  $\bar{x} \approx 349.343$  and  $s \approx 15.072$

	Rad(Norm)	d/c(Real)
1-Variable		
$\bar{x}$	=349.342857	
$\Sigma x$	=12227	
$\Sigma x^2$	=4.2791E+06	
$\sigma x$	=14.8553845	
$s x$	=15.0722629	
$n$	=35	

**b** Since  $n = 35$  is sufficiently large,  $\bar{X}_n$  is approximately normally distributed. {Central Limit Theorem}

$\sigma$  is unknown and  $T = \frac{\bar{X}_n - \mu}{\frac{s_{n-1}}{\sqrt{n}}} \sim t_{34}$

For a 95% confidence interval,  $\alpha = 0.05$ .  $\therefore \frac{\alpha}{2} = 0.025$

Using technology,  $t_{34, 0.025} \approx 2.032$

The 95% confidence interval for  $\mu$  is  $\bar{x} - t_{34, 0.025} \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{34, 0.025} \frac{s}{\sqrt{n}}$

$$\therefore 349.343 - 2.032 \times \frac{15.072}{\sqrt{35}} \leq \mu \leq 349.343 + 2.032 \times \frac{15.072}{\sqrt{35}}$$

$$\therefore 344.2 \leq \mu \leq 354.5$$

So, we are 95% confident that the population mean completion time lies between 344.2 seconds and 354.5 seconds.

**108** Let  $\mu$  be the population mean time Eugene takes to walk the 5 km route.

The hypotheses that should be considered are:

$H_0: \mu = 40$  {the average time is 40 minutes}

$H_1: \mu < 40$  {the average time is below 40 minutes}

**109**

	Retain $H_0$	Reject $H_0$
$H_0$ true		A
$H_0$ false	B	

**a** A represents a Type I error.

**b** B represents a Type II error.

**110 a**  $H_0: \mu = 26$  {the average summer temperature is  $26^\circ\text{C}$ }

$H_1: \mu > 26$  {the average summer temperature exceeds  $26^\circ\text{C}$ }

**b i**  $\bar{x} = 26.6^\circ\text{C}$ ,  $\sigma = 3.85^\circ\text{C}$ ,  $n = 60$

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{26.6 - 26}{\frac{3.85}{\sqrt{60}}} \approx 1.21$$

**ii**  $p\text{-value} = P(Z \geq z)$  where  $Z \sim N(0, 1^2)$

$$\approx P(Z \geq 1.21)$$

$$\approx 0.114$$

**c** The significance level is  $\alpha = 0.05$ .

Since  $p\text{-value} > 0.05 = \alpha$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% significance level.

We therefore accept  $H_0$ .

Since we have accepted  $H_0$ , we conclude that the average summer temperature is  $26^\circ\text{C}$ . Gregory's claim is not valid.



**111** Step 1: Let  $\mu$  be the population mean diameter of the tennis balls manufactured by the company.

The hypotheses to be considered are:

$$H_0: \mu = 6.541 \quad \{\text{the mean diameter is 6.541 cm}\}$$

$$H_1: \mu < 6.541 \quad \{\text{the mean diameter is below 6.541 cm}\}$$

Step 2: The significance level is  $\alpha = 0.05$ .

Step 3:  $\bar{x} = 6.548$  cm,  $s = 0.173$  cm,  $n = 500$

$$\begin{aligned} \text{The value of the test statistic is } t &= \frac{6.548 - 6.541}{\frac{0.173}{\sqrt{500}}} \\ &\approx 0.905 \end{aligned}$$

Step 4: Since  $H_1: \mu < 6.541$ ,  $p\text{-value} = P(T \leq t)$  where  $T \sim t_{499}$

$$\approx P(T \leq 0.905)$$

$$\approx 0.817 \quad \{\text{using technology}\}$$

Step 5: Since  $p\text{-value} > 0.05 = \alpha$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% level of significance. We therefore accept  $H_0$ .

Step 6: Since we have accepted  $H_0$ , there is insufficient evidence to conclude that the tennis balls produced by this company do not meet the minimum international standard diameter.

**112** The data values are matched in pairs, so we used the paired  $t$ -test.

Step 1: Let  $\mu_D$  be the population mean difference between the times after and before the program.

The hypotheses to be considered are

$$H_0: \mu_D = 0 \quad \{\text{the program made no difference}\}$$

$$H_1: \mu_D < 0 \quad \{\text{the program improved running times}\}$$

Step 2: The significance level is  $\alpha = 0.01$ .

Step 3:

Before program ( $x_i$ )	29.4	33.6	27.2	25.3	35.7	35.2	29.8	37.8	40.1	34.1
After program ( $y_i$ )	28.2	31.9	28.1	26.7	35.2	35.3	28.2	35.3	39.3	35.6
$d_i = y_i - x_i$	-1.2	-1.7	0.9	1.4	-0.5	0.1	-1.6	-2.5	-0.8	1.5

	List 1	List 2	List 3	List 4
SUB				
1	29.4	28.2	-1.2	
2	33.6	31.9	-1.7	
3	27.2	28.1	0.9	
4	25.3	26.7	1.4	

1-Sample tTest	
Data	: List
$\mu_0$	: 0
List	: List3
Freq	: 1
Save Res	: None

1-Sample tTest	
$\mu$	< 0
t	= -1.0087216
p	= 0.16972852
$\bar{x}$	= -0.44
sx	= 1.37937184
n	= 10

The observed value of the test statistic  $\approx -1.01$ .

Step 4: From the screenshots above, the  $p\text{-value} \approx 0.170$ .

Step 5: Since the  $p\text{-value} > 0.01 = \alpha$  we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 1% level of significance. We therefore accept  $H_0$ .

Step 6: Since we have accepted  $H_0$ , we conclude that the mean difference between the running times before and after the program is 0.

The coach's claim is not valid.

**113**

	Sample mean	Sample standard deviation
Before change	183	5.83
After change	184	2.35

Step 1: Let  $\mu_1$  be the population mean weight before the change in fertiliser, and  $\mu_2$  be the population mean weight after the change in fertiliser.

The hypotheses to be considered are:

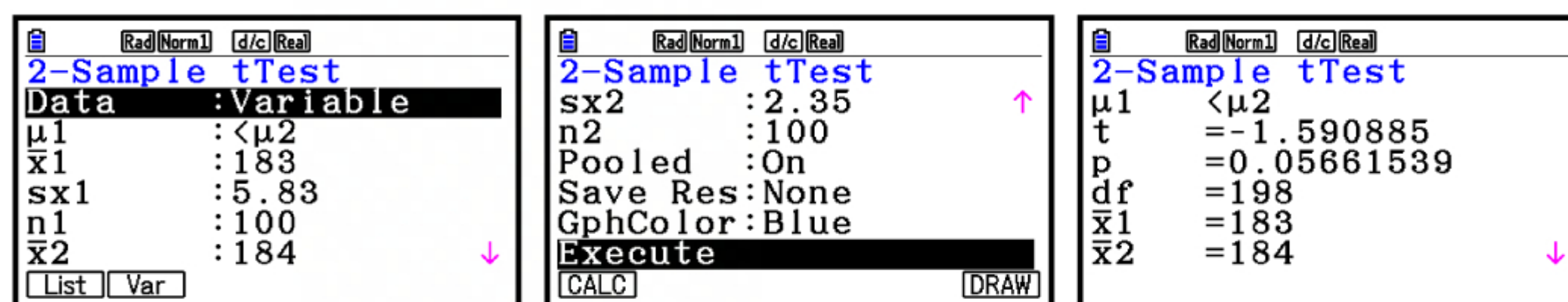
$$H_0: \mu_1 = \mu_2 \quad \{\text{the new fertiliser is no better than the old fertiliser}\}$$

$$H_1: \mu_1 < \mu_2 \quad \{\text{the new fertiliser is better than the old fertiliser}\}$$

Step 2: The significance level is  $\alpha = 0.05$ .



Step 3:



Using technology, the value of the test statistic is  $t \approx -1.59$ .

Step 4: From the screenshots above, the  $p$ -value  $\approx 0.0566$ .

Step 5: Since the  $p$ -value  $> 0.05 = \alpha$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% level of significance. We therefore accept  $H_0$ .

Step 6: Since we have accepted  $H_0$ , we cannot conclude that the change in fertiliser has increased the mean weight of potatoes.

- 114 a**  $H_0: \lambda = 200$  {the number of customers entering the supermarket per hour is 200}  
 $H_1: \lambda > 200$  {the number of customers entering the supermarket per hour exceeds 200}

- b** The significance level is  $\alpha = 0.05$ .

The test statistic is  $t = 653$ .

$$n \times \lambda_0 = 3 \times 200 = 600$$

So the null distribution is  $T \sim \text{Po}(600)$ .

$$\begin{aligned} \therefore p\text{-value} &= P(T \geq t) \\ &= P(T \geq 653) \\ &\approx 0.0170 \end{aligned}$$

Since  $p\text{-value} < 0.05 = \alpha$ , we have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% level of significance. We therefore accept  $H_1$ .

Since we have accepted  $H_1$ , we conclude that the number of customers entering the supermarket exceeds 200 per hour.

The supermarket should be upgraded.

- 115 a** Let  $p$  be the probability that the coin lands on heads.

- i**  $H_0: p = 0.5$  {the coin is fair}  
 $H_1: p \neq 0.5$  {the coin is biased}

- ii**  $H_0: p = 0.5$  {the coin is fair}  
 $H_1: p > 0.5$  {the coin is biased towards heads}

- b** The significance level is  $\alpha = 0.05$ .

The test statistic is  $x = 32$ .

The null distribution is  $X \sim (50, 0.5)$ .

$$\begin{aligned} \therefore p\text{-value} &= P(X \geq x) \\ &= P(X \geq 32) \\ &\approx 0.0325 \end{aligned}$$

Since  $p\text{-value} < 0.05 = \alpha$ , we have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% level of significance. We therefore accept  $H_1$ .

Since we have accepted  $H_1$ , we conclude that the coin is biased towards heads. Erin's suspicion is justified.

**116**

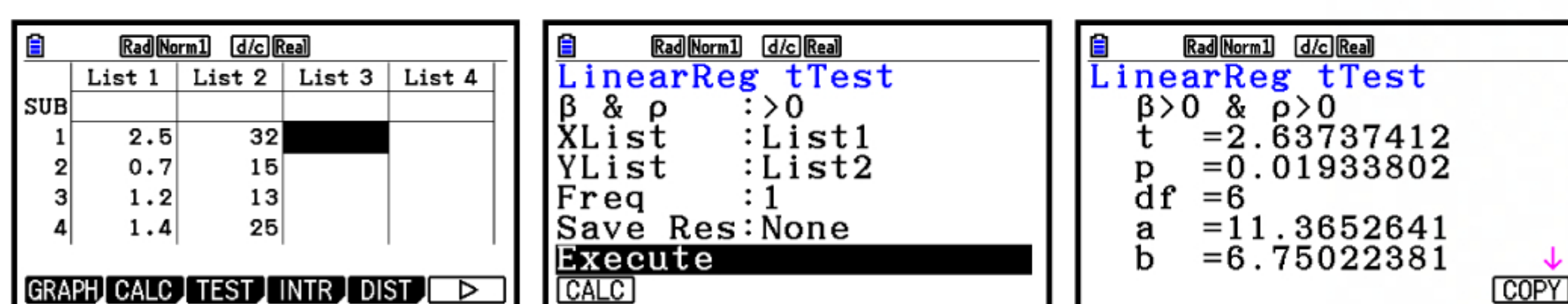
Weight ( $x$ kg)	2.5	0.7	1.2	1.4	2.9	0.8	1.6	1.8
Volume ( $y \times 1000 \text{ cm}^3$ )	32	15	13	25	30	21	26	16

- a** Let  $\rho$  be the population product-moment correlation coefficient between the variables.

The hypotheses to be considered are:

- $H_0: \rho = 0$  {there is no correlation between the variables}  
 $H_1: \rho > 0$  {the variables are positively correlated}

- b**



The test statistic  $\approx 2.64$  and the  $p$ -value  $\approx 0.0193$ .



- c The significance level is  $\alpha = 0.02$ .

Since  $p\text{-value} < 0.02 = \alpha$ , we have enough evidence to reject  $H_0$  in favour of  $H_1$ . We therefore accept  $H_1$ .

Since we have accepted  $H_1$ , we conclude that the variables are positively correlated.

- 117 a  $P(\text{Type I error}) = \alpha = 0.03$

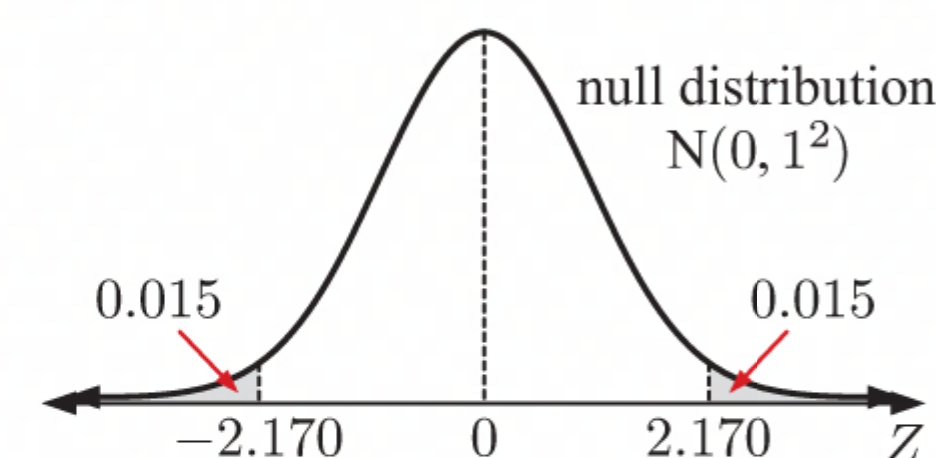
- b i  $H_0: \mu = 20.3, H_1: \mu \neq 20.3$

$H_0$  is retained if  $-z_{0.015} < Z < z_{0.015}$  where  $z_{0.015} \approx 2.170$ .

The true distribution of  $Z$  is  $N\left(\frac{20.2 - 20.3}{\frac{13.5}{\sqrt{20}}}, 1^2\right) \equiv N\left(\frac{-0.1\sqrt{20}}{13.5}, 1^2\right)$

$$\begin{aligned}\therefore \beta &= P(\text{Type II error}) \\ &= P(\text{Retain } H_0 \mid H_0 \text{ false}) \\ &\approx P\left(-2.170 < Z < 2.170 \mid Z \sim N\left(\frac{-0.1\sqrt{20}}{13.5}, 1^2\right)\right) \\ &\approx 0.9699\end{aligned}$$

- ii Power  $= 1 - \beta \approx 0.0301$



- 118 a Since  $X$  is a discrete random variable with values  $0, 1, 2, \dots$ , so is  $t$ .

$\therefore$  the critical region is  $\{t \mid t \geq 55\}$ .

- b A Type I error is rejecting  $H_0$  when  $H_0$  is in fact true. This means deciding  $\lambda > 3$  when in fact  $\lambda = 3$  and  $X \sim \text{Po}(3)$ .

$$\begin{aligned}P(\text{Type I error}) &= P(\text{Reject } H_0 \mid H_0 \text{ true}) \\ &= P(T \geq 55 \mid T \sim \text{Po}(45)) \\ &\approx 0.0816\end{aligned}$$

- c A Type II error is accepting  $H_0$  when  $H_0$  is in fact false. This means accepting  $\lambda = 3$  when in fact  $\lambda = 3.5$ .

$$\begin{aligned}P(\text{Type II error}) &= P(\text{Retain } H_0 \mid H_0 \text{ false}) \\ &= P(T < 55 \mid T \sim \text{Po}(52.5)) \\ &\approx 0.617\end{aligned}$$

119

Type of cookie	Number eaten	Expected frequency
choc-chip	789	$2143 \times 0.35 = 750.05$
oatmeal	542	$2143 \times 0.25 = 535.75$
shortbread	423	$2143 \times 0.2 = 428.6$
butter	389	$2143 \times 0.2 = 428.6$
Total	2143	

We are making a claim about population proportions, so a  $\chi^2$  goodness of fit test is appropriate.

Step 1: Let  $p_1, p_2, p_3$ , and  $p_4$  be the population proportion of choc-chip, oatmeal, shortbread, and butter cookies respectively.

The hypotheses that should be tested are:

$$H_0: p_1 = 0.35, p_2 = 0.25, p_3 = 0.2, p_4 = 0.2$$

$$H_1: \text{at least one of } p_1 \neq 0.35, p_2 \neq 0.25, p_3 \neq 0.2, \text{ or } p_4 \neq 0.2.$$

Step 2: The significance level is  $\alpha = 0.01$ .

Step 3:  $df = 4 - 1 = 3$

Using technology, the value of the test statistic is  $\chi^2_{\text{calc}} \approx 5.83$ .

Step 4: From the screenshots above, the  $p\text{-value} \approx 0.120$ .

Step 5: Since  $p\text{-value} > 0.01 = \alpha$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 1% level of significance. We therefore accept  $H_0$ .

Step 6: Since we have accepted  $H_0$ , we cannot conclude that the proportions of cookies eaten are different to the proportions of cookies that Brianna provides. Brianna should not change the proportion of cookies she provides.



- 120 a  $H_0$ : the data is from  $N(174, 6^2)$   
 $H_1$ : the data is not from  $N(174, 6^2)$

b

Height ( $H$ cm)	Frequency	Probability	Expected frequency
$160 \leq H < 165$	18	$\approx 0.066\,81$	$\approx 0.066\,81 \times 200 \approx 13.3614$
$165 \leq H < 170$	33	$\approx 0.185\,69$	$\approx 0.185\,69 \times 200 \approx 37.1371$
$170 \leq H < 175$	59	$\approx 0.313\,69$	$\approx 0.313\,69 \times 200 \approx 62.7383$
$175 \leq H < 180$	61	$\approx 0.275\,16$	$\approx 0.275\,16 \times 200 \approx 55.0322$
$180 \leq H < 185$	18	$\approx 0.125\,28$	$\approx 0.125\,28 \times 200 \approx 25.0557$
$185 \leq H < 190$	11	$\approx 0.033\,38$	$\approx 0.033\,38 \times 200 \approx 6.6753$

c  $df = 6 - 1 = 5$

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Using technology, the value of the test statistic is  $\chi^2_{\text{calc}} \approx 7.73$ .

- d Since  $\chi^2_{\text{calc}} < \chi^2_{\text{crit}} = 9.24$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$ . We therefore accept  $H_0$ .  
Since we have accepted  $H_0$ , we conclude that the data is from the normal distribution  $N(174, 6^2)$ .

121

Challenges completed	Number of participants
0	22
1	63
2	67
3	45
4	3

a mean  $\bar{x} = \frac{0(22) + 1(63) + \dots + 4(3)}{200}$   
 $= \frac{344}{200}$   
 $= 1.72$   
b  $p \approx \frac{\bar{x}}{n} \approx \frac{1.72}{4} \approx 0.43$

- c Step 1:  $H_0$ : the data is from a binomial distribution.  
 $H_1$ : the data is *not* from a binomial distribution.  
Step 2: The significance level is  $\alpha = 0.05$ .  
Step 3:  $X \sim B(4, 0.43)$

Challenges completed ( $x$ )	$P(X = x)$	$f_{\text{obs}}$	$f_{\text{exp}}$	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
0	$\approx 0.105\,56$	22	$\approx 0.105\,56 \times 200 \approx 21.1120$	$\approx 0.0374$
1	$\approx 0.318\,53$	63	$\approx 0.318\,53 \times 200 \approx 63.7064$	$\approx 0.0078$
2	$\approx 0.360\,44$	67	$\approx 0.360\,44 \times 200 \approx 72.0888$	$\approx 0.3592$
3	$\approx 0.181\,28$	45	$\approx 0.181\,28 \times 200 \approx 36.2552$	$\approx 2.1093$
4	$\approx 0.034\,19$	3	$\approx 0.034\,19 \times 200 \approx 6.8376$	$\approx 2.1539$
Total				$\approx 4.6675$

So,  $\chi^2_{\text{calc}} \approx 4.67$ .

Step 4:  $df = 5 - 1 - 1 = 3$

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Using technology,  $p$ -value  $\approx 0.198$ .

- Step 5: Since  $p\text{-value} > 0.05 = \alpha$ , we do not have sufficient evidence to reject  $H_0$  in favour of  $H_1$ . We therefore accept  $H_0$ .  
Step 6: Since we have accepted  $H_0$ , we conclude that the data is binomially distributed.



- 122 a**  $H_0$ : Preferred milk flavour and age are independent.  
 $H_1$ : Preferred milk flavour and age are not independent.

**b**

Using technology, the value of the test statistic is  $\chi^2_{\text{calc}} \approx 10.7$ .

- c** Since  $\chi^2_{\text{calc}} > \chi^2_{\text{crit}} = 7.81$ , we have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% significance level.  
 We conclude that *preferred milk flavour* is not independent of *age* at a 5% significance level.

**123**

	Variety A	Variety B	Variety C	Sum
Fertiliser	65	48	75	188
No fertiliser	40	54	58	152
Sum	105	102	133	340

- a**  $H_0$ : The effect of fertiliser is independent of the variety of orange.  
 $H_1$ : The effect of fertiliser is not independent of the variety of orange.

**b**

	Variety A	Variety B	Variety C
Fertiliser	$\frac{188 \times 105}{340} \approx 58.1$	$\frac{188 \times 102}{340} \approx 56.4$	73.5
No fertiliser	$\frac{152 \times 105}{340} \approx 46.9$	45.6	59.5

**c**

Using technology,  $\chi^2_{\text{calc}} \approx 4.72$ .

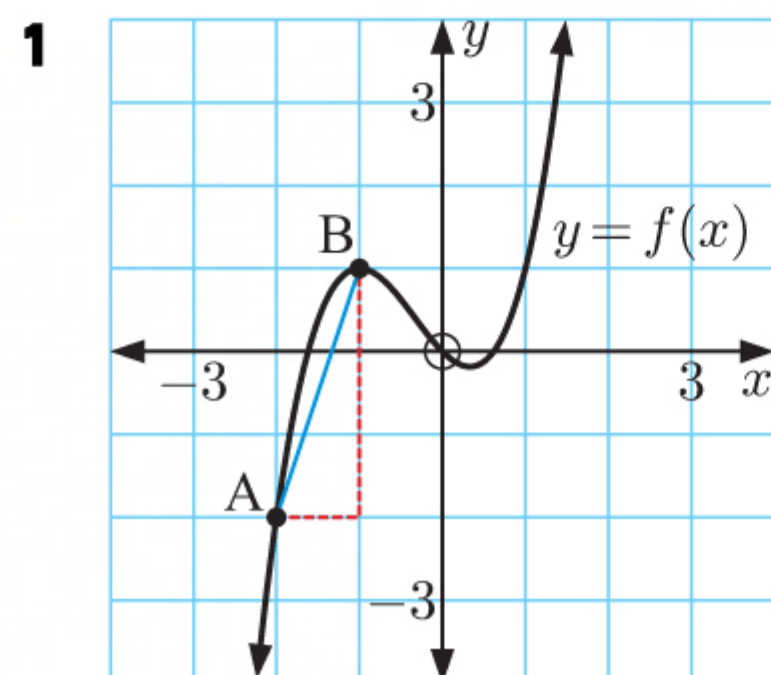
- d**  $df = (2 - 1)(3 - 1) = 2$   
**e** From the screenshots in **c**,  $p$ -value  $\approx 0.0944$ .  
**f** The significance level is  $\alpha = 0.05$ .

Since  $p\text{-value} > 0.05 = \alpha$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% significance level.  
 We therefore accept  $H_0$ .

We conclude that the *effect of the fertiliser* is independent of the *variety of orange*.



# TOPIC 5 SKILL BUILDER QUESTIONS

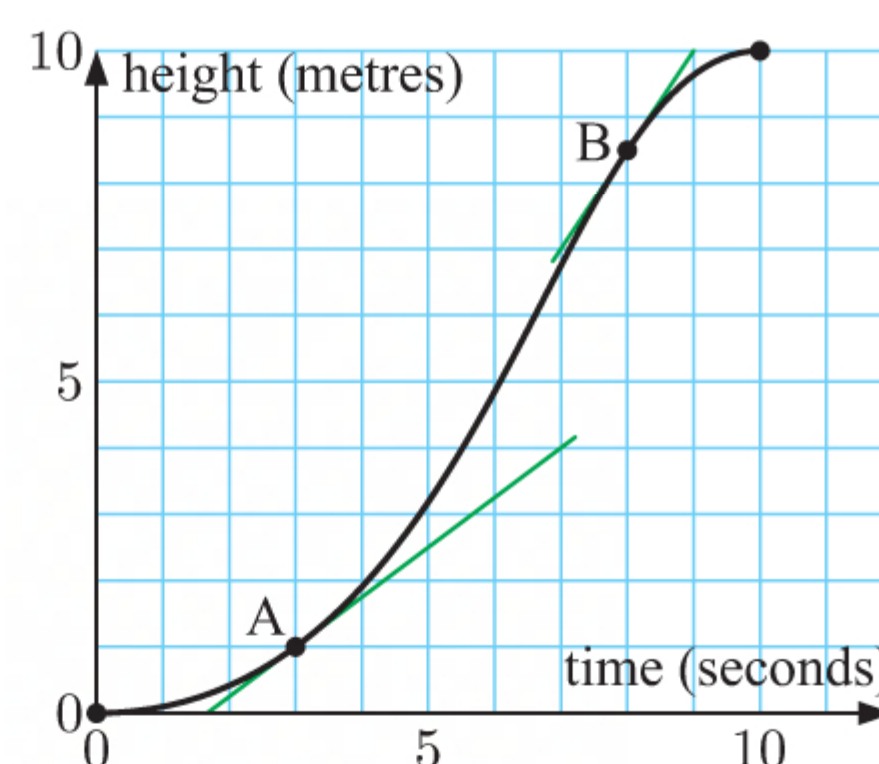


Average rate of change = gradient of chord [AB]

$$\begin{aligned}
 &= \frac{1 - (-2)}{-1 - (-2)} \\
 &= \frac{3}{1} \\
 &= 3
 \end{aligned}$$

- 2 a The tangent at A has gradient  $\frac{4-1}{7-3} = \frac{3}{4}$ .  
 $\therefore$  the elevator's instantaneous speed after 3 seconds is  $0.75 \text{ m s}^{-1}$ .

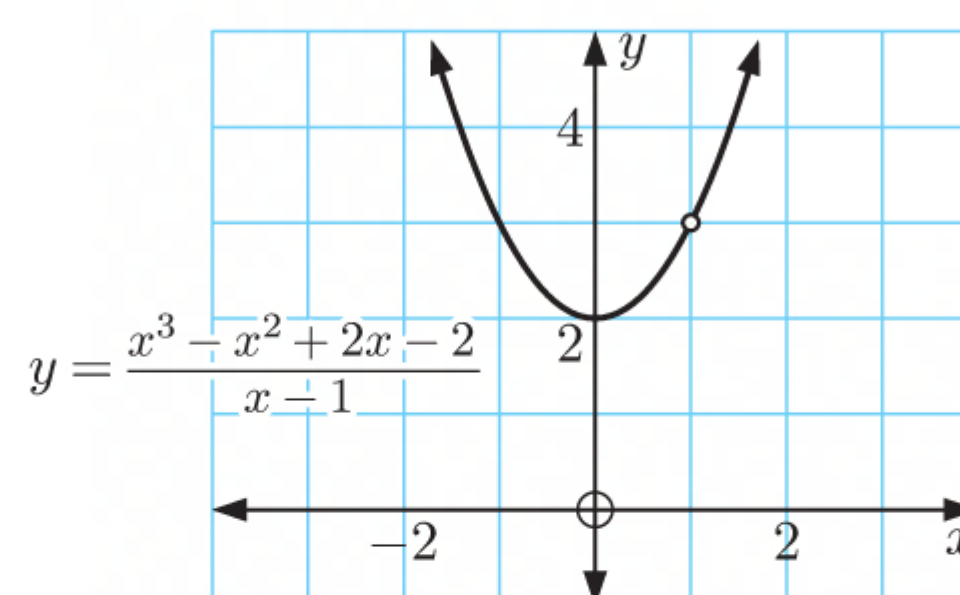
- b The tangent at B has gradient  $\frac{10-7}{9-7} = \frac{3}{2}$ .  
 $\therefore$  the elevator's instantaneous speed after 8 seconds is  $1.5 \text{ m s}^{-1}$ .



- 3 a When  $x = 1$ , the denominator of  $\frac{x^3 - x^2 + 2x - 2}{x - 1}$  is zero. So, the function is undefined at  $x = 1$ .

- b From the graph, we can see that  $y \rightarrow 3$  as  $x \rightarrow 1$  from either direction.

So,  $\lim_{x \rightarrow 1} f(x) = 3$ .



4 a

$x$	50	100	200	500	1000
$\frac{\ln x}{x}$	$\approx 0.078\,24$	$\approx 0.046\,05$	$\approx 0.026\,49$	$\approx 0.012\,43$	$\approx 0.006\,91$

- b We predict that  $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0$ .

- 5 a The tangent at  $x = -1$  passes through  $(-1, 0)$  and  $(0, 3)$ .

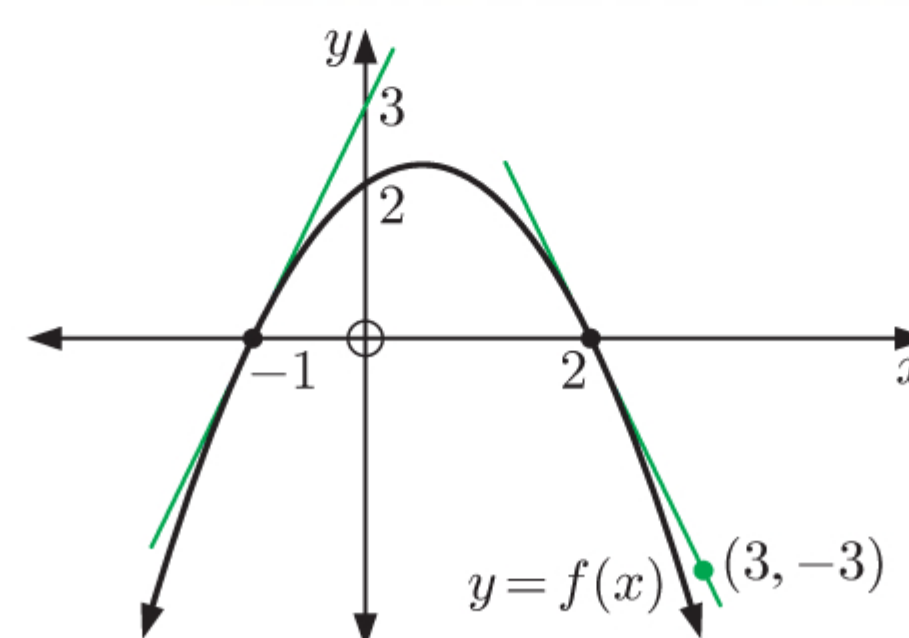
$\therefore f'(-1) = \text{gradient of the tangent}$

$$\begin{aligned}
 &= \frac{3-0}{0-(-1)} \\
 &= \frac{3}{1} \\
 &= 3
 \end{aligned}$$

- b The tangent at  $x = 2$  passes through  $(2, 0)$  and  $(3, -3)$ .

$\therefore f'(2) = \text{gradient of the tangent}$

$$\begin{aligned}
 &= \frac{-3-0}{3-2} \\
 &= \frac{-3}{1} \\
 &= -3
 \end{aligned}$$



- 6 a  $\frac{d}{dx}(6 - 3x + 2x^2) = 0 - 3(1) + 2(2x)$   
 $= -3 + 4x$

- c  $\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{x^2}\right) = \frac{d}{dx}(x^{-1} + x^{-2})$   
 $= -x^{-2} - 2x^{-3}$   
 $= -\frac{1}{x^2} - \frac{2}{x^3}$

- b  $\frac{d}{dx}\left(\frac{1}{2}x^2 + 3x - 5\right) = \frac{1}{2}(2x) + 3(1) - 0$   
 $= x + 3$

- d  $\frac{d}{dx}\left(\frac{2x^2 + x + 1}{x}\right) = \frac{d}{dx}(2x + 1 + x^{-1})$   
 $= 2(1) + 0 - x^{-2}$   
 $= 2 - \frac{1}{x^2}$



$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad y &= 5x^3 - 3x^2 + 4x + 7 \\ \therefore \frac{dy}{dx} &= 5(3x^2) - 3(2x) + 4(1) \\ &= 15x^2 - 6x + 4 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= \frac{3}{\sqrt{x}} - 2\sqrt{x} \\ &= 3x^{-\frac{1}{2}} - 2x^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= 3\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) - 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) \\ &= -\frac{3}{2}x^{-\frac{3}{2}} - x^{-\frac{1}{2}} \\ &= -\frac{3}{2x\sqrt{x}} - \frac{1}{\sqrt{x}} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= \frac{2x - x^2}{\sqrt{x}} \\ &= \frac{2x - x^2}{x^{\frac{1}{2}}} \\ &= 2x^{\frac{1}{2}} - x^{\frac{3}{2}} \\ \therefore \frac{dy}{dx} &= 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \frac{3}{2}x^{\frac{1}{2}} \\ &= x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{x}} - \frac{3}{2}\sqrt{x} \end{aligned}$$

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad y &= (x^2 - 3x)^5 \\ \therefore \frac{dy}{dx} &= 5(x^2 - 3x)^4(2x - 3) \quad \{\text{chain rule}\} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= \frac{3}{(x^2 + 3)^3} = 3(x^2 + 3)^{-3} \\ \therefore \frac{dy}{dx} &= 3 \times (-3)(x^2 + 3)^{-4} \times (2x) \quad \{\text{chain rule}\} \\ &= -18x(x^2 + 3)^{-4} \\ &= \frac{-18x}{(x^2 + 3)^4} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= \sqrt{x^2 - 3x} = (x^2 - 3x)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \frac{1}{2}(x^2 - 3x)^{-\frac{1}{2}}(2x - 3) \quad \{\text{chain rule}\} \\ &= \frac{2x - 3}{2\sqrt{x^2 - 3x}} \end{aligned}$$

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad f(x) &= ax^2 + bx^3, \quad f'(1) = -5, \quad \text{and} \quad f'(-1) = -13 \\ \therefore f'(x) &= 2ax + 3bx^2 \end{aligned}$$

$$\begin{aligned} \text{But } f'(1) &= -5, \quad \text{so} \quad 2a(1) + 3b(1)^2 = -5 \\ \therefore 2a + 3b &= -5 \end{aligned}$$

$$\begin{aligned} \text{and } f'(-1) &= -13, \quad \text{so} \quad 2a(-1) + 3b(-1)^2 = -13 \\ \therefore -2a + 3b &= -13 \end{aligned}$$

$$\text{Solving the system of equations} \quad \begin{cases} 2a + 3b = -5 \\ -2a + 3b = -13 \end{cases}$$

simultaneously gives  $a = 2$ ,  $b = -3$ .

$$\begin{aligned} \mathbf{b} \quad f(x) &= ax + \frac{b}{x^2}, \quad f(1) = 8, \quad \text{and} \quad f'(1) = -7 \\ &= ax + bx^{-2} \\ \therefore f'(x) &= a - 2bx^{-3} \\ &= a - \frac{2b}{x^3} \end{aligned}$$

$$\begin{aligned} \text{But } f(1) &= 8, \quad \text{so} \quad a(1) + \frac{b}{1^2} = 8 \\ \therefore a + b &= 8 \end{aligned}$$

$$\begin{aligned} \text{and } f'(1) &= -7, \quad \text{so} \quad a - \frac{2b}{1^3} = -7 \\ \therefore a - 2b &= -7 \end{aligned}$$

$$\text{Solving the system of equations} \quad \begin{cases} a + b = 8 \\ a - 2b = -7 \end{cases}$$

simultaneously gives  $a = 3$ ,  $b = 5$ .



$$\begin{aligned} 10 \quad a \quad y &= 3x - 2x^2 \\ \therefore \frac{dy}{dx} &= 3 - 2(2x) \\ &= 3 - 4x \end{aligned}$$

When  $x = 4$ ,  $\frac{dy}{dx} = 3 - 16 = -13$ .

So, the tangent has gradient  $-13$ .

$$\begin{aligned} 11 \quad f(x) &= ax^2 + bx - 7 \\ \therefore f'(x) &= 2ax + b \end{aligned}$$

Now at  $(-1, -10)$ , the tangent has gradient 1.

$$\begin{aligned} \therefore f(-1) &= -10 \quad \text{and} \quad f'(-1) = 1 \\ \therefore a(-1)^2 + b(-1) - 7 &= -10 \quad \text{and} \quad 2a(-1) + b = 1 \\ \therefore a - b &= -3 \quad \text{and} \quad -2a + b = 1 \end{aligned}$$

Solving the system of equations  $\begin{cases} a - b = -3 \\ -2a + b = 1 \end{cases}$  simultaneously gives  $a = 2$ ,  $b = 5$ .

$$\begin{aligned} 12 \quad a \quad y &= 3x^2 + 5x + 1 \\ \therefore \frac{dy}{dx} &= 6x + 5 \end{aligned}$$

$$\begin{aligned} \therefore \text{the tangent has gradient } 11 \text{ when } 6x + 5 &= 11 \\ \therefore 6x &= 6 \\ \therefore x &= 1 \end{aligned}$$

When  $x = 1$ ,  $y = 3(1)^2 + 5(1) + 1 = 9$

So, the tangent has gradient 11 at the point  $(1, 9)$ .

$$\begin{aligned} 13 \quad f(x) &= (ax + b)^c \\ \therefore f'(x) &= c(ax + b)^{c-1}(a) \quad \{\text{chain rule}\} \\ &= ac(ax + b)^{c-1} \end{aligned}$$

Now  $f'(x) = 81x^2 + 108x + 36$  is a polynomial of degree 2, so we must have  $c - 1 = 2$   
 $\therefore c = 3$

$$\begin{aligned} \therefore 3a(ax + b)^2 &= 81x^2 + 108x + 36 \\ \therefore 3a(a^2x^2 + 2abx + b^2) &= 81x^2 + 108x + 36 \\ \therefore 3a^3x^2 + 6a^2bx + 3ab^2 &= 81x^2 + 108x + 36 \end{aligned}$$

Equating coefficients gives:

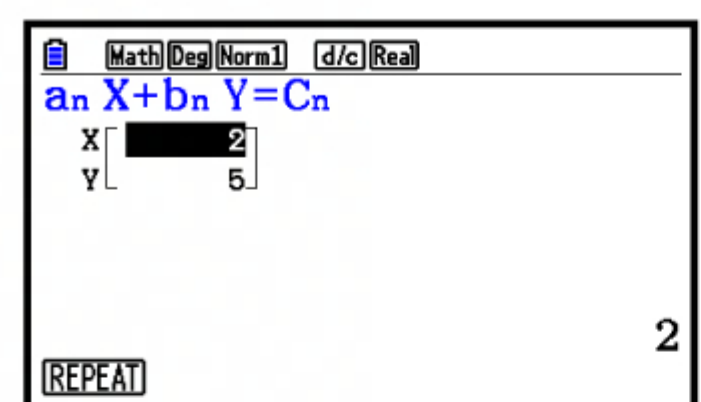
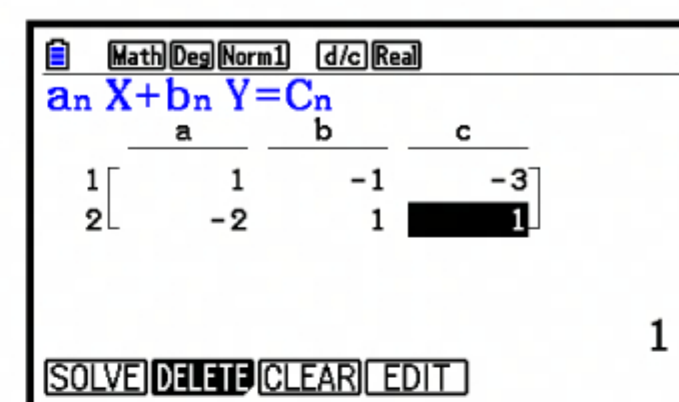
$$\begin{aligned} 3a^3 &= 81 \quad \dots (1) \\ 6a^2b &= 108 \quad \dots (2) \\ 3ab^2 &= 36 \quad \dots (3) \end{aligned}$$

$$\begin{aligned} \text{From (1), } 3a^3 &= 81 \\ \therefore a^3 &= 27 \\ \therefore a &= 3 \end{aligned}$$

$$\begin{aligned} b \quad y &= \frac{x^2 + 4x - 1}{x^2} \\ &= 1 + \frac{4}{x} - \frac{1}{x^2} \\ &= 1 + 4x^{-1} - x^{-2} \\ \therefore \frac{dy}{dx} &= -4x^{-2} + 2x^{-3} \\ &= -\frac{4}{x^2} + \frac{2}{x^3} \end{aligned}$$

When  $x = 1$ ,  $\frac{dy}{dx} = -4 + 2 = -2$ .

So, the tangent has gradient  $-2$ .



$$\begin{aligned} b \quad f(x) &= \frac{1}{2}x^3 - 4x - 2 \\ \therefore f'(x) &= \frac{3}{2}x^2 - 4 \end{aligned}$$

$$\begin{aligned} \therefore \text{the tangent has gradient } \frac{1}{2} \text{ when } \frac{3}{2}x^2 - 4 &= \frac{1}{2} \\ \therefore 3x^2 - 8 &= 1 \\ \therefore 3x^2 &= 9 \\ \therefore x^2 &= 3 \\ \therefore x &= \pm\sqrt{3} \end{aligned}$$

$$f(-\sqrt{3}) = \frac{1}{2}(-\sqrt{3})^3 - 4(-\sqrt{3}) - 2 \approx 2.33$$

$$f(\sqrt{3}) = \frac{1}{2}(\sqrt{3})^3 - 4(\sqrt{3}) - 2 \approx -6.33$$

So, the tangent has gradient  $\frac{1}{2}$  at the points  $(-\sqrt{3}, 2.33)$  and  $(\sqrt{3}, -6.33)$ .



Substituting  $a = 3$  into (2) gives  $6(3)^2b = 108$   
 $\therefore 54b = 108$   
 $\therefore b = 2$

Check:  $3ab^2 = 3(3)(2)^2 = 36$  ✓

**14 a**  $y = x^2\sqrt{x^2 + 2x} = x^2(x^2 + 2x)^{\frac{1}{2}}$   
 $\therefore \frac{dy}{dx} = 2x(x^2 + 2x)^{\frac{1}{2}} + x^2 \times \frac{1}{2}(x^2 + 2x)^{-\frac{1}{2}}(2x + 2)$  {product rule}  
 $= 2x\sqrt{x^2 + 2x} + \frac{x^2(x + 1)}{\sqrt{x^2 + 2x}}$

**b**  $y = \sqrt{x}(2x + 3)^4 = x^{\frac{1}{2}} \times (2x + 3)^4$   
 $\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(2x + 3)^4 + x^{\frac{1}{2}} \times 4(2x + 3)^3(2)$  {product rule}  
 $= \frac{(2x + 3)^4}{2\sqrt{x}} + 8\sqrt{x}(2x + 3)^3$

**c**  $y = (2x + 1)^3(x - 5)^2$   
 $\therefore \frac{dy}{dx} = 3(2x + 1)^2(2)(x - 5)^2 + (2x + 1)^3 \times 2(x - 5)(1)$  {product rule}  
 $= 6(2x + 1)^2(x - 5)^2 + 2(2x + 1)^3(x - 5)$

**15**  $y = (x - 2)^2(2x - 1)$   
 $\therefore \frac{dy}{dx} = 2(x - 2)(2x - 1) + (x - 2)^2(2)$  {product rule}  
 $= 2(x - 2)[(2x - 1) + (x - 2)]$   
 $= 2(x - 2)(3x - 3)$   
 $= 6(x - 2)(x - 1)$

Now  $\frac{dy}{dx} = 36$  where  $6(x - 2)(x - 1) = 36$   
 $\therefore (x - 2)(x - 1) = 6$   
 $\therefore x^2 - 3x + 2 = 6$   
 $\therefore x^2 - 3x - 4 = 0$   
 $\therefore (x + 1)(x - 4) = 0$   
 $\therefore x = -1$  or  $4$

**16 a**  $y = \frac{x^3}{x^2 - 1}$   
 $\therefore \frac{dy}{dx} = \frac{3x^2(x^2 - 1) - (2x)x^3}{(x^2 - 1)^2}$  {quotient rule}  
 $= \frac{3x^4 - 3x^2 - 2x^4}{(x^2 - 1)^2}$   
 $= \frac{x^4 - 3x^2}{(x^2 - 1)^2}$

When  $x = 2$ ,  $\frac{dy}{dx} = \frac{(2)^4 - 3(2)^2}{(2^2 - 1)^2} = \frac{16 - 12}{9} = \frac{4}{9}$ .

So, the tangent has gradient  $\frac{4}{9}$ .

**b**  $y = \frac{\sqrt{x}}{2x + 5} = \frac{x^{\frac{1}{2}}}{2x + 5}$   
 $\therefore \frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}(2x + 5) - (2) \times x^{\frac{1}{2}}}{(2x + 5)^2}$  {quotient rule}  
 $= \frac{2x + 5 - 4x}{2\sqrt{x}(2x + 5)^2}$   
 $= \frac{5 - 2x}{2\sqrt{x}(2x + 5)^2}$

When  $x = 4$ ,

$$\frac{dy}{dx} = \frac{5 - 2(4)}{2\sqrt{4}(2(4) + 5)^2} = \frac{-3}{2 \times 2 \times 169} = \frac{-3}{676}.$$

So, the tangent has gradient  $-\frac{3}{676}$ .

**17 a**  $f(t) = 20te^{-0.1t}$   
 $\therefore f'(t) = 20(1)e^{-0.1t} + 20t(-0.1e^{-0.1t})$  {product rule}  
 $= 20e^{-0.1t} - 2te^{-0.1t}$

**b**  $f(t) = \frac{100}{1 + 7e^{-\frac{t}{4}}} = 100\left(1 + 7e^{-\frac{t}{4}}\right)^{-1}$   
 $\therefore f'(t) = -100\left(1 + 7e^{-\frac{t}{4}}\right)^{-2} \times \left(-\frac{7}{4}e^{-\frac{t}{4}}\right)$  {chain rule}  
 $= \frac{175e^{-\frac{t}{4}}}{\left(1 + 7e^{-\frac{t}{4}}\right)^2}$



$$\begin{aligned}
 \text{c} \quad f(t) &= \frac{t+9}{e^t} \\
 \therefore f'(t) &= \frac{(1)e^t - e^t(t+9)}{(e^t)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{1 - (t+9)}{e^t} \\
 &= \frac{-t-8}{e^t}
 \end{aligned}$$

$$\begin{aligned}
 \text{18} \quad f(x) &= e^{ax+2} + x^2 \quad \text{and} \quad f(2) = f'(2) \\
 \therefore f'(x) &= ae^{ax+2} + 2x \\
 \text{Now } f(2) &= e^{2a+2} + 4 \quad \text{and} \quad f'(2) = ae^{2a+2} + 4 \\
 \therefore e^{2a+2} + 4 &= ae^{2a+2} + 4 \\
 \therefore e^{2a+2} &= ae^{2a+2} \\
 \therefore (a-1)e^{2a+2} &= 0 \\
 \therefore a-1 &= 0 \quad \{e^x > 0 \text{ for all } x\} \\
 \therefore a &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{19 a} \quad f(x) &= e^x \ln x \\
 \therefore f'(x) &= e^x \times \frac{1}{x} + e^x \ln x \quad \{\text{product rule}\} \\
 &= e^x \left( \frac{1}{x} + \ln x \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= \ln(2x+3) \\
 \therefore f'(x) &= \frac{2}{2x+3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f(x) &= [\ln(x^2+5)]^2 \\
 \therefore f'(x) &= 2 \ln(x^2+5) \times \left( \frac{2x}{x^2+5} \right) \quad \{\text{chain rule}\} \\
 &= \frac{4x \ln(x^2+5)}{x^2+5}
 \end{aligned}$$

$$\text{20 a} \quad \frac{d}{dx}(3 \sin(x-4)) = 3 \cos(x-4)$$

$$\begin{aligned}
 \text{b} \quad \frac{d}{dx}(12x - 2 \cos \frac{x}{3}) &= 12 - 2(-\sin \frac{x}{3} \times \frac{1}{3}) \\
 &= 12 + \frac{2}{3} \sin \frac{x}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \frac{d}{dx}(x^2 \sin 3x) &= (2x) \sin 3x + x^2(3 \cos 3x) \quad \{\text{product rule}\} \\
 &= 2x \sin 3x + 3x^2 \cos 3x
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \frac{d}{dx}((\sin x)e^{\cos x}) &= (\cos x)e^{\cos x} + (\sin x)((-\sin x)e^{\cos x}) \quad \{\text{product rule}\} \\
 &= (\cos x)e^{\cos x} - (\sin^2 x)e^{\cos x} \\
 &= e^{\cos x}(\cos x - \sin^2 x)
 \end{aligned}$$

$$\begin{aligned}
 \text{21 a} \quad y &= \tan 2x \\
 \therefore \frac{dy}{dx} &= \frac{2}{\cos^2 2x}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= \tan(3x-4) \\
 \therefore \frac{dy}{dx} &= \frac{3}{\cos^2(3x-4)}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= \tan^2 x \\
 \therefore \frac{dy}{dx} &= \frac{2 \tan x}{\cos^2 x}
 \end{aligned}$$

$$\begin{aligned}
 \text{22 a} \quad f(x) &= \sqrt{\sin(2x+1)} = (\sin(2x+1))^{\frac{1}{2}} \\
 \therefore f'(x) &= \frac{1}{2}(\sin(2x+1))^{-\frac{1}{2}}(2 \cos(2x+1)) \quad \{\text{chain rule}\} \\
 &= \frac{\cos(2x+1)}{\sqrt{\sin(2x+1)}}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= \cos \frac{x}{2} \sin \frac{x}{3} \\
 \therefore f'(x) &= \left(-\frac{1}{2} \sin \frac{x}{2}\right) \sin \frac{x}{3} + \cos \frac{x}{2} \left(\frac{1}{3} \cos \frac{x}{3}\right) \quad \{\text{product rule}\} \\
 &= -\frac{1}{2} \sin \frac{x}{2} \sin \frac{x}{3} + \frac{1}{3} \cos \frac{x}{2} \cos \frac{x}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f(x) &= \ln\left(\frac{\sin x}{x}\right) = \ln(\sin x) - \ln x \\
 \therefore f'(x) &= \frac{\cos x}{\sin x} - \frac{1}{x} \\
 &= \frac{1}{\tan x} - \frac{1}{x}
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{23} \quad f(x) &= \cos^4 x \\
 \therefore f'(x) &= 4 \cos^3 x (-\sin x) \quad \{\text{chain rule}\} \\
 &= -4 \cos^3 x \sin x \\
 f'\left(\frac{3\pi}{4}\right) &= -4 \cos^3\left(\frac{3\pi}{4}\right) \sin \frac{3\pi}{4} \\
 &= -4 \left(-\frac{1}{\sqrt{2}}\right)^3 \left(\frac{1}{\sqrt{2}}\right) \\
 &= 1
 \end{aligned}$$

So, the gradient of the tangent is 1.

$$\mathbf{24} \quad \mathbf{a} \quad y = \frac{3}{x^2} = 3x^{-2}$$

$$\therefore \frac{dy}{dx} = -6x^{-3}$$

$$\therefore \frac{d^2y}{dx^2} = 18x^{-4} = \frac{18}{x^4}$$

$$\mathbf{c} \quad y = \frac{x+3}{6-x}$$

$$\therefore \frac{dy}{dx} = \frac{(1)(6-x) - (-1)(x+3)}{(6-x)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{6-x+x+3}{(6-x)^2}$$

$$= \frac{9}{(6-x)^2}$$

$$= 9(6-x)^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = -18(6-x)^{-3} \times (-1) \quad \{\text{chain rule}\}$$

$$= \frac{18}{(6-x)^3}$$

$$\mathbf{25} \quad f(x) = \ln(\cos x)$$

$$\mathbf{a} \quad f\left(\frac{\pi}{4}\right) = \ln\left(\cos \frac{\pi}{4}\right)$$

$$= \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$= \ln\left(2^{-\frac{1}{2}}\right)$$

$$= -\frac{1}{2} \ln 2$$

$$\mathbf{b} \quad f'(x) = \frac{-\sin x}{\cos x} \quad \{\text{chain rule}\}$$

$$= -\tan x$$

$$\therefore f'\left(\frac{\pi}{4}\right) = -\tan \frac{\pi}{4}$$

$$= -1$$

$$\mathbf{c} \quad f'(x) = \frac{-\sin x}{\cos x} \quad \{\text{from b}\}$$

$$\therefore f''(x) = \frac{-\cos x(\cos x) - (-\sin x)(-\sin x)}{\cos^2 x} \quad \{\text{quotient rule}\}$$

$$= \frac{-\cos^2 x - \sin^2 x}{\cos^2 x}$$

$$= \frac{-1}{\cos^2 x}$$

$$\therefore f''\left(\frac{\pi}{4}\right) = \frac{-1}{\cos^2\left(\frac{\pi}{4}\right)}$$

$$= \frac{-1}{\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= -2$$

$$\mathbf{26} \quad \mathbf{a} \quad y = x^2 + 2x - 5$$

$$\therefore \frac{dy}{dx} = 2x + 2$$

$$\begin{aligned}
 \text{Now when } x = 1, \quad y &= 1^2 + 2(1) - 5 \quad \text{and} \quad \frac{dy}{dx} = 2(1) + 2 \\
 &= -2 \qquad \qquad \qquad = 4
 \end{aligned}$$

$\therefore$  the point of contact is  $(1, -2)$  and the gradient of the tangent is 4.

$\therefore$  the tangent has equation  $y = 4(x - 1) - 2$

which is  $y = 4x - 6$ .



$$\begin{aligned}
 \mathbf{b} \quad y &= 3 - \frac{2}{x} \\
 &= 3 - 2x^{-1} \\
 \therefore \frac{dy}{dx} &= -2(-x^{-2}) \\
 &= \frac{2}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now when } x = -2, \quad y &= 3 - \frac{2}{-2} \quad \text{and} \quad \frac{dy}{dx} = \frac{2}{(-2)^2} \\
 &= 4 \qquad \qquad \qquad = \frac{1}{2}
 \end{aligned}$$

$\therefore$  the point of contact is  $(-2, 4)$  and the gradient of the tangent is  $\frac{1}{2}$ .

$$\begin{aligned}
 \therefore \text{ the tangent has equation } y &= \frac{1}{2}(x + 2) + 4 \\
 \text{which is } y &= \frac{1}{2}x + 5.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{27} \quad \mathbf{a} \quad g(x) &= -x \cos x \\
 \therefore g'(x) &= (-1) \cos x - x(-\sin x) \quad \{\text{product rule}\} \\
 &= -\cos x + x \sin x
 \end{aligned}$$

$$\mathbf{b} \quad \text{Since } g\left(\frac{\pi}{3}\right) = -\frac{\pi}{3} \cos \frac{\pi}{3} = -\frac{\pi}{3} \times \frac{1}{2} = -\frac{\pi}{6}, \text{ the point of contact is } \left(\frac{\pi}{3}, -\frac{\pi}{6}\right).$$

$$\begin{aligned}
 \text{Now } g'\left(\frac{\pi}{3}\right) &= -\cos \frac{\pi}{3} + \frac{\pi}{3} \sin \frac{\pi}{3} \\
 &= -\frac{1}{2} + \frac{\pi}{3} \times \frac{\sqrt{3}}{2} \\
 &= -\frac{1}{2} + \frac{\pi\sqrt{3}}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{The tangent has equation } y &= g'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + g\left(\frac{\pi}{3}\right) \\
 \therefore y &= \left(-\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\right)\left(x - \frac{\pi}{3}\right) - \frac{\pi}{6} \\
 \therefore y &= \left(-\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\right)x + \frac{\pi}{6} - \frac{\pi^2\sqrt{3}}{6} - \frac{\pi}{6} \\
 \therefore y &= \left(-\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\right)x - \frac{\pi^2\sqrt{3}}{6}
 \end{aligned}$$

$$\mathbf{28} \quad \text{Since } f(2) = \ln(2(2) + 3) = \ln 7, \text{ the point of contact is } (2, \ln 7).$$

$$\text{Now } f'(x) = \frac{2}{2x+3}, \text{ so at } x = 2 \text{ the tangent has gradient } f'(2) = \frac{2}{2(2)+3} = \frac{2}{7}.$$

$$\begin{aligned}
 \text{The tangent has equation } y &= f'(2)(x - 2) + f(2) \\
 \therefore y &= \frac{2}{7}(x - 2) + \ln 7 \\
 \therefore y &= \frac{2}{7}x - \frac{4}{7} + \ln 7
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{29} \quad \mathbf{a} \quad f(x) &= -x^2 + 4x \\
 \therefore f'(x) &= -2x + 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Since } f(k) &= -k^2 + 4k, \text{ the point of contact is } (k, -k^2 + 4k). \\
 \text{Now } f'(k) &= -2k + 4.
 \end{aligned}$$

$$\begin{aligned}
 \text{The tangent has equation } y &= f'(k)(x - k) + f(k) \\
 \therefore y &= (-2k + 4)(x - k) - k^2 + 4k \\
 \therefore y &= (-2k + 4)x + 2k^2 - 4k - k^2 + 4k \\
 \therefore y &= (-2k + 4)x + k^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \text{The tangent passes through } (4, 9), \text{ so } &(-2k + 4)(4) + k^2 = 9 \\
 \therefore -8k + 16 + k^2 &= 9 \\
 \therefore k^2 - 8k + 7 &= 0 \\
 \therefore (k - 1)(k - 7) &= 0 \\
 \therefore k &= 1 \text{ or } 7
 \end{aligned}$$

The gradient of the tangent is positive, so  $-2k + 4 > 0$ .

$$\text{Now if } k = 1, \quad -2k + 4 = -2 + 4 = 2 \quad \checkmark$$

$$\text{if } k = 7, \quad -2k + 4 = -14 + 4 = -10 \quad \times$$

So  $k = 1$ .



$$30 \quad y = x^3 + 2x + 1$$

$$\therefore \frac{dy}{dx} = 3x^2 + 2$$

$$\begin{aligned} \text{Now when } x = -1, \quad y &= (-1)^3 + 2(-1) + 1 \quad \text{and} \quad \frac{dy}{dx} = 3(-1)^2 + 2 \\ &= -1 - 2 + 1 \quad \quad \quad = 3 + 2 \\ &= -2 \quad \quad \quad = 5 \end{aligned}$$

$\therefore$  the point of contact is  $(-1, -2)$  and the gradient of the tangent is 5.

$$\begin{aligned} \therefore \text{ the tangent has equation } y &= 5(x + 1) - 2 \\ \text{which is } y &= 5x + 3. \end{aligned}$$

$$\text{The curve meets the tangent again when } x^3 + 2x + 1 = 5x + 3$$

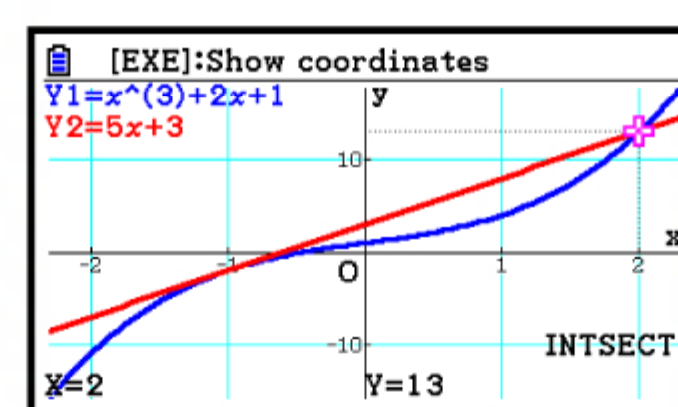
$$\therefore x^3 - 3x - 2 = 0$$

$$\therefore (x + 1)^2(x - 2) = 0 \quad \{(x + 1)^2 \text{ must be a factor of this cubic}\}$$

$$\therefore x = -1 \text{ or } 2$$

$$\text{When } x = 2, \quad y = 2^3 + 2(2) + 1 = 13$$

$\therefore$  the tangent meets the curve again at  $(2, 13)$ .



$$31 \quad \mathbf{a} \quad y = \frac{a}{x} - x^2 + 1 = ax^{-1} - x^2 + 1$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -ax^{-2} - 2x \\ &= \frac{-a}{x^2} - 2x \end{aligned}$$

Now the gradient of the tangent at  $x = 2$  is  $-5$ .

$$\therefore \frac{-a}{2^2} - 2(2) = -5$$

$$\therefore \frac{-a}{4} - 4 = -5$$

$$\therefore \frac{-a}{4} = -1$$

$$\therefore a = 4$$

$$\mathbf{b} \quad y = \frac{a}{x} - x^2 + 1 = \frac{4}{x} - x^2 + 1 \quad \{\text{from a}\}$$

$$\begin{aligned} \text{When } x = 2, \quad y &= \frac{4}{2} - 2^2 + 1 \\ &= 2 - 4 + 1 \\ &= -1 \end{aligned}$$

So, the point of contact is  $(2, -1)$ .

$$\text{The equation of tangent is } y = -5(x - 2) + (-1)$$

$$\therefore y = -5x + 10 - 1$$

$$\therefore y = -5x + 9$$

$$32 \quad \mathbf{a} \quad f(x) = x^3 - 2x$$

$$\therefore f'(x) = 3x^2 - 2$$

$$\text{Now } f(1) = 1^3 - 2(1) = -1 \text{ and}$$

$$f'(1) = 3(1)^2 - 2 = 1$$

$\therefore$  the point of contact is  $(1, -1)$  and the gradient of the normal is  $-1$ .

$$\begin{aligned} \therefore \text{ the equation of the normal is } y &= -(x - 1) - 1 \\ \text{which is } y &= -x. \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f(x) &= \frac{3}{x} - \frac{6}{x^2} \\ &= 3x^{-1} - 6x^{-2} \end{aligned}$$

$$\begin{aligned} \therefore f'(x) &= -3x^{-2} + 12x^{-3} \\ &= -\frac{3}{x^2} + \frac{12}{x^3} \end{aligned}$$

$$\text{Now } f(2) = \frac{3}{2} - \frac{6}{4} = 0 \text{ and } f'(2) = -\frac{3}{4} + \frac{12}{8} = \frac{3}{4}$$

$\therefore$  the point of contact is  $(2, 0)$  and the gradient of the normal is  $-\frac{4}{3}$ .

$$\begin{aligned} \therefore \text{ the equation of the normal is } y &= -\frac{4}{3}(x - 2) + 0 \\ \text{which is } y &= -\frac{4}{3}x + \frac{8}{3}. \end{aligned}$$



**33 a**  $f(x) = ax^2 + bx$

**i** When  $x = 2$ ,  $2 + 3y = -4$   
 $\therefore 3y = -6$   
 $\therefore y = -2$

$\therefore f(2) = -2$   
 $\therefore a(2)^2 + b(2) = -2$   
 $\therefore 4a + 2b = -2$

**b** Solving the system of equations  $\begin{cases} 4a + 2b = -2 \\ 4a + b = 3 \end{cases}$   
 simultaneously gives  $a = 2$ ,  $b = -5$ .

**c** From **a ii**, the normal  $L_1$  has gradient  $-\frac{1}{3}$ .

$\therefore$  the tangent  $L_2$  has gradient  $-\frac{1}{3}$ .

Now from **b**,  $f(x) = 2x^2 - 5x$  and  $f'(x) = 4x - 5$ .

$\therefore$  the tangent has gradient  $-\frac{1}{3}$  when  $4x - 5 = -\frac{1}{3}$   
 $\therefore 12x - 15 = -1$   
 $\therefore 12x = 14$   
 $\therefore x = \frac{14}{12} = \frac{7}{6}$

$f\left(\frac{7}{6}\right) = 2\left(\frac{7}{6}\right)^2 - 5\left(\frac{7}{6}\right) = -\frac{28}{9}$   
 $\therefore$  Q has coordinates  $\left(\frac{7}{6}, -\frac{28}{9}\right)$ .

**34 a i**  $f(x)$  is increasing for  $\frac{2}{3} \leq x \leq 2$ .

**ii**  $f(x)$  is decreasing for  $x \leq \frac{2}{3}$  and  $x \geq 2$ .

**b**  $f(x) = -x^3 + 4x^2 - 4x$   
 $\therefore f'(x) = -3x^2 + 8x - 4$

From the graph, the zeros of  $f'(x)$  should be  $\frac{2}{3}$  and 2.

Check:  $f'\left(\frac{2}{3}\right) = -3\left(\frac{2}{3}\right)^2 + 8\left(\frac{2}{3}\right) - 4$  and  $f'(2) = -3(2)^2 + 8(2) - 4$   
 $= -\frac{4}{3} + \frac{16}{3} - 4$   $= -12 + 16 - 4$   
 $= \frac{12}{3} - 4$   $= 0$  ✓  
 $= 0$  ✓

$f'(x)$  has sign diagram  $\begin{array}{c} \leftarrow - \quad + \quad - \rightarrow \\ \frac{2}{3} \quad 2 \end{array} \begin{array}{c} f'(x) \\ x \end{array}$

This is consistent with our observations in **a**.

**35 a**  $f(x) = 5 - 3x$

$\therefore f'(x) = -3$

$\therefore f'(x) < 0$  for all  $x$ .

$\therefore f(x)$  is decreasing for all  $x$ .

**b**  $f(x) = 2x^2 - 7x + 6$

$\therefore f'(x) = 4x - 7$

which is 0 when  $x = \frac{7}{4}$

$\therefore f'(x)$  has sign diagram  $\begin{array}{c} \leftarrow - \quad + \rightarrow \\ \frac{7}{4} \end{array} \begin{array}{c} f'(x) \\ x \end{array}$

So,  $f(x)$  is increasing for  $x \geq \frac{7}{4}$  and decreasing for  $x \leq \frac{7}{4}$ .

**ii**  $f'(x) = 2ax + b$

Now the normal has equation  $x + 3y = -4$

$\therefore 3y = -4 - x$

$\therefore y = -\frac{4}{3} - \frac{1}{3}x$

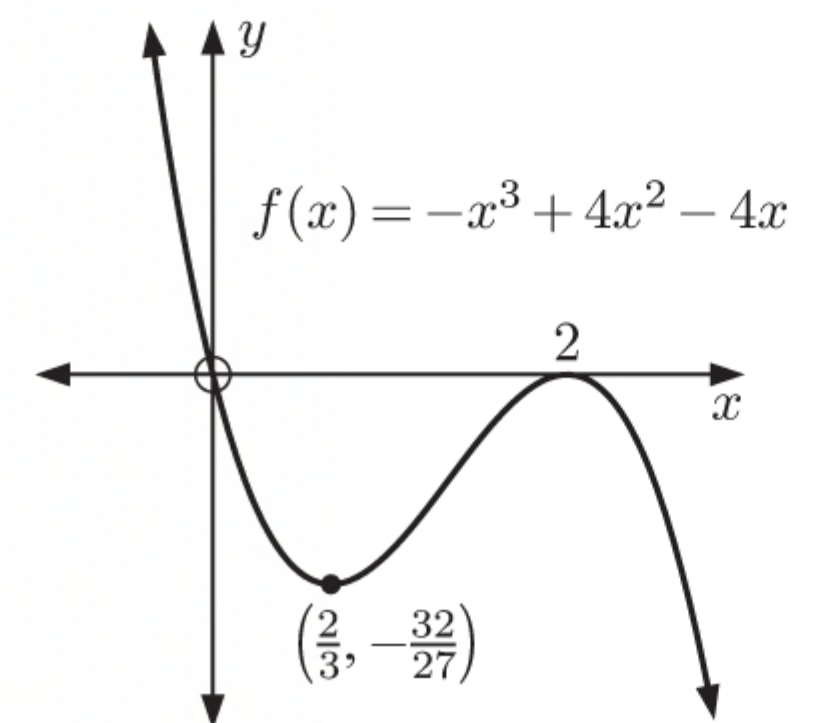
$\therefore$  the normal has gradient  $-\frac{1}{3}$  at  $x = 2$ .

$\therefore$  the tangent has gradient 3 at  $x = 2$

$\therefore f'(2) = 3$

$\therefore 2a(2) + b = 3$

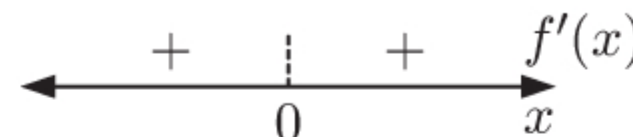
$\therefore 4a + b = 3$





**c**  $f(x) = -\frac{1}{x} = -x^{-1}$

$\therefore f'(x) = x^{-2} = \frac{1}{x^2}$  which has sign diagram



So,  $f(x)$  is increasing for all  $x \neq 0$ .

**d**  $f(x) = 2x^3 - 9x^2 + 7x + 6$

$\therefore f'(x) = 6x^2 - 18x + 7$

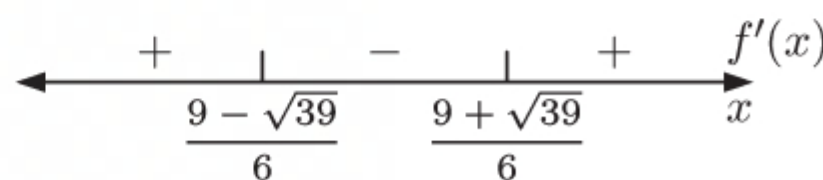
$f'(x) = 0$  when  $6x^2 - 18x + 7 = 0$

$$\therefore x = \frac{18 \pm \sqrt{(-18)^2 - 4(6)(7)}}{2(6)}$$

$$\therefore x = \frac{18 \pm \sqrt{156}}{12}$$

$$\therefore x = \frac{9 \pm \sqrt{39}}{6}$$

$f'(x)$  has sign diagram

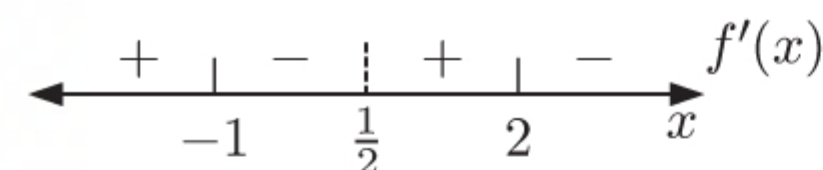


So,  $f(x)$  is increasing for  $x \leq \frac{9 - \sqrt{39}}{6}$  and  $x \geq \frac{9 + \sqrt{39}}{6}$ , and decreasing for  $\frac{9 - \sqrt{39}}{6} \leq x \leq \frac{9 + \sqrt{39}}{6}$ .

**36 a**  $f(x) = \ln\left(\frac{1-2x}{x^2+2}\right) = \ln(1-2x) - \ln(x^2+2)$

$$\begin{aligned} \therefore f'(x) &= \frac{-2}{1-2x} - \frac{2x}{x^2+2} \\ &= \frac{-2(x^2+2) - 2x(1-2x)}{(1-2x)(x^2+2)} \\ &= \frac{-2x^2 - 4 - 2x + 4x^2}{(x^2+2)(1-2x)} \\ &= \frac{2x^2 - 2x - 4}{(x^2+2)(1-2x)} \\ &= \frac{2(x-2)(x+1)}{(x^2+2)(1-2x)} \end{aligned}$$

**b** The sign diagram of  $f'(x)$  is



$f(x)$  is decreasing whenever  $f'(x) \leq 0$ .

But  $f(x)$  is undefined when  $1 - 2x \leq 0$  which is when  $x \geq \frac{1}{2}$ .

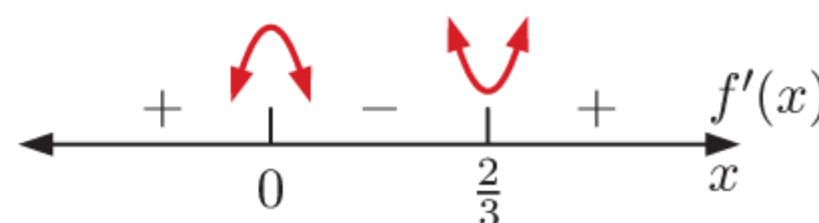
$\therefore f(x)$  is decreasing for  $-1 \leq x < \frac{1}{2}$ .

**37 a**  $f(x) = x^3 - x^2$

$\therefore f'(x) = 3x^2 - 2x$   
 $= x(3x - 2)$

$\therefore f'(x) = 0$  when  $x = 0$  or  $\frac{2}{3}$ .

$\therefore$  the sign diagram for  $f'(x)$  is



$$\begin{aligned} f(0) &= 0^3 - 0^2 = 0 \quad \text{and} \quad f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^2 \\ &= \frac{8}{27} - \frac{4}{9} \\ &= -\frac{4}{27} \end{aligned}$$

So, there is a local maximum at  $(0, 0)$  and a local minimum at  $\left(\frac{2}{3}, -\frac{4}{27}\right)$ .

**b**  $f(x) = x^4 - 2x^3 + 4x^2 - 8$

$\therefore f'(x) = 4x^3 - 6x^2 + 8x$

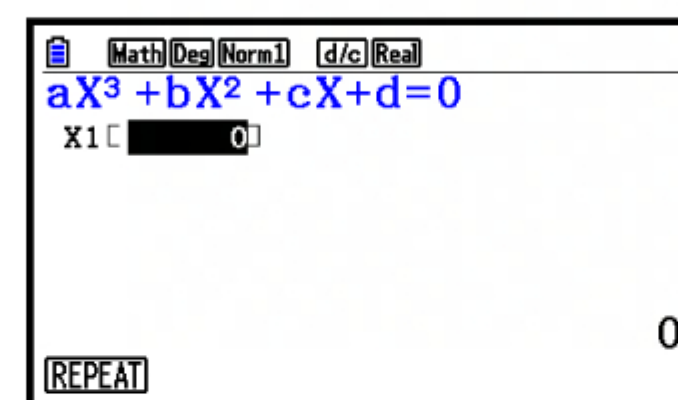
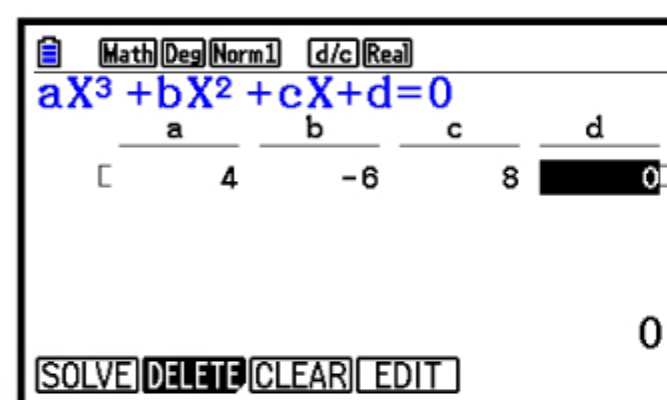
Using technology, the only real zero of  $f'(x)$  is 0.

$f'(x)$  has sign diagram



$f(0) = -8$

So, there is a local minimum at  $(0, -8)$ .





$$\begin{aligned}
 \text{c} \quad f(x) &= 2x + \frac{6}{x} \\
 &= 2x + 6x^{-1} \\
 \therefore f'(x) &= 2 - 6x^{-2} \\
 &= 2 - \frac{6}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{which is 0 when } 2 - \frac{6}{x^2} &= 0 \\
 \therefore \frac{6}{x^2} &= 2 \\
 \therefore 2x^2 &= 6 \\
 \therefore x^2 &= 3 \\
 \therefore x &= \pm\sqrt{3}
 \end{aligned}$$

$$f'(x) \text{ has sign diagram } \begin{array}{ccccccc} & + & & - & & - & + \\ & \downarrow & & \downarrow & & \downarrow & \\ & -\sqrt{3} & & 0 & & \sqrt{3} & \\ & \leftarrow & & & & & \rightarrow \end{array} \begin{array}{c} f'(x) \\ x \end{array}$$

$$\begin{aligned}
 f(-\sqrt{3}) &= -2\sqrt{3} - \frac{6}{\sqrt{3}} \quad \text{and} \quad f(\sqrt{3}) = 2\sqrt{3} + \frac{6}{\sqrt{3}} \\
 &= -4\sqrt{3} \qquad \qquad \qquad = 4\sqrt{3}
 \end{aligned}$$

So, there is a local maximum at  $(-\sqrt{3}, -4\sqrt{3})$  and a local minimum at  $(\sqrt{3}, 4\sqrt{3})$ .

$$\begin{aligned}
 \text{38 a} \quad f(x) &= 2x^3 + ax + b \\
 \therefore f'(x) &= 6x^2 + a
 \end{aligned}$$

Now  $f(x)$  has a stationary point at  $(1, 1)$ .

$$\begin{aligned}
 \text{So, } f'(1) &= 0 \quad \text{and} \quad f(1) = 1 \\
 \therefore 6(1)^2 + a &= 0 \quad \therefore 2(1)^3 - 6(1) + b = 1 \\
 \therefore a &= -6 \quad \therefore 2 - 6 + b = 1 \\
 & \quad \therefore b = 5
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \text{From a, } f(x) &= 2x^3 - 6x + 5 \quad \text{and} \quad f'(x) = 6x^2 - 6 \\
 \text{Stationary points occur where } f'(x) &= 0
 \end{aligned}$$

$$\begin{aligned}
 \therefore 6x^2 - 6 &= 0 \\
 \therefore 6x^2 &= 6 \\
 \therefore x^2 &= 1 \\
 \therefore x &= \pm 1
 \end{aligned}$$

$f'(x)$  has sign diagram

$$\begin{array}{ccccccc} & + & & - & & + & \\ & \downarrow & & \downarrow & & \downarrow & \\ & -1 & & 1 & & & \\ & \leftarrow & & & & & \rightarrow \end{array} \begin{array}{c} f'(x) \\ x \end{array}$$

$$\begin{aligned}
 f(-1) &= 2(-1)^3 - 6(-1) + 5 \\
 &= -2 + 6 + 5 \\
 &= 9
 \end{aligned}$$

So,  $(1, 1)$  is a local minimum and  $(-1, 9)$  is a local maximum.

$$\begin{aligned}
 \text{39 a} \quad f(x) &= x^3 - 2x^2, \quad -1 \leq x \leq 1 \\
 \therefore f'(x) &= 3x^2 - 4x \\
 &= x(3x - 4)
 \end{aligned}$$

which is 0 when  $x = 0$  or  $\frac{4}{3}$ .

$$\text{The sign diagram of } f'(x) \text{ is } \begin{array}{ccccccc} & + & & - & & + & \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & \frac{4}{3} & & & \\ & \leftarrow & & & & & \rightarrow \end{array} \begin{array}{c} f'(x) \\ x \end{array}$$

$\therefore$  there is a local maximum at  $x = 0$ , and a local minimum at  $x = \frac{4}{3}$ .

Critical value ( $x$ )	$f(x)$
$-1$ (end point)	$-3$
$0$ (local maximum)	$0$
$1$ (end point)	$-1$

The greatest of these values is 0 when  $x = 0$ .

The least of these values is  $-3$  when  $x = -1$ .



$$\mathbf{b} \quad f(x) = x^2 - \frac{27}{x} = x^2 - 27x^{-1}, \quad -6 \leq x \leq -1$$

$$\therefore f'(x) = 2x + 27x^{-2}$$

$$\text{which is 0 when } 2x + 27x^{-2} = 0$$

$$\therefore 2x = -\frac{27}{x^2}$$

$$\therefore x^3 = -\frac{27}{2}$$

$$\therefore x = -\frac{3}{\sqrt[3]{2}}$$

The sign diagram of  $f'(x)$  is

$\therefore$  there is a local minimum at  $x = -\frac{3}{\sqrt[3]{2}}$ .

Critical value ( $x$ )	$f(x)$
-6 (end point)	40.5
$-\frac{3}{\sqrt[3]{2}}$ (local minimum)	$\approx 17.0$
-1 (end point)	28

The greatest of these values is 40.5 when  $x = -6$ .

The least of these values is  $\approx 17.0$  when  $x = -\frac{3}{\sqrt[3]{2}}$ .

$$\mathbf{c} \quad f(x) = x^3 - 6x^2 + 12x - 10, \quad 0 \leq x \leq 5$$

$$\therefore f'(x) = 3x^2 - 12x + 12$$

$$\text{which is 0 when } 3x^2 - 12x + 12 = 0$$

$$\therefore 3(x^2 - 4x + 4) = 0$$

$$\therefore 3(x - 2)^2 = 0$$

$$\therefore x = 2$$

The sign diagram of  $f'(x)$  is

$\therefore$  there is a stationary inflection at  $x = 2$ .

Critical value ( $x$ )	$f(x)$
0 (end point)	-10
5 (end point)	25

The greatest of these values is 25 when  $x = 5$ .

The least of these values is -10 when  $x = 0$ .

$$\mathbf{40} \quad \mathbf{a} \quad y = xe^{-x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (1)e^{-x} + x(-e^{-x}) \quad \{\text{product rule}\} \\ &= e^{-x} - xe^{-x} \end{aligned}$$

Stationary points occur where  $\frac{dy}{dx} = 0$

$$\therefore e^{-x} - xe^{-x} = 0$$

$$\therefore e^{-x}(1 - x) = 0$$

$$\therefore 1 - x = 0 \quad \{e^{-x} > 0\}$$

$$\therefore x = 1$$

$$\text{When } x = 1, y = (1)e^{-1} = \frac{1}{e}$$

Now  $f'(x)$  has sign diagram

So,  $(1, \frac{1}{e})$  is a local maximum.



$$\begin{aligned}
 \mathbf{b} \quad y &= \frac{x-3}{x^2-5} \\
 \therefore \frac{dy}{dx} &= \frac{(1)(x^2-5) - (2x)(x-3)}{(x^2-5)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{x^2-5-2x^2+6x}{(x^2-5)^2} \\
 &= \frac{-x^2+6x-5}{(x^2-5)^2}
 \end{aligned}$$

Stationary points occur where  $\frac{dy}{dx} = 0$

$$\begin{aligned}
 \therefore -x^2 + 6x - 5 &= 0 \\
 \therefore x^2 - 6x + 5 &= 0 \\
 \therefore (x-5)(x-1) &= 0 \\
 \therefore x &= 1 \text{ or } 5
 \end{aligned}$$

When  $x = 1$ ,  $y = \frac{1-3}{1^2-5} = \frac{-2}{-4} = \frac{1}{2}$ .

When  $x = 5$ ,  $y = \frac{5-3}{5^2-5} = \frac{2}{20} = \frac{1}{10}$ .

Now  $f'(x)$  has sign diagram

So,  $(1, \frac{1}{2})$  is a local minimum and  $(5, \frac{1}{10})$  is a local maximum.

$$\begin{aligned}
 \mathbf{41 \ a} \quad f(x) &= \frac{e^{3x}}{kx}, \quad k \neq 0 \\
 \therefore f'(x) &= \frac{3e^{3x}(kx) - ke^{3x}}{(kx)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{ke^{3x}(3x-1)}{k^2x^2} \\
 &= \frac{e^{3x}(3x-1)}{kx^2} \quad \{k \neq 0\}
 \end{aligned}$$

Stationary points occur where  $f'(x) = 0$

$$\begin{aligned}
 \therefore e^{3x}(3x-1) &= 0 \\
 \therefore 3x-1 &= 0 \quad \{e^{3x} > 0 \text{ for all } x\} \\
 \therefore x &= \frac{1}{3}
 \end{aligned}$$

So, the stationary point has  $x$ -coordinate  $\frac{1}{3}$ .

**b i** If the stationary point is a local minimum, then the sign diagram of  $f'(x)$  should be



This occurs when  $k > 0$ .

**ii** Using **b i**, if the stationary point is a local maximum, then  $k < 0$ .

**Note:** It is not possible for the stationary point to be an inflection point because the factor  $(3x-1)$  in  $f'(x)$  is raised to an odd power. So, the sign of  $f'(x)$  will always be different on either side of  $x = \frac{1}{3}$ .

**c** The stationary point has  $y$ -coordinate  $-\frac{e}{2}$ .

$$\therefore f\left(\frac{1}{3}\right) = -\frac{e}{2} \quad \{\text{using a}\}$$

$$\therefore \frac{e^{3(\frac{1}{3})}}{k(\frac{1}{3})} = -\frac{e}{2}$$

$$\therefore \frac{3e}{k} = -\frac{e}{2}$$

$$\therefore \frac{k}{2} = -3$$

$$\therefore k = -6$$

Since  $k < 0$ , the stationary point is a local maximum.



**d**  $g(x) = -f(2x)$

$$f(x) \xrightarrow[\text{reflection in } x\text{-axis}]{\text{horizontal stretch scale factor } \frac{1}{2}} -f(x) \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} -f(2x)$$

So, a reflection in the  $x$ -axis, followed by a horizontal stretch with scale factor  $\frac{1}{2}$  maps  $y = f(x)$  onto  $y = g(x)$ .

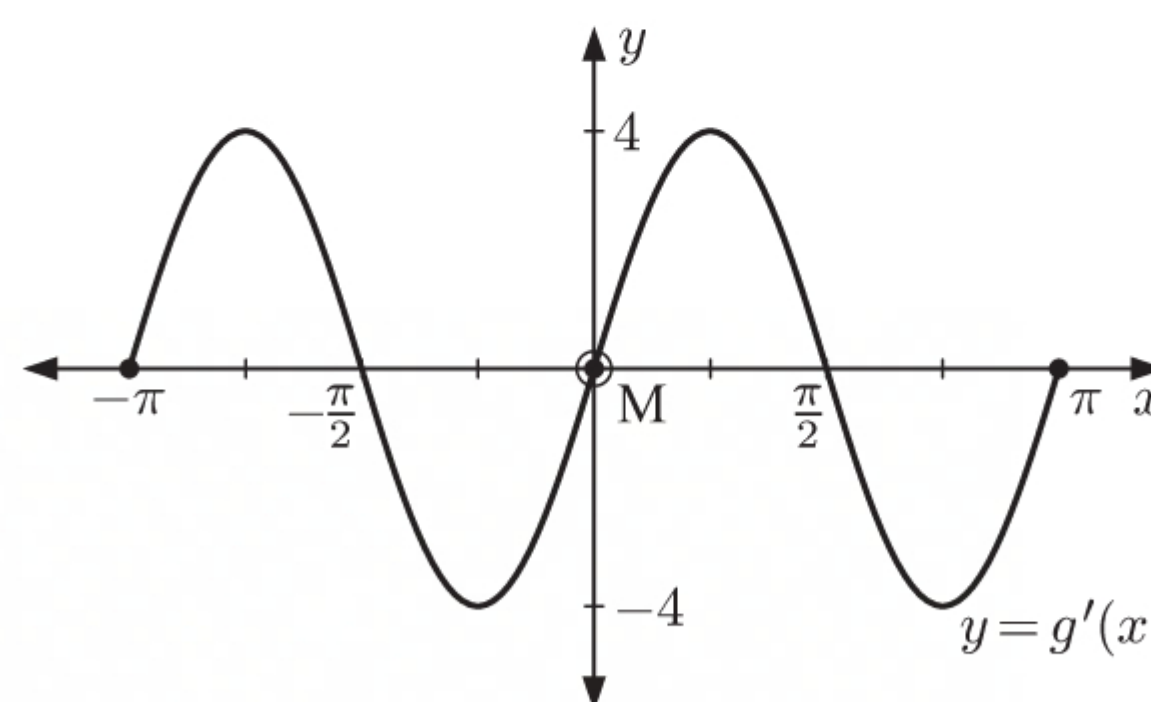
The stationary point of  $f(x)$  is  $(\frac{1}{3}, -\frac{e}{2})$ , so for the stationary point of  $g(x)$ :

$$\left(\frac{1}{3}, -\frac{e}{2}\right) \xrightarrow[\text{reflection in } x\text{-axis}]{\text{horizontal stretch scale factor } \frac{1}{2}} \left(\frac{1}{3}, \frac{e}{2}\right) \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} \left(\frac{1}{6}, \frac{e}{2}\right)$$

So, the stationary point of  $g(x)$  is  $(\frac{1}{6}, \frac{e}{2})$  which is a local minimum due to the reflection in the  $x$ -axis.

**42 a**  $g(x) = 3 - 2\cos 2x$   
 $\therefore g'(x) = -2(-2\sin 2x)$   
 $= 4\sin 2x$

**b, d**



**c** From **b**, the graph of  $y = g'(x)$  cuts the  $x$ -axis 5 times.

$\therefore$  there are 5 solutions to  $g'(x) = 0$  for  $-\pi \leq x \leq \pi$ .

**43 a**  $f(x) = x^2 + 3x + 5$   
 $\therefore f'(x) = 2x + 3$   
 $\therefore f''(x) = 2$

**i**  $f'(x) = 0$  when  $x = -\frac{3}{2}$

The sign diagram of  $f'(x)$  is

$$\begin{array}{c} \leftarrow - \quad | \quad + \rightarrow \\ \quad \quad -\frac{3}{2} \quad x \end{array} \quad f'(x)$$

So,  $f(x)$  is increasing for  $x \geq -\frac{3}{2}$ .

**ii** Using **i**,  $f(x)$  is decreasing for  $x \leq -\frac{3}{2}$ .

**iii**  $f''(x) = 2 > 0$  for all  $x \in \mathbb{R}$ .

So,  $f(x)$  is concave upwards for all  $x \in \mathbb{R}$ .

**iv** Using **iii**, there are no intervals in which  $f(x)$  is concave downwards.

**b**  $f(x) = e^{-x^2}$   
 $\therefore f'(x) = -2xe^{-x^2}$  {chain rule}  
 $\therefore f''(x) = -2e^{-x^2} - 2x(-2xe^{-x^2})$  {product rule}  
 $= -2e^{-x^2} + 4x^2e^{-x^2}$   
 $= 2e^{-x^2}(2x^2 - 1)$

**i**  $f'(x) = 0$  when  $x = 0$

The sign diagram of  $f'(x)$  is

$$\begin{array}{c} \leftarrow + \quad | \quad - \rightarrow \\ \quad \quad 0 \quad x \end{array} \quad f'(x)$$

So,  $f(x)$  is increasing for  $x \leq 0$ .

**ii** Using **i**,  $f(x)$  is decreasing for  $x \geq 0$ .

**iii**  $f''(x) = 0$  when  $2e^{-x^2}(2x^2 - 1) = 0$

$$\therefore 2x^2 - 1 = 0 \quad \left\{ e^{-x^2} > 0 \text{ for all } x \right\}$$

$$\therefore x^2 = \frac{1}{2}$$

$$\therefore x = \pm \frac{1}{\sqrt{2}}$$



The sign diagram for  $f''(x)$  is

So,  $f(x)$  is concave upwards for  $x \leq -\frac{1}{\sqrt{2}}$  and  $x \geq \frac{1}{\sqrt{2}}$ .

**iv** Using **iii**,  $f(x)$  is concave downwards for  $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$ .

**c**  $f(x) = x \ln(x^2)$

$$\therefore f'(x) = (1) \ln(x^2) + x \left( \frac{2x}{x^2} \right) \quad \{\text{product rule}\}$$

$$= \ln(x^2) + 2$$

$$\therefore f''(x) = \frac{2x}{x^2} = \frac{2}{x}$$

**i**  $f'(x) = 0$  when  $\ln(x^2) + 2 = 0$

$$\therefore \ln(x^2) = -2$$

$$\therefore x^2 = e^{-2}$$

$$\therefore x = \pm e^{-1} = \pm \frac{1}{e}$$

The sign diagram of  $f'(x)$  is

So,  $f(x)$  is increasing for  $x \leq -\frac{1}{e}$  and  $x \geq \frac{1}{e}$ .

**ii** Using **i**,  $f(x)$  is decreasing for  $-\frac{1}{e} \leq x < 0$  and  $0 < x \leq \frac{1}{e}$ .

**iii** The sign diagram of  $f''(x)$  is

So,  $f(x)$  is concave upwards for  $x > 0$ .

**iv** Using **iii**,  $f(x)$  is concave downwards for  $x < 0$ .

**44 a**  $f(x) = xe^{1-2x^2}$

$$\therefore f'(x) = (1)e^{1-2x^2} + x(-4xe^{1-2x^2}) \quad \{\text{product rule}\}$$

$$= e^{1-2x^2} - 4x^2e^{1-2x^2}$$

$$= e^{1-2x^2}(1 - 4x^2)$$

$$\therefore f''(x) = -4xe^{1-2x^2}(1 - 4x^2) + e^{1-2x^2}(-8x) \quad \{\text{product rule}\}$$

$$= -4xe^{1-2x^2}(1 - 4x^2 + 2)$$

$$= -4xe^{1-2x^2}(3 - 4x^2)$$

**b** Stationary points occur where  $f'(x) = 0$

$$\therefore e^{1-2x^2}(1 - 4x^2) = 0 \quad \{\text{using a}\}$$

$$\therefore 1 - 4x^2 = 0 \quad \left\{ e^{1-2x^2} > 0 \text{ for all } x \right\}$$

$$\therefore x^2 = \frac{1}{4}$$

$$\therefore x = \pm \frac{1}{2}$$

The sign diagram of  $f'(x)$  is

Now  $f(-\frac{1}{2}) = (-\frac{1}{2})e^{1-2(\frac{1}{4})} = -\frac{1}{2}e^{\frac{1}{2}} = -\frac{\sqrt{e}}{2}$  and

$$f(\frac{1}{2}) = (\frac{1}{2})e^{1-2(\frac{1}{4})} = \frac{1}{2}e^{\frac{1}{2}} = \frac{\sqrt{e}}{2}$$

So  $(-\frac{1}{2}, -\frac{\sqrt{e}}{2})$  is a local minimum and  $(\frac{1}{2}, \frac{\sqrt{e}}{2})$  is a local maximum.



**c** Inflection points occur where  $f''(x) = 0$

$$\therefore -4xe^{1-2x^2}(3-4x^2) = 0 \quad \{\text{using a}\}$$

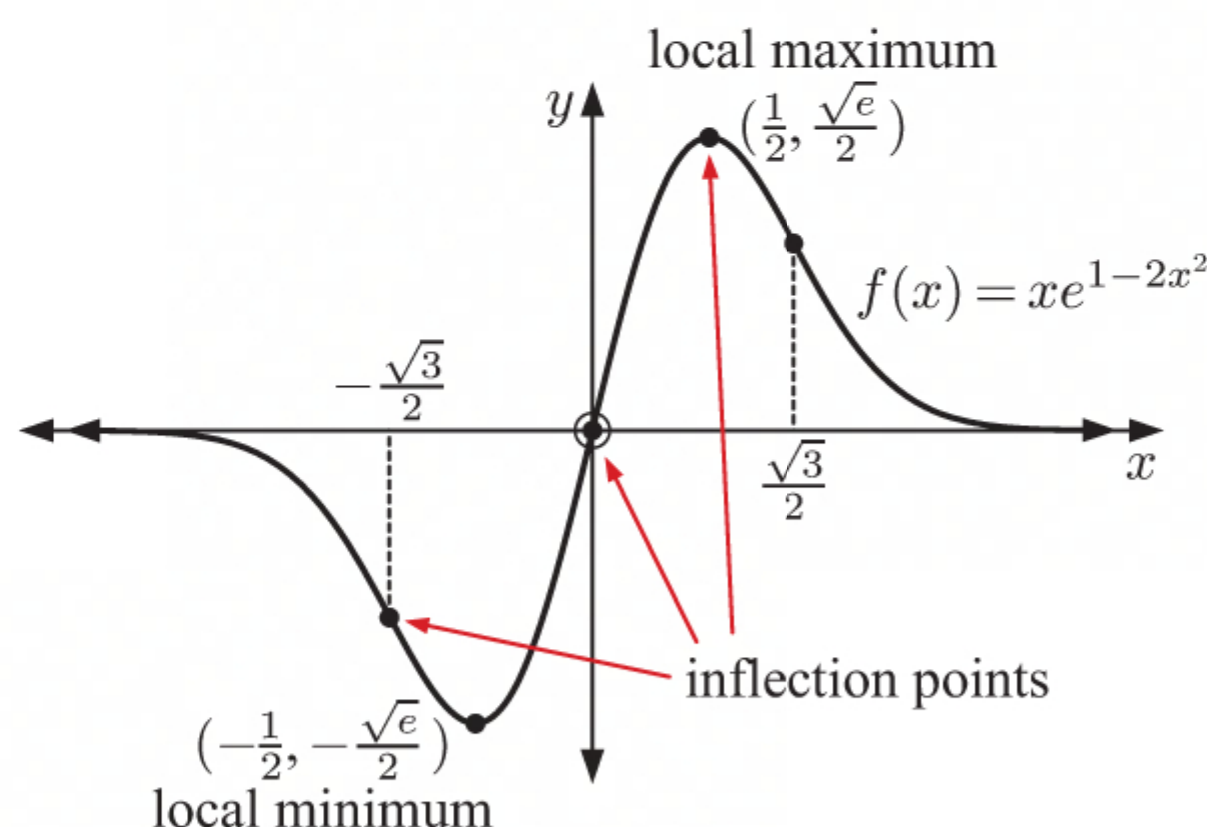
$$\therefore x(3-4x^2) = 0 \quad \left\{ e^{1-2x^2} > 0 \text{ for all } x \right\}$$

$$\therefore x = 0 \quad \text{or} \quad x^2 = \frac{3}{4}$$

$$\therefore x = \pm \frac{\sqrt{3}}{2}$$

So, the  $x$ -coordinates of the inflection points are  $0$ ,  $-\frac{\sqrt{3}}{2}$ , and  $\frac{\sqrt{3}}{2}$ .

**d**



**45 a**  $C(x) = -0.2x^2 + 4x + 10$  for  $0 \leq x \leq 10$

$$C(5) = -0.2(5)^2 + 4(5) + 10 = \$25$$

Hence it costs \$25 to produce 5 bracelets.

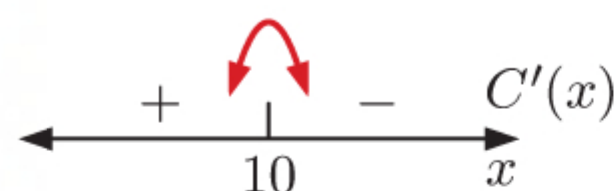
**c** If  $C$  is measured in dollars and  $x$  is the number of bracelets produced, then  $C'(x)$  has units “dollars per bracelet”.

**e**  $C'(x) = 0$  when  $-0.4x + 4 = 0$

$$\therefore -0.4x = -4$$

$$\therefore x = 10$$

$C'(x)$  has sign diagram:



$\therefore$  there is a maximum when  $x = 10$ .

$$C(10) = -0.2(10)^2 + 4(10) + 10 = 30$$

$\therefore$  the maximum value of  $C(x)$  on  $0 \leq x \leq 10$  is \$30.

**b**  $C'(x) = -0.2(2x) + 4$

$$= -0.4x + 4$$

**d**  $C'(5) = -0.4(5) + 4$

$$= 2 \text{ dollars per bracelet}$$

$\therefore$  when 5 bracelets are produced, the cost is increasing by \$2 per bracelet.

**46 a i** 2:30 pm is 30 minutes after 2 pm

$$H(30) = 10 \cos \pi + 200$$

$$= -10 + 200$$

$$= 190$$

$\therefore$  at 2:30 pm, the minute hand's tip is 190 cm above ground level.

**iii** 2:51 pm is 51 minutes after 2 pm

$$H(51) = 10 \cos\left(\frac{51}{30}\pi\right) + 200$$

$$\approx 206$$

$\therefore$  at 2:51 pm, the minute hand's tip is about 206 cm above ground level.

**b**  $H(t) = 10 \cos\left(\frac{\pi}{30}t\right) + 200$

$$\therefore H'(t) = -\frac{\pi}{3} \sin\left(\frac{\pi}{30}t\right)$$

$$\therefore H'(7) = -\frac{\pi}{3} \sin\left(\frac{7}{30}\pi\right) \approx -0.701$$

At 2:07 pm, the height of the minute hand's tip is decreasing at about 0.701 cm per minute.

**ii** 2:45 pm is 45 minutes after 2 pm

$$H(45) = 10 \cos \frac{3\pi}{2} + 200$$

$$= 200$$

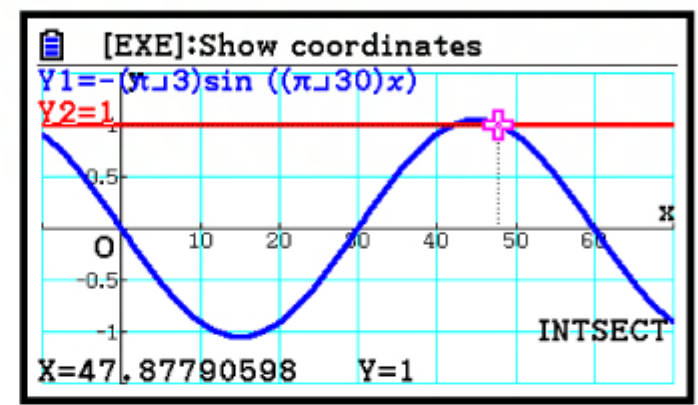
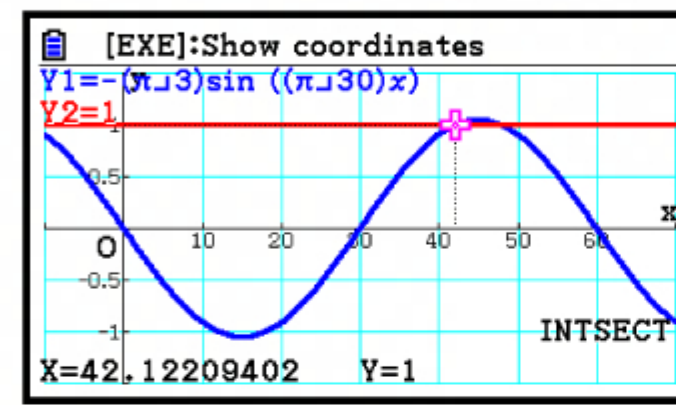
$\therefore$  at 2:45 pm, the minute hand's tip is 200 cm above ground level.



**c**  $H'(t) = 1$  when  $-\frac{\pi}{3} \sin\left(\frac{\pi}{30}t\right) = 1$

Using technology,  $t \approx 42.1$  or  $47.9$   $\{0 \leq t \leq 60\}$

$\therefore$  the height of the minute hand's tip is increasing at 1 cm per minute at about 2:42 pm and 2:48 pm.



**47**  $W(t) = 100e^{-\frac{t}{20}}$ ,  $t \geq 0$

**a**  $W(0) = 100e^0$   
 $= 100$

$\therefore$  the initial amount of radioactive substance present is 100 grams.

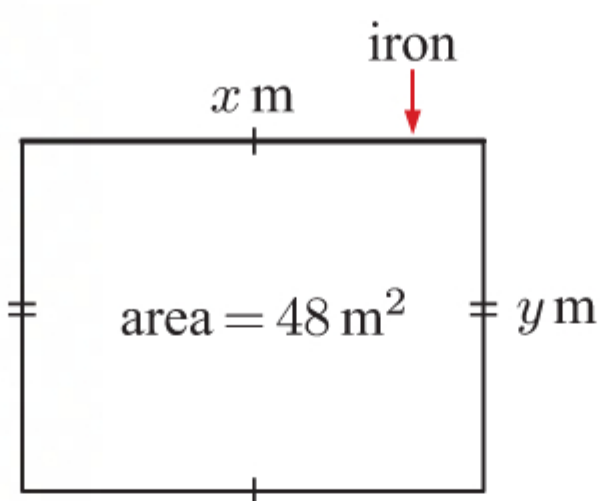
**c**  $W'(t) = 100\left(-\frac{1}{20}e^{-\frac{t}{20}}\right)$   
 $= -5e^{-\frac{t}{20}}$

Since  $e^{-\frac{t}{20}} > 0$  for all  $t$ ,  $W'(t) < 0$  for all  $t$ .

So, the weight of the radioactive substance is decreasing for all  $t \geq 0$ .

**e** As  $t \rightarrow \infty$ ,  $e^{-\frac{t}{20}} \rightarrow 0$   
 $\therefore W(t) \rightarrow 0$

**48 a**



Let the length of the side adjacent to the corrugated iron fence be  $y$  m.

Now  $\text{area} = xy$

$\therefore xy = 48$

$\therefore y = \frac{48}{x}$

Cost of fencing  $= 18(2y + x) + 30x$

$\therefore C = 18\left(\frac{2 \times 48}{x} + x\right) + 30x$

$\therefore C = \frac{2 \times 48 \times 18}{x} + 18x + 30x$

$\therefore C = \frac{2 \times 48 \times 18}{x} + 48x$

$\therefore C = 48\left(\frac{36}{x} + x\right)$  dollars

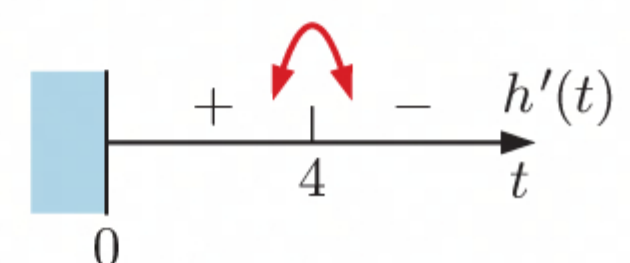
**49 a**  $h(t) = 100 + 32t - 4t^2$

$\therefore h'(t) = 32 - 8t$

**b**  $h'(t) = 0$  when  $32 = 8t$

$\therefore t = 4$

Sign diagram for  $h'(t)$ :



So, the height is maximised when  $t = 4$  seconds.

Now  $h(4) = 100 + 32(4) - 4(4)^2 = 164$

$\therefore$  the maximum height is 164 m.

**b**  $W(t) = \frac{100}{2} = 50$

$\therefore 100e^{-\frac{t}{20}} = 50$

$\therefore e^{-\frac{t}{20}} = \frac{50}{100} = \frac{1}{2}$

$\therefore -\frac{t}{20} = \ln\left(\frac{1}{2}\right) = -\ln 2$

$\therefore t = 20 \ln 2 \approx 13.9$

$\therefore$  it will take about 13.9 hours for half of the mass to decay.

**d**  $W'(3) = -5e^{-\frac{3}{20}} \approx -4.30$

The weight of the radioactive substance is decreasing at about 4.30 grams per day after 3 days.

**b**  $\frac{dC}{dx} = 48\left(\frac{-36}{x^2} + 1\right)$

Now  $C$  is minimised when  $\frac{dC}{dx} = 0$

$\therefore 48\left(\frac{-36}{x^2} + 1\right) = 0$

$\therefore \frac{-36}{x^2} + 1 = 0$

$\therefore \frac{36}{x^2} = 1$

$\therefore x^2 = 36$

$\therefore x = 6$   $\{x > 0\}$

The sign diagram of  $\frac{dC}{dx}$  is

$\therefore C$  is minimised when  $x = 6$ .

When  $x = 6$ ,  $y = \frac{48}{6} = 8$ .

The dimensions that minimise the cost of fencing are 6 m  $\times$  8 m, where one of the 6 m sides is fenced with corrugated iron.



- 50 a** Surface area = area of triangles + area of rectangles

$$= 2\left(\frac{1}{2}x^2\right) + 2(xy)$$

$$= x^2 + 2xy$$

$$\therefore x^2 + 2xy = 27$$

- b** From **a**,  $x^2 + 2xy = 27$

$$\therefore 2xy = 27 - x^2$$

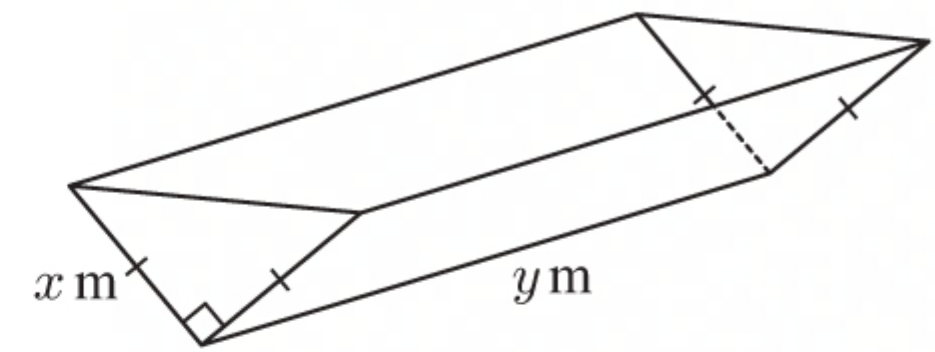
$$\therefore y = \frac{27}{2x} - \frac{x}{2}$$

Now volume = cross-sectional area  $\times$  depth

$$\therefore V = \frac{1}{2}x^2y$$

$$\therefore V = \frac{1}{2}x^2\left(\frac{27}{2x} - \frac{x}{2}\right)$$

$$\therefore V = \frac{27x}{4} - \frac{x^3}{4}$$



**c**  $\frac{dV}{dx} = \frac{27}{4} - \frac{3}{4}x^2$

Now  $V$  is maximised when  $\frac{dV}{dx} = 0$

$$\therefore \frac{27}{4} - \frac{3}{4}x^2 = 0$$

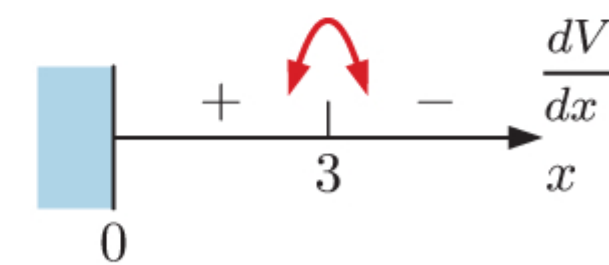
$$\therefore 27 - 3x^2 = 0$$

$$\therefore 3x^2 = 27$$

$$\therefore x^2 = 9$$

$$\therefore x = 3 \quad \{x > 0\}$$

The sign diagram of  $\frac{dV}{dx}$  is



$\therefore V$  is maximised when  $x = 3$ .

When  $x = 3$ ,  $y = \frac{27}{2(3)} - \frac{3}{2}$

$$= \frac{9}{2} - \frac{3}{2}$$

$$= \frac{6}{2}$$

$$= 3$$

So,  $V$  is maximised when  $x = y = 3$ .

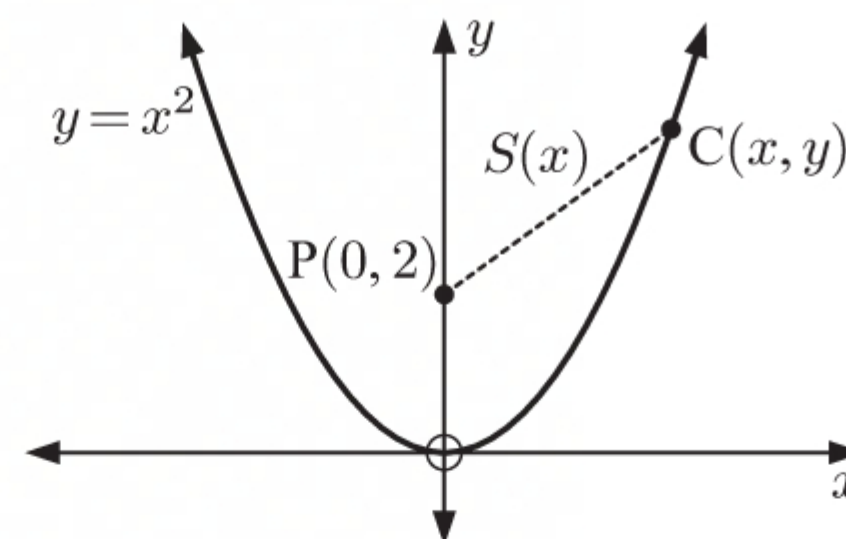
- 51 a**  $C$  has coordinates  $(x, x^2)$ .

Now  $S(x) = CP$

$$= \sqrt{(x-0)^2 + (x^2-2)^2}$$

$$= \sqrt{x^2 + x^4 - 4x^2 + 4}$$

$$= \sqrt{x^4 - 3x^2 + 4}$$



**b**  $S^2 = x^4 - 3x^2 + 4$

$$\therefore \frac{d}{dx}(S^2) = 4x^3 - 6x$$

which is 0 when  $4x^3 - 6x = 0$

$$\therefore 2x(2x^2 - 3) = 0$$

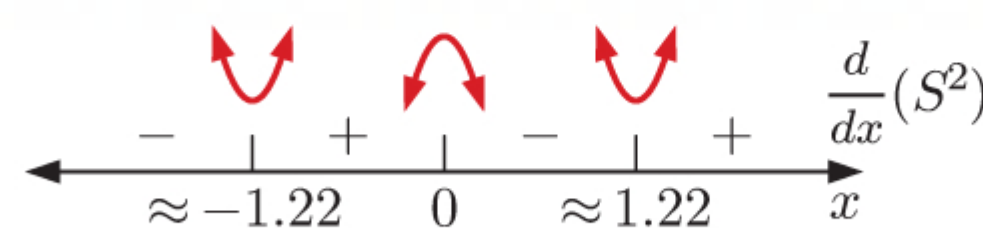
$$\therefore x = 0 \quad \text{or} \quad 2x^2 - 3 = 0$$

$$\therefore x^2 = \frac{3}{2}$$

$$\therefore x = \pm\sqrt{\frac{3}{2}}$$

$$\therefore x \approx -1.22 \quad \text{or} \quad 1.22$$

The sign diagram of  $\frac{d}{dx}(S^2)$  is



$\therefore$  there is a local maximum at  $x = 0$ , and local minima at  $x \approx -1.22$  and  $x \approx 1.22$ .

So,  $S^2$  is minimised when  $x \approx -1.22$  or  $x \approx 1.22$ .



<i>Critical point (x)</i>	$S^2$	$S$
-2 (end point)	8	$\approx 2.83$
$\approx -1.22$ (local minimum)	$\approx 1.75$	$\approx 1.32$
0 (local maximum)	4	2
$\approx 1.22$ (local minimum)	$\approx 1.75$	$\approx 1.32$
2 (end point)	8	$\approx 2.83$

The greatest distance between the comet and the observer is  $\approx 2.83$  units when  $x = \pm 2$ .

The shortest distance between the comet and the observer is  $\approx 1.32$  units when  $x = \pm 1.22$ .

**52** C has coordinates  $(x, \cos x)$ .

Let  $A$  be the area of rectangle ABCD.

$$\therefore A = 2x \cos x \text{ units}^2, \quad 0 \leq x \leq \frac{\pi}{2}$$

$$\begin{aligned} \text{Now } \frac{dA}{dx} &= 2 \cos x + 2x(-\sin x) \quad \{\text{product rule}\} \\ &= 2 \cos x - 2x \sin x \\ &= 2(\cos x - x \sin x) \end{aligned}$$

$$A \text{ is maximised when } \frac{dA}{dx} = 0$$

$$\therefore 2(\cos x - x \sin x) = 0$$

$$\therefore \cos x - x \sin x = 0$$

$$\therefore \cos x = x \sin x$$

$$\therefore x \approx 0.860 \quad \{\text{using technology}\}$$

The sign diagram of  $\frac{dA}{dx}$  is

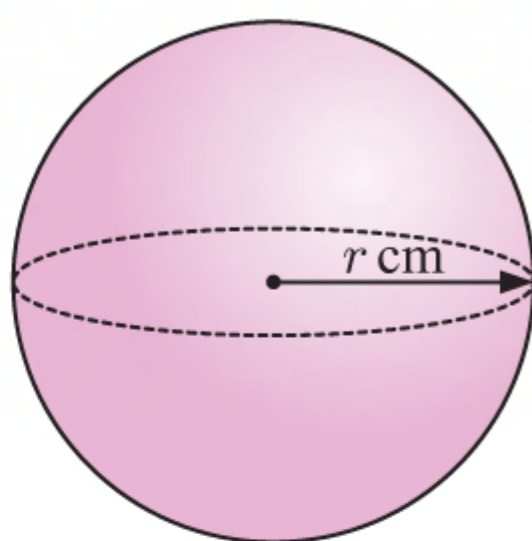
$\therefore A$  is maximised when  $x \approx 0.860$ .

$\therefore$  when ABCD has maximum area, C has coordinates  $(\approx 0.860, \approx \cos 0.860)$  which is  $(\approx 0.860, \approx 0.652)$ .

**53 a** The radius decreases at a constant rate of  $\frac{8 \text{ cm}}{5 \text{ min}} = 1.6 \text{ cm min}^{-1}$

$$\therefore \frac{dr}{dt} = -1.6$$

**b**



$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \therefore \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \end{aligned}$$

$$\text{But } \frac{dr}{dt} = -1.6$$

$$\therefore \frac{dV}{dt} = 4\pi r^2(-1.6) = -6.4\pi r^2$$

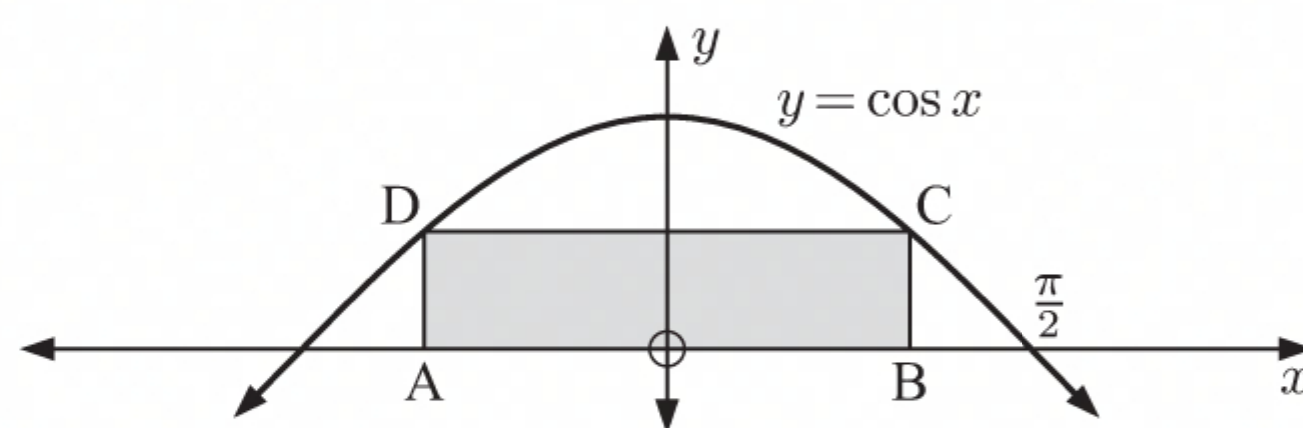
$$\text{At time } t = 2.5, \quad r = 8 - 1.6 \times 2.5 = 4 \text{ cm}$$

$$\begin{aligned} \therefore \frac{dV}{dt} &= -6.4\pi \times 4^2 \\ &\approx -322 \end{aligned}$$

$\therefore$  the volume is decreasing at about  $322 \text{ cm}^3 \text{ min}^{-1}$ .

$$\begin{aligned} \text{c The average change in volume} &= \frac{V(5) - V(1)}{5 - 1} \\ &= \frac{0 - \frac{4}{3}\pi(6.4)^3}{4} \quad \{\text{when } t = 1, r = 6.4\} \\ &= -\frac{\pi}{3}(6.4)^3 \\ &\approx -275 \end{aligned}$$

$\therefore$  on average, the volume is decreasing at about  $275 \text{ cm}^3 \text{ min}^{-1}$ .





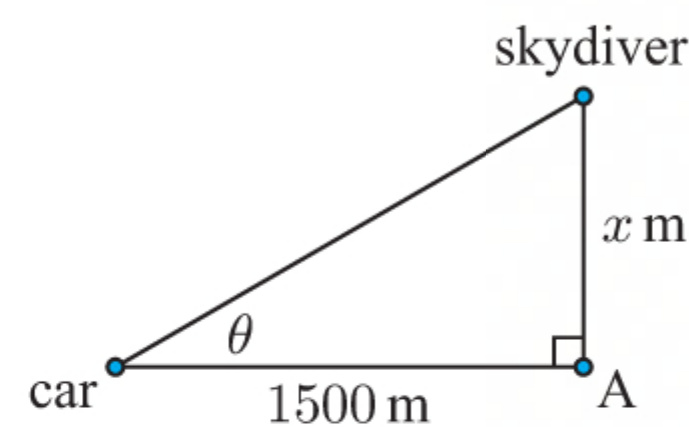
$$\begin{aligned}
 54 \quad \tan \theta &= \frac{x}{1500} \\
 \therefore x &= 1500 \tan \theta \\
 \therefore \frac{dx}{d\theta} &= \frac{1500}{\cos^2 \theta}
 \end{aligned}$$

$$\text{Now } \frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} \quad \{\text{chain rule}\}$$

$$\text{When } \theta = \frac{\pi}{6}, \quad \frac{dx}{dt} = -50 \quad \text{and} \quad \frac{dx}{d\theta} = \frac{1500}{\cos^2 \frac{\pi}{6}} = 2000$$

$$\therefore \frac{d\theta}{dt} = \frac{-50}{2000} = -0.025$$

So, the angle of elevation from the car to the skydiver is decreasing at 0.025 radians per minute when  $\theta = \frac{\pi}{6}$ .



$$55 \quad \mathbf{a} \quad \begin{array}{|c|c|c|} \hline i & x_i & f(x_i) \\ \hline 0 & 1 & 13 \\ 1 & 1.2 & 13.288 \\ 2 & 1.4 & 13.384 \\ 3 & 1.6 & 13.336 \\ 4 & 1.8 & 13.192 \\ 5 & 2 & 13 \\ \hline \end{array} \quad n = 5, \quad a = 1, \quad b = 2, \quad f(x) = x^3 - 6x^2 + 11x + 7$$

$$h = \frac{b-a}{n} = \frac{1}{5}, \quad x_i = 1 + \frac{1}{5}i$$

Using the trapezoidal rule, the area

$$\begin{aligned}
 &\approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)) \\
 &\approx 13.24 \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_1^2 f(x) dx &= \int_1^2 (x^3 - 6x^2 + 11x + 7) dx \\
 &= \left[ \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 + 7x \right]_1^2 \\
 &= \left( \frac{1}{4}(2)^4 - 2(2)^3 + \frac{11}{2}(2)^2 + 7(2) \right) - \left( \frac{1}{4}(1)^4 - 2(1)^3 + \frac{11}{2}(1)^2 + 7(1) \right) \\
 &= 24 - \frac{43}{4} \\
 &= \frac{53}{4} \\
 &= 13.25
 \end{aligned}$$

The actual area between  $y = f(x)$  and the  $x$ -axis for  $1 \leq x \leq 2$  is 13.25 units<sup>2</sup>.

$$\begin{aligned}
 \mathbf{c} \quad \text{Percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{|13.24 - 13.25|}{13.25} \times 100\% \\
 &= \frac{0.01}{13.25} \times 100\% \\
 &\approx 0.0755\%
 \end{aligned}$$

$$56 \quad \mathbf{a} \quad \frac{d}{dx}(x^2) = 2x$$

$\therefore$  the antiderivative of  $2x$  is  $x^2$ .

$$\mathbf{b} \quad \frac{d}{dx}(x^3) = 3x^2$$

$$\therefore \frac{d}{dx}\left(\frac{1}{9}x^3\right) = \frac{x^2}{3}$$

$\therefore$  the antiderivative of  $\frac{x^2}{3}$  is  $\frac{1}{9}x^3$ .

$$\mathbf{c} \quad \frac{3}{x^2} = 3x^{-2}$$

$$\frac{d}{dx}(x^{-1}) = -x^{-2}$$

$$\therefore \frac{d}{dx}(-3x^{-1}) = 3x^{-2}$$

$\therefore$  the antiderivative of  $\frac{3}{x^2}$  is  $-3x^{-1} = -\frac{3}{x}$ .



$$\begin{aligned}
 \mathbf{57} \quad \frac{d}{dx} \left( \frac{2}{x^2} - 3x \right) &= \frac{d}{dx} (2x^{-2} - 3x) \\
 &= -4x^{-3} - 3 \\
 &= -\frac{4}{x^3} - 3
 \end{aligned}$$

$$\therefore \frac{d}{dx} \left( -2 \left( \frac{2}{x^2} - 3x \right) \right) = \frac{8}{x^3} + 6$$

$$\therefore \text{the antiderivative of } \frac{8}{x^3} + 6 \text{ is } -2 \left( \frac{2}{x^2} - 3x \right) = -\frac{4}{x^2} + 6x.$$

$$\therefore \int \left( \frac{8}{x^3} + 6 \right) dx = -\frac{4}{x^2} + 6x + c$$

$$\mathbf{58} \quad \mathbf{a} \quad \int -3 dx = -3x + c$$

$$\begin{aligned}
 \mathbf{b} \quad \int \left( \frac{3}{x^2} + 2x^3 - 4 \right) dx &= \int (3x^{-2} + 2x^3 - 4) dx \\
 &= \frac{3}{-1} x^{-1} + \frac{2}{4} x^4 - 4x + c \\
 &= -\frac{3}{x} + \frac{1}{2} x^4 - 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int \left( \frac{1}{x} + 2x \right)^2 dx &= \int \left( \left( \frac{1}{x} \right)^2 + 2 \left( \frac{1}{x} \right) (2x) + (2x)^2 \right) dx \\
 &= \int \left( \frac{1}{x^2} + 4 + 4x^2 \right) dx \\
 &= \int (x^{-2} + 4 + 4x^2) dx \\
 &= \frac{1}{-1} x^{-1} + 4x + \frac{4}{3} x^3 + c \\
 &= -\frac{1}{x} + 4x + \frac{4}{3} x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{59} \quad \mathbf{a} \quad \frac{dy}{dx} &= 4x \\
 \therefore y &= \int 4x dx \\
 &= \frac{4}{2} x^2 + c \\
 &= 2x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{dy}{dx} &= x^2 + \frac{1}{2}x + \frac{1}{3} \\
 \therefore y &= \int \left( x^2 + \frac{1}{2}x + \frac{1}{3} \right) dx \\
 &= \frac{1}{3} x^3 + \frac{1}{2} \left( \frac{1}{2} x^2 \right) + \frac{1}{3} x + c \\
 &= \frac{1}{3} x^3 + \frac{1}{4} x^2 + \frac{1}{3} x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \frac{dy}{dx} &= \frac{3x^4 + 5}{x^3} \\
 &= 3x + 5x^{-3} \\
 \therefore y &= \int (3x + 5x^{-3}) dx \\
 &= \frac{3}{2} x^2 + \frac{5}{-2} x^{-2} + c \\
 &= \frac{3}{2} x^2 - \frac{5}{2x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{60} \quad \mathbf{a} \quad \int (x\sqrt{x} - 5 \cos x) dx &= \int (x^{\frac{3}{2}} - 5 \cos x) dx \\
 &= \frac{2}{5} x^{\frac{5}{2}} - 5 \sin x + c \\
 &= \frac{2}{5} x^2 \sqrt{x} - 5 \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int \left( \sin x + \frac{1}{\sqrt[3]{x}} \right) dx &= \int (\sin x + x^{-\frac{1}{3}}) dx \\
 &= -\cos x + \frac{3}{2} x^{\frac{2}{3}} + c \\
 &= -\cos x + \frac{3}{2} \sqrt[3]{x^2} + c
 \end{aligned}$$

$$\mathbf{61} \quad \mathbf{a} \quad \int (x-3)^2 dx = \frac{1}{3} (x-3)^3 + c$$

$$\begin{aligned}
 \mathbf{b} \quad \int \frac{x^2 + 3x + 5}{\sqrt[3]{x}} dx &= \int \frac{x^2 + 3x + 5}{x^{\frac{1}{3}}} dx \\
 &= \int (x^{\frac{5}{3}} + 3x^{\frac{2}{3}} + 5x^{-\frac{1}{3}}) dx \\
 &= \frac{3}{8} x^{\frac{8}{3}} + \frac{9}{5} x^{\frac{5}{3}} + \frac{15}{2} x^{\frac{2}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{62} \quad f'(x) &= 4x - 3x^2 \\
 \therefore f(x) &= \int (4x - 3x^2) dx \\
 &= \frac{4}{2} x^2 - x^3 + c \\
 &= 2x^2 - x^3 + c
 \end{aligned}$$

$$\text{Now } f(3) = -2, \text{ so } 2(3)^2 - 3^3 + c = -2$$

$$\therefore 18 - 27 + c = -2$$

$$\therefore -9 + c = -2$$

$$\therefore c = 7$$

$$\therefore f(x) = 2x^2 - x^3 + 7$$



$$\mathbf{63 \ a} \quad f''(x) = e^x + 2x - 1, \quad f'(0) = 4, \quad f(0) = 1$$

$$\begin{aligned} \therefore f'(x) &= \int (e^x + 2x - 1) dx \\ &= e^x + x^2 - x + c \end{aligned}$$

$$\begin{aligned} \text{Now } f'(0) &= 4, \quad \therefore e^0 + 0^2 - 0 + c = 4 \\ &\therefore 1 + c = 4 \\ &\therefore c = 3 \end{aligned}$$

$$\therefore f'(x) = e^x + x^2 - x + 3$$

$$\begin{aligned} \therefore f(x) &= \int (e^x + x^2 - x + 3) dx \\ &= e^x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x + d \end{aligned}$$

$$\text{Now } f(0) = 1,$$

$$\begin{aligned} \therefore e^0 + \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 + 3(0) + d &= 1 \\ \therefore 1 + d &= 1 \\ \therefore d &= 0 \end{aligned}$$

$$\therefore f(x) = e^x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x$$

$$\mathbf{b} \quad f''(x) = 2 + \sin x, \quad f'(\pi) = 1, \quad f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$\begin{aligned} \therefore f'(x) &= \int (2 + \sin x) dx \\ &= 2x - \cos x + c \end{aligned}$$

$$\begin{aligned} \text{Now } f'(\pi) &= 1, \quad \therefore 2\pi - \cos \pi + c = 1 \\ &\therefore 2\pi - (-1) + c = 1 \\ &\therefore 2\pi + 1 + c = 1 \\ &\therefore 2\pi + c = 0 \\ &\therefore c = -2\pi \end{aligned}$$

$$\therefore f'(x) = 2x - \cos x - 2\pi$$

$$\begin{aligned} \therefore f(x) &= \int (2x - \cos x - 2\pi) dx \\ &= x^2 - \sin x - 2\pi x + d \end{aligned}$$

$$\text{Now } f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4},$$

$$\begin{aligned} \therefore \frac{\pi^2}{4} - \sin \frac{\pi}{2} - 2\pi\left(\frac{\pi}{2}\right) + d &= \frac{\pi^2}{4} \\ \therefore -1 - \pi^2 + d &= 0 \\ \therefore d &= \pi^2 + 1 \end{aligned}$$

$$\therefore f(x) = x^2 - \sin x - 2\pi x + \pi^2 + 1$$

$$\mathbf{c} \quad f''(x) = \frac{2}{\sqrt{x}} + 3x = 2x^{-\frac{1}{2}} + 3x, \quad f(1) = -\frac{19}{3}, \quad f(4) = \frac{64}{3}$$

$$\begin{aligned} \therefore f'(x) &= \int (2x^{-\frac{1}{2}} + 3x) dx \\ &= 4x^{\frac{1}{2}} + \frac{3}{2}x^2 + c \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= \int (4x^{\frac{1}{2}} + \frac{3}{2}x^2 + c) dx \\ &= \frac{8}{3}x^{\frac{3}{2}} + \frac{x^3}{2} + cx + d \end{aligned}$$

$$\text{Now } f(1) = -\frac{19}{3}$$

$$\begin{aligned} \therefore \frac{8}{3}(1)^{\frac{3}{2}} + \frac{1^3}{2} + c(1) + d &= -\frac{19}{3} \\ \therefore \frac{8}{3} + \frac{1}{2} + c + d &= -\frac{19}{3} \\ \therefore \frac{19}{6} + c + d &= -\frac{19}{3} \\ \therefore c + d &= -\frac{19}{2} \quad \dots (1) \end{aligned}$$

and

$$f(4) = \frac{64}{3}$$

$$\begin{aligned} \therefore \frac{8}{3}(4)^{\frac{3}{2}} + \frac{4^3}{2} + c(4) + d &= \frac{64}{3} \\ \therefore \frac{64}{3} + \frac{64}{2} + 4c + d &= \frac{64}{3} \\ \therefore 32 + 4c + d &= 0 \\ \therefore 4c + d &= -32 \quad \dots (2) \end{aligned}$$

$$\begin{aligned} 4c + 4d &= -38 \quad \{(1) \times 4\} \\ -4c - d &= 32 \quad \{(2) \times -1\} \\ \hline \text{Adding, } 3d &= -6 \\ \therefore d &= -2 \end{aligned}$$

$$\begin{aligned} \text{Substituting } d = -2 \text{ into (1) gives } c - 2 &= -\frac{19}{2} \\ \therefore c &= -\frac{15}{2} \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= \frac{8}{3}x^{\frac{3}{2}} + \frac{1}{2}x^3 - \frac{15}{2}x - 2 \\ &= \frac{8}{3}x\sqrt{x} + \frac{1}{2}x^3 - \frac{15}{2}x - 2 \end{aligned}$$

$$\begin{aligned} \mathbf{64 \ a} \quad &\int (3x - 5)^3 dx \\ &= \frac{1}{3} \frac{(3x - 5)^4}{4} + c \\ &= \frac{1}{12} (3x - 5)^4 + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad &\int \frac{2}{\sqrt{4-x}} dx \\ &= \int 2(4-x)^{-\frac{1}{2}} dx \\ &= 2 \times \frac{1}{-\frac{1}{2}} \frac{(4-x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= -4(4-x)^{\frac{1}{2}} + c \\ &= -4\sqrt{4-x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad &\int (e^{2x} + 3e^{-x+2}) dx \\ &= \frac{1}{2}e^{2x} + \frac{3}{-1}e^{-x+2} + c \\ &= \frac{1}{2}e^{2x} - 3e^{-x+2} + c \end{aligned}$$



$$\begin{aligned}
 \mathbf{65} \quad \mathbf{a} \quad & \int (2 \sin(x-3) + e^{3x}) dx \\
 &= 2(-\cos(x-3)) + \frac{1}{3}e^{3x} + c \\
 &= -2 \cos(x-3) + \frac{1}{3}e^{3x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{2}{5x-1} dx \\
 &= \frac{2}{5} \ln|5x-1| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \cos(5-7x) dx \\
 &= -\frac{1}{7} \sin(5-7x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{12}{\sqrt[3]{x}} dx \\
 &= \int 12x^{-\frac{1}{3}} dx \\
 &= 12 \times \frac{3}{2} x^{\frac{2}{3}} + c \\
 &= 18x^{\frac{2}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int (e^x - 4)^2 dx \\
 &= \int (e^{2x} - 8e^x + 16) dx \\
 &= \frac{1}{2}e^{2x} - 8e^x + 16x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int \left( 5 \sin x + \frac{3}{\cos^2 x} \right) dx \\
 &= -5 \cos x + 3 \tan x + c
 \end{aligned}$$

$$\mathbf{66} \quad f'(x) = \frac{4}{5-x}, \quad f(4) = 6$$

$$\begin{aligned}
 \therefore f(x) &= \int \frac{4}{5-x} dx \\
 &= -4 \ln|5-x| + c
 \end{aligned}$$

$$\text{Now } f(4) = 6, \quad \therefore -4 \ln|5-4| + c = 6$$

$$\therefore -4 \ln 1 + c = 6$$

$$\therefore c = 6$$

$$\therefore f(x) = -4 \ln|5-x| + 6$$

$$\begin{aligned}
 \mathbf{67} \quad \mathbf{a} \quad & \int 3x^2(5+x^3)^4 dx \\
 &= \int u^4 \frac{du}{dx} dx \quad \left\{ u = 5+x^3 \quad \therefore \frac{du}{dx} = 3x^2 \right\} \\
 &= \int u^4 du \\
 &= \frac{1}{5} u^5 + c \\
 &= \frac{1}{5} (5+x^3)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \frac{e^{\frac{1}{x}}}{x^2} dx \\
 &= \int e^u \left( -\frac{du}{dx} \right) dx \quad \left\{ u = \frac{1}{x} \quad \therefore \frac{du}{dx} = -\frac{1}{x^2} \right\} \\
 &= \int -e^u du \\
 &= -e^u + c \\
 &= -e^{\frac{1}{x}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{68} \quad \mathbf{a} \quad & \int \sqrt{x^2+3x-1} (2x+3) dx \\
 &= \int \sqrt{u} \frac{du}{dx} dx \\
 &\quad \left\{ u = x^2+3x-1 \quad \therefore \frac{du}{dx} = 2x+3 \right\} \\
 &= \int u^{\frac{1}{2}} du \\
 &= \frac{2}{3} u^{\frac{3}{2}} + c \\
 &= \frac{2}{3} (x^2+3x-1)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int x e^{x^2+2} dx \\
 &= \int e^u \times \frac{1}{2} \frac{du}{dx} dx \quad \left\{ u = x^2+2 \quad \therefore \frac{du}{dx} = 2x \right\} \\
 &= \int \frac{1}{2} e^u du \\
 &= \frac{1}{2} e^u + c \\
 &= \frac{1}{2} e^{x^2+2} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{2(\ln x)^2}{x} dx \\
 &= \int 2u^2 \frac{du}{dx} dx \quad \left\{ u = \ln x \quad \therefore \frac{du}{dx} = \frac{1}{x} \right\} \\
 &= \int 2u^2 du \\
 &= \frac{2}{3} u^3 + c \\
 &= \frac{2}{3} (\ln x)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{e^x+2}{e^x+2x} dx \\
 &= \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \left\{ u = e^x+2x \quad \therefore \frac{du}{dx} = e^x+2 \right\} \\
 &= \int \frac{1}{u} du \\
 &= \ln|u| + c \\
 &= \ln|e^x+2x| + c
 \end{aligned}$$



$$\begin{aligned}
 \text{c} \quad & \int \frac{6-8x}{2x^2-3x+2} dx \\
 &= \int \frac{1}{u} \left( -2 \frac{du}{dx} \right) dx \quad \left\{ u = 2x^2 - 3x + 2 \quad \therefore \frac{du}{dx} = 4x - 3 \right\} \\
 &= \int -\frac{2}{u} du \\
 &= -2 \ln|u| + c \\
 &= -2 \ln|2x^2 - 3x + 2| + c
 \end{aligned}$$

$$\begin{aligned}
 69 \quad \text{a} \quad & \int_0^1 (x^2 + x) dx \\
 &= \left[ \frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_0^1 \\
 &= \left( \frac{1}{3} + \frac{1}{2} \right) - 0 \\
 &= \frac{5}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int_{-2}^1 (x^3 - 2x^2 - 4x + 9) dx \\
 &= \left[ \frac{1}{4}x^4 - \frac{2}{3}x^3 - 2x^2 + 9x \right]_{-2}^1 \\
 &= \left( \frac{1}{4} - \frac{2}{3} - 2 + 9 \right) - \left( \frac{(-2)^4}{4} - \frac{2}{3}(-2)^3 - 2(-2)^2 + 9(-2) \right) \\
 &= \frac{79}{12} - \left( -\frac{50}{3} \right) \\
 &= \frac{93}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int_3^5 \left( \frac{8}{x^2} + 3x \right) dx = \int_3^5 (8x^{-2} + 3x) dx \\
 &= \left[ -8x^{-1} + \frac{3}{2}x^2 \right]_3^5 \\
 &= \left( -\frac{8}{5} + \frac{3}{2} \times 25 \right) - \left( -\frac{8}{3} + \frac{3}{2} \times 9 \right) \\
 &= \frac{359}{10} - \frac{65}{6} \\
 &= \frac{376}{15}
 \end{aligned}$$

$$\begin{aligned}
 70 \quad \text{a} \quad & \int_{-1}^1 3x^2 dx = [x^3]_{-1}^1 \\
 &= 1^3 - (-1)^3 \\
 &= 1 + 1 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int_2^3 \frac{5x^3 - 2x}{x^5} dx = \int_2^3 (5x^{-2} - 2x^{-4}) dx \\
 &= \left[ -5x^{-1} + \frac{2}{3}x^{-3} \right]_2^3 \\
 &= \left( -\frac{5}{3} + \frac{2}{3} \times \frac{1}{27} \right) - \left( -\frac{5}{2} + \frac{2}{3} \times \frac{1}{8} \right) \\
 &= -\frac{133}{81} - \left( -\frac{29}{12} \right) \\
 &= \frac{251}{324}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int_{-3}^0 (1 - 3x)^2 dx = \int_{-3}^0 (1 - 6x + 9x^2) dx \\
 &= \left[ x - 3x^2 + 3x^3 \right]_{-3}^0 \\
 &= 0 - (-3 - 3(-3)^2 + 3(-3)^3) \\
 &= 0 - (-111) \\
 &= 111
 \end{aligned}$$

$$71 \quad \text{a} \quad \int_0^3 2^x dx \approx 10.09886529$$

$$\int_0^3 2^x dx \approx 10.10$$

$$\text{b} \quad \int_{-1.5}^{2.4} \frac{5x+20}{x+6} dx \approx 13.25845691$$

$$\int_{-1.5}^{2.4} \frac{5x+20}{x+6} dx \approx 13.26$$

$$\text{c} \quad \int_{-2}^{-1} \ln(-x) dx \approx 0.3862943611$$

$$\int_{-2}^{-1} \ln(-x) dx \approx 0.3863$$

$$\text{d} \quad \int_3^7 2\sqrt{x} e^{-2x} dx \approx 4.624433678 \times 10^{-3}$$

$$\int_3^7 2\sqrt{x} e^{-2x} dx \approx 0.004624$$



$$\begin{aligned}
\mathbf{72} \quad & \int_0^{\frac{\pi}{2}} (\sin 3x + 5 \cos x) dx \\
&= \left[ -\frac{1}{3} \cos 3x + 5 \sin x \right]_0^{\frac{\pi}{2}} \\
&= \left( -\frac{1}{3} \cos \frac{3\pi}{2} + 5 \sin \frac{\pi}{2} \right) - \left( -\frac{1}{3} \cos 0 + 5 \sin 0 \right) \\
&= 0 + 5 - \left( -\frac{1}{3} + 0 \right) \\
&= 5 + \frac{1}{3} \\
&= \frac{16}{3}
\end{aligned}$$

$$\begin{aligned}
\mathbf{73} \quad & \int_a^{2a} \sqrt{x} dx = 2 \\
& \therefore \int_a^{2a} x^{\frac{1}{2}} dx = 2 \\
& \therefore \left[ \frac{2}{3} x^{\frac{3}{2}} \right]_a^{2a} = 2 \\
& \therefore \frac{2}{3} (2a)^{\frac{3}{2}} - \frac{2}{3} a^{\frac{3}{2}} = 2 \\
& \therefore 2^{\frac{3}{2}} \times a^{\frac{3}{2}} - a^{\frac{3}{2}} = 3 \\
& \therefore a^{\frac{3}{2}} (2^{\frac{3}{2}} - 1) = 3 \\
& \therefore a^{\frac{3}{2}} = \frac{3}{2^{\frac{3}{2}} - 1} \\
& \therefore a = \left( \frac{3}{2^{\frac{3}{2}} - 1} \right)^{\frac{2}{3}}
\end{aligned}$$

$$\begin{aligned}
\mathbf{74} \quad & y = x\sqrt{4-x} = x(4-x)^{\frac{1}{2}} \\
& \therefore \frac{dy}{dx} = (1)(4-x)^{\frac{1}{2}} + x \left( \frac{1}{2} (4-x)^{-\frac{1}{2}} (-1) \right) \quad \{\text{product rule}\} \\
&= \sqrt{4-x} - \frac{x}{2\sqrt{4-x}} \\
&= \frac{2(4-x) - x}{2\sqrt{4-x}} \\
&= \frac{8-2x-x}{2\sqrt{4-x}} \\
&= \frac{8-3x}{2\sqrt{4-x}} \quad \dots (*) \\
& \therefore \int_0^2 \frac{8-3x}{\sqrt{4-x}} dx = [x\sqrt{4-x}]_0^2 \quad \{\text{using } (*)\} \\
&= 2\sqrt{4-2} - 0\sqrt{4-0} \\
&= 2\sqrt{2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{75} \quad \mathbf{a} \quad & \int_1^5 \frac{2x^3+1}{x^2} dx = \int_1^5 (2x + x^{-2}) dx \\
&= \left[ x^2 - \frac{1}{x} \right]_1^5 \\
&= \left( 25 - \frac{1}{5} \right) - (1 - 1) \\
&= \frac{124}{5}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad & \int_{-1}^1 e^x (2 - 3e^{-x})^2 dx \\
&= \int_{-1}^1 e^x (4 - 12e^{-x} + 9e^{-2x}) dx \\
&= \int_{-1}^1 (4e^x - 12 + 9e^{-x}) dx \\
&= [4e^x - 12x - 9e^{-x}]_{-1}^1 \\
&= (4e^1 - 12 - 9e^{-1}) - (4e^{-1} + 12 - 9e^1) \\
&= 13e - 24 - 13e^{-1} \\
&= 13 \left( e - \frac{1}{e} \right) - 24
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad & \int_0^2 \frac{3}{5-2x} dx = \left[ \frac{3}{-2} \ln |5-2x| \right]_0^2 \\
&= -\frac{3}{2} \ln |5-4| + \frac{3}{2} \ln |5-0| \\
&= -\frac{3}{2} \ln 1 + \frac{3}{2} \ln 5 \\
&= \frac{3}{2} \ln 5
\end{aligned}$$

$$\begin{aligned}
\mathbf{d} \quad & \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{4}{\cos^2\left(\frac{x}{2}\right)} dx = \left[ 8 \tan \frac{x}{2} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= 8 \tan \frac{\pi}{4} - 8 \tan \frac{\pi}{6} \\
&= 8 - \frac{8}{\sqrt{3}}
\end{aligned}$$



**76 a** Let  $u = x^2 - 8 \quad \therefore \frac{du}{dx} = 2x$

When  $x = 3$ ,  $u = 1$

When  $x = 5$ ,  $u = 17$

$$\begin{aligned} \therefore \int_3^5 \frac{x}{x^2-8} dx &= \int_1^{17} \frac{1}{u} \left( \frac{1}{2} \frac{du}{dx} \right) dx \\ &= \frac{1}{2} \int_1^{17} \frac{1}{u} du \\ &= \frac{1}{2} [\ln |u|]_1^{17} \\ &= \frac{1}{2} (\ln 17 - \ln 1) \\ &= \frac{1}{2} \ln 17 \end{aligned}$$

**c** Let  $u = \sqrt{x} \quad \therefore \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

When  $x = 1$ ,  $u = 1$

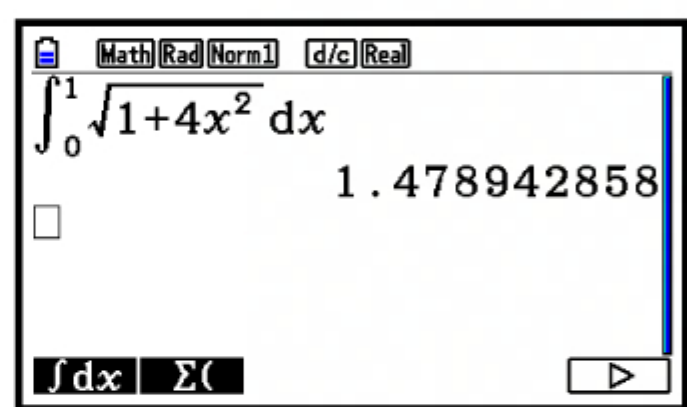
When  $x = 4$ ,  $u = 2$

$$\begin{aligned} \therefore \int_1^4 \frac{3e^{\sqrt{x}}}{\sqrt{x}} dx &= \int_1^2 3e^u \left( 2 \frac{du}{dx} \right) dx \\ &= 6 \int_1^2 e^u du \\ &= 6 [e^u]_1^2 \\ &= 6(e^2 - e) \\ &= 6e(e - 1) \end{aligned}$$

**77 a**  $f(x) = x^2, \quad 0 \leq x \leq 1$

$\therefore f'(x) = 2x$

$$\begin{aligned} \text{So, } L &= \int_0^1 \sqrt{1 + (2x)^2} dx \\ &= \int_0^1 \sqrt{1 + 4x^2} dx \\ &\approx 1.48 \text{ units} \end{aligned}$$



**b** Let  $u = x^2 + 1 \quad \therefore \frac{du}{dx} = 2x$

When  $x = 0$ ,  $u = 1$

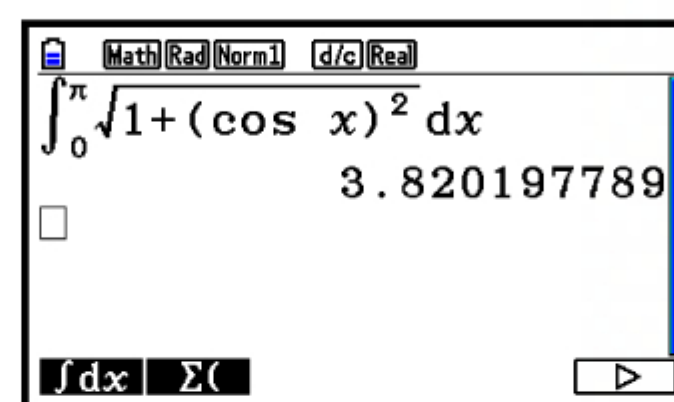
When  $x = 2$ ,  $u = 5$

$$\begin{aligned} \therefore \int_0^2 x\sqrt{x^2+1} dx &= \int_1^5 \sqrt{u} \left( \frac{1}{2} \frac{du}{dx} \right) dx \\ &= \frac{1}{2} \int_1^5 u^{\frac{1}{2}} du \\ &= \frac{1}{2} \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_1^5 \\ &= \frac{1}{2} \left[ \frac{2}{3} (5)^{\frac{3}{2}} - \frac{2}{3} (1)^{\frac{3}{2}} \right] \\ &= \frac{5\sqrt{5}}{3} - \frac{1}{3} \\ &= \frac{5\sqrt{5}-1}{3} \end{aligned}$$

**b**  $f(x) = \sin x, \quad 0 \leq x \leq \pi$

$\therefore f'(x) = \cos x$

$$\begin{aligned} \text{So, } L &= \int_0^\pi \sqrt{1 + \cos^2 x} dx \\ &\approx 3.82 \text{ units} \end{aligned}$$

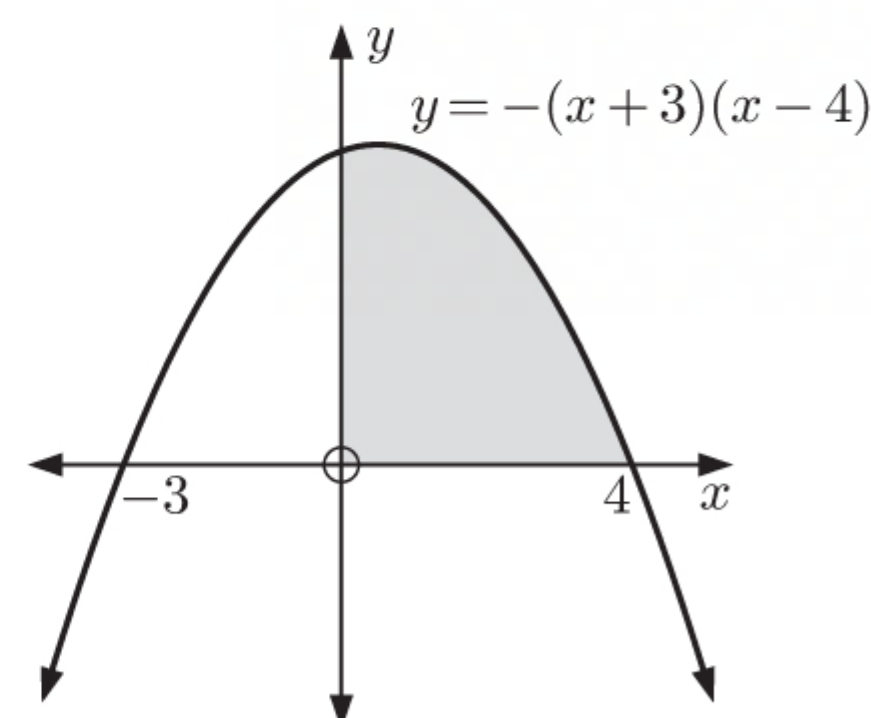


**78 a** The  $x$ -intercepts of  $y = -(x+3)(x-4)$  are  $-3$  and  $4$ .

$\therefore$  shaded area  $= \int_0^4 -(x+3)(x-4) dx$

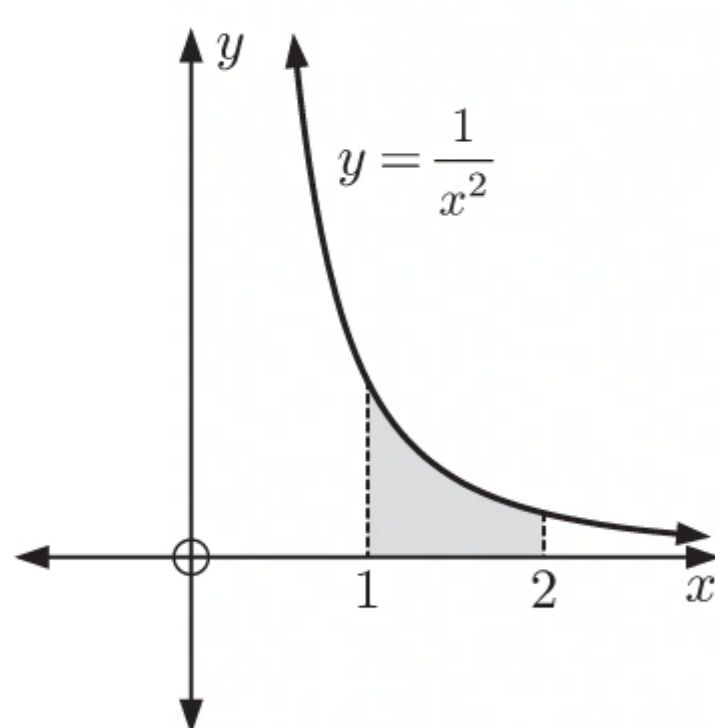
**b** Shaded area  $= \int_0^4 -(x+3)(x-4) dx$

$$\begin{aligned} &= \int_0^4 -(x^2 - x - 12) dx \\ &= \int_0^4 (-x^2 + x + 12) dx \\ &= \left[ -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 12x \right]_0^4 \\ &= \left( -\frac{4^3}{3} + \frac{4^2}{2} + 12(4) \right) - 0 \\ &= \frac{104}{3} = 34\frac{2}{3} \text{ units}^2 \end{aligned}$$

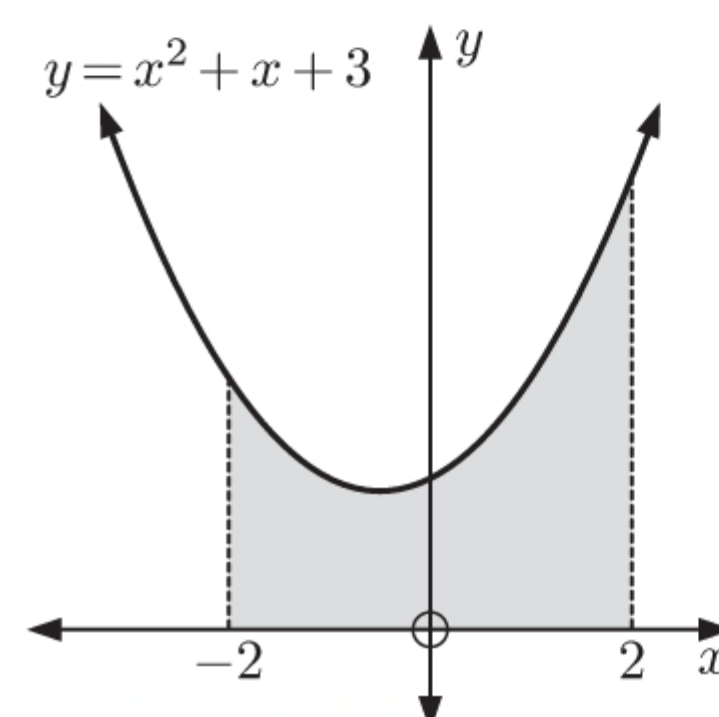




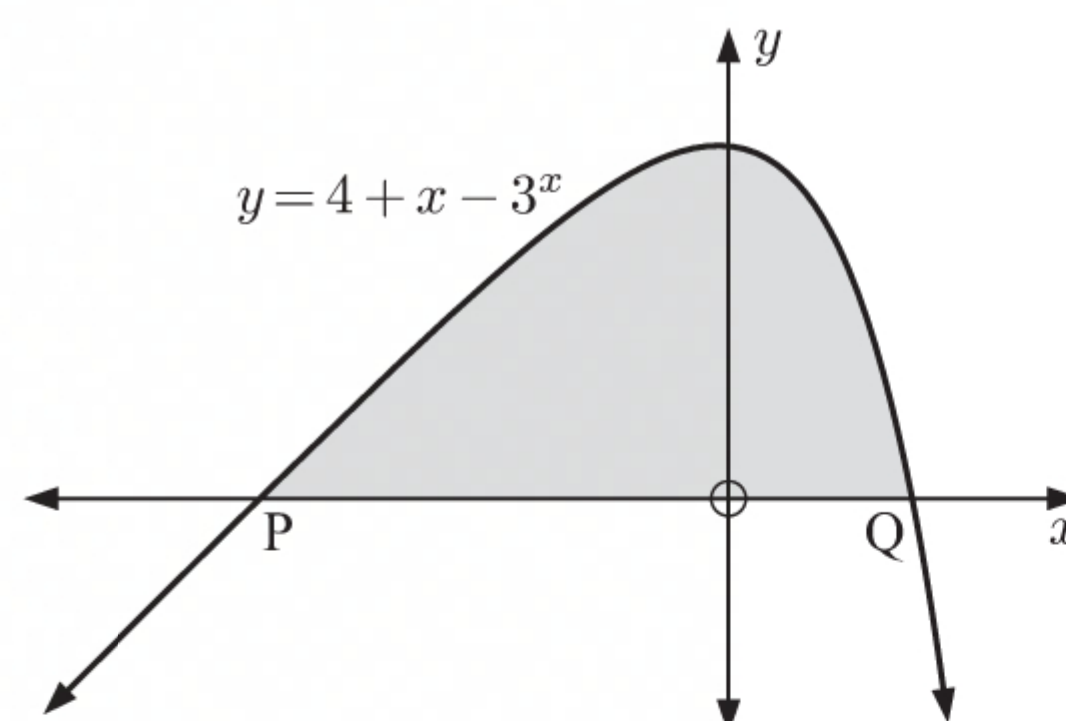
**79 a** Area =  $\int_1^2 \frac{1}{x^2} dx$   
 $= \int_1^2 x^{-2} dx$   
 $= [-x^{-1}]_1^2$   
 $= -\frac{1}{2} - \left(-\frac{1}{1}\right)$   
 $= \frac{1}{2} \text{ units}^2$



**b** Area =  $\int_{-2}^2 (x^2 + x + 3) dx$   
 $= \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x\right]_{-2}^2$   
 $= \left(\frac{2^3}{3} + \frac{2^2}{2} + 3(2)\right) - \left(\frac{(-2)^3}{3} + \frac{(-2)^2}{2} + 3(-2)\right)$   
 $= \frac{32}{3} - \left(-\frac{20}{3}\right)$   
 $= \frac{52}{3}$   
 $= 17\frac{1}{3} \text{ units}^2$



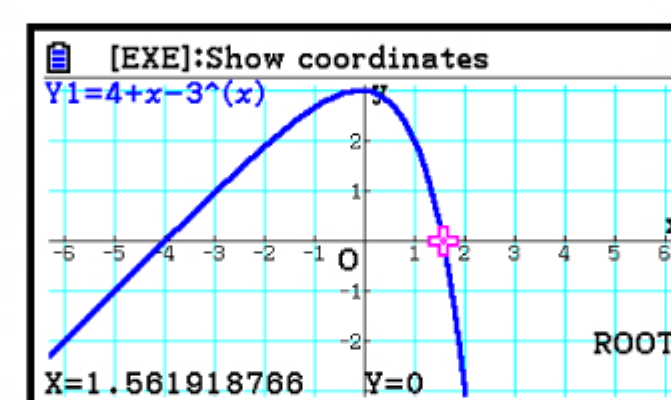
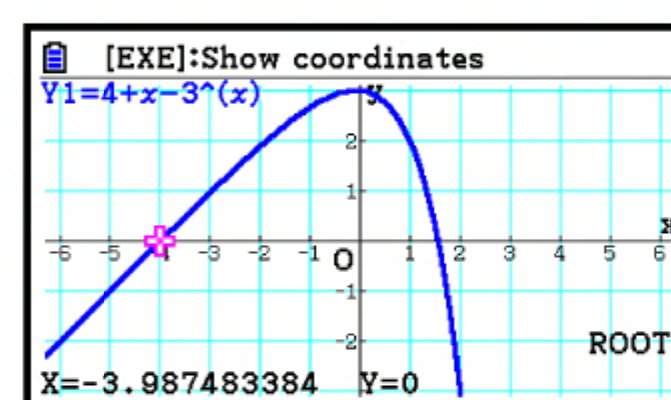
**80 a** P and Q are the x-intercepts of  $y = 4 + x - 3^x$ .



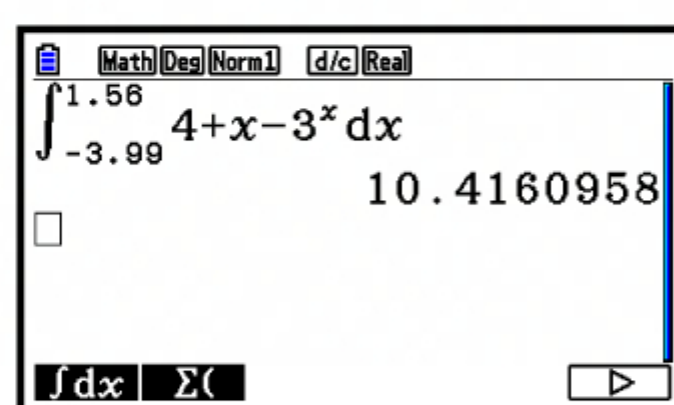
When  $y = 0$ ,  $4 + x - 3^x = 0$

$\therefore x \approx -3.99$  or  $1.56$   
{technology}

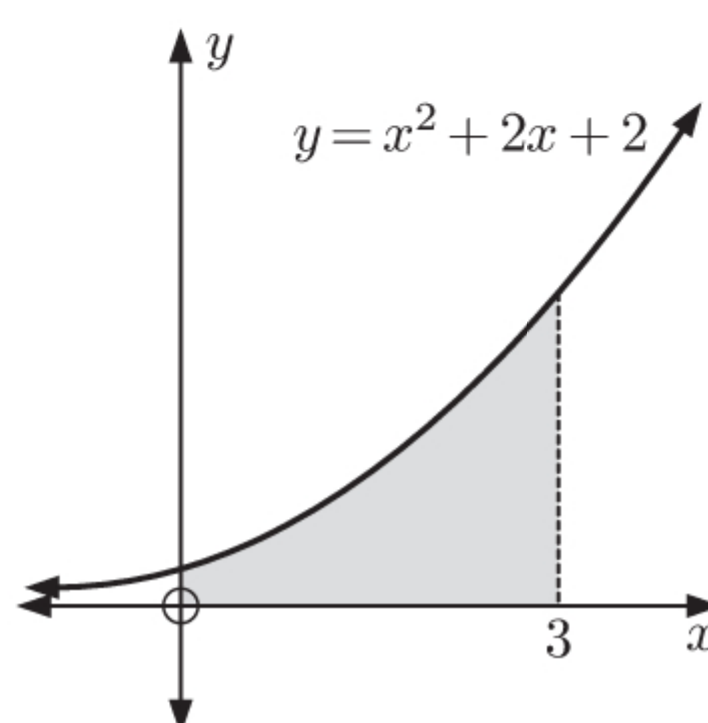
$\therefore$  P is  $(-3.99, 0)$  and Q is  $(1.56, 0)$ .



**b** Area  $\approx \int_{-3.99}^{1.56} (4 + x - 3^x) dx$   
 $\approx 10.4 \text{ units}^2$  {technology}



**81 a** Area =  $\int_0^3 (x^2 + 2x + 2) dx$   
 $= \left[\frac{1}{3}x^3 + x^2 + 2x\right]_0^3$   
 $= \left(\frac{3^3}{3} + 3^2 + 2(3)\right) - 0$   
 $= 24 \text{ units}^2$





**b**  $y = 4 - x^2$

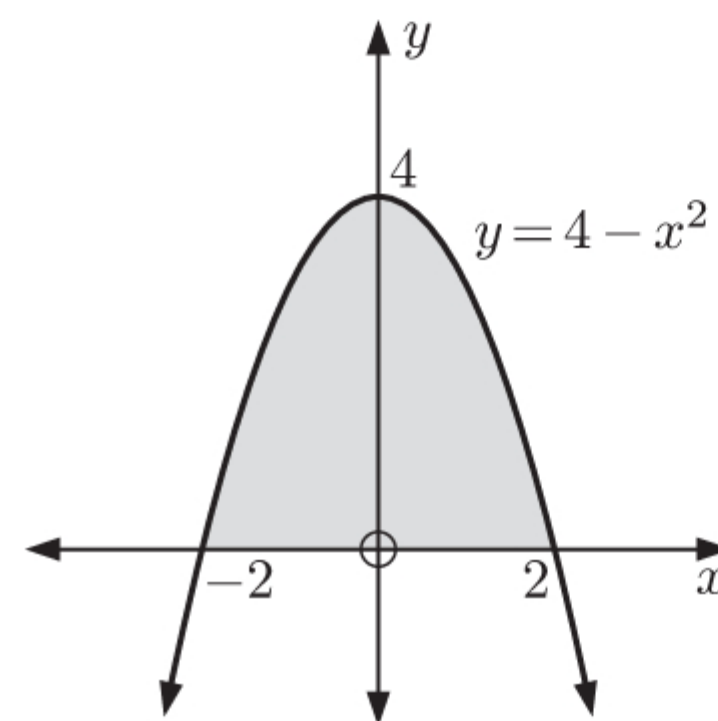
When  $y = 0$ ,  $4 - x^2 = 0$

$$\therefore x^2 = 4$$

$$\therefore x = \pm 2$$

$\therefore$  the  $x$ -intercepts are  $-2$  and  $2$ .

$$\begin{aligned} \therefore \text{area} &= \int_{-2}^2 (4 - x^2) dx \\ &= \left[ 4x - \frac{1}{3}x^3 \right]_{-2}^2 \\ &= \left( 4(2) - \frac{2^3}{3} \right) - \left( 4(-2) - \frac{(-2)^3}{3} \right) \\ &= \frac{16}{3} - \left( -\frac{16}{3} \right) \\ &= \frac{32}{3} \\ &= 10\frac{2}{3} \text{ units}^2 \end{aligned}$$



**c**  $y = (2x + 5)^2$

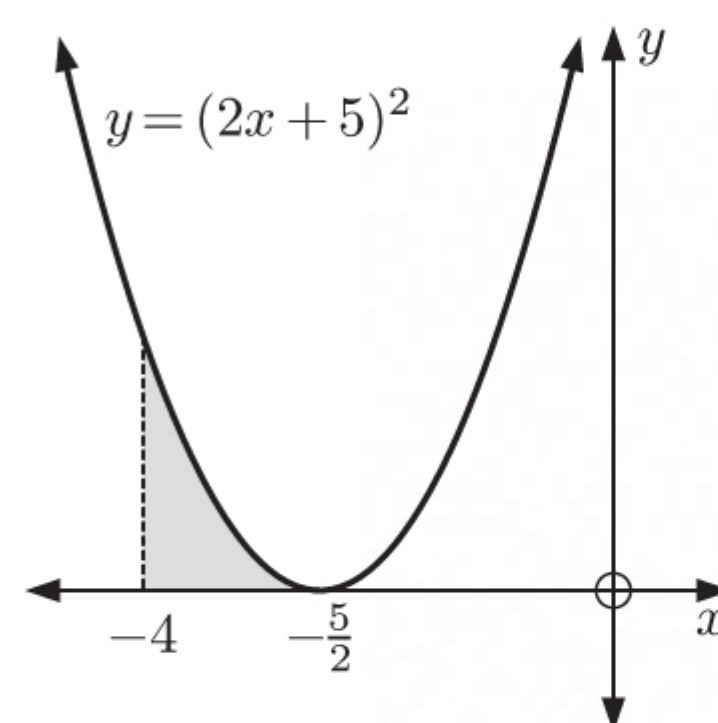
When  $y = 0$ ,  $(2x + 5)^2 = 0$

$$\therefore 2x + 5 = 0$$

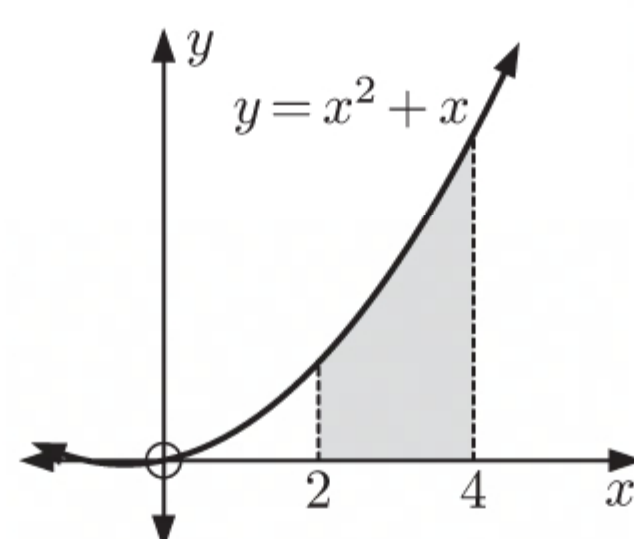
$$\therefore x = -\frac{5}{2}$$

$\therefore$  the  $x$ -intercept is  $-\frac{5}{2}$ .

$$\begin{aligned} \therefore \text{area} &= \int_{-4}^{-\frac{5}{2}} (2x + 5)^2 dx \\ &= \int_{-4}^{-\frac{5}{2}} (4x^2 + 20x + 25) dx \\ &= \left[ \frac{4}{3}x^3 + 10x^2 + 25x \right]_{-4}^{-\frac{5}{2}} \\ &= \left( \frac{4}{3}\left(-\frac{5}{2}\right)^3 + 10\left(-\frac{5}{2}\right)^2 + 25\left(-\frac{5}{2}\right) \right) - \left( \frac{4}{3}(-4)^3 + 10(-4)^2 + 25(-4) \right) \\ &= -\frac{125}{6} - \left( -\frac{76}{3} \right) \\ &= \frac{9}{2} \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$

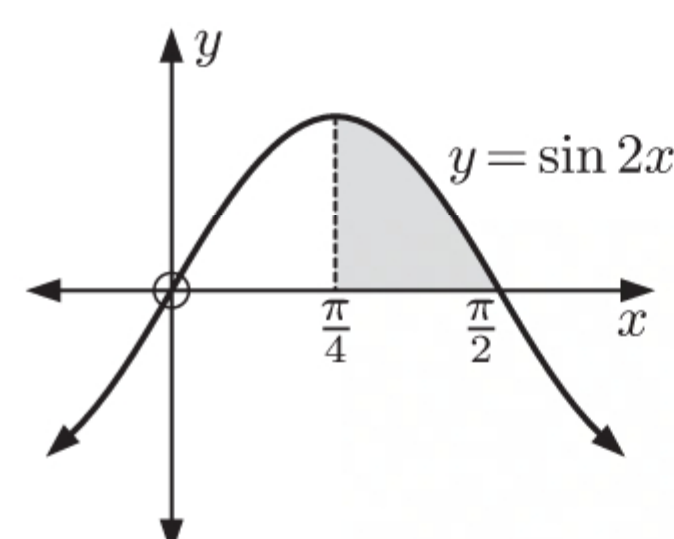


**82 a**



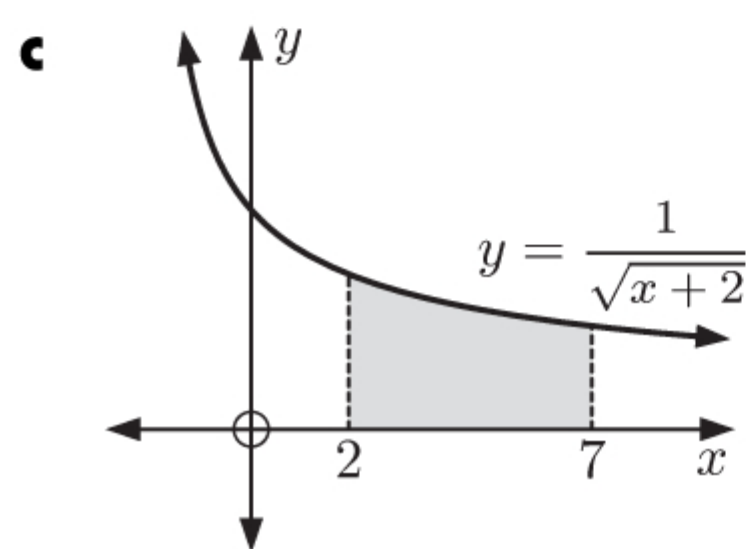
$$\begin{aligned} \text{Area} &= \int_2^4 (x^2 + x) dx \\ &= \left[ \frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_2^4 \\ &= \left( \frac{1}{3}(4)^3 + \frac{1}{2}(4)^2 \right) - \left( \frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 \right) \\ &= \frac{64}{3} + 8 - \frac{8}{3} - 2 \\ &= \frac{56}{3} + 6 \\ &= \frac{74}{3} \\ &= 24\frac{2}{3} \text{ units}^2 \end{aligned}$$

**b**

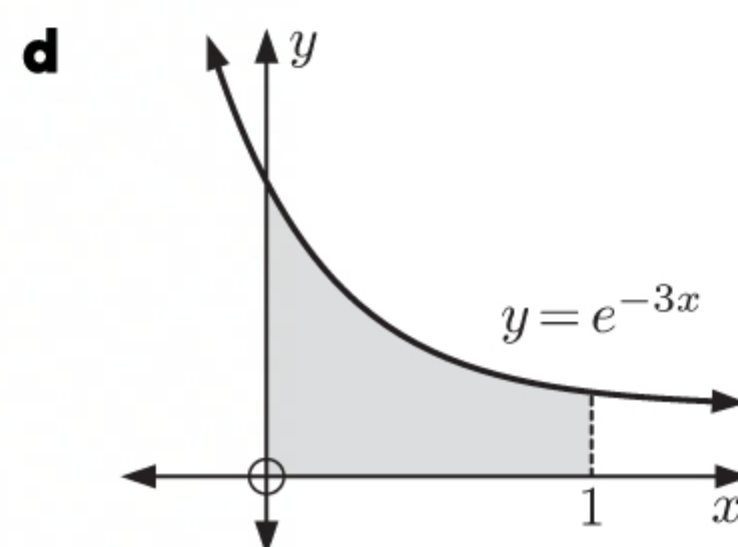


$$\begin{aligned} \text{Area} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2x dx \\ &= \left[ -\frac{1}{2} \cos 2x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= -\frac{1}{2} \cos \pi + \frac{1}{2} \cos \frac{\pi}{2} \\ &= -\frac{1}{2}(-1) + \frac{1}{2}(0) \\ &= \frac{1}{2} \text{ units}^2 \end{aligned}$$



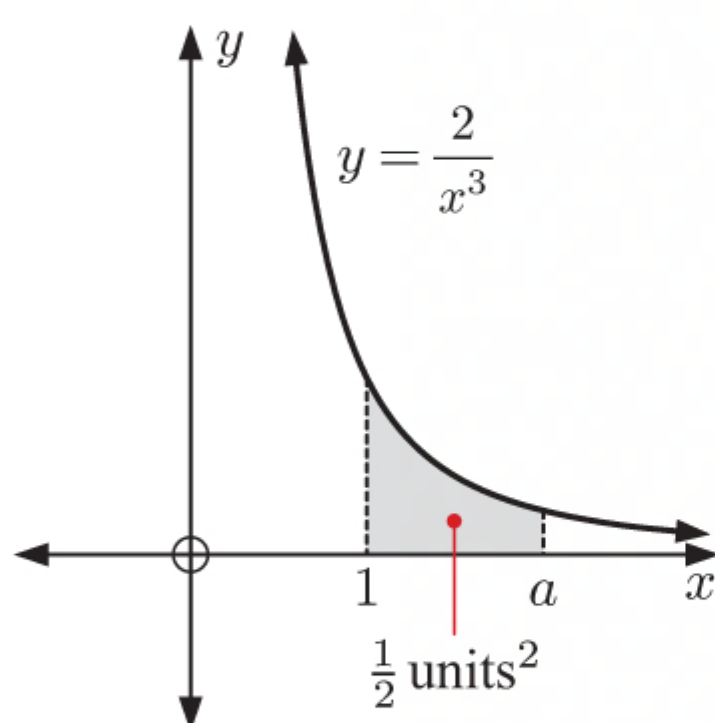


$$\begin{aligned}
 \text{Area} &= \int_2^7 \frac{1}{\sqrt{x+2}} dx \\
 &= \int_2^7 (x+2)^{-\frac{1}{2}} dx \\
 &= \left[ 2(x+2)^{\frac{1}{2}} \right]_2^7 \\
 &= 2\sqrt{7+2} - 2\sqrt{2+2} \\
 &= 2\sqrt{9} - 2\sqrt{4} \\
 &= 2(3) - 2(2) \\
 &= 2 \text{ units}^2
 \end{aligned}$$



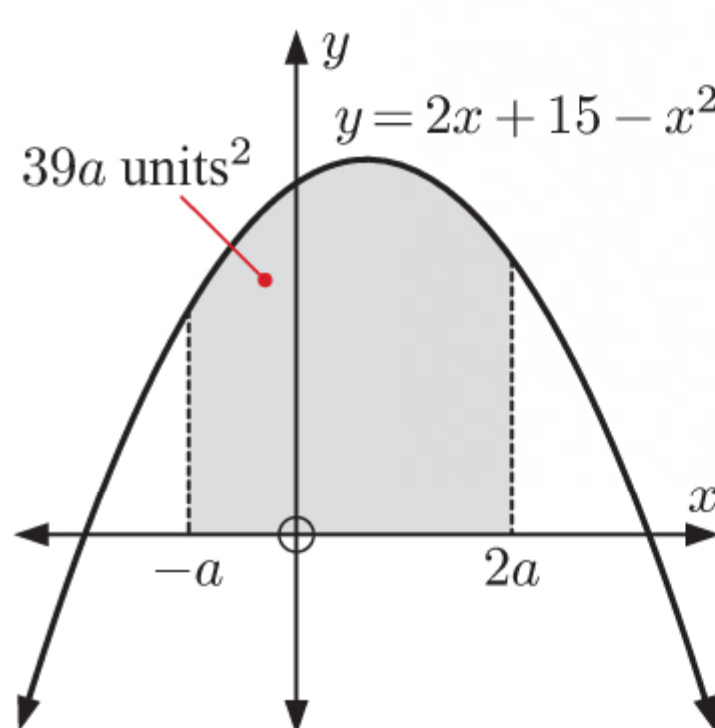
$$\begin{aligned}
 \text{Area} &= \int_0^1 e^{-3x} dx \\
 &= \left[ -\frac{1}{3}e^{-3x} \right]_0^1 \\
 &= -\frac{1}{3}e^{-3} + \frac{1}{3}e^0 \\
 &= -\frac{1}{3} \frac{1}{e^3} + \frac{1}{3} \\
 &= \frac{1}{3} \left( 1 - \frac{1}{e^3} \right) \text{ units}^2
 \end{aligned}$$

83 a



$$\begin{aligned}
 \text{Area} &= \frac{1}{2} \text{ units}^2 \\
 \therefore \int_1^a \frac{2}{x^3} dx &= \frac{1}{2} \\
 \therefore \int_1^a 2x^{-3} dx &= \frac{1}{2} \\
 \therefore \left[ -x^{-2} \right]_1^a &= \frac{1}{2} \\
 \therefore -\frac{1}{a^2} + \frac{1}{1^2} &= \frac{1}{2} \\
 \therefore -\frac{1}{a^2} &= -\frac{1}{2} \\
 \therefore a^2 &= 2 \\
 \therefore a &= \pm\sqrt{2} \\
 \therefore a &= \sqrt{2} \quad \{a > 1\}
 \end{aligned}$$

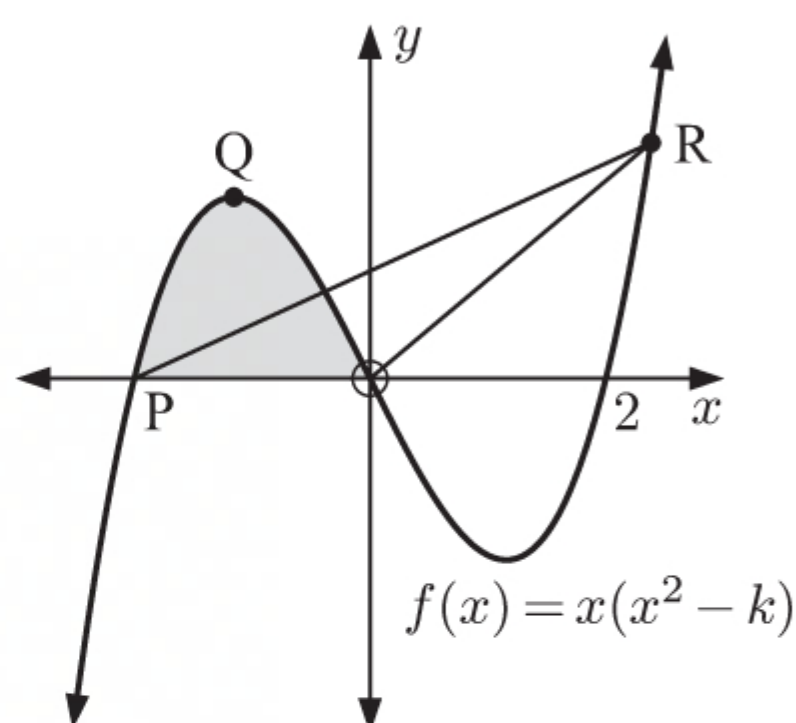
b



$$\begin{aligned}
 \text{Area} &= 39a \text{ units}^2 \\
 \therefore \int_{-a}^{2a} (2x + 15 - x^2) dx &= 39a \\
 \therefore \left[ x^2 + 15x - \frac{1}{3}x^3 \right]_{-a}^{2a} &= 39a \\
 \therefore \left( 4a^2 + 30a - \frac{8}{3}a^3 \right) - \left( a^2 - 15a + \frac{1}{3}a^3 \right) &= 39a \\
 \therefore 3a^2 + 45a - 3a^3 &= 39a \\
 \therefore 3a^2 + 6a - 3a^3 &= 0 \\
 \therefore -3a(a^2 - a - 2) &= 0 \\
 \therefore a(a-2)(a+1) &= 0 \\
 \therefore a &= -1, 0, \text{ or } 2 \\
 \therefore a &= 2 \quad \{a > 0\}
 \end{aligned}$$

84 a 2 is an  $x$ -intercept.

$$\begin{aligned}
 \therefore f(2) &= 0 \\
 \therefore 2(2^2 - k) &= 0 \\
 \therefore 4 - k &= 0 \\
 \therefore k &= 4
 \end{aligned}$$





**b** From **a**,  $f(x) = x(x^2 - 4)$ .

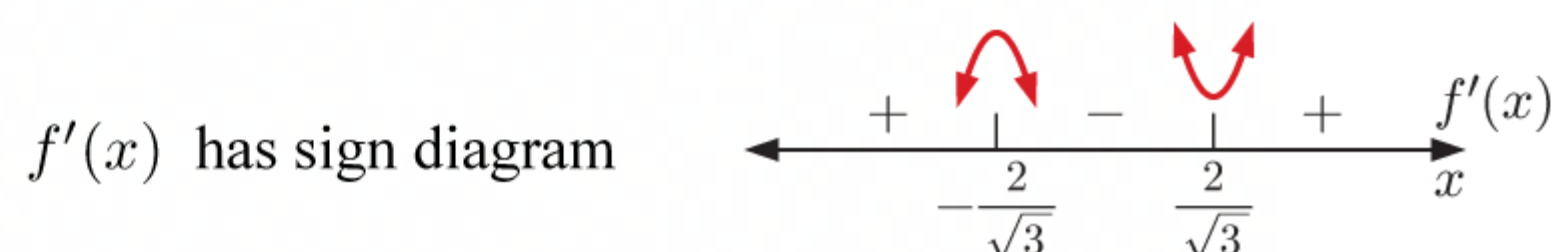
P is the other  $x$ -intercept of  $y = f(x)$ .

$$\begin{aligned} f(x) = 0 \text{ when } x(x^2 - 4) &= 0 \\ \therefore x = 0 \text{ or } x^2 &= 4 \\ \therefore x &= \pm 2 \end{aligned}$$

$\therefore$  P has coordinates  $(-2, 0)$ .

**d** Stationary points occur where  $f'(x) = 0$

$$\begin{aligned} \therefore 3x^2 - 4 &= 0 \\ \therefore x^2 &= \frac{4}{3} \\ \therefore x &= \pm \frac{2}{\sqrt{3}} \end{aligned}$$

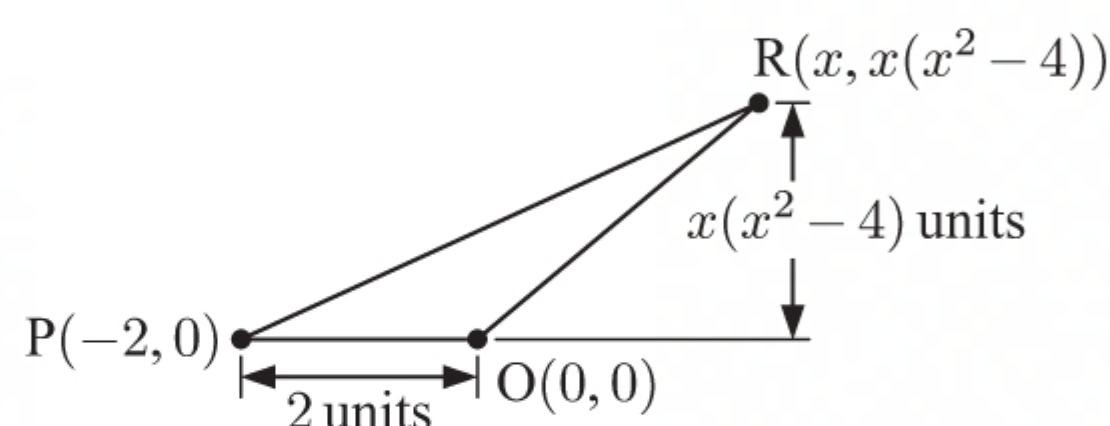


$\therefore$  the local maximum Q has  $x$ -coordinate  $-\frac{2}{\sqrt{3}}$ .

$$\begin{aligned} \text{e } \int f(x) dx &= \int x(x^2 - 4) dx \\ &= \int (x^3 - 4x) dx \\ &= \frac{1}{4}x^4 - 2x^2 + c \end{aligned}$$

$$\begin{aligned} \text{f Shaded area} &= \int_{-2}^0 f(x) dx \\ &= \left[ \frac{1}{4}x^4 - 2x^2 \right]_{-2}^0 \\ &= 0 - \left( \frac{(-2)^4}{4} - 2(-2)^2 \right) \\ &= 0 - (-4) \\ &= 4 \text{ units}^2 \end{aligned}$$

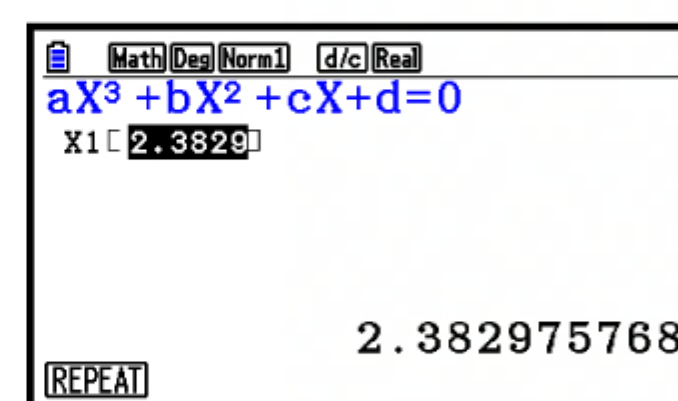
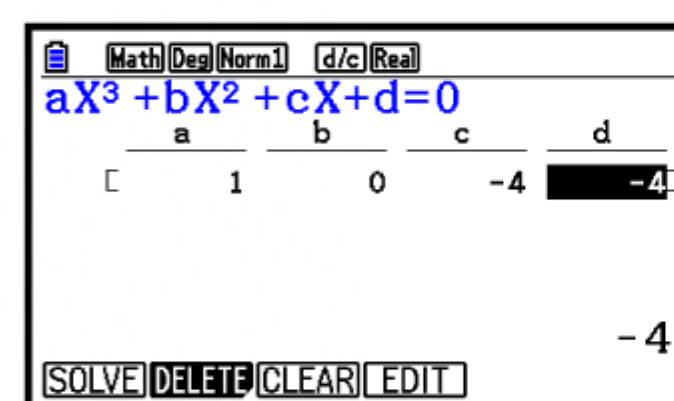
**g** R has coordinates  $(x, x(x^2 - 4))$ .



$$\begin{aligned} \text{Area of } \triangle POR &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times 2 \times x(x^2 - 4) \\ &= x(x^2 - 4) \end{aligned}$$

Shaded area = area of  $\triangle POR$

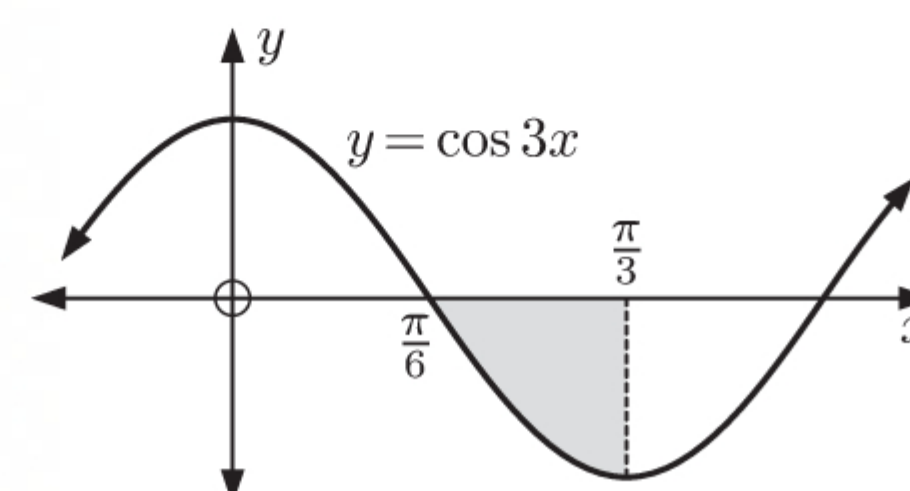
$$\begin{aligned} \therefore 4 &= x(x^2 - 4) \\ \therefore 4 &= x^3 - 4x \\ \therefore x^3 - 4x - 4 &= 0 \\ \therefore x &\approx 2.38 \quad \{\text{technology}\} \\ \therefore \text{R has coordinates } &(\approx 2.38, 4). \end{aligned}$$



**85 a** The first  $x$ -intercept of  $y = \cos 3x$  is given by  $\cos 3x = 0$

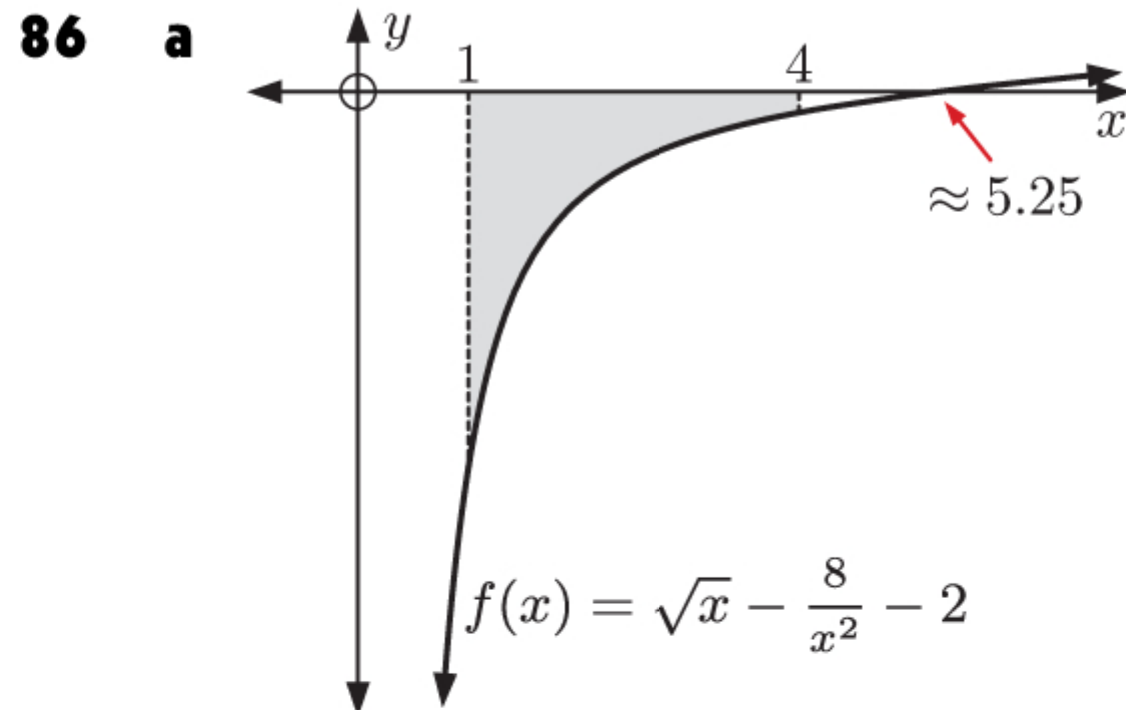
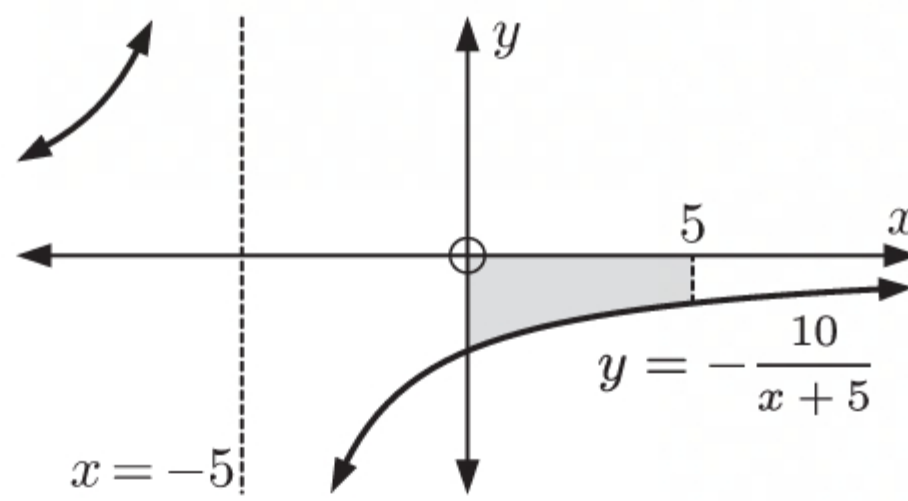
$$\begin{aligned} \therefore 3x &= \frac{\pi}{2} \\ \therefore x &= \frac{\pi}{6} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x dx \\ &= - \left[ \frac{1}{3} \sin 3x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= - \left( \frac{1}{3} \sin \pi - \frac{1}{3} \sin \frac{\pi}{2} \right) \\ &= - \left( \frac{1}{3}(0) - \frac{1}{3}(1) \right) \\ &= \frac{1}{3} \text{ units}^2 \end{aligned}$$





$$\begin{aligned}
 \text{b Area} &= - \int_0^5 -\frac{10}{x+5} dx \\
 &= [10 \ln|x+5|]_0^5 \\
 &= 10 \ln 10 - 10 \ln 5 \\
 &= 10 \ln 2 \text{ units}^2 \\
 &\approx 6.93 \text{ units}^2
 \end{aligned}$$



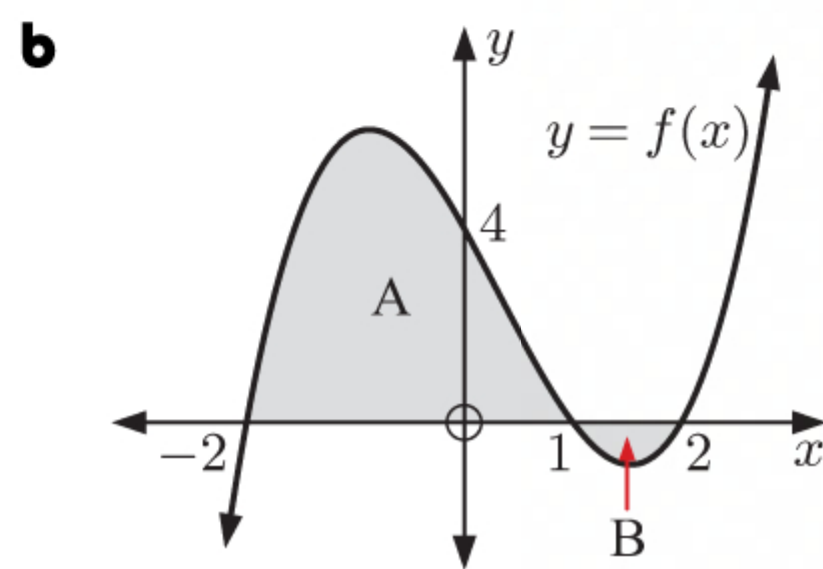
**b** Shaded area  $= - \int_1^4 \left( \sqrt{x} - \frac{8}{x^2} - 2 \right) dx$

$$\begin{aligned}
 &= - \int_1^4 \left( x^{\frac{1}{2}} - 8x^{-2} - 2 \right) dx \\
 &= - \left[ \frac{2}{3} x^{\frac{3}{2}} + 8x^{-1} - 2x \right]_1^4 \\
 &= - \left( \frac{16}{3} + 2 - 8 \right) + \left( \frac{2}{3} + 8 - 2 \right) \\
 &= \frac{22}{3} = 7\frac{1}{3} \text{ units}^2
 \end{aligned}$$

**87**  $f(x) = x^3 - x^2 - 4x + 4$

**a**  $\int_{-2}^2 f(x) dx = \int_{-2}^2 (x^3 - x^2 - 4x + 4) dx$

$$\begin{aligned}
 &= \left[ \frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x \right]_{-2}^2 \\
 &= \left( \frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 - 2(2)^2 + 4(2) \right) - \left( \frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 2(-2)^2 + 4(-2) \right) \\
 &= -\frac{8}{3} + 8 - \frac{8}{3} + 8 \\
 &= \frac{32}{3}
 \end{aligned}$$



For  $1 \leq x \leq 2$ , the graph of  $y = f(x)$  is below the  $x$ -axis, so  $\int_1^2 f(x) dx$  is negative.

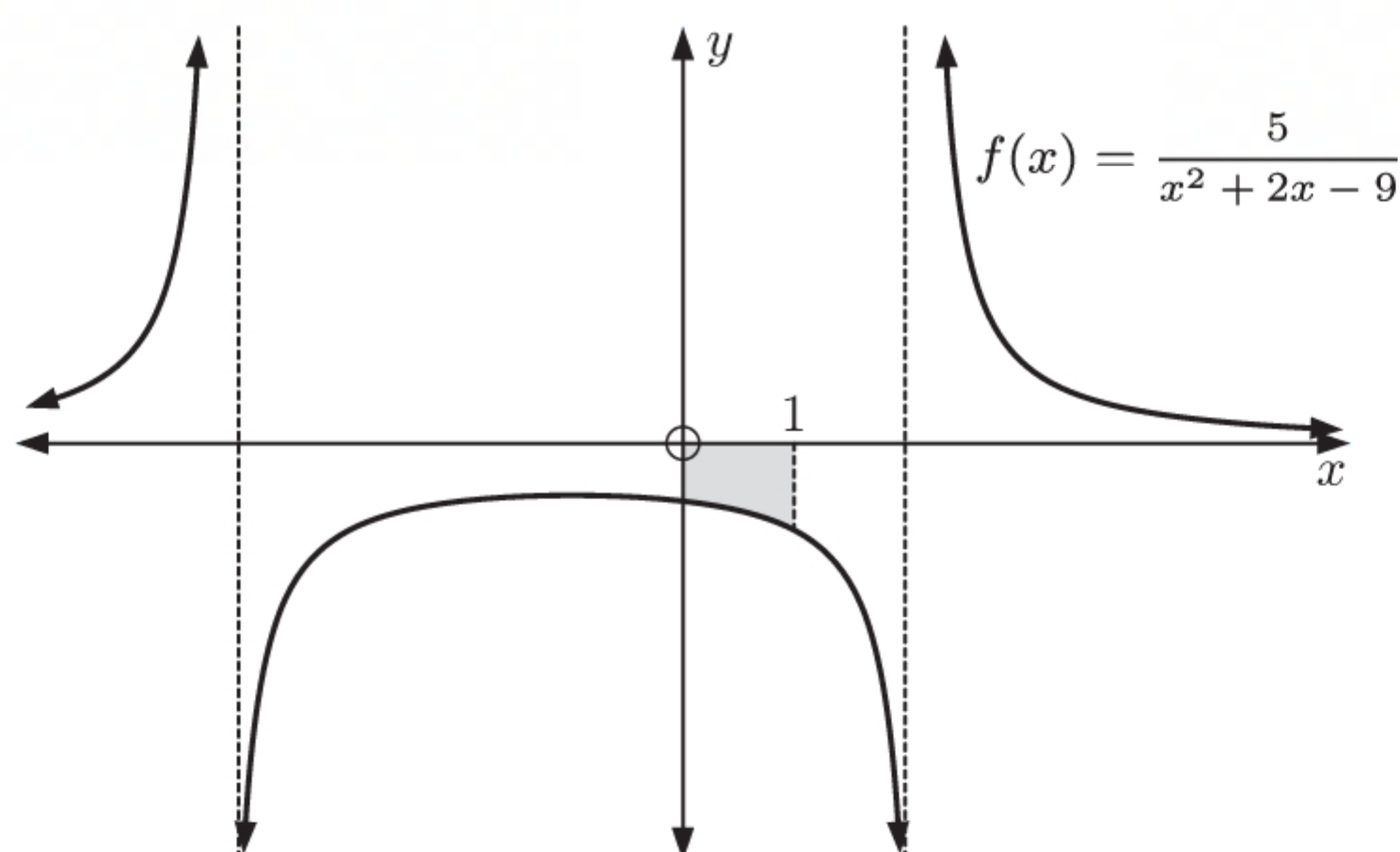
$\int_{-2}^2 f(x) dx$  is the *difference* of areas A and B.

**c** Shaded area  $= \int_{-2}^1 f(x) dx + \int_1^2 -f(x) dx$

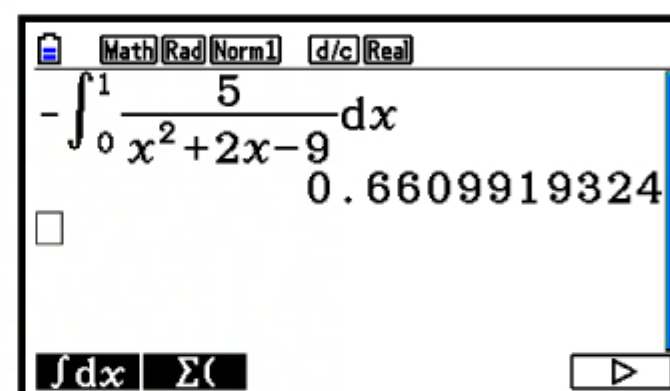
$$\begin{aligned}
 &= \int_{-2}^1 f(x) dx - \int_1^2 f(x) dx \\
 &= \int_{-2}^1 (x^3 - x^2 - 4x + 4) dx - \int_1^2 (x^3 - x^2 - 4x + 4) dx \\
 &= \left[ \frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x \right]_{-2}^1 - \left[ \frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x \right]_1^2 \\
 &= \left( \frac{1}{4}(1)^4 - \frac{1}{3}(1)^3 - 2(1)^2 + 4(1) \right) - \left( \frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 2(-2)^2 + 4(-2) \right) \\
 &\quad - \left( \frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 - 2(2)^2 + 4(2) \right) + \left( \frac{1}{4}(1)^4 - \frac{1}{3}(1)^3 - 2(1)^2 + 4(1) \right) \\
 &= \left( \frac{1}{4} - \frac{1}{3} - 2 + 4 \right) - \left( \frac{16}{4} + \frac{8}{3} - 8 - 8 \right) - \left( \frac{16}{4} - \frac{8}{3} - 8 + 8 \right) + \left( \frac{1}{4} - \frac{1}{3} - 2 + 4 \right) \\
 &= \left( -\frac{1}{12} + 2 \right) - (-4) - (-4) + \left( -\frac{1}{12} + 2 \right) \\
 &= -\frac{1}{6} + 12 \\
 &= \frac{71}{6} = 11\frac{5}{6} \text{ units}^2
 \end{aligned}$$



88 a



b Shaded area  $= -\int_0^1 \frac{5}{x^2 + 2x - 9} dx$   
 $\approx 0.661$  {using technology}



89  $\frac{d}{dx}(\ln(x^2 + 9)) = \frac{2x}{x^2 + 9}$

 Now, shaded area  $= \ln 3$  units<sup>2</sup>

$$\therefore \int_0^k \frac{x}{x^2 + 9} dx = \ln 3$$

$$\therefore \int_0^k \frac{2x}{x^2 + 9} dx = 2 \ln 3$$

$$\therefore [\ln(x^2 + 9)]_0^k = \ln(3^2)$$

$$\therefore \ln(k^2 + 9) - \ln 9 = \ln 9$$

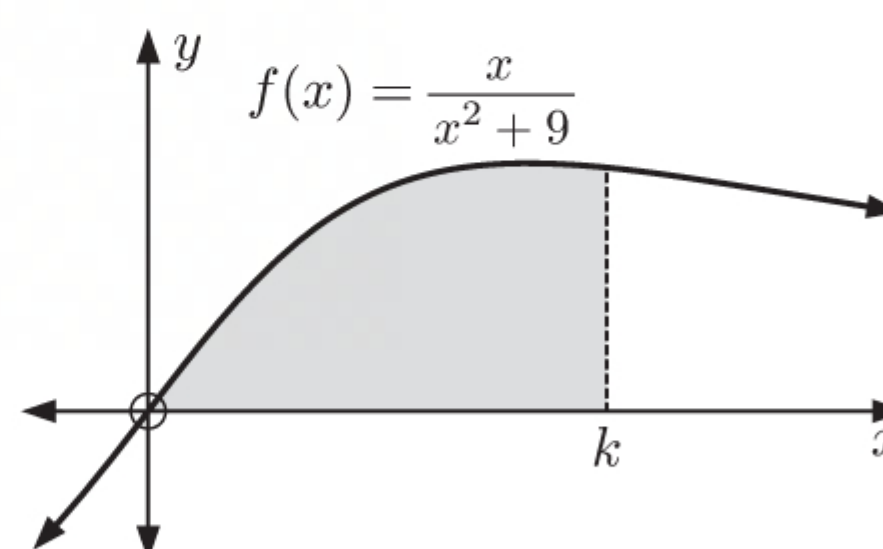
$$\therefore \ln(k^2 + 9) = 2 \ln 9 = \ln 81$$

$$\therefore k^2 + 9 = 81$$

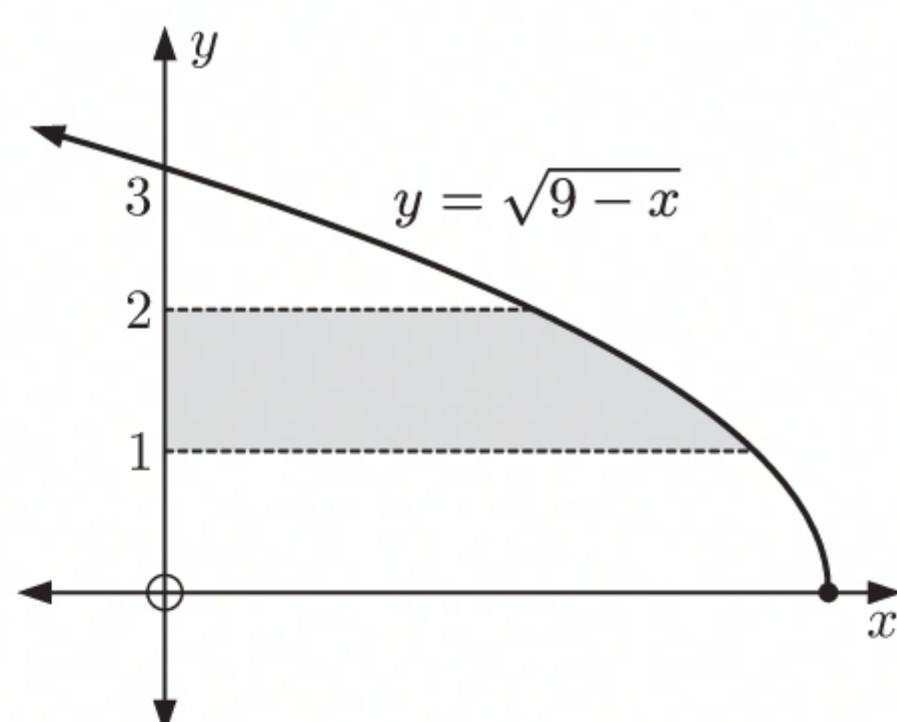
$$\therefore k^2 = 72$$

$$\therefore k = \sqrt{72} \quad \{k > 0\}$$

$$\therefore k = 6\sqrt{2}$$



90 a



$$y = \sqrt{9-x}$$

$$\therefore y^2 = 9-x$$

$$\therefore x = 9-y^2$$

$$\therefore f^{-1}(y) = 9-y^2$$

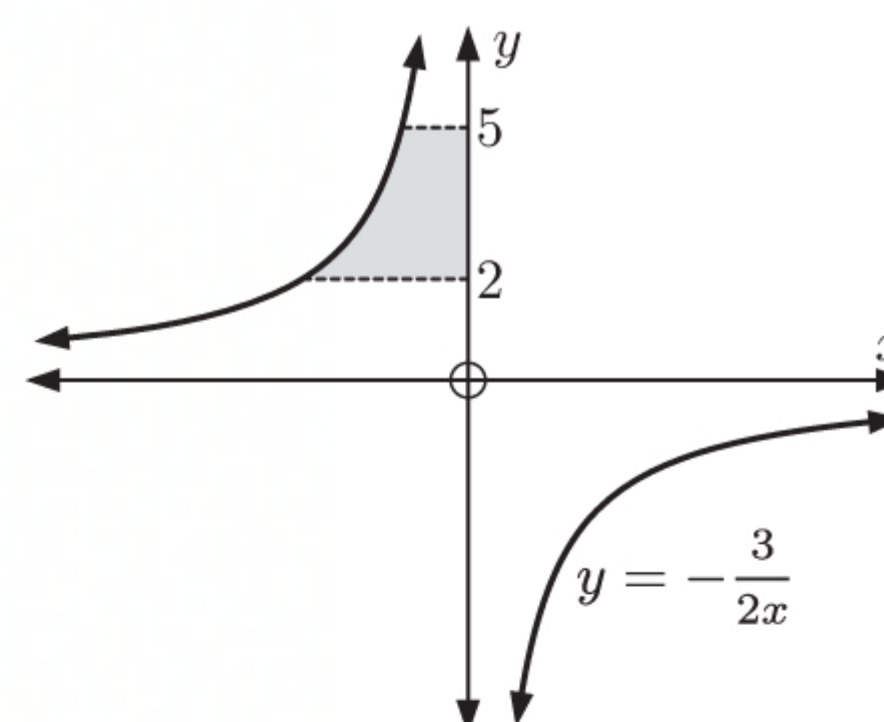
$$\therefore \text{area} = \int_1^2 (9-y^2) dy$$

$$= \left[ 9y - \frac{1}{3}y^3 \right]_1^2$$

$$= \left( 18 - \frac{8}{3} \right) - \left( 9 - \frac{1}{3} \right)$$

$$= \frac{20}{3} = 6\frac{2}{3} \text{ units}^2$$

b



$$y = -\frac{3}{2x}$$

$$\therefore x = -\frac{3}{2y}$$

$$\therefore f^{-1}(y) = -\frac{3}{2y}$$

$$\therefore \text{area} = -\int_2^5 -\frac{3}{2y} dy$$

$$= \frac{3}{2} \int_2^5 \frac{1}{y} dy$$

$$= \frac{3}{2} [\ln|y|]_2^5$$

$$= \frac{3}{2} (\ln 5 - \ln 2)$$

$$= \frac{3}{2} \ln \frac{5}{2} \text{ units}^2$$



**91** When  $x = 0$ ,  $y = 8$

$\therefore$  the  $y$ -intercept is 8.

Now  $y = x^2 + 8$

$\therefore x^2 = y - 8$

$\therefore x = \sqrt{y - 8}$

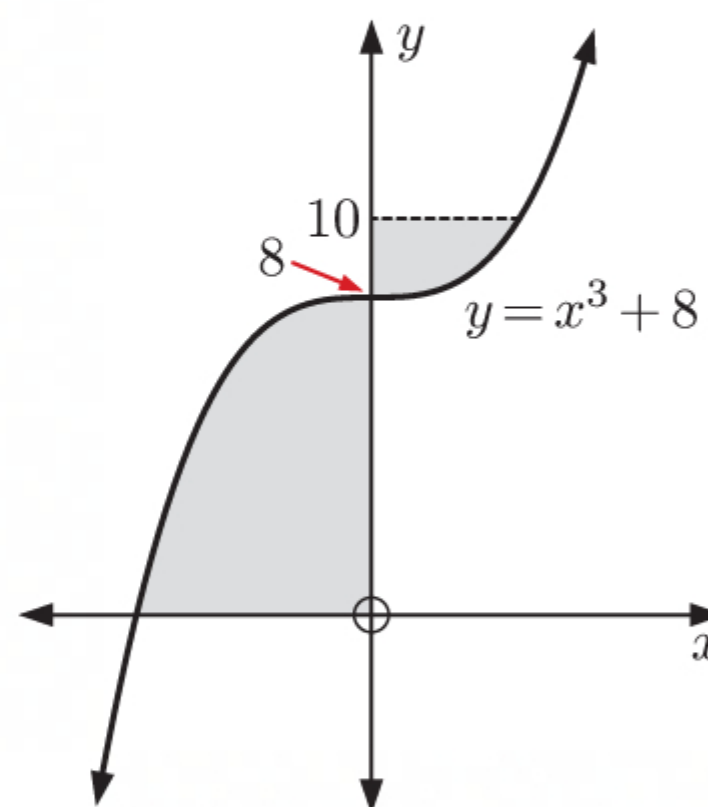
$\therefore f^{-1}(y) = (y - 8)^{\frac{1}{2}}$

$\therefore \text{area} = -\int_0^8 (y - 8)^{\frac{1}{2}} dy + \int_8^{10} (y - 8)^{\frac{1}{2}} dy$

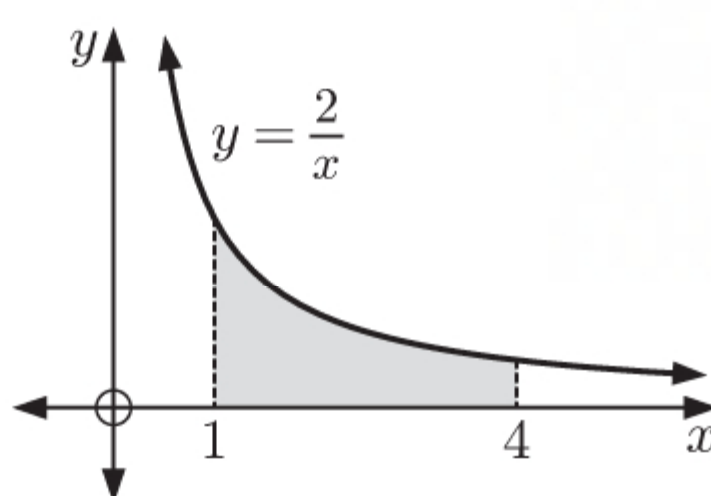
$= -\left[\frac{2}{3}(y - 8)^{\frac{3}{2}}\right]_0^8 + \left[\frac{2}{3}(y - 8)^{\frac{3}{2}}\right]_8^{10}$

$= -\left(0 - \frac{2}{3}(-8)^{\frac{3}{2}}\right) + \left(\frac{2}{3}(2)^{\frac{3}{2}} - 0\right)$

$= \left(12 + \frac{2}{3}\sqrt{2}\right) \text{ units}^2$



**92 a** Volume  $= \pi \int_1^4 y^2 dx$   
 $= \pi \int_1^4 \left(\frac{2}{x}\right)^2 dx$   
 $= 4\pi \int_1^4 x^{-2} dx$   
 $= 4\pi [-x^{-1}]_1^4$   
 $= 4\pi \left(-\frac{1}{4} - (-1)\right)$   
 $= 4\pi \left(\frac{3}{4}\right)$   
 $= 3\pi \text{ units}^3$



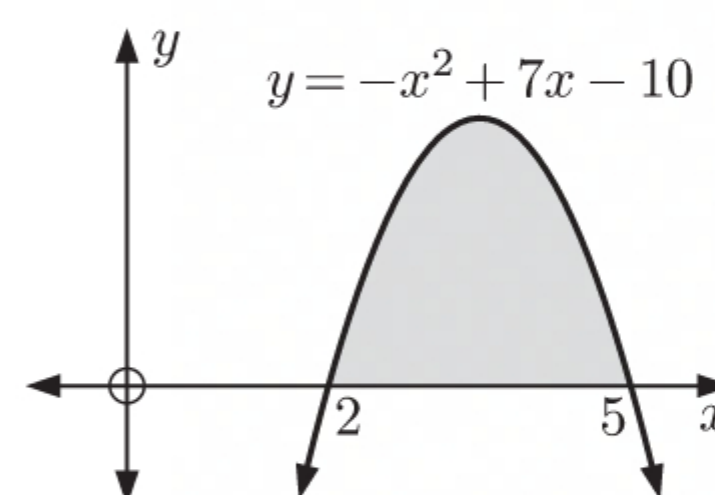
**b** When  $y = 0$ ,  $-x^2 + 7x - 10 = 0$

$\therefore x^2 - 7x + 10 = 0$

$\therefore (x - 5)(x - 2) = 0$

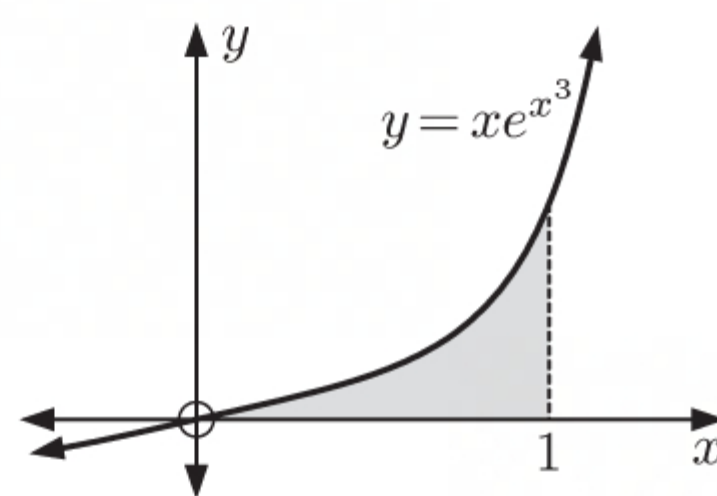
$\therefore x = 5 \text{ or } 2$

Volume  $= \pi \int_2^5 y^2 dx$   
 $= \pi \int_2^5 (-x^2 + 7x - 10)^2 dx$   
 $= \pi \int_2^5 ((-x^2 + 7x)^2 - 20(-x^2 + 7x) + 100) dx$   
 $= \pi \int_2^5 (x^4 - 14x^3 + 49x^2 + 20x^2 - 140x + 100) dx$   
 $= \pi \int_2^5 (x^4 - 14x^3 + 69x^2 - 140x + 100) dx$   
 $= \pi \left[\frac{1}{5}x^5 - \frac{7}{2}x^4 + 23x^3 - 70x^2 + 100x\right]_2^5$   
 $= \pi \left[(625 - \frac{4375}{2} + 2875 - 1750 + 500) - (\frac{32}{5} - 56 + 184 - 280 + 200)\right]$   
 $= \pi \left(\frac{125}{2} - \frac{272}{5}\right)$   
 $= \frac{81\pi}{10} \text{ units}^3$





$$\begin{aligned}
 \text{c Volume} &= \int_0^1 y^2 dx \\
 &= \int_0^1 (xe^{x^3})^2 dx \\
 &= \int_0^1 x^2 e^{2x^3} dx
 \end{aligned}$$



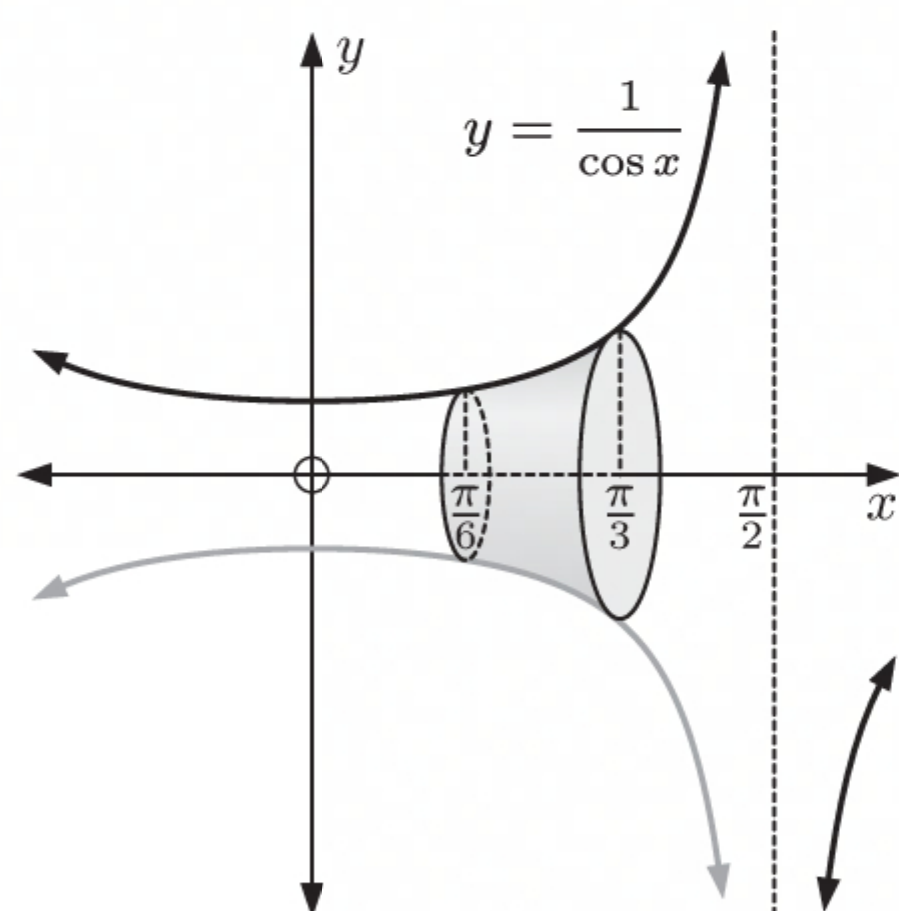
$$\text{Let } u = 2x^3 \quad \therefore \frac{du}{dx} = 6x^2$$

$$\text{When } x = 0, u = 0$$

$$\text{When } x = 1, u = 2$$

$$\begin{aligned}
 \therefore \text{volume} &= \int_0^2 e^u \left( \frac{1}{6} \frac{du}{dx} \right) dx \\
 &= \frac{1}{6} \int_0^2 e^u du \\
 &= \frac{1}{6} [e^u]_0^2 \\
 &= \frac{1}{6} (e^2 - 1) \text{ units}^3
 \end{aligned}$$

93

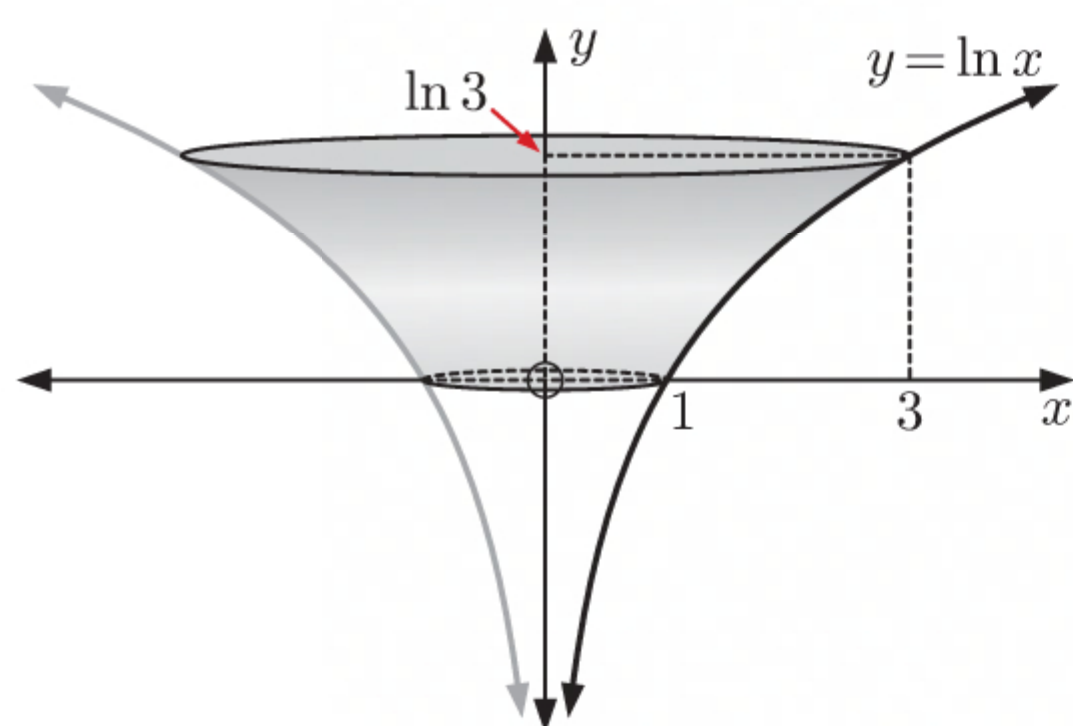


$$\begin{aligned}
 \text{Volume} &= \pi \int_{\pi/6}^{\pi/3} y^2 dx \\
 &= \pi \int_{\pi/6}^{\pi/3} \frac{1}{\cos^2 x} dx \\
 &= \pi [\tan x]_{\pi/6}^{\pi/3} \\
 &= \pi (\tan \frac{\pi}{3} - \tan \frac{\pi}{6}) \\
 &= \pi \left( \sqrt{3} - \frac{1}{\sqrt{3}} \right) \text{ units}^3
 \end{aligned}$$

94

$$y = \ln x$$

$$\therefore x = e^y$$



$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\ln 3} x^2 dy \\
 &= \pi \int_0^{\ln 3} e^{2y} dy \\
 &= \pi \left[ \frac{1}{2} e^{2y} \right]_0^{\ln 3} \\
 &= \frac{1}{2} \pi (e^{2 \ln 3} - e^0) \\
 &= \frac{1}{2} \pi (3^2 - 1) \\
 &= \frac{1}{2} \pi (9 - 1) \\
 &= 4\pi \text{ units}^3
 \end{aligned}$$

$$\text{95 When } x = 0, 0 + \frac{y^2}{9} = 1$$

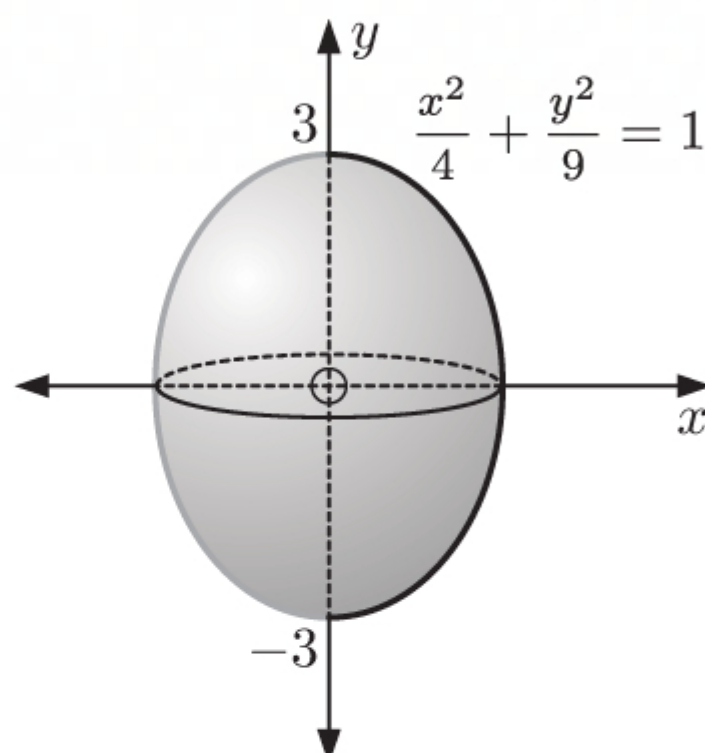
$$\therefore y^2 = 9$$

$$\therefore y = \pm 3$$

$$\text{Now } \frac{x^2}{4} + \frac{y^2}{9} = 1, x \geq 0$$

$$\therefore \frac{x^2}{4} = 1 - \frac{y^2}{9}$$

$$\therefore x^2 = 4 \left( 1 - \frac{y^2}{9} \right)$$





$$\begin{aligned}
 \text{Volume} &= \pi \int_{-3}^3 x^2 dy \\
 &= \pi \int_{-3}^3 4 \left( 1 - \frac{y^2}{9} \right) dy \\
 &= 4\pi \int_{-3}^3 \left( 1 - \frac{y^2}{9} \right) dy \\
 &= 4\pi \left[ y - \frac{y^3}{27} \right]_{-3}^3 \\
 &= 4\pi [(3 - 1) - (-3 + 1)] \\
 &= 4\pi (2 - (-2)) \\
 &= 16\pi \text{ units}^3
 \end{aligned}$$

**96**  $G(t) = \frac{2.5}{t+1}$  metres per year

**a**  $G(t) > 0$  for all  $t \geq 0$ .

$\therefore$  the rate at which the tree grows is always positive.

$\therefore$  the tree is always growing taller.

**b i**  $\int_0^5 G(t) dt = \int_0^5 \frac{2.5}{t+1} dt$

$$\begin{aligned}
 &= [2.5 \ln|t+1|]_0^5 \\
 &= 2.5 \ln 6 - 2.5 \ln 1 \\
 &= 2.5 \ln 6 \approx 4.48 \text{ metres}
 \end{aligned}$$

The tree grew about 4.48 metres in the first 5 years of its lifetime.

**c**  $\int_0^{15} G(t) dt = \int_0^{15} \frac{2.5}{t+1} dt$

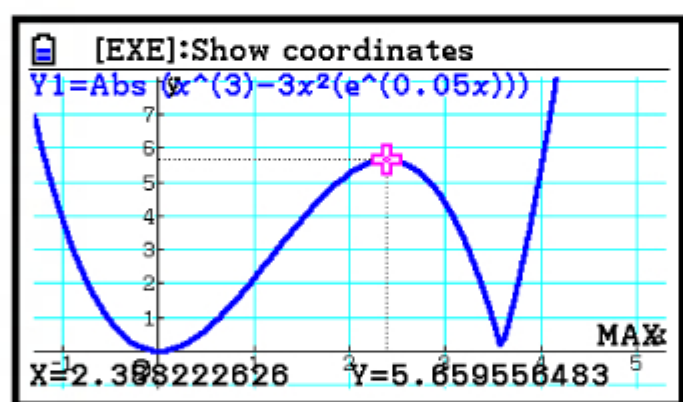
$$\begin{aligned}
 &= [2.5 \ln|t+1|]_0^{15} \\
 &= 2.5 \ln 16 - 2.5 \ln 1 \\
 &= 2.5 \ln 16 \approx 6.93 \text{ metres}
 \end{aligned}$$

The tree grew about 6.93 metres over its entire lifetime.

**97**  $v(t) = t^3 - 3t^2 e^{0.05t}$ ,  $t \geq 0$  seconds

**a** speed =  $|v(t)|$

Using technology, the maximum of  $|v(t)|$  over  $0 \leq t \leq 4$  is  $\approx 5.66 \text{ m s}^{-1}$  when  $t \approx 2.39$ .



**98**  $s(t) = 12t - 3t^3 + 1$  cm,  $t \geq 0$  seconds

**a**  $v(t) = s'(t)$

$$\begin{aligned}
 &= 12 - 9t^2 \text{ cm s}^{-1} \\
 a(t) &= v'(t) \\
 &= -18t \text{ cm s}^{-2}
 \end{aligned}$$

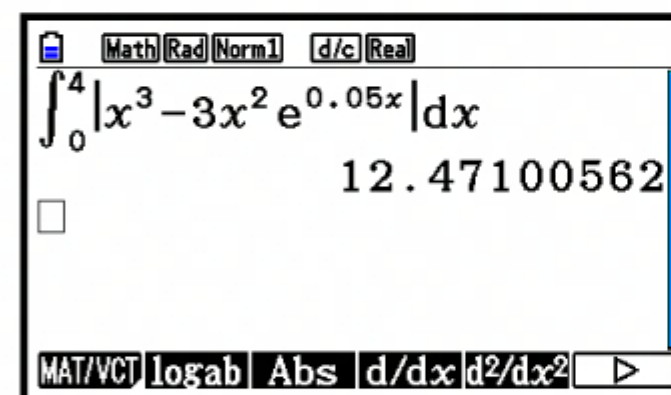
**ii**  $\int_5^{10} G(t) dt = \int_5^{10} \frac{2.5}{t+1} dt$

$$\begin{aligned}
 &= [2.5 \ln|t+1|]_5^{10} \\
 &= 2.5 \ln 11 - 2.5 \ln 6 \\
 &= 2.5 \ln\left(\frac{11}{6}\right) \approx 1.52 \text{ metres}
 \end{aligned}$$

The tree grew about 1.52 metres between the 5th and 10th years of its lifetime.

**b** Total distance =  $\int_0^4 |v(t)| dt$


$$\begin{aligned}
 &= \int_0^4 |t^3 - 3t^2 e^{0.05t}| dt \\
 &\approx 12.5 \text{ m}
 \end{aligned}$$





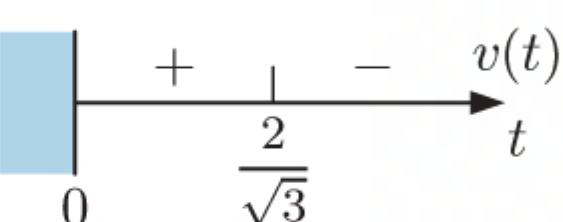
$$\begin{aligned} \text{b i } v(1) &= 12 - 9(1)^2 \\ &= 3 \text{ m s}^{-1} \\ \text{speed} &= |v(1)| = 3 \text{ m s}^{-1} \end{aligned}$$

$$\begin{aligned} \text{ii } v(2) &= 12 - 9(2)^2 \\ &= 12 - 9(4) \\ &= 12 - 36 \\ &= -24 \text{ m s}^{-1} \\ \therefore \text{speed} &= |v(2)| = 24 \text{ m s}^{-1} \end{aligned}$$

**c i** The sign diagram of  $a(t)$  is 

$\therefore$  the velocity of the particle is always decreasing.

$$\begin{aligned} \text{ii } v(t) = 0 \text{ when } 12 - 9t^2 &= 0 \\ \therefore 9t^2 &= 12 \\ \therefore t^2 &= \frac{4}{3} \\ \therefore t &= \frac{2}{\sqrt{3}} \quad \{t \geq 0\} \end{aligned}$$

The sign diagram of  $v(t)$  is 

The speed of the particle is decreasing when  $v(t)$  and  $a(t)$  have opposite sign, that is when  $0 \leq t \leq \frac{2}{\sqrt{3}}$ .

**99**  $v = 2\sqrt{t} - t \text{ m s}^{-1}, \quad t \geq 0$

**a** When  $t = 5$ ,  $v = 2\sqrt{5} - 5 \approx -0.528 \text{ m s}^{-1}$   
 $\therefore \text{speed} = |v| \approx 0.528 \text{ m s}^{-1}$

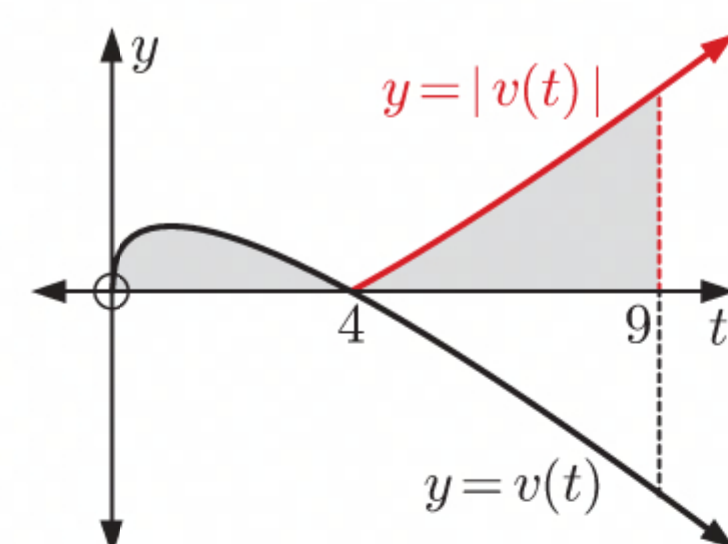
**b**  $a = \frac{dv}{dt}$   
 $= \frac{2}{2\sqrt{t}} - 1$   
 $= \frac{1}{\sqrt{t}} - 1 \text{ m s}^{-2}$

**c** The direction of motion changes when  $v = 0$   
 $\therefore 2\sqrt{t} - t = 0$   
 $\therefore 2\sqrt{t} = t$   
 $\therefore 4t = t^2$   
 $\therefore t^2 - 4t = 0$   
 $\therefore t(t - 4) = 0$   
 $\therefore t = 0 \text{ or } 4$

Now  $v = 0$  when  $t = 0$  means that the particle was initially stationary. So, it does not make sense to talk about the direction of motion changing when there is nothing to compare its current direction to.

$\therefore$  the direction of motion changes at  $t = 4$  seconds.

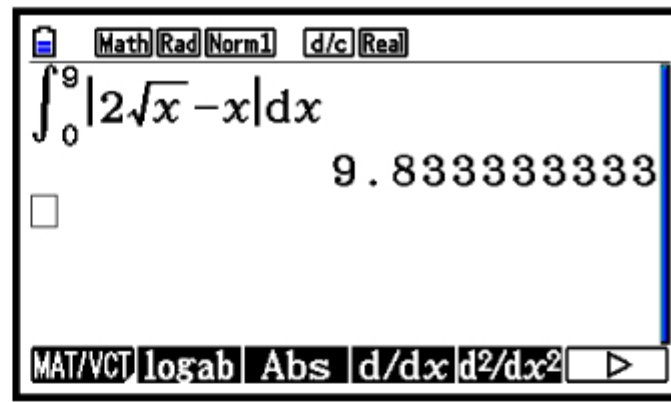
**d** Total distance  $= \int_0^9 |v| dt$   
 $= \int_0^4 v dt + \int_4^9 -v dt$   
 $= \int_0^4 v dt - \int_4^9 v dt$   
 $= \int_0^4 (2\sqrt{t} - t) dt - \int_4^9 (2\sqrt{t} - t) dt$   
 $= \left[ \frac{4}{3}t^{\frac{3}{2}} - \frac{1}{2}t^2 \right]_0^4 - \left[ \frac{4}{3}t^{\frac{3}{2}} - \frac{1}{2}t^2 \right]_4^9$   
 $= \left( \frac{4}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(4)^2 \right) - 0 - \left( \frac{4}{3}(9)^{\frac{3}{2}} - \frac{1}{2}(9)^2 \right) + \left( \frac{4}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(4)^2 \right)$   
 $= \frac{32}{3} - 8 - 36 + \frac{81}{2} + \frac{32}{3} - 8$   
 $= \frac{64}{3} + \frac{81}{2} - 52$   
 $= \frac{59}{6}$   
 $= 9\frac{5}{6} \text{ m}$





Alternatively, using technology:

$$\int_0^9 |v| dt \approx 9.833 \approx 9\frac{5}{6} \text{ m}$$



**100**  $v = \frac{20}{\sqrt{2t+1}} \text{ m s}^{-1}, \quad 0 \leq t \leq 10$

**a** The brakes are applied when  $t = 0$

$$\begin{aligned} \therefore v &= \frac{20}{\sqrt{0+1}} \\ &= 20 \text{ m s}^{-1} \end{aligned}$$

$\therefore$  the speed of the truck when the brakes are applied is  $20 \text{ m s}^{-1}$ .

**c** The truck has acceleration  $a = -2.5 \text{ m s}^{-1}$  when

$$\begin{aligned} \frac{-20}{(2t+1)^{\frac{3}{2}}} &= -2.5 \\ \therefore (2t+1)^{\frac{3}{2}} &= \frac{20}{2.5} = 8 \\ \therefore 2t+1 &= 8^{\frac{2}{3}} = 4 \\ \therefore 2t &= 3 \\ \therefore t &= \frac{3}{2} \text{ seconds} \end{aligned}$$

**b**  $v = 20(2t+1)^{-\frac{1}{2}} \text{ m s}^{-1}$

$$\begin{aligned} \text{Now } a &= \frac{dv}{dt} \\ &= 20 \times \left(-\frac{1}{2}\right)(2t+1)^{-\frac{3}{2}}(2) \quad \{\text{chain rule}\} \\ &= \frac{-20}{(2t+1)^{\frac{3}{2}}} \text{ m s}^{-2} \end{aligned}$$

**d** Distance travelled  $= \int_0^{10} |v| dt$

$$\begin{aligned} &= \int_0^{10} \frac{20}{\sqrt{2t+1}} dt \\ &= \int_0^{10} 20(2t+1)^{-\frac{1}{2}} dt \\ &= \left[40(2t+1)^{\frac{1}{2}}\right]_0^{10} \\ &= 40\sqrt{20+1} - 40\sqrt{1} \\ &= 40\sqrt{21} - 40 \\ &= 40(\sqrt{21} - 1) \\ &\approx 143 \text{ metres} \end{aligned}$$

**101**  $v = 2s + \frac{1}{s} = 2s + s^{-1} \text{ m s}^{-1}$

**a**  $a = v \frac{dv}{ds}$

$$\begin{aligned} &= (2s + s^{-1})(2 - s^{-2}) \\ &= 4s - 2s^{-1} + 2s^{-1} - s^{-3} \\ &= 4s - \frac{1}{s^3} \text{ m s}^{-2} \end{aligned}$$

**b** Initially,  $s = 0.5$

$$\therefore v = 2(0.5) + \frac{1}{0.5} = 3 \text{ m s}^{-1}$$

$$\text{and } a = 4(0.5) - \frac{1}{(0.5)^3} = -6 \text{ m s}^{-2}$$

**c** Initially,  $v > 0$  and  $a < 0$ .

$\therefore$  the speed of the particle is decreasing initially.

**d i** When  $v = 8.25$ ,  $2s + \frac{1}{s} = 8.25$

$$\begin{aligned} \therefore 2s^2 + 1 &= 8.25s \\ \therefore 2s^2 - 8.25s + 1 &= 0 \\ \therefore 8s^2 - 33s + 4 &= 0 \\ \therefore (8s-1)(s-4) &= 0 \\ \therefore s &= 4 \quad \{s \geq 0.5\} \end{aligned}$$

So, the displacement of the particle is 4 m to the right of the origin when its velocity is  $8.25 \text{ m s}^{-1}$ .

**ii** When  $v = a$ ,  $2s + \frac{1}{s} = 4s - \frac{1}{s^3}$

$$\begin{aligned} \therefore 2s^4 + s^2 &= 4s^4 - 1 \\ \therefore 2s^4 - s^2 - 1 &= 0 \\ \therefore (2s^2 + 1)(s^2 - 1) &= 0 \\ \therefore s^2 &= 1 \quad \{2s^2 + 1 > 0 \text{ for all } s\} \\ \therefore s &= 1 \quad \{s \geq 0.5\} \end{aligned}$$

So, the displacement of the particle is 1 m to the right of the origin when its velocity is equal to its acceleration.



**102**  $\frac{dy}{dx} = e^x - 2x$ ,  $y(0) = 1$

$y(0) = 1$  gives us  $x_0 = 0$  and  $y_0 = 1$ .

**a i**

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	1	1	0.5	1.5
2	0.5	1.5	0.6487	1	1.8244

$\therefore y(1) \approx 1.8244$

**ii**

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	1	1	0.25	1.25
2	0.25	1.25	0.7840	0.5	1.4460
3	0.5	1.4460	0.6487	0.75	1.6082
4	0.75	1.6082	0.6170	1	1.7624

$\therefore y(1) \approx 1.7624$

**b** Using the Fundamental Theorem of Calculus,  $y(1) = y(0) + \int_0^1 \frac{dy}{dx} dx$

$$\begin{aligned}
 &= 1 + \int_0^1 (e^x - 2x) dx \\
 &= 1 + [e^x - x^2]_0^1 \\
 &= 1 + (e - 1) - (1 - 0) \\
 &= e - 1 \\
 &\approx 1.7183
 \end{aligned}$$

The accuracy of Euler's method was improved by decreasing the step size.

**103**  $\frac{dy}{dx} = x - y$ ,  $y(1) = 2$

$y(1) = 2$  gives us  $x_0 = 1$  and  $y_0 = 2$ .

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	1	2	-1	1.2	1.8
2	1.2	1.8	-0.6	1.4	1.68
3	1.4	1.68	-0.28	1.6	1.624
4	1.6	1.624	-0.024	1.8	1.6192
5	1.8	1.6192	0.1808	2	1.65536

$\therefore y(2) \approx 1.65536$

**104 a**  $\frac{dy}{dx} = 3x + 7x^3$

$$\begin{aligned}
 \therefore y &= \int (3x + 7x^3) dx \\
 &= \frac{3}{2}x^2 + \frac{7}{4}x^4 + c
 \end{aligned}$$

Now  $y(0) = 5$ , so  $0 + 0 + c = 5$   
 $\therefore c = 5$

So, the particular solution is  $y = \frac{3}{2}x^2 + \frac{7}{4}x^4 + 5$ .

**b**  $\frac{dy}{dx} = \cos 2x - 3e^x$

$$\begin{aligned}
 \therefore y &= \int (\cos 2x - 3e^x) dx \\
 &= \frac{1}{2} \sin 2x - 3e^x + c
 \end{aligned}$$

Now  $y(0) = -2$ , so  $\frac{1}{2} \sin 0 - 3e^0 + c = -2$   
 $\therefore c = 1$

So, the particular solution is  $y = \frac{1}{2} \sin 2x - 3e^x + 1$ .



$$\begin{aligned}
 \text{c} \quad \frac{dy}{dx} &= \frac{2 \sin 3x}{4 + \cos 3x} \\
 \therefore y &= \int \frac{2 \sin 3x}{4 + \cos 3x} dx \\
 &= \int \frac{1}{u} \left( -\frac{2}{3} \frac{du}{dx} \right) dx \quad \left\{ u = 4 + \cos 3x, \quad \frac{du}{dx} = -3 \sin 3x \right\} \\
 &= -\frac{2}{3} \int \frac{1}{u} du \\
 &= -\frac{2}{3} \ln |u| + c \\
 &= -\frac{2}{3} \ln(4 + \cos 3x) + c \quad \{4 + \cos 3x > 0 \text{ for all } x\}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } y\left(\frac{\pi}{2}\right) &= 2 \ln 2, \text{ so } -\frac{2}{3} \ln(4 + \cos \frac{3\pi}{2}) + c = 2 \ln 2 \\
 \therefore -\frac{2}{3} \ln 4 + c &= \ln 4 \\
 \therefore c &= \frac{5}{3} \ln 4
 \end{aligned}$$

So, the particular solution is  $y = -\frac{2}{3} \ln(4 + \cos 3x) + \frac{5}{3} \ln 4$ .

$$\begin{aligned}
 \text{105 a} \quad \frac{dy}{dt} &= \frac{\cos t}{\sqrt{1 - \sin t}} \\
 \therefore y &= \int \frac{\cos t}{\sqrt{1 - \sin t}} dt \\
 &= \int \frac{1}{\sqrt{u}} \left( -\frac{du}{dt} \right) dt \quad \left\{ u = 1 - \sin t, \quad \frac{du}{dt} = -\cos t \right\} \\
 &= - \int u^{-\frac{1}{2}} du \\
 &= -2u^{\frac{1}{2}} + c \\
 &= -2\sqrt{1 - \sin t} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad 4x \frac{dN}{dx} &= \left(2x - \frac{3}{x}\right)^2 \\
 \therefore 4x \frac{dN}{dx} &= 4x^2 - 12 + \frac{9}{x^2} \\
 \therefore \frac{dN}{dx} &= x - \frac{3}{x} + \frac{9}{4x^3} \\
 \therefore N &= \int \left(x - \frac{3}{x} + \frac{9}{4}x^{-3}\right) dx \\
 &= \frac{1}{2}x^2 - 3 \ln |x| - \frac{9}{8}x^{-2} + c \\
 &= \frac{1}{2}x^2 - 3 \ln |x| - \frac{9}{8x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \frac{5}{s} + e^{-2s} - \frac{dQ}{ds} &= 0 \\
 \therefore \frac{dQ}{ds} &= \frac{5}{s} + e^{-2s} \\
 \therefore Q &= \int \left(\frac{5}{s} + e^{-2s}\right) ds \\
 &= 5 \ln |s| - \frac{1}{2}e^{-2s} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{106 a} \quad \frac{dy}{dx} &= xy^2 \\
 \therefore \frac{1}{y^2} \frac{dy}{dx} &= x \\
 \therefore \int \frac{1}{y^2} \frac{dy}{dx} dx &= \int x dx \\
 \therefore \int y^{-2} dy &= \int x dx \\
 \therefore -y^{-1} &= \frac{1}{2}x^2 + c \\
 \therefore \frac{1}{y} &= c - \frac{1}{2}x^2 \\
 \therefore y &= \frac{1}{c - \frac{1}{2}x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \frac{dy}{dx} &= 5\sqrt{y} \\
 \therefore \frac{1}{\sqrt{y}} \frac{dy}{dx} &= 5 \\
 \therefore \int \frac{1}{\sqrt{y}} \frac{dy}{dx} dx &= \int 5 dx \\
 \therefore \int y^{-\frac{1}{2}} dy &= \int 5 dx \\
 \therefore 2y^{\frac{1}{2}} &= 5x + c \\
 \therefore y^{\frac{1}{2}} &= \frac{5}{2}x + c \\
 \therefore y &= \left(\frac{5}{2}x + c\right)^2
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{c} \quad & \frac{dy}{dx} = \frac{xy}{x^2 + 1} \\
 & \therefore \frac{1}{y} \frac{dy}{dx} = \frac{x}{x^2 + 1} \\
 & \therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int \frac{x}{x^2 + 1} dx \\
 & \therefore \int \frac{1}{y} dy = \frac{1}{2} \int \frac{2x}{x^2 + 1} dx \\
 & \therefore \ln|y| = \frac{1}{2} \ln|x^2 + 1| + c \\
 & \quad = \frac{1}{2} \ln(x^2 + 1) + c \quad \{x^2 + 1 > 0 \text{ for all } x\} \\
 & \therefore |y| = e^c \sqrt{x^2 + 1} \\
 & \therefore y = \pm e^c \sqrt{x^2 + 1} \\
 & \quad = A\sqrt{x^2 + 1} \quad \{A = \pm e^c\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{107} \quad \mathbf{a} \quad & \frac{dP}{dz} = -3P^2z \\
 & \therefore \frac{1}{P^2} \frac{dP}{dz} = -3z \\
 & \therefore \int \frac{1}{P^2} \frac{dP}{dz} dz = \int -3z dz \\
 & \therefore \int P^{-2} dP = -3 \int z dz \\
 & \therefore \frac{P^{-1}}{-1} = -3 \left( \frac{z^2}{2} \right) + c \\
 & \therefore -\frac{1}{P} = -\frac{3z^2}{2} + c \\
 & \text{But } P(2) = 1, \text{ so } -1 = -6 + c \\
 & \quad \therefore c = 5 \\
 & \therefore -\frac{1}{P} = -\frac{3}{2}z^2 + 5 \\
 & \therefore P = \frac{1}{\frac{3}{2}z^2 - 5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{108} \quad & \frac{dy}{dx} = \frac{xy}{x-1} \\
 & \therefore \frac{1}{y} \frac{dy}{dx} = \frac{x}{x-1} \\
 & \therefore \frac{1}{y} \frac{dy}{dx} = \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1} \\
 & \therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int \left( 1 + \frac{1}{x-1} \right) dx \\
 & \therefore \ln|y| = x + \ln|x-1| + c \\
 & \text{But, when } x = 2, y = 2 \\
 & \quad \therefore \ln 2 = 2 + c \\
 & \quad \therefore c = \ln 2 - 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{dy}{dx} = x + \frac{1}{3}xy = \frac{1}{3}x(3 + y) \\
 & \frac{1}{3+y} \frac{dy}{dx} = \frac{1}{3}x \\
 & \therefore \int \frac{1}{3+y} \frac{dy}{dx} dx = \int \frac{1}{3}x dx \\
 & \therefore \int \frac{1}{3+y} dy = \frac{1}{3} \int x dx \\
 & \therefore \ln|y+3| = \frac{1}{3} \frac{x^2}{2} + c \\
 & \quad = \frac{1}{6}x^2 + c \\
 & \therefore |y+3| = e^c e^{\frac{1}{6}x^2} \\
 & \therefore y+3 = \pm e^c e^{\frac{1}{6}x^2} \\
 & \therefore y+3 = Ae^{\frac{1}{6}x^2} \quad \{A = \pm e^c\} \\
 & \therefore y = Ae^{\frac{1}{6}x^2} - 3 \\
 & \text{But when } x = 1, y = 2, \text{ so } 2 = Ae^{\frac{1}{6}} - 3 \\
 & \quad \therefore A = 5e^{-\frac{1}{6}} \\
 & \therefore y = 5e^{-\frac{1}{6}} e^{\frac{1}{6}x^2} - 3 \\
 & \quad = 5e^{\frac{1}{6}(x^2-1)} - 3
 \end{aligned}$$

$$\begin{aligned}
 & \text{So, } \ln|y| = x + \ln|x-1| + \ln 2 - 2 \\
 & \therefore \ln|y| - \ln|2(x-1)| = x - 2 \\
 & \therefore \ln \left| \frac{y}{2(x-1)} \right| = x - 2 \\
 & \therefore \left| \frac{y}{2(x-1)} \right| = e^{x-2} \\
 & \therefore \frac{y}{2(x-1)} = \pm e^{x-2} \\
 & \text{But } x = 2, y = 2 \text{ does not satisfy the negative solution} \\
 & \therefore y = 2(x-1)e^{x-2}.
 \end{aligned}$$



**109**

$$\frac{dy}{dx} = \frac{2x}{\sin y}$$

$$\therefore \sin y \frac{dy}{dx} = 2x$$

$$\therefore \int \sin y \frac{dy}{dx} dx = \int 2x dx$$

$$\therefore \int \sin y dy = \int 2x dx$$

$$\therefore -\cos y = x^2 + c$$

$$\therefore \cos y = c - x^2$$

But  $y(0) = \frac{\pi}{6}$ , so  $\cos \frac{\pi}{6} = c - 0$

$$\therefore c = \frac{\sqrt{3}}{2}$$

So, the particular solution is  $\cos y = \frac{\sqrt{3}}{2} - x^2$  which is defined where

$$-1 \leq \cos y \leq 1$$

$$\therefore -1 \leq \frac{\sqrt{3}}{2} - x^2 \leq 1$$

$$\therefore -1 \leq x^2 - \frac{\sqrt{3}}{2} \leq 1$$

$$\therefore -1 + \frac{\sqrt{3}}{2} \leq x^2 \leq 1 + \frac{\sqrt{3}}{2}$$

$$\therefore 0 \leq x^2 \leq 1 + \frac{\sqrt{3}}{2} \quad \left\{ -1 + \frac{\sqrt{3}}{2} < 0 \right\}$$

$$\therefore -\sqrt{1 + \frac{\sqrt{3}}{2}} \leq x \leq \sqrt{1 + \frac{\sqrt{3}}{2}}$$

**110 a**

$$\frac{dv}{dt} = -kv, \quad k > 0$$

$$\therefore \int \frac{1}{v} \frac{dv}{dt} dt = \int -k dt$$

$$\therefore \int \frac{1}{v} dv = - \int k dt$$

$$\therefore \ln v = -kt + c \quad \{v > 0\}$$

$$\therefore v(t) = e^{-kt+c}$$

Now  $v(0) = 100$ , so  $\ln 100 = c$  and  $v(2) = 40$ ,

$$\text{so } \ln 40 = -2k + \ln 100$$

$$\therefore 2k = \ln 100 - \ln 40$$

$$\therefore 2k = \ln \frac{5}{2}$$

$$\therefore k = \frac{1}{2} \ln \left( \frac{5}{2} \right) \quad \text{as required.}$$

**b** Since  $v > 0$  for all  $t$ , there is no change in direction.

$\therefore$  in the first 2 seconds, the object travels a distance

$$= \int_0^2 v(t) dt$$

$$= \int_0^2 e^{-kt+c} dt$$

$$= \left[ \frac{1}{-k} e^{-kt+c} \right]_0^2$$

$$= \frac{1}{-k} [v(t)]_0^2$$

$$= \frac{1}{-k} (v(2) - v(0))$$

$$= \frac{1}{-k} (40 - 100)$$

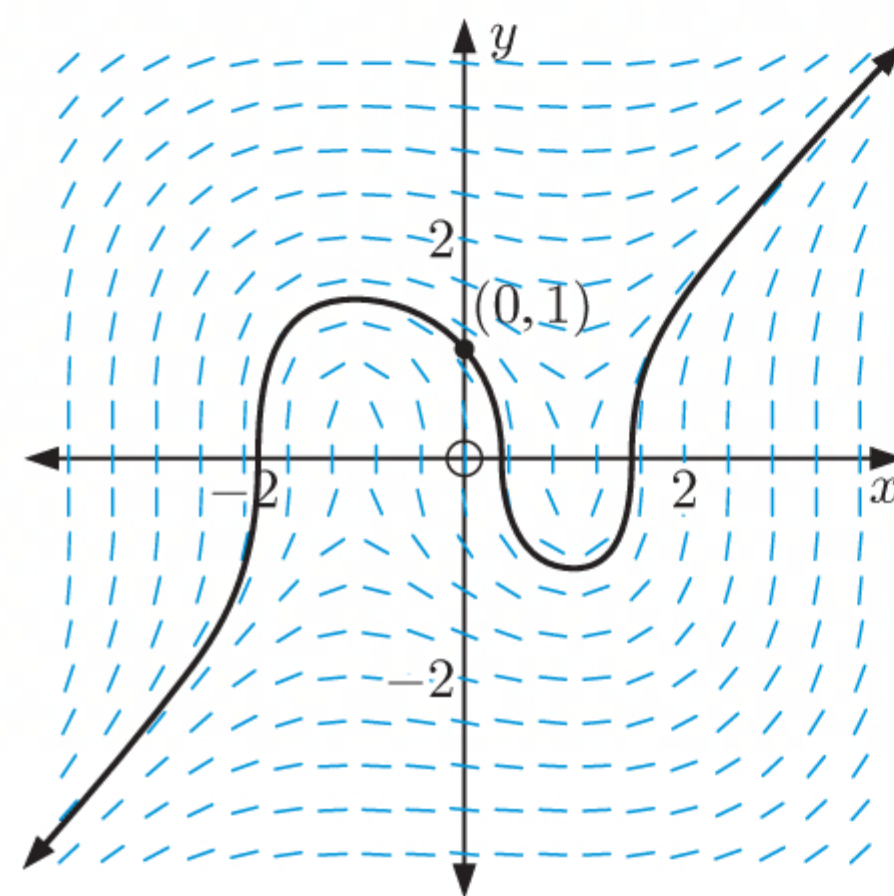
$$= \frac{-60}{-\frac{1}{2} \ln \left( \frac{5}{2} \right)} \quad \{\text{from a}\}$$

$$\approx 131 \text{ m}$$

**111 a**

At  $(0, 1)$ ,  $\frac{dy}{dx} = \frac{0^2 - 1}{1^2} = -1$

$\therefore$  the gradient of the tangent to the solution curve at  $(0, 1)$  is  $-1$ .

**b**



$$\mathbf{c} \quad \frac{dy}{dx} = \frac{x^2 - 1}{y^2}$$

$$\therefore y^2 \frac{dy}{dx} = x^2 - 1$$

$$\therefore \int y^2 \frac{dy}{dx} dx = \int (x^2 - 1) dx$$

$$\therefore \int y^2 dy = \int (x^2 - 1) dx$$

$$\therefore \frac{1}{3}y^3 = \frac{1}{3}x^3 - x + c$$

The solution curve in **b** passes through  $(0, 1)$

$$\therefore \frac{1}{3} = \frac{1}{3}(0)^3 - 0 + c$$

$$\therefore c = \frac{1}{3}$$

So, the equation of the solution curve is  $\frac{1}{3}y^3 = \frac{1}{3}x^3 - x + \frac{1}{3}$

$$\text{or } y^3 = x^3 - 3x + 1.$$

**112 a** Equilibrium points occur where  $\dot{x} = \dot{y} = 0$ .

Now when  $\dot{x} = 0$

$$2y - x + 2 = 0$$

$$\therefore x = 2y + 2 \quad \dots (*)$$

and when  $\dot{y} = 0$

$$y - 4x = 0$$

$$\therefore y - 4(2y + 2) = 0 \quad \{\text{using } (*)\}$$

$$\therefore -7y - 8 = 0$$

$$\therefore y = -\frac{8}{7} \quad \text{and} \quad x = 2\left(-\frac{8}{7}\right) + 2 = -\frac{2}{7}$$

$\therefore \left(-\frac{2}{7}, -\frac{8}{7}\right)$  is the equilibrium point.

From the phase portrait, the equilibrium point is a centre.

$$\mathbf{b} \quad \text{At } (0, 0), \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 - 0 + 2 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$\therefore$  the solution curves rotate clockwise.

**113 a** Using the given eigenvalues and eigenvectors, a general solution to the system is  $\mathbf{x} = Ae^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + Be^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

$$\mathbf{b} \quad \mathbf{i} \quad \text{When } t = 0, \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \dot{\mathbf{x}} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\mathbf{ii} \quad \text{When } t = 0, \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore A \begin{pmatrix} -2 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

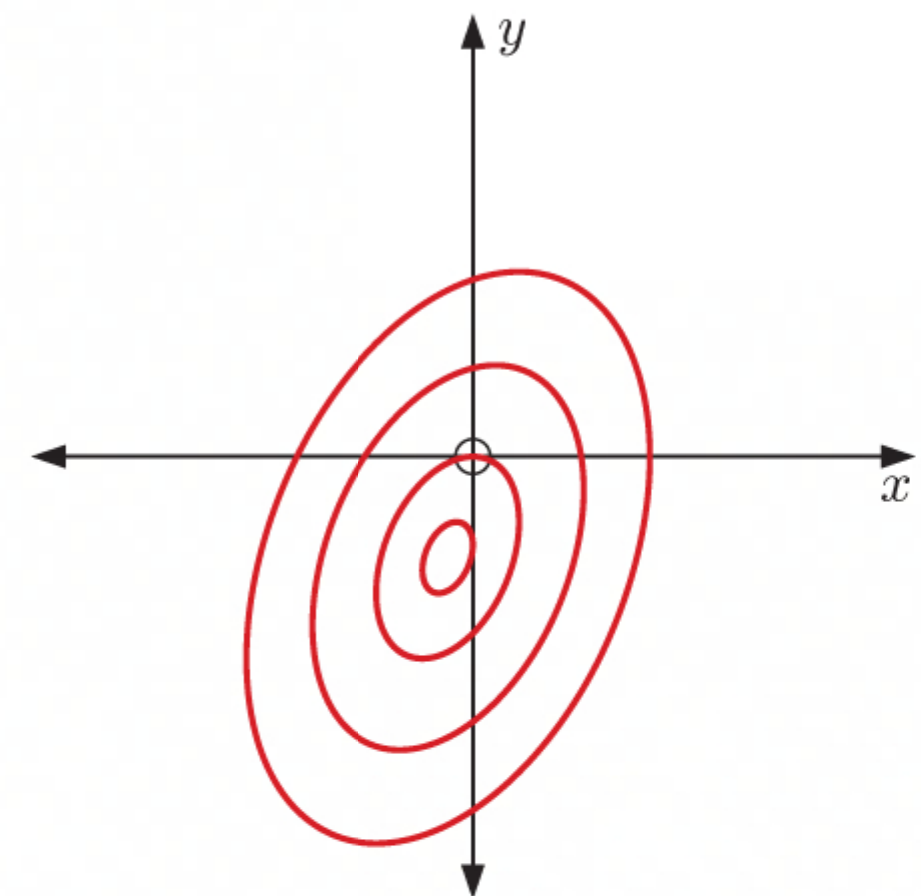
$$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

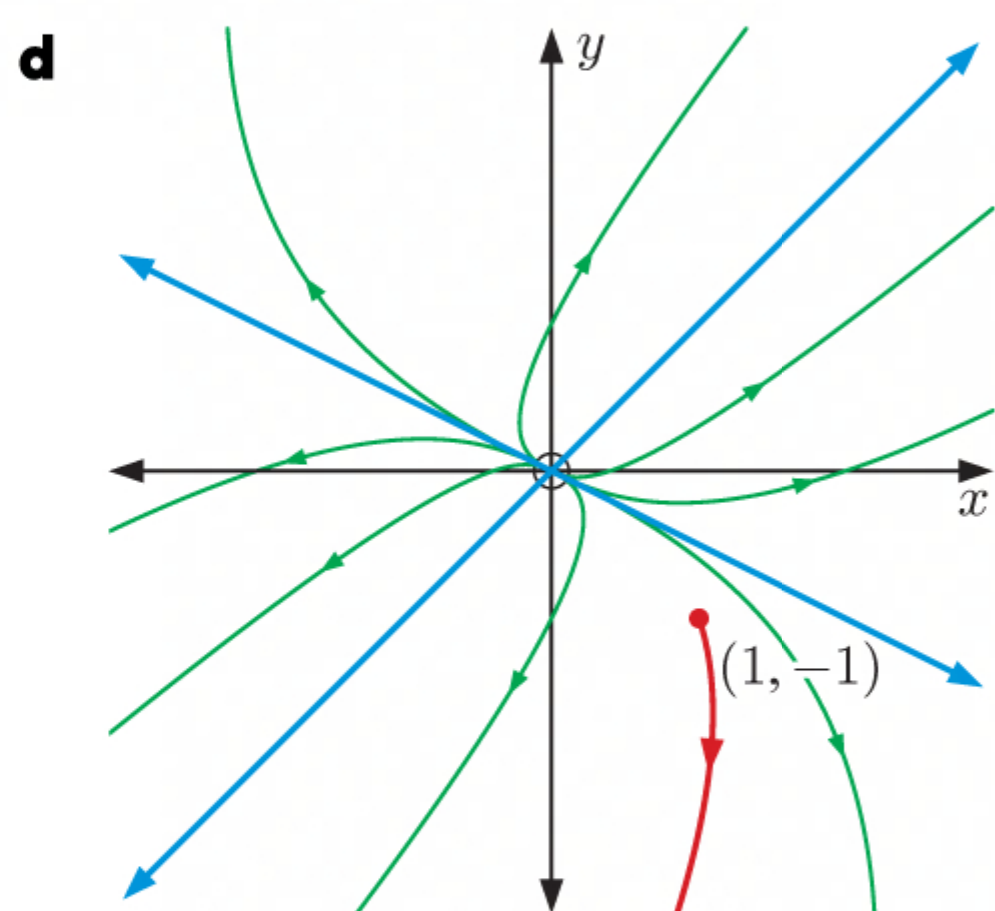
The particular solution is  $\mathbf{x} = -\frac{2}{3}e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \frac{1}{3}e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

**c** The eigenvalues  $\lambda_1 = 2$  and  $\lambda_2 = 5$  satisfy  $\lambda_2 > \lambda_1 > 0$ .

$\therefore$  the equilibrium point at  $(0, 0)$  is an unstable fixed point.







**e** As  $t \rightarrow \infty$ , trajectories become parallel to  $k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $k \in \mathbb{R}$ .

**114 a** Let  $\mathbf{A} = \begin{pmatrix} 1 & -6 \\ 6 & -1 \end{pmatrix}$ .

If  $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$  then  $\begin{vmatrix} \lambda - 1 & 6 \\ -6 & \lambda + 1 \end{vmatrix} = 0$

$$\therefore (\lambda - 1)(\lambda + 1) + 36 = 0$$

$$\therefore \lambda^2 - 1 + 36 = 0$$

$$\therefore \lambda^2 = -35$$

$$\therefore \lambda = \pm i\sqrt{35}$$

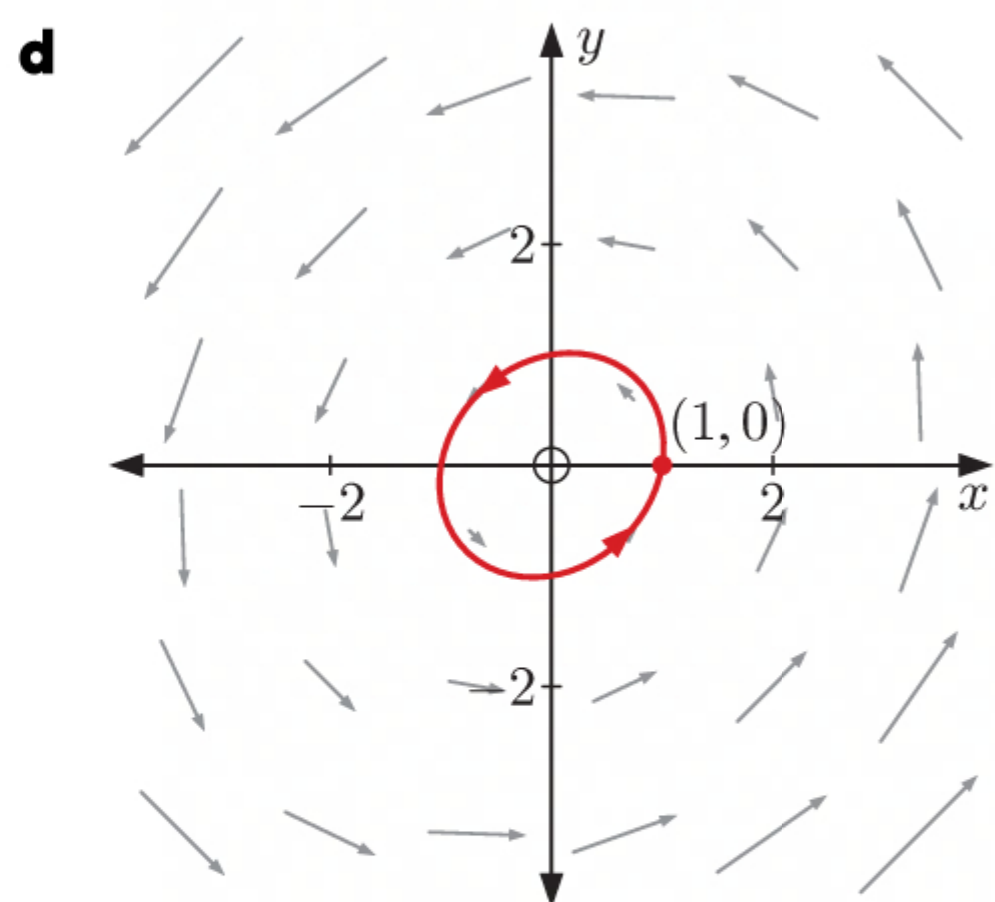
The eigenvalues are  $\pm i\sqrt{35}$  which are purely imaginary.

$\therefore$  the equilibrium point at  $O$  is a centre.

**b** When  $t = 0$ ,  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore$  the initial trajectory vector is  $\dot{\mathbf{x}} = \begin{pmatrix} 1 & -6 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ .

**c** Using **b**, the solution curves rotate anticlockwise.



**115 a** Let  $y = \frac{dx}{dt}$

$$\therefore \frac{dy}{dt} = \frac{d^2x}{dt^2}$$

$$\therefore \frac{dy}{dt} - 3y - 4x = 0$$

$$\therefore \frac{dy}{dt} = 4x + 3y$$

$$\therefore \text{the system is } \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = 4x + 3y \end{cases}.$$

**b** The system has matrix form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  where  $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}$ .



$$\begin{aligned}
 \text{c} \quad \text{If } \det(\lambda \mathbf{I} - \mathbf{A}) = 0 \text{ then } & \begin{vmatrix} \lambda & -1 \\ -4 & \lambda - 3 \end{vmatrix} = 0 \\
 & \therefore \lambda(\lambda - 3) - 4 = 0 \\
 & \therefore \lambda^2 - 3\lambda - 4 = 0 \\
 & \therefore (\lambda - 4)(\lambda + 1) = 0 \\
 & \therefore \lambda = 4 \text{ or } -1
 \end{aligned}$$

The eigenvalues are 4 and  $-1$ .

For  $\lambda_1 = 4$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  where  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{aligned}
 \therefore \begin{pmatrix} 4 & -1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 \therefore 4a - b &= 0
 \end{aligned}$$

Letting  $a = t$ ,  $t \neq 0$ , then  $b = 4t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$  choosing  $t = 1$ ,  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$  is an eigenvector corresponding to  $\lambda_1 = 4$ .

For  $\lambda_2 = -1$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  where  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{aligned}
 \therefore \begin{pmatrix} -1 & -1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 \therefore -a - b &= 0
 \end{aligned}$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = -t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$  choosing  $t = 1$ ,  $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$  is an eigenvector corresponding to  $\lambda_2 = -1$ .

**d** The eigenvalues satisfy  $\lambda_1 > 0 > \lambda_2$ .

$\therefore$  the equilibrium point at  $O$  is a saddle point.

**e i** Using the eigenvalues and eigenvectors in **c**, a general solution to the system is  $\mathbf{x} = Ae^{4t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + Be^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ .

$$\text{When } t = 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

$$\therefore A \begin{pmatrix} 1 \\ 4 \end{pmatrix} + B \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\text{The particular solution is } \mathbf{x} = -e^{4t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 2e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

**ii** As  $t \rightarrow \infty$ ,  $e^{-t} \rightarrow 0$

$$\therefore \mathbf{x} \rightarrow -e^{4t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\therefore x \rightarrow -\infty \quad \text{and} \quad \frac{dx}{dt} = y \rightarrow -\infty$$



**116 a** When  $t = 0$ ,  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\therefore$  the initial trajectory is  $\dot{\mathbf{x}} = \begin{pmatrix} 3(3)(-1) + 4(-1) - 3^2(-1) \\ 3(3) - (-1)^2 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$ .

**b i** Using step size  $h = 0.1$ ,

$$t_i = t_{i-1} + 0.1,$$

$$x_i = x_{i-1} + 0.1(3x_{i-1}y_{i-1} + 4y_{i-1} - x_{i-1}^2y_{i-1}),$$

and  $y_i = y_{i-1} + 0.1(3x_{i-1} - y_{i-1}^2)$ .

Using technology,  $x_5 \approx 3.92$  and  $y_5 \approx 2.60$ .

**ii** Using step size  $h = 0.1$ ,  $x_i \approx 4.00$  and  $y_i \approx 3.46$  for large  $i$ .

So, the equilibrium point P has coordinates  $(4.00, 3.46)$ .

**c** From the phase portrait, we see that P( $x, y$ ) has  $x > 0$  and  $y > 0$ .

Equilibrium points occur where  $\dot{x} = \dot{y} = 0$ .

Now  $\dot{x} = 0$  where  $3xy + 4y - x^2y = 0$

$$\therefore y(3x + 4 - x^2) = 0$$

$$\therefore x^2 - 3x - 4 = 0 \quad \{y > 0\}$$

$$\therefore (x - 4)(x + 1) = 0$$

$$\therefore x = 4 \quad \{x > 0\}$$

Also  $\dot{y} = 0$  where  $3x - y^2 = 0$

$$\therefore y^2 = 3(4) = 12$$

$$\therefore y = 2\sqrt{3} \quad \{y > 0\}$$

$\therefore$  P has coordinates  $(4, 2\sqrt{3})$ .

Now  $2\sqrt{3} \approx 3.46$  so our approximation in **b ii** is very accurate.

**117 a** If  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ ,  $t \geq 0$ , then  $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ .

When  $t = 0$ ,  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

Now  $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

$$\begin{aligned} \therefore \left| \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \right| &= \sqrt{(-\sin t)^2 + \cos^2 t} \\ &= \sqrt{\sin^2 t + \cos^2 t} \\ &= 1 \end{aligned}$$

So, the cat starts at  $(1, 0)$  and moves anticlockwise around the circle centred at  $(0, 0)$  with radius 1 unit at 1 unit per second.

$\therefore$  the position of the cat at time  $t$  is  $(\cos t, \sin t)$ .

**b** The dog is always swimming directly towards the cat.

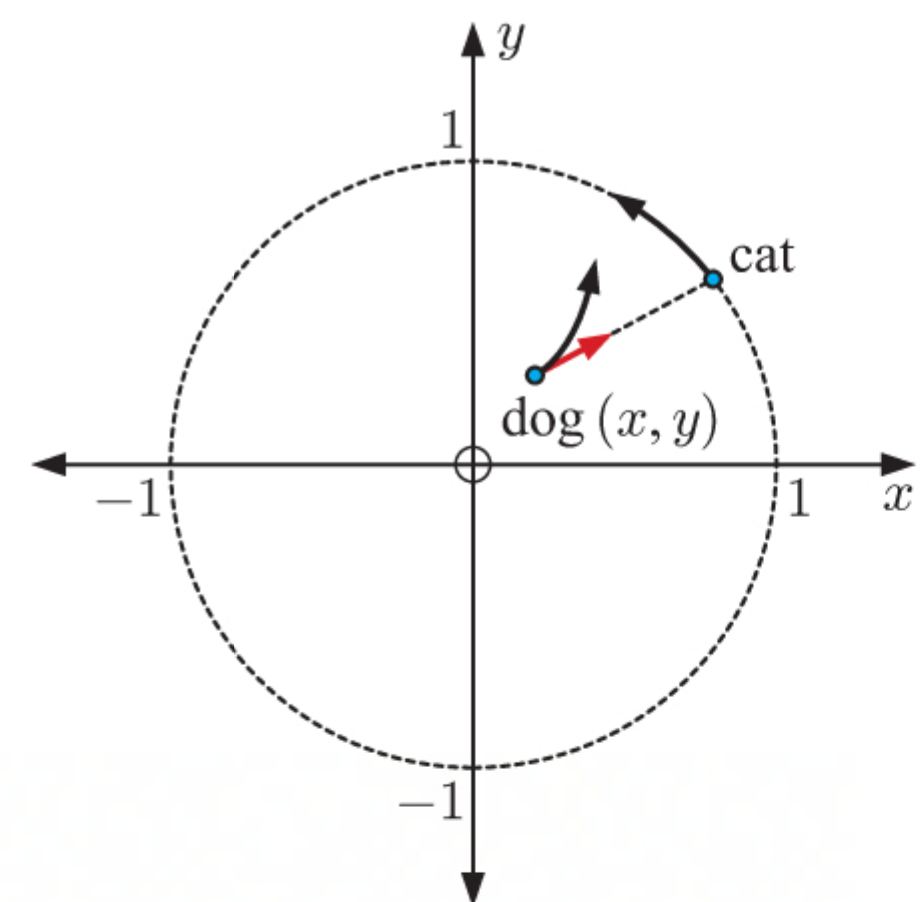
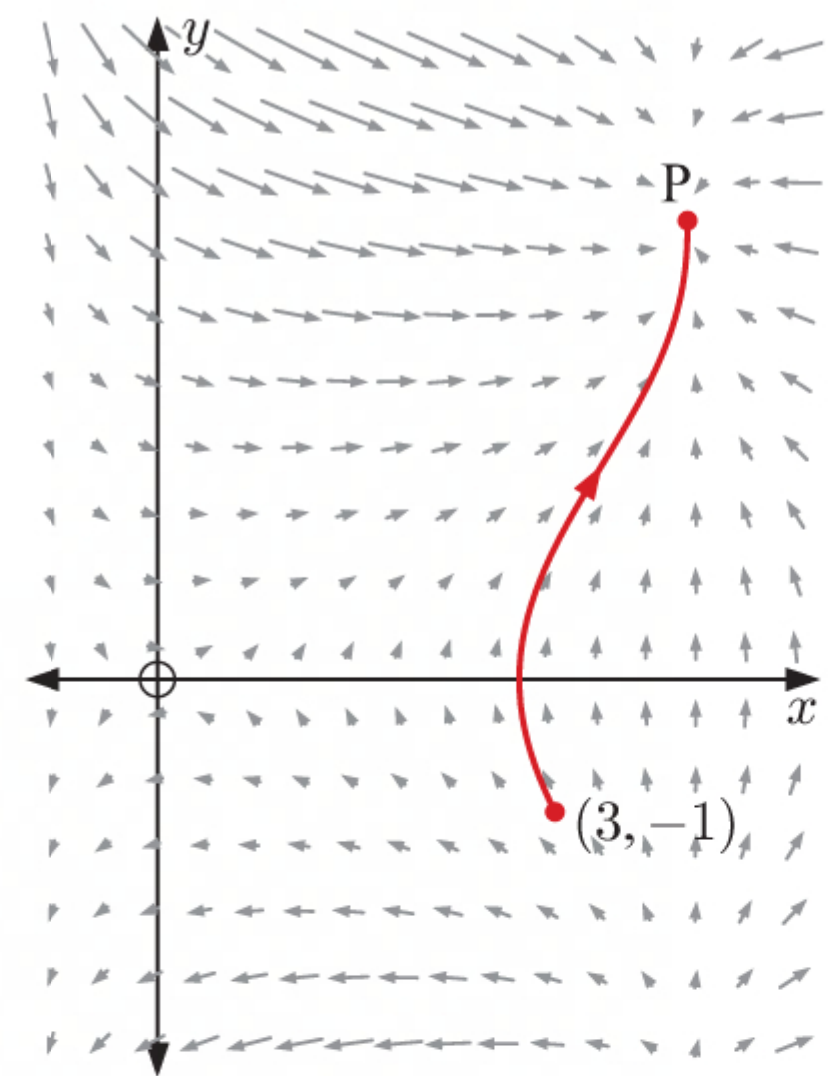
$\therefore$  at time  $t$ , the dog is swimming in the direction  $\mathbf{b} = \begin{pmatrix} \cos t - x \\ \sin t - y \end{pmatrix}$ .

The dog swims at  $D$  units per second.

$\therefore$  the dog has velocity vector  $\mathbf{v} = \frac{D}{|\mathbf{b}|} \mathbf{b} = \frac{D}{\sqrt{(\cos t - x)^2 + (\sin t - y)^2}} \begin{pmatrix} \cos t - x \\ \sin t - y \end{pmatrix}$

So, the position  $(x, y)$  of the dog at time  $t$  satisfies

$$\begin{cases} \dot{x} = \frac{\cos t - x}{\sqrt{(\cos t - x)^2 + (\sin t - y)^2}} D \\ \dot{y} = \frac{\sin t - y}{\sqrt{(\cos t - x)^2 + (\sin t - y)^2}} D. \end{cases}$$





$$\text{c If } D = 0.5, \quad \begin{cases} \dot{x} = \frac{0.5(\cos t - x)}{\sqrt{(\cos t - x)^2 + (\sin t - y)^2}} \\ \dot{y} = \frac{0.5(\sin t - y)}{\sqrt{(\cos t - x)^2 + (\sin t - y)^2}} \end{cases}$$

At time  $t_0 = 0$ ,  $x_0 = 0$  and  $y_0 = 0$ .

i Using step size  $h = 0.5$ ,  $t_i = t_{i-1} + 0.5$ ,

$$x_i = x_{i-1} + 0.25 \left( \frac{\cos t_{i-1} - x_{i-1}}{\sqrt{(\cos t_{i-1} - x_{i-1})^2 + (\sin t_{i-1} - y_{i-1})^2}} \right),$$

$$\text{and } y_i = y_{i-1} + 0.25 \left( \frac{\sin t_{i-1} - y_{i-1}}{\sqrt{(\cos t_{i-1} - x_{i-1})^2 + (\sin t_{i-1} - y_{i-1})^2}} \right).$$

Using technology,

$t$ (seconds)	Position	$t$ (seconds)	Position
1	(0.449, 0.152)	11	(-0.495, -0.118)
2	(0.340, 0.606)	12	(-0.160, -0.469)
3	(-0.140, 0.671)	13	(0.316, -0.381)
4	(-0.490, 0.330)	14	(0.497, 0.0680)
5	(-0.456, -0.157)	15	(0.214, 0.461)
6	(-0.0514, -0.427)	16	(-0.269, 0.433)
7	(0.393, -0.237)	17	(-0.507, 0.0105)
8	(0.450, 0.242)	18	(-0.282, -0.418)
9	(0.0625, 0.531)	19	(0.201, -0.462)
10	(-0.390, 0.356)	20	(0.498, -0.0793)

ii Using i,

**(1)** after 18 seconds, the dog is about  $\sqrt{(-0.282)^2 + (-0.418)^2} \approx 0.504$  units from the centre of the pond.

**(2)** after 19 seconds, the dog is about  $\sqrt{(0.201)^2 + (-0.462)^2} \approx 0.504$  units from the centre of the pond.

**(3)** after 20 seconds, the dog is about  $\sqrt{(0.498)^2 + (-0.0793)^2} \approx 0.504$  units from the centre of the pond.

iii In the long term, we predict that the dog will swim anticlockwise in a circle centred at  $(0, 0)$  with radius about 0.504 units.



# MIXED QUESTIONS

## MIXED QUESTIONS SET 1

1  $x$  could be from 5.5 cm to 6.5 cm.

$y$  could be from 3.5 cm to 4.5 cm.

$z$  could be from 6.5 cm to 7.5 cm.

$\theta$  could be from  $21.5^\circ$  to  $22.5^\circ$ .

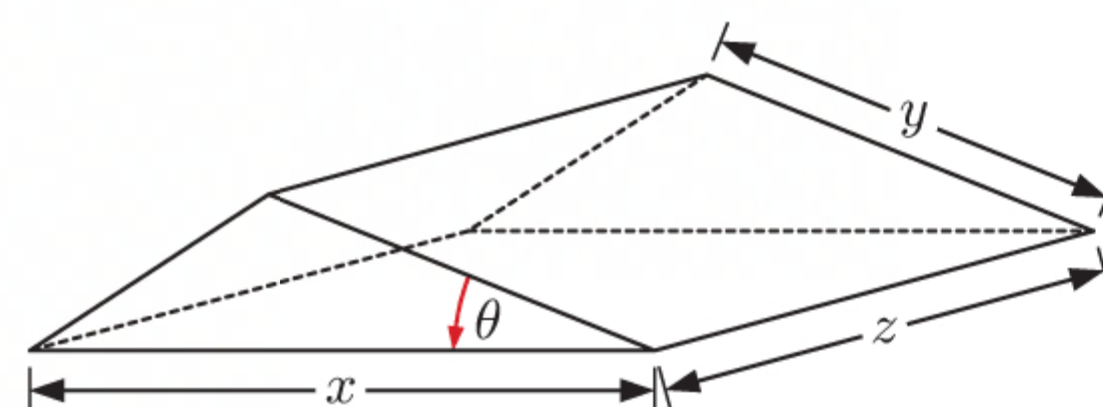
Now volume of prism = area of cross-section  $\times$  length

$$\begin{aligned} &= \frac{1}{2}xy \sin \theta \times z \\ &= \frac{xyz \sin \theta}{2} \end{aligned}$$

$\therefore$  the upper boundary of the volume is  $\frac{6.5 \times 4.5 \times 7.5 \sin 22.5^\circ}{2} \approx 42.0 \text{ cm}^3$

and the lower boundary of the volume is  $\frac{5.5 \times 3.5 \times 6.5 \sin 21.5^\circ}{2} \approx 22.9 \text{ cm}^3$ .

The volume of the prism is between about  $22.9 \text{ cm}^3$  and about  $42.0 \text{ cm}^3$ .



2 a  $H(t) = 80 - 5t^2 \text{ m}$   
 $\therefore H(0) = 80 - 5(0)^2 = 80 \text{ m}$   
 The initial height is 80 m.

b The toy hits the ground when  $H(t) = 0$

$$\therefore 80 - 5t^2 = 0$$

$$\therefore 80 = 5t^2$$

$$\therefore t^2 = 16$$

$$\therefore t = \pm 4$$

But  $t > 0$ , so  $t = 4$  seconds

So, the toy hits the ground after 4 seconds.

c  $H'(t) = -10t \text{ m s}^{-1}$   
 $H'(2) = -10(2) = -20 \text{ m s}^{-1}$

After 2 seconds of flight, the toy aeroplane is travelling at  $20 \text{ m s}^{-1}$  towards the ground.

3 a Let  $u_0$  be the original value of the car.

Since the value of the car depreciates by 10% each year, the value of the car after 3 years is

$$u_0 \times (1 - 0.1)^3 = u_0 \times (0.9)^3.$$

$$\therefore u_0 \times (0.9)^3 = 26\,244$$

$$\begin{aligned} \therefore u_0 &= \frac{26\,244}{(0.9)^3} \\ &= 36\,000 \end{aligned}$$

$\therefore$  the original value of the car is \$36 000.

c For the value of the car to fall below \$10 000, we need to find  $n$  such that  $u_n < 10\,000$ .

Using technology, the first term less than 10 000 is  $u_{13} \approx 9150.72$ .

So, in the 13th year the value of the car falls below \$10 000.

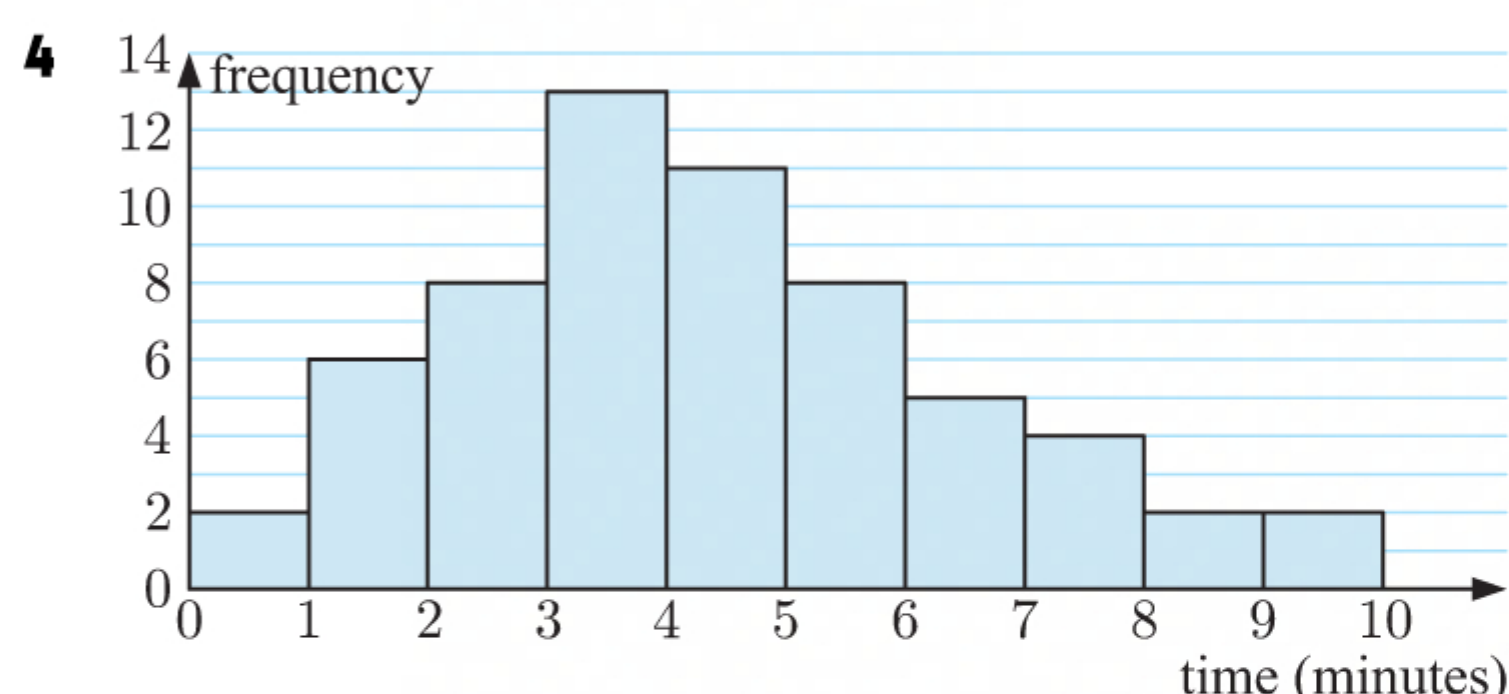
b Let  $u_n$  be the value of the car after  $n$  years.

$$\begin{aligned} \therefore u_n &= u_0 \times (1 - d)^n \\ &= 36\,000 \times (0.9)^n \end{aligned}$$

which describes a geometric sequence with

$u_0 = 36\,000$  and  $r = 0.9$ .

Math (Norm1) d/c (Real)	
Y1=36000*0.9^(x)	
X	Y1
10	12552
11	11297
12	10167
13	9150.7
9150.716982	
FORMULA DELETE ROW EDIT GRAPH-CON GRAPH-PLT	



a The modal class is  $3 \leq t < 4$  where  $t$  is the time in minutes.



**b**

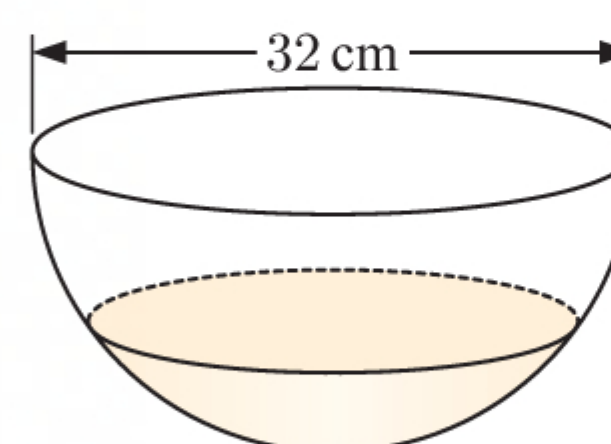
Duration of call ( $t$ min)	Frequency ( $f$ )	Midpoint ( $x$ )	Product ( $xf$ )
$0 \leq t < 1$	2	0.5	1
$1 \leq t < 2$	6	1.5	9
$2 \leq t < 3$	8	2.5	20
$3 \leq t < 4$	13	3.5	45.5
$4 \leq t < 5$	11	4.5	49.5
$5 \leq t < 6$	8	5.5	44
$6 \leq t < 7$	5	6.5	32.5
$7 \leq t < 8$	4	7.5	30
$8 \leq t < 9$	2	8.5	17
$9 \leq t < 10$	2	9.5	19
<i>Total</i>	$\sum f = 61$		$\sum xf = 267.5$

**c**  $\bar{x} = \frac{\sum xf}{\sum f}$   
 $= \frac{267.5}{61}$   
 $\approx 4.39$

$\therefore$  the mean length of a phone call is about 4.39 minutes.

**d**  $P(\geq 6 \text{ minutes}) \approx \frac{5 + 4 + 2 + 2}{61}$   
 $\approx \frac{13}{61}$   
 $\approx 0.213$

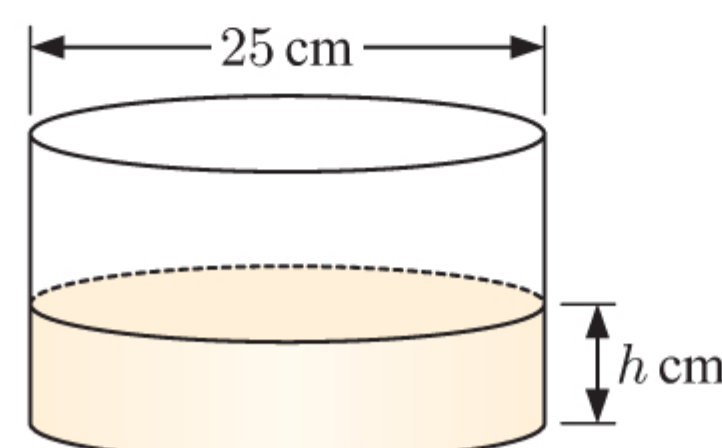
**5 a**  $V = \frac{1}{2} \times \text{volume of sphere}$   
 $= \frac{1}{2} \times \frac{4}{3} \pi r^3$   
 $= \frac{2}{3} \times \pi \times \left(\frac{32}{2}\right)^3 \text{ cm}^3$   
 $= \frac{8192}{3} \pi \text{ cm}^3$   
 $\approx 8580 \text{ cm}^3$



The capacity of the bowl is about 8580 mL or 8.58 L.

**b i** When 20% full, the bowl contains  $\frac{8192}{3} \pi \times 0.2 \approx 1720 \text{ mL}$  or about 1.72 L of cake batter.

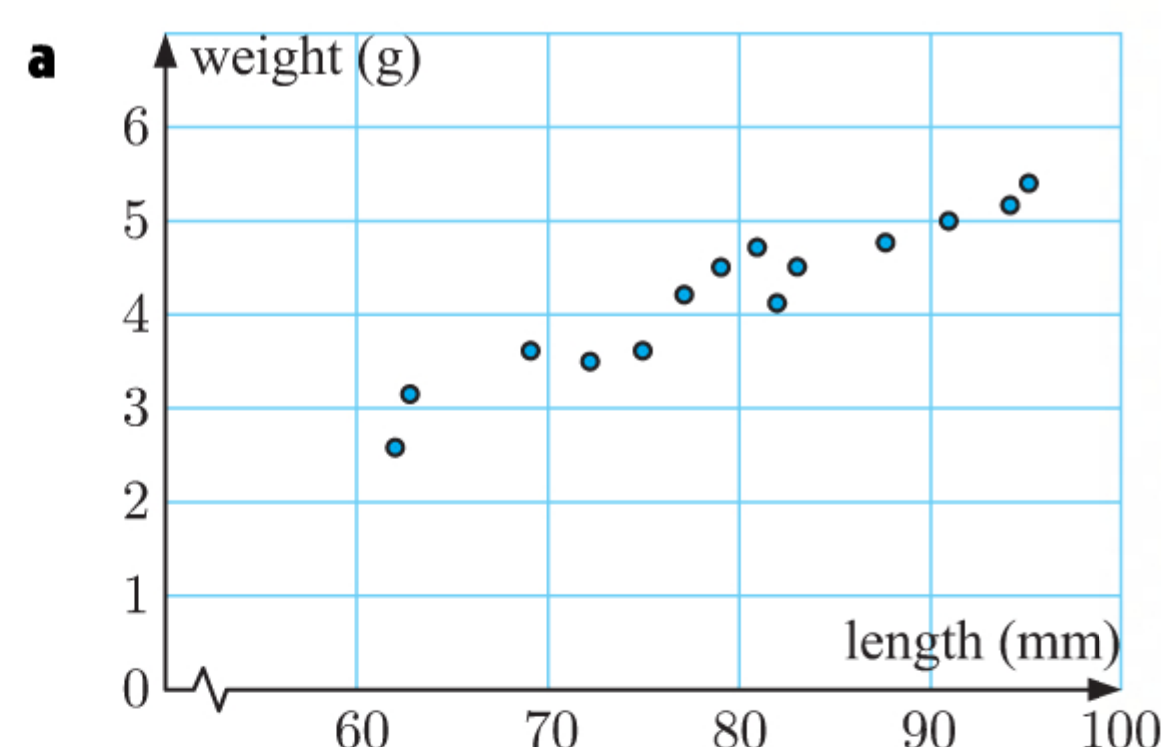
**ii**  $V = \frac{8192}{3} \pi \times 0.2 \text{ cm}^3$   
 $\therefore \pi \times \left(\frac{25}{2}\right)^2 \times h = \frac{8192}{3} \pi \times 0.2$   
 $\therefore h = \frac{8192 \times 0.2}{3 \times (12.5)^2}$   
 $\approx 3.50 \text{ cm}$



The cake batter will reach about 3.50 cm up the tin.

**6**

Length (mm)	95	83	91	82	75	62	79	63	81	69	94	88	72	77
Weight (g)	5.4	4.5	5.0	4.1	3.7	2.6	4.5	3.1	4.7	3.7	5.1	4.8	3.6	4.2



**b**

```

LinearReg(ax+b)
a = 0.07291841
b = -1.5723113
r = 0.96187039
r^2 = 0.92519465
MSE = 0.05184722
y = ax + b
    
```

So,  $r \approx 0.962$ .

**c** There is very strong, positive, linear correlation between *length* and *weight*.

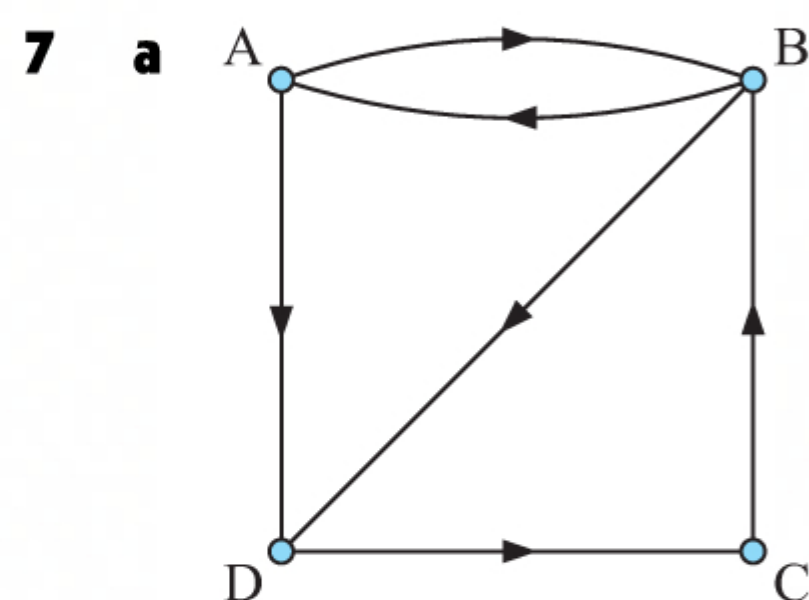
**d** From the screenshot in **b**, the equation of the least squares regression line is  $y \approx 0.0729x - 1.57$ .

**e i** When  $x = 110 \text{ mm}$ ,  
 $y \approx 0.0729 \times 110 - 1.57$   
 $\approx 6.45 \text{ g}$

**ii** When  $x = 70 \text{ mm}$ ,  
 $y \approx 0.0729 \times 70 - 1.57$   
 $\approx 3.53 \text{ g}$

**f** The prediction in **e ii** is more likely to be reliable, as it is an interpolation.





**b**  $A = \begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$

**c** Using technology,  $A + A^2 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

There are 0s in the matrix  $A + A^2$ , which indicates that there are pairs of cities which you cannot travel between in at most 2 flights.

**d** Using technology,  $A + A^2 + A^3 = \begin{pmatrix} 1 & 3 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

There are no 0s in the matrix  $A + A^2 + A^3$ , which indicates that it is possible to travel between any two cities in at most 3 flights.

**e** If direct flights from C to D and D to A are added, the adjacency matrix becomes  $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ .

Using technology,  $A + A^2 = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}$

There are no 0s in the matrix  $A + A^2$ , which indicates that it is now possible to travel between any two cities in at most 2 flights.

**8**  $f(x) = \ln(x\sqrt{1-2x})$

**a**  $\ln(x\sqrt{1-2x})$  is defined when

$$x\sqrt{1-2x} > 0$$

$$\therefore x > 0 \quad \text{and} \quad 1 - 2x > 0$$

$$\therefore x > 0 \quad \text{and} \quad 2x < 1$$

$$\therefore x > 0 \quad \text{and} \quad x < \frac{1}{2}$$

So, the domain is  $\{x \mid 0 < x < \frac{1}{2}\}$ .

**b**  $f(x) = \ln(x(1-2x)^{\frac{1}{2}})$

$$\begin{aligned} \therefore f'(x) &= \frac{(1-2x)^{\frac{1}{2}} + \frac{1}{2}x(1-2x)^{-\frac{1}{2}}(-2)}{x(1-2x)^{\frac{1}{2}}} \times \frac{(1-2x)^{\frac{1}{2}}}{(1-2x)^{\frac{1}{2}}} \\ &= \frac{(1-2x) - x}{x(1-2x)} \\ &= \frac{1-3x}{x(1-2x)} \end{aligned}$$

**c** At the point(s) where the normal has gradient  $-\frac{6}{5}$ , the tangent has gradient  $\frac{5}{6}$ .

Now  $f'(x) = \frac{5}{6}$  where  $\frac{1-3x}{x(1-2x)} = \frac{5}{6}$  {from **b**}

$$\therefore 6(1-3x) = 5x(1-2x)$$

$$\therefore 6 - 18x = 5x - 10x^2$$

$$\therefore 10x^2 - 23x + 6 = 0$$

$$\therefore (10x-3)(x-2) = 0$$

$$\therefore x = \frac{3}{10} \quad \{0 < x < \frac{1}{2}\}$$

$$\begin{aligned} f\left(\frac{3}{10}\right) &= \ln\left(\frac{3}{10}\sqrt{1-2\left(\frac{3}{10}\right)}\right) \\ &= \ln\left(\frac{3}{10}\sqrt{1-\frac{3}{5}}\right) \\ &= \ln\left(\frac{3}{10}\sqrt{\frac{2}{5}}\right) \\ &\approx -1.66 \end{aligned}$$

$\therefore$  the normal to  $y = f(x)$  has gradient  $-\frac{6}{5}$  at  $\left(\frac{3}{10}, -1.66\right)$ .



9  $\frac{dy}{dx} = \frac{x}{1+x^2}$

$y(0) = 2$  gives us  $x_0 = 0$  and  $y_0 = 2$ .

**a**

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	2	0	0.2	2
2	0.2	2	0.1923	0.4	2.0385
3	0.4	2.0385	0.3448	0.6	2.1074
4	0.6	2.1074	0.4412	0.8	2.1957
5	0.8	2.1957	0.4878	1	2.2932

$\therefore y(1) \approx 2.2932$

**b** Using technology to estimate  $y(1)$  using Euler's method with step size 0.005 for 200 steps, we get  $y(1) \approx 2.3453$ .

**c**  $y(1) = y(0) + \int_0^1 \frac{dy}{dx} dx$  {Fundamental Theorem of Calculus}

$$= 2 + \int_0^1 \frac{x}{1+x^2} dx$$

$$= 2 + \int_0^2 \frac{1}{u} \left( \frac{1}{2} \frac{du}{dx} \right) dx \quad \begin{cases} u = 1 + x^2, & \frac{du}{dx} = 2x \\ \text{When } x = 0, & u = 1 \\ \text{When } x = 1, & u = 2 \end{cases}$$

$$= 2 + \frac{1}{2} \int_1^2 \frac{1}{u} du$$

$$= 2 + \frac{1}{2} [\ln |u|]_1^2$$

$$= 2 + \frac{1}{2} (\ln 2 - 0)$$

$$= 2 + \frac{1}{2} \ln 2$$

$$\approx 2.3466$$

The accuracy of Euler's method was improved by decreasing the step size.

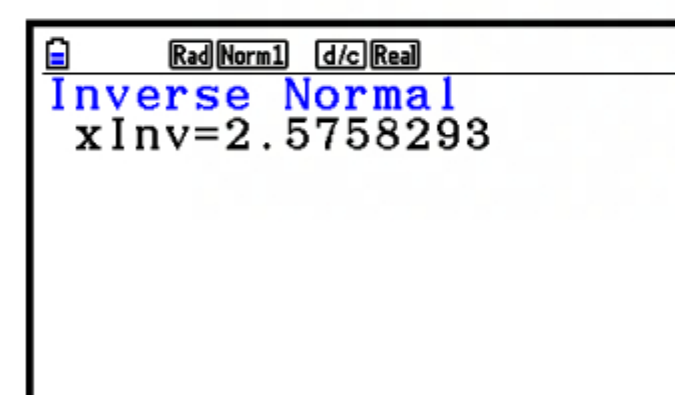
10 **a**  $H_0: \mu = 27.3$  {water consumption has not changed}

$H_1: \mu \neq 27.3$  {water consumption has changed}

**b** We know the population is normally distributed with known standard deviation, so a  $Z$ -test is appropriate.

The significance level is  $\alpha = 0.01$ .

**i** The critical value  $c$  satisfies  $P(Z \leq -c \text{ or } Z \geq c) = 0.01$   
 $\therefore P(Z \geq c) = 0.005$   
 $\therefore c = z_{0.005} \approx 2.58$



**ii** The critical region is  $\{z \mid z \leq -c \text{ or } z \geq c\} = \{z \mid z \leq -2.58 \text{ or } z \geq 2.58\}$ .

**iii** The acceptance region is  $\{z \mid -c \leq z \leq c\} = \{-2.58 < z < 2.58\}$ .

**c i**  $\bar{x} = 29.6$  kL,  $\sigma = 5$  kL,  $n = 50$

The test statistic is  $z = \frac{29.6 - 27.3}{\frac{5}{\sqrt{50}}} \approx 3.25$

**ii** 3.25 is in the critical region, so there is sufficient evidence to reject  $H_0$ .

So, there is sufficient evidence to conclude that water consumption has changed this year.



## MIXED QUESTIONS SET 2

1 Let W denote Wollongong, P denote Picton, and C denote Canberra.

a North-west is in the direction  $315^\circ$ . South-west is in the direction  $225^\circ$ .

$$\widehat{PWN} = 360^\circ - 315^\circ = 45^\circ$$

$$\widehat{WPN} = 180^\circ - 45^\circ = 135^\circ \quad \{\text{co-interior angles}\}$$

$$\widehat{CPW} = 225^\circ - 135^\circ = 90^\circ$$

$\therefore \triangle CPW$  is right angled at P.

$$CW^2 = 36^2 + 210^2 \quad \{\text{Pythagoras}\}$$

$$\begin{aligned} \therefore CW &= \sqrt{36^2 + 210^2} \quad \{\text{as } CW > 0\} \\ &\approx 213 \text{ km} \end{aligned}$$

So, Canberra is about 213 km from Wollongong.

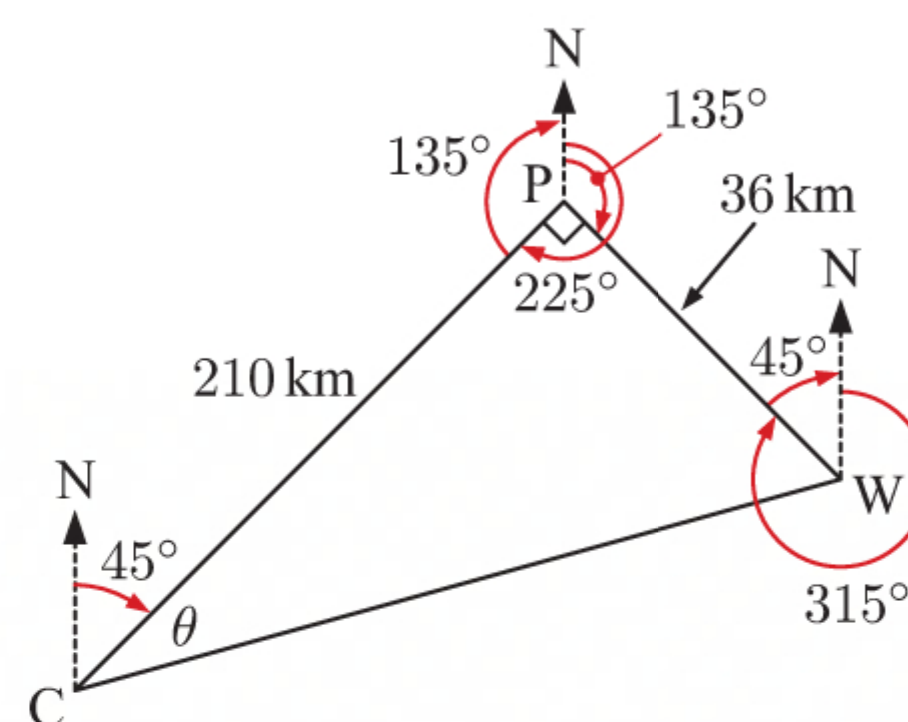
b  $\widehat{CPN} = 360^\circ - 225^\circ = 135^\circ$

$$\widehat{NCP} = 180^\circ - 135^\circ = 45^\circ \quad \{\text{co-interior angles}\}$$

$$\tan \theta = \frac{36}{210}$$

$$\therefore \theta = \tan^{-1}\left(\frac{36}{210}\right) \approx 9.73^\circ$$

$\therefore$  the bearing of Wollongong from Canberra  $\approx 45^\circ + 9.73^\circ \approx 054.7^\circ$



2  $f'(x) = \frac{a}{x^2} + bx^2, \quad f(-1) = -7, \quad f(1) = 7, \quad f(2) = 26$   
 $= ax^{-2} + bx^2$

$$\begin{aligned} \therefore f(x) &= \int (ax^{-2} + bx^2) dx \\ &= -ax^{-1} + \frac{b}{3}x^3 + c \\ &= -\frac{a}{x} + \frac{b}{3}x^3 + c \end{aligned}$$

Now  $f(-1) = -7,$

$f(1) = 7,$  and

$f(2) = 26$

$$\therefore -\frac{a}{-1} + \frac{b}{3}(-1)^3 + c = -7$$

$$\therefore -\frac{a}{1} + \frac{b}{3}(1)^3 + c = 7$$

$$\therefore -\frac{a}{2} + \frac{b}{3}(2)^3 + c = 26$$

$$\therefore a - \frac{1}{3}b + c = -7$$

$$\therefore -a + \frac{1}{3}b + c = 7$$

$$\therefore -\frac{1}{2}a + \frac{8}{3}b + c = 26$$

So we have the system of equations 
$$\begin{cases} a - \frac{1}{3}b + c = -7 \\ -a + \frac{1}{3}b + c = 7 \\ -\frac{1}{2}a + \frac{8}{3}b + c = 26 \end{cases}$$

Solving simultaneously gives  $a = -4, b = 9, c = 0$ .

$$\begin{aligned} \therefore f(x) &= -\frac{-4}{x} + \frac{9}{3}x^3 \\ &= \frac{4}{x} + 3x^3 \end{aligned}$$

	a	b	c	d
1	1	-0.333	1	-7
2	-1	0.3333	1	7
3	-0.5	2.6666	1	26
				26

	X	Y	Z
	-4	9	0
			-4

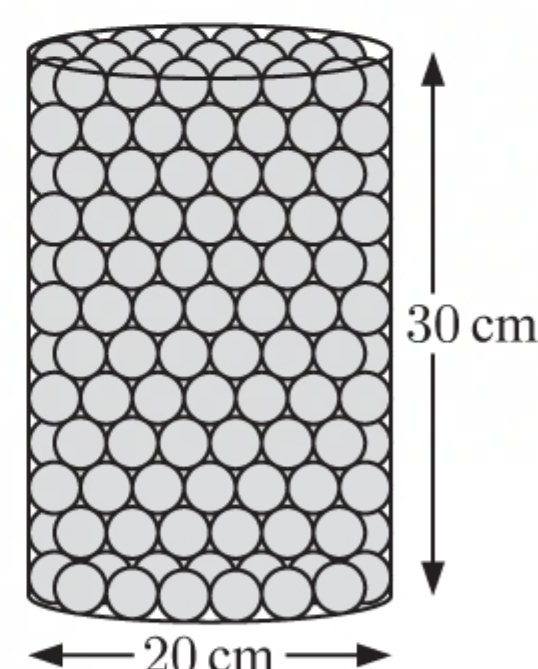
3 a Volume of jar  $= \pi\left(\frac{20}{2}\right)^2 \times 30 = 3000\pi \text{ cm}^3$

$$\begin{aligned} \text{Total volume of marbles} &= 0.6 \times \text{volume of jar} \\ &= 0.6 \times 3000\pi \\ &= 1800\pi \text{ cm}^3 \end{aligned}$$

Volume of 1 marble with radius  $r \text{ cm} = \frac{4}{3}\pi r^3 \text{ cm}^3$

$$\begin{aligned} \therefore \text{number of marbles } N &= \frac{\text{total volume of marbles}}{\text{volume of 1 marble}} \\ &= \frac{1800\pi}{\frac{4}{3}\pi r^3} \\ &= \frac{1350}{r^3} \end{aligned}$$

$\therefore$  the model is  $N = \frac{1350}{r^3}$ .





**b** When  $r = 1.5$ ,  $N = \frac{1350}{(1.5)^3} = 400$  marbles.

**c** The estimate in **b** is an underestimate because the actual percentage of the jar's volume that the marbles occupy is greater than Julie's assumption.

$$\begin{aligned} \text{d Percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\ &= \frac{|400 - 426|}{426} \times 100\% \\ &= \frac{26}{426} \times 100\% \\ &\approx 6.10\% \end{aligned}$$

**4**  $X \sim N(40, (6.8)^2)$

**a i**

Normal C.D	Variable
Data	: Variable
Lower	: $-9 \times 10^9$
Upper	: 25
$\sigma$	: 6.8
$\mu$	: 40
Save Res	: None
None	: LIST

Normal C.D	Variable
p	: 0.01369611
z: Low	: $-1.324 \times 10^9$
z: Up	: $-2.2058824$

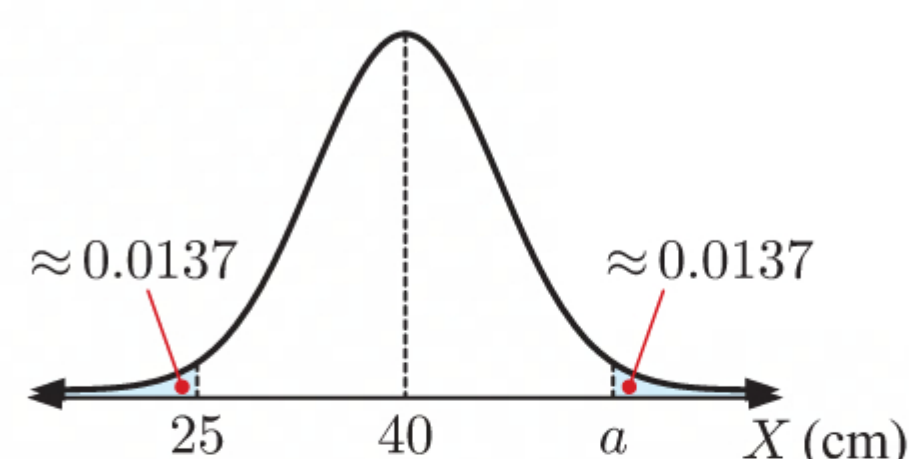
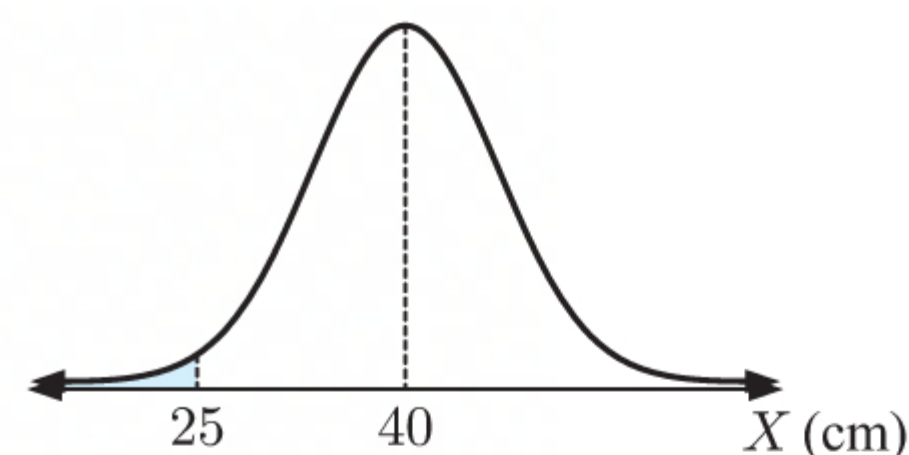
$$P(X < 25) \approx 0.0137$$

**ii** 25 is 15 less than 40.

By symmetry,  $P(X < 25) = P(X > 40 + 15)$

$$\therefore P(X < 25) = P(X > 55)$$

$$\therefore a = 55$$



**b**

Normal C.D	Variable
Data	: Variable
Lower	: 35
Upper	: $9 \times 10^9$
$\sigma$	: 6.8
$\mu$	: 40
Save Res	: None
None	: LIST

Normal C.D	Variable
p	: 0.7689198
z: Low	: $-0.7352941$
z: Up	: $1.3235 \times 10^9$

$$P(X > 35) \approx 0.769$$

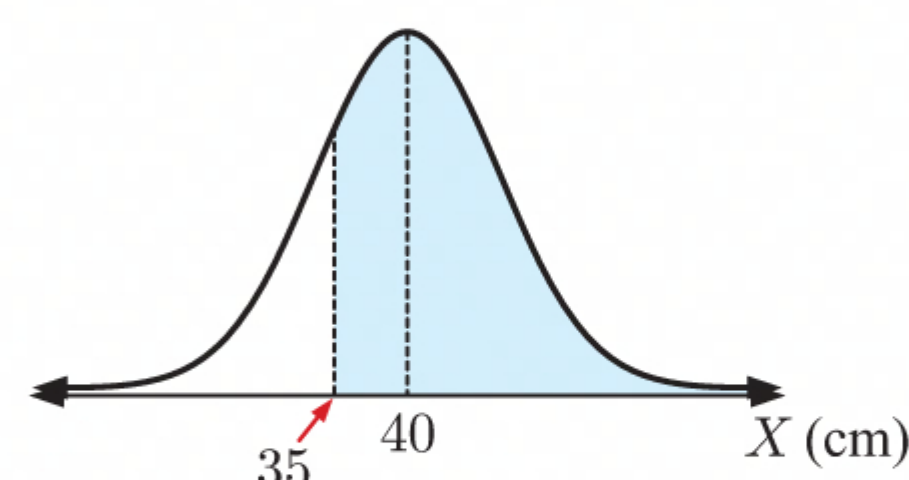
Let Y be the number of maize plants more than 35 cm high.

$n = 6$ , so  $Y = 0, 1, 2, 3, 4, 5$ , or  $6$  and  $p \approx 0.769$ .

$Y \sim B(6, 0.769)$

So,  $P(Y = 4) \approx 0.280$

BinomialPD	Variable
BinomialPD	: (4, 6, 0.769)
	: 0.2799113946
Bpd	: Bcd
	: InvB



**5**

$x$	-2	0	3	5
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	$k$	$\frac{1}{12}$

**a** X can only take the values -2, 0, 3, or 5.

$\therefore X$  is a discrete random variable.

**b** Since this is a probability distribution,  $P(X = -2) + P(X = 0) + P(X = 3) + P(X = 5) = 1$

$$\therefore \frac{1}{3} + \frac{1}{6} + k + \frac{1}{12} = 1$$

$$\therefore k + \frac{7}{12} = 1$$

$$\therefore k = \frac{5}{12}$$

**c** Since  $P(X = 3)$  is the greatest probability, 3 is the mode of the distribution.

$$P(X = -2) = \frac{1}{3} \approx 0.333$$

$$P(X = -2) + P(X = 0) = \frac{1}{3} + \frac{1}{6} = 0.5$$

Since  $P(X = -2) + P(X = 0) \geq 0.5$ , the median is 0.



$$\begin{aligned} \text{d } E(X) &= -2\left(\frac{1}{3}\right) + 0\left(\frac{1}{6}\right) + 3\left(\frac{5}{12}\right) + 5\left(\frac{1}{12}\right) \\ &= -\frac{2}{3} + 0 + \frac{5}{4} + \frac{5}{12} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - \mu^2 \\ &= \frac{1}{3}(-2)^2 + \frac{1}{6}(0)^2 + \frac{5}{12}(3)^2 + \frac{1}{12}(5)^2 - (1)^2 \\ &= \frac{4}{3} + 0 + \frac{45}{12} + \frac{25}{12} - 1 \\ &= \frac{37}{6} \end{aligned}$$

$$\begin{aligned} \text{Standard deviation of } X &= \sqrt{\text{Var}(X)} \\ &= \sqrt{\frac{37}{6}} \\ &\approx 2.48 \end{aligned}$$

**6**

Distance ( $d$ km)	5	10	20
Signal strength ( $S$ units)	40	2.5	0.156 25

**a**  $S \propto \frac{1}{d^4}$ , so  $S = \frac{k}{d^4}$  where  $k$  is a constant.

Using the first point,  $40 = \frac{k}{5^4}$   
 $\therefore k = 25\,000$   
 $\therefore S = \frac{25\,000}{d^4}$

**c** If  $d$  is tripled, then

$d$  is multiplied by 3  
 $\therefore d^4$  is multiplied by  $3^4 = 81$   
 $\therefore \frac{1}{d^4}$  is multiplied by  $\frac{1}{81}$   
 $\therefore S$  is multiplied by  $\frac{1}{81} \approx 0.0123$

So, the signal strength is decreased by about 98.8%.

**7 a** A has direction vector  $\mathbf{a} = \begin{pmatrix} 10 \\ 18 \\ 3 \end{pmatrix}$ . B has direction vector  $\mathbf{b} = \begin{pmatrix} 15 \\ k \\ 4 \end{pmatrix}$ ,  $k \in \mathbb{R}$ .

Their paths are perpendicular, so  $\mathbf{a} \cdot \mathbf{b} = 0$

$$\begin{aligned} \therefore \begin{pmatrix} 10 \\ 18 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 15 \\ k \\ 4 \end{pmatrix} &= 0 \\ \therefore 150 + 18k + 12 &= 0 \\ \therefore 18k &= -162 \\ \therefore k &= -9 \end{aligned}$$

**b** B has direction vector  $\mathbf{b} = \begin{pmatrix} 15 \\ -9 \\ 4 \end{pmatrix}$  and C has direction vector  $\mathbf{c} = \begin{pmatrix} 12 \\ -7 \\ 2 \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b} \cdot \mathbf{c}|}{|\mathbf{b}| |\mathbf{c}|} \\ &= \frac{|180 + 63 + 8|}{\sqrt{322} \sqrt{197}} \\ &= \frac{251}{\sqrt{322} \sqrt{197}} \\ \therefore \theta &= \cos^{-1} \left( \frac{251}{\sqrt{322} \sqrt{197}} \right) \\ \therefore \theta &\approx 4.74^\circ \end{aligned}$$

$\therefore$  the angle between the paths of aeroplanes B and C is about  $4.74^\circ$ .

**8**  $z = x^a b^y$

Now  $7.48 = 1.5^a \times 2.3^b$  and

$$\begin{aligned} \therefore \log(7.48) &= \log(1.5^a \times 2.3^b) \\ \therefore \log(7.48) &= \log(1.5^a) + \log(2.3^b) \end{aligned}$$

$$\therefore a \log(1.5) + b \log(2.3) = \log(7.48) \quad \dots (1)$$

Solving (1) and (2) simultaneously using technology gives  $a \approx 2.56$ ,  $b \approx 1.17$ .

$$0.983 = 1.1^a \times 0.8^b$$

$$\begin{aligned} \therefore \log(0.983) &= \log(1.1^a \times 0.8^b) \\ \therefore \log(0.983) &= \log(1.1^a) + \log(0.8^b) \end{aligned}$$

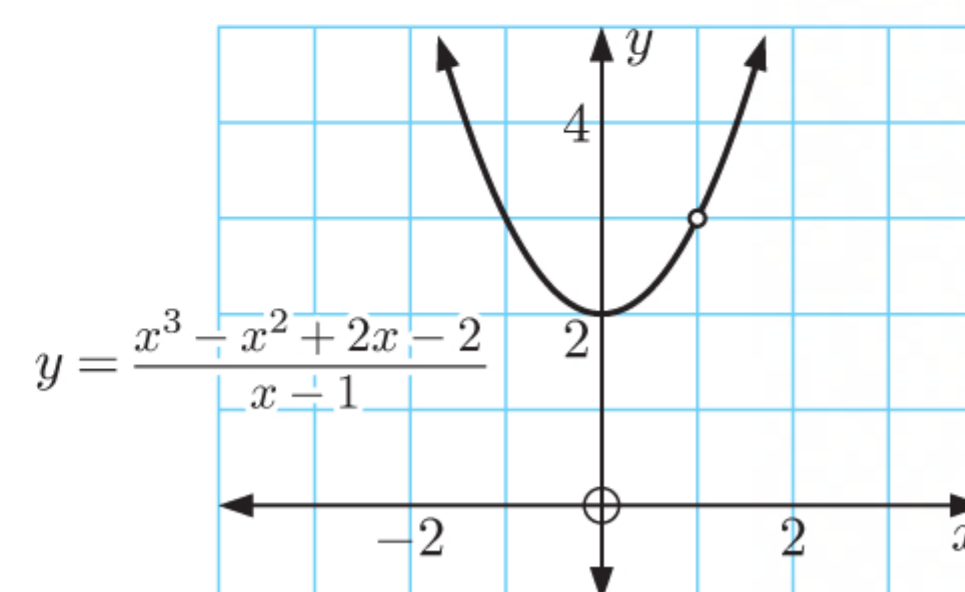
$$\therefore a \log(1.1) + b \log(0.8) = \log(0.983) \quad \dots (2)$$

Math	Eng	Norm1	d/c	Real
$a_n X + b_n Y = C_n$				
	a	b	c	
1	0.176	0.3617	0.8739	
2	0.0413	-0.096	-7E-3	
log 0.983				

Math	Eng	Norm1	d/c	Real
$a_n X + b_n Y = C_n$				
	x	y		
	2.5593	1.17		
2.559351087				
REPEAT				



- 9 a When  $x = 1$ , the denominator of  $\frac{x^3 - x^2 + 2x - 2}{x - 1}$  is zero. So, the function is undefined at  $x = 1$ .
- b From the graph, we can see that  $y \rightarrow 3$  as  $x \rightarrow 1$  from either direction.
- So,  $\lim_{x \rightarrow 1} f(x) = 3$ .



10	Number of houses sold ( $x$ )	0	1	2	3	4	5	6
	Frequency	10	28	33	13	6	7	3

- a The mean  $\bar{x} = \frac{0 \times 10 + 1 \times 28 + 2 \times 33 + 3 \times 13 + 4 \times 6 + 5 \times 7 + 6 \times 3}{10 + 28 + 33 + 13 + 6 + 7 + 3}$
- $$= \frac{210}{100}$$
- $$= 2.1 \text{ houses}$$
- b  $H_0$ : The data is from  $\text{Po}(2.1)$ .
- c The null distribution is  $X \sim \text{Po}(2.1)$ .

Number of houses sold ( $x$ )	$P(X = x)$	$f_{\text{exp}}$
0	$\approx 0.12246$	$\approx 12.246$
1	$\approx 0.25716$	$\approx 25.716$
2	$\approx 0.27002$	$\approx 27.002$
3	$\approx 0.18901$	$\approx 18.901$
4	$\approx 0.09923$	$\approx 9.923$
5	$\approx 0.04168$	$\approx 4.168$
6	$\approx 0.01459$	$\approx 1.459$
$\geq 7$	$\approx 0.00586$	$\approx 0.586$

There are expected frequencies less than 5, so we combine  $x = 5$ ,  $x = 6$ , and  $x \geq 7$ , into  $x \geq 5$  giving us the expected frequency table:

Number of houses sold ( $x$ )	0	1	2	3	4	$\geq 5$
$f_{\text{obs}}$	10	28	33	13	6	10
$f_{\text{exp}}$	$\approx 12.246$	$\approx 25.716$	$\approx 27.002$	$\approx 18.901$	$\approx 9.923$	$\approx 6.213$

- d  $df = 6 - 1 - 1 = 4$

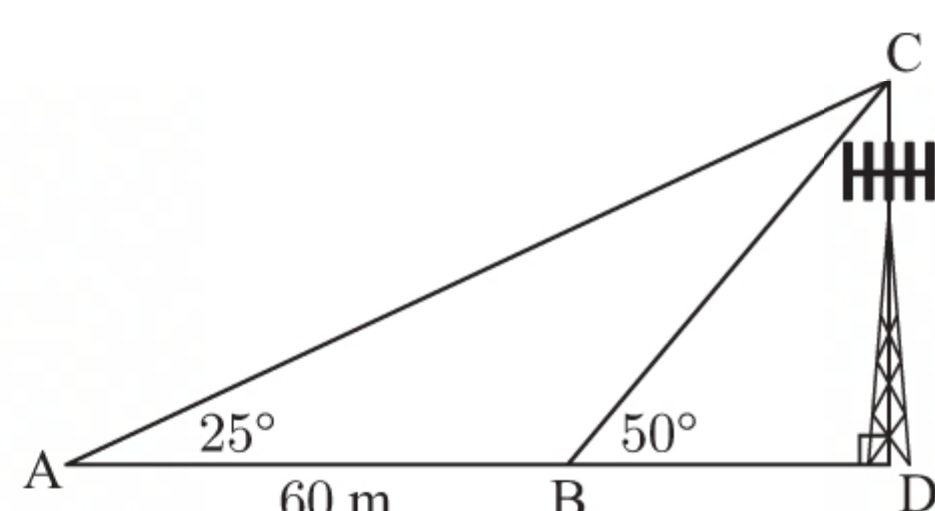
Des	Norm1	d/c	Real
Sub	List 1	List 2	List 3
4	13	18.901	
5	6	9.9231	
6	10	6.2126	
7			

Using technology, the test statistic  $\chi^2_{\text{calc}} \approx 7.65$ , and the  $p$ -value  $\approx 0.105$ .

- f Since  $p\text{-value} > \alpha = 0.1$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$ . We therefore accept  $H_0$ , and conclude that the data is from the Poisson distribution  $\text{Po}(2.1)$ .

## MIXED QUESTIONS SET 3

- 1 a  $\widehat{ACB} = 50^\circ - 25^\circ$  {exterior angle of a triangle}
- $$= 25^\circ$$



- b  $\triangle ACB$  is isosceles {base angles equal}

$$\therefore BC = AB = 60 \text{ m}$$

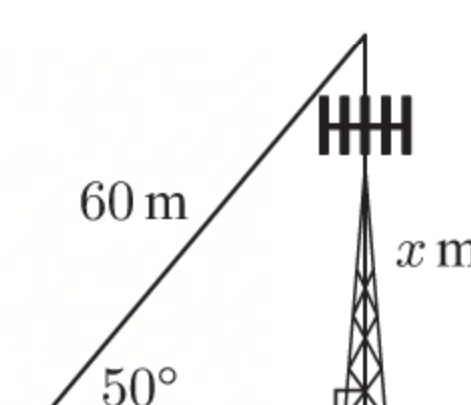
Let the tower have height  $x$  m.

$$\therefore \sin 50^\circ = \frac{x}{60}$$

$$\therefore x = 60 \sin 50^\circ$$

$$\therefore x \approx 46.0$$

The tower is about 46.0 m high.





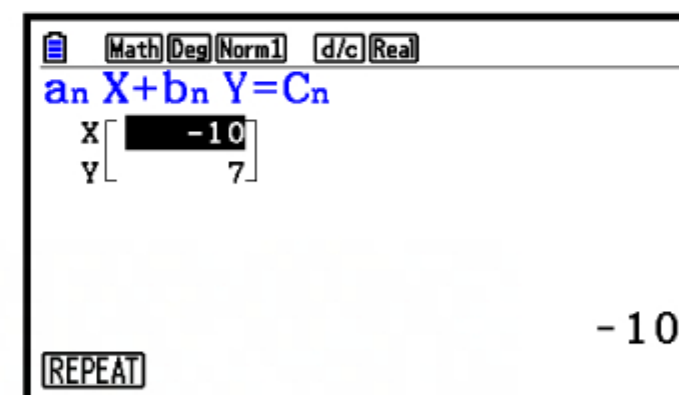
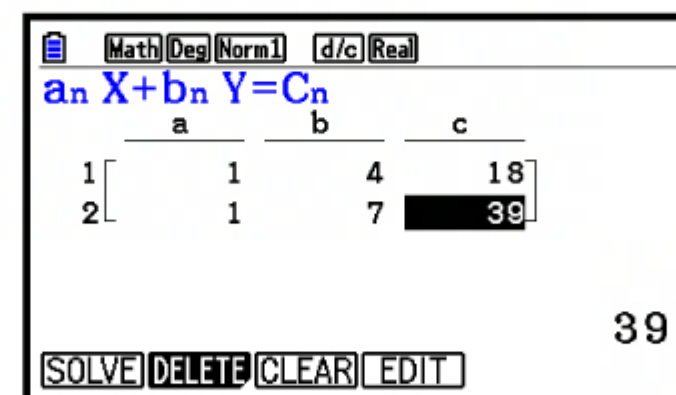
**2 a**  $u_n = u_1 + (n-1)d$

Now  $u_5 = 18 \quad \therefore u_1 + 4d = 18$

and  $u_8 = 39 \quad \therefore u_1 + 7d = 39$

Solving the system of equations  $\begin{cases} u_1 + 4d = 18 \\ u_1 + 7d = 39 \end{cases}$

simultaneously gives  $u_1 = -10$  and  $d = 7$ .



**b**  $u_{12} = u_1 + 11d$

$= -10 + 11(7) \quad \{\text{using a}\}$

$= -10 + 77$

$= 67$

**c**  $S_{10} = \frac{10}{2}(2u_1 + 9d)$   
 $= 5(2(-10) + 9(7))$   
 $= 5(-20 + 63)$   
 $= 5(43)$   
 $= 215$

**3 a** The tangent to the curve  $y = ax^3 - bx^2$  at the point where  $x = 3$  is  $y = x - 6$ .

$\therefore$  the tangent has gradient 1, and the point of contact is  $(3, 3 - 6)$  which is  $(3, -3)$ .

Now  $f(x) = ax^3 - bx^2$

$\therefore f'(x) = 3ax^2 - 2bx$

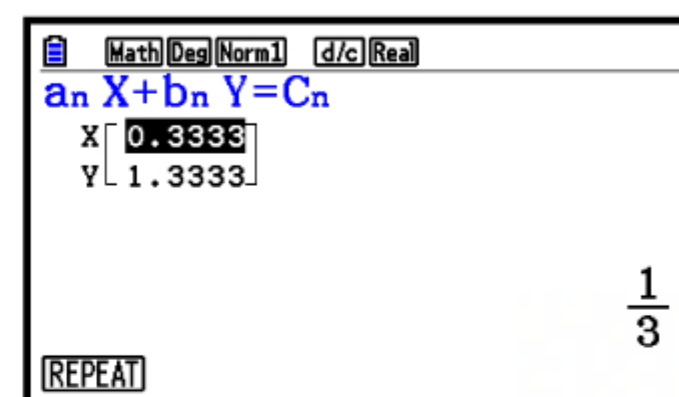
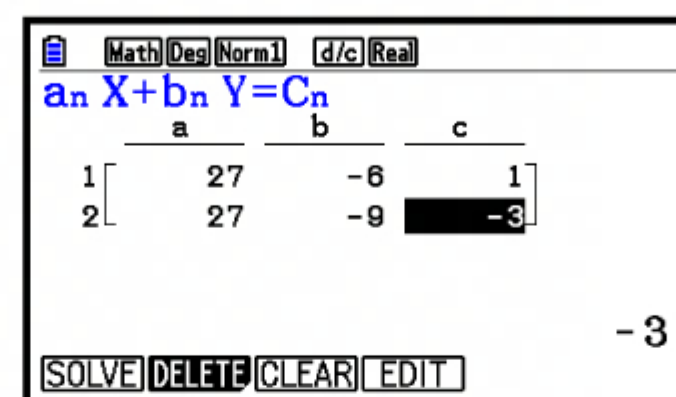
So,  $f'(3) = 1$  and  $f(3) = -3$

$\therefore 3a(3)^2 - 2b(3) = 1 \quad \therefore a(3)^3 - b(3)^2 = -3$

$\therefore 27a - 6b = 1 \quad \therefore 27a - 9b = -3$

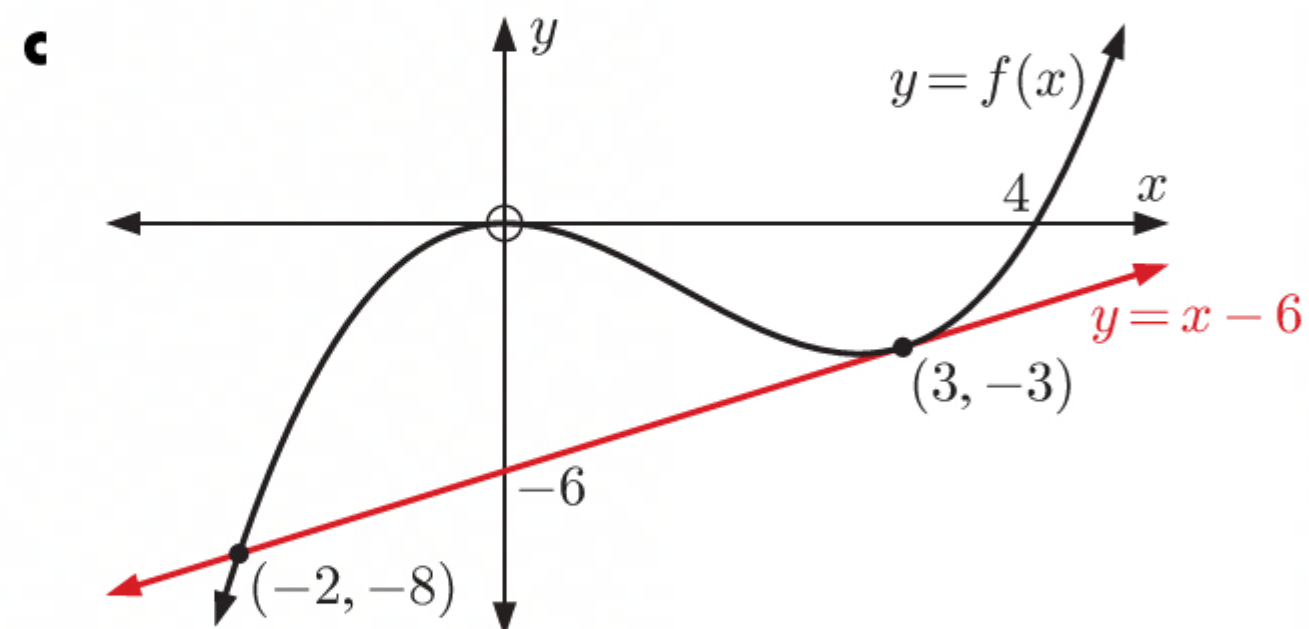
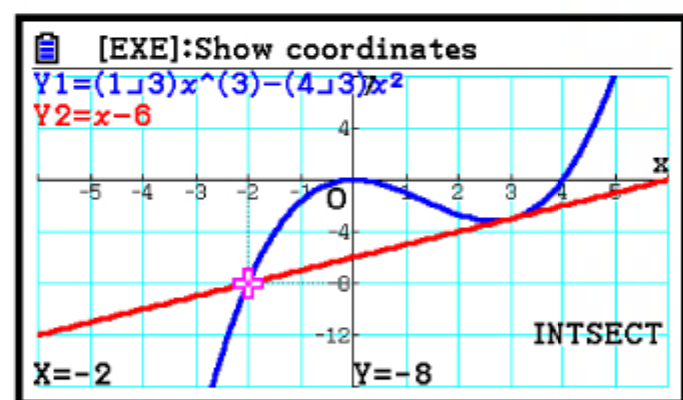
Solving the system of equations  $\begin{cases} 27a - 6b = 1 \\ 27a - 9b = -3 \end{cases}$

simultaneously gives  $a = \frac{1}{3}$  and  $b = \frac{4}{3}$ .



**b** From **a**,  $f(x) = \frac{1}{3}x^3 - \frac{4}{3}x^2$ .

Using technology, the tangent  $y = x - 6$  meets the curve  $y = f(x)$  again at  $(-2, -8)$ .



**4 a i**  $(2, 2)$  lies in cell B, so it is closest to supermarket B.

**ii**  $(-5, -3)$  lies in cell A, so it is closest to supermarket A.

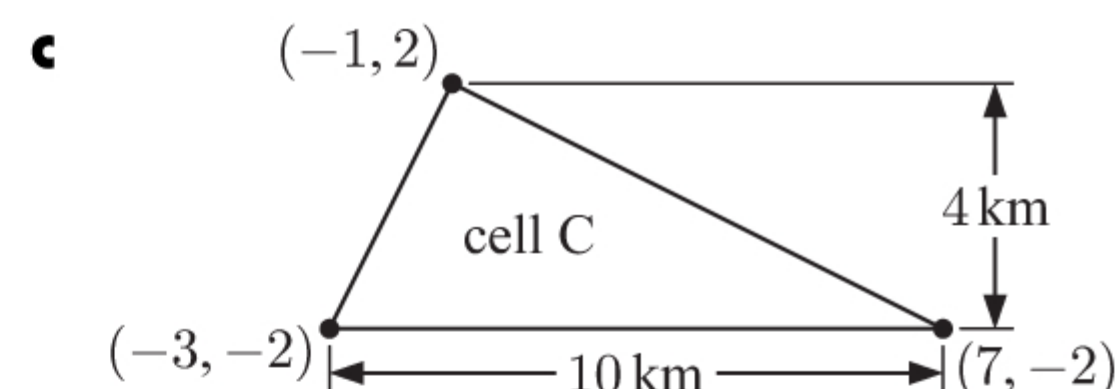
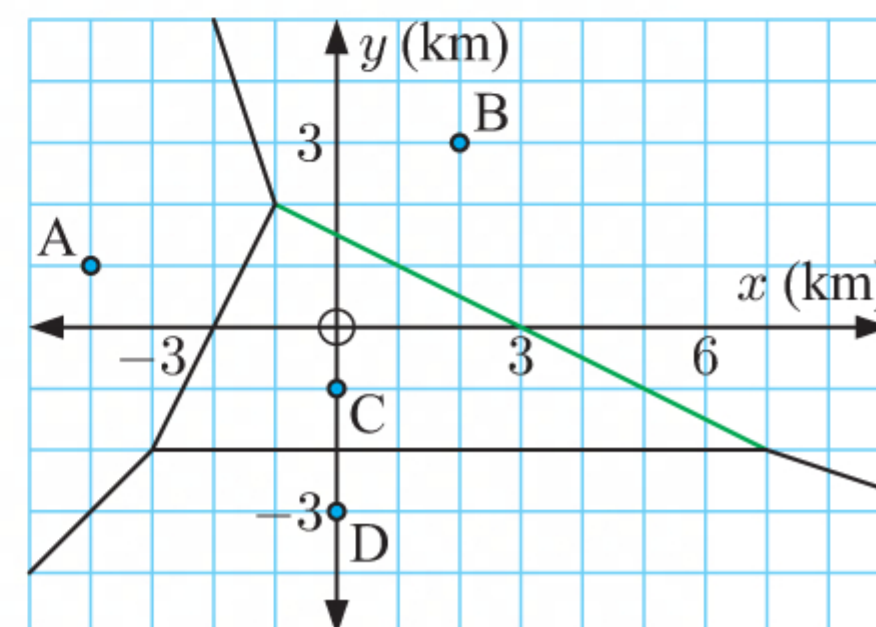
**iii**  $(6, -4)$  lies in cell D, so it is closest to supermarket D.

**b** The green edge passes through  $(-1, 2)$  and  $(7, -2)$ .

The gradient  $= \frac{-2 - 2}{7 - (-1)} = \frac{-4}{8} = \frac{-1}{2}$ .

$\therefore$  its equation is  $x + 2y = (-1) + 2(2)$

or  $x + 2y - 3 = 0$



Area of cell C  $= \frac{1}{2} \times \text{base} \times \text{height}$   
 $= \frac{1}{2} \times 10 \times 4$   
 $= 20 \text{ km}^2$

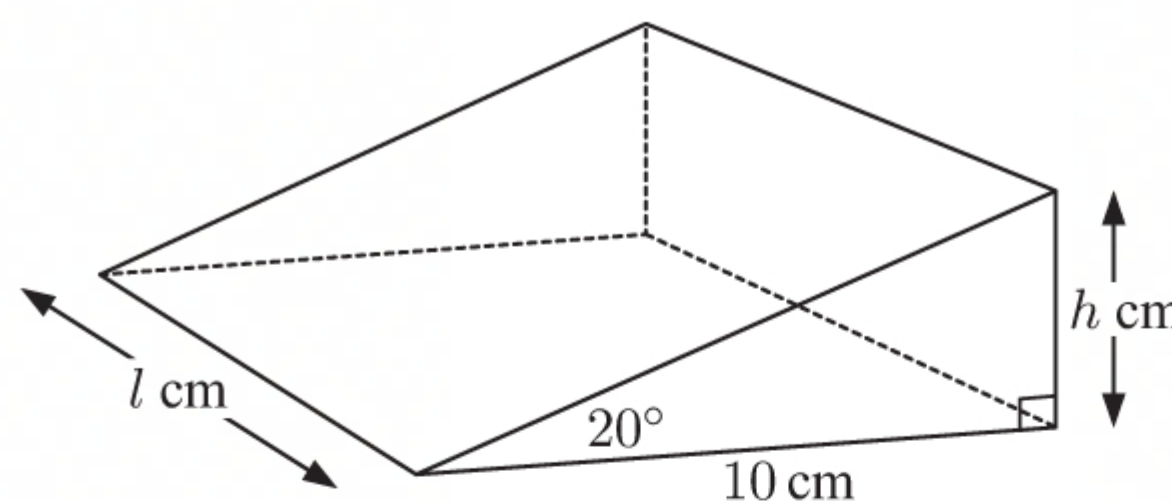


**d** Jennifer's home is equally closest to supermarkets A, C, and D.

$\therefore$  Jennifer's home is located at the vertex adjacent to cells A, C, and D.

$\therefore$  Jennifer's home is located at  $(-3, -2)$ .

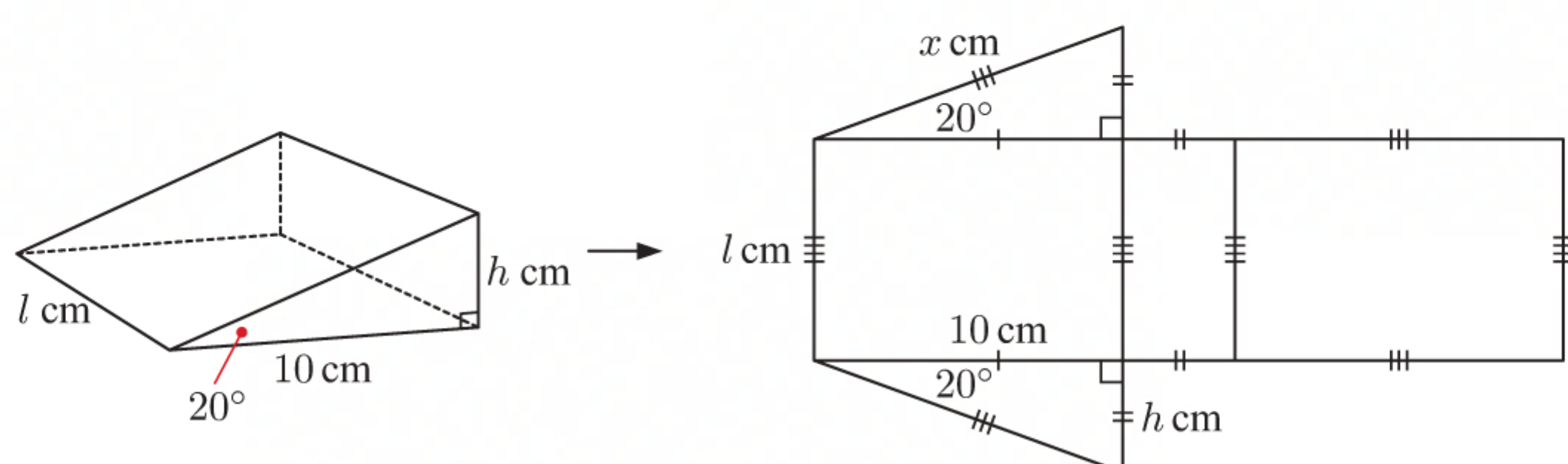
**5 a**  $\tan 20^\circ = \frac{h}{10}$   
 $\therefore h = 10 \tan 20^\circ$   
 $\approx 3.640$



**b** Area of triangular end  $= \frac{1}{2} \times \text{base} \times \text{height}$   
 $= \frac{1}{2} \times 10 \times h$   
 $= 5 \times 10 \tan 20^\circ \quad \{\text{from a}\}$   
 $= 50 \tan 20^\circ \text{ cm}^2$   
 $\approx 18.2 \text{ cm}^2$

**c** Volume of door-stop  $= 60 \text{ cm}^3$   
 $\therefore \text{area of triangular end} \times \text{length} = 60$   
 $\therefore 50 \tan 20^\circ \times l = 60 \quad \{\text{from b}\}$   
 $\therefore l = \frac{60}{50 \tan 20^\circ}$   
 $\therefore l = \frac{6}{5 \tan 20^\circ}$   
 $\therefore l \approx 3.30$

**d**



Let the hypotenuse of the triangular end be  $x$  cm.

$$\cos 20^\circ = \frac{10}{x}$$

$$\therefore x = 10 \cos 20^\circ$$

Surface area  $= (10 \times l) + (h \times l) + (x \times l) + 2 \times (\frac{1}{2} \times 10 \times h)$   
 $= (10 + h + x) \times l + 10h$   
 $= (10 + 10 \tan 20^\circ + 10 \cos 20^\circ) \times \frac{6}{5 \tan 20^\circ} + 10 \times 10 \tan 20^\circ \quad \{\text{from a and c}\}$   
 $\approx 112 \text{ cm}^2$

**6 a** Let  $\mu_D$  be the population mean difference between the new mattress and the old mattress sleep times.

The hypotheses to be considered are:

$H_0: \mu_D = 0$  {there is no difference between the new and old mattresses}

$H_1: \mu_D > 0$  {the new mattress increases sleep times}

**b**

Old mattress ( $x_i$ )	7.2	6.9	6.5	7.1	7.8	8.3	7.1	6.7	6.2	7.4
New mattress ( $y_i$ )	7.6	7.5	6.2	7.4	8.2	8.1	7.4	7.5	6.9	7.5
$d_i = y_i - x_i$	0.4	0.6	-0.3	0.3	0.4	-0.2	0.3	0.8	0.7	0.1

	List 1	List 2	List 3	List 4
SUB				
1	7.2	7.6	0.4	
2	6.9	7.5	0.6	
3	6.5	6.2	-0.3	
4	7.1	7.4	0.3	
				0.4

	List
1-Sample tTest	
Data	:List
$\mu$	:> $\mu_0$
$\mu_0$	:0
List	:List3
Freq	:1
Save Res	:None
LIST	

	Value
1-Sample tTest	
$\mu$	>0
t	=2.72004252
p	=0.01180308
$\bar{x}$	=0.31
sx	=0.36040101
n	=10

Using technology, the test statistic  $\approx 2.72$  and  $p$ -value  $\approx 0.0118$ .

**c** The significance level is  $\alpha = 0.05$ .

Since  $p$ -value  $< 0.05 = \alpha$ , we have enough evidence to reject  $H_0$  in favour of  $H_1$ . We therefore accept  $H_1$ .

Since we have accepted  $H_1$ , we conclude that the new mattress increases sleep times.



**7**  $x_1(t) = 0.6 + 5t$ ,  $y_1(t) = 10 + 5t - 4.9t^2$ ,  $t \geq 0$

**a** The target hits the ground when  $y_1(t) = 0$

$$\therefore 10 + 5t - 4.9t^2 = 0$$

$$\therefore 4.9t^2 - 5t - 10 = 0$$

$$\therefore t = \frac{5 \pm \sqrt{(-5)^2 - 4(4.9)(-10)}}{9.8}$$

$$\therefore t \approx 2.027 \quad \{\text{as } t \geq 0\}$$

The total flight time  $\approx 2.03$  seconds.

**b**  $x_1(0) = 0.6$  and  $x_1(2.027) \approx 0.6 + 5(2.027) \approx 10.7$

$\therefore$  the target travelled  $\approx 10.7 - 0.6 \approx 10.1$  m horizontally.

**c i**  $x_2(t) = x_1(t-1)$  and  $y_2(t) = y_1(t-1)$ ,  $t \geq 1$

$$\therefore x_2(t) = 0.6 + 5(t-1) \quad \text{and} \quad y_2(t) = 10 + 5(t-1) - 4.9(t-1)^2, \quad t \geq 1$$

**ii**  $D^2 = [x_1(t) - x_2(t)]^2 + [y_1(t) - y_2(t)]^2$

$$= [\cancel{0.6} + 5t - \cancel{0.6} - 5(t-1)]^2 + [\cancel{10} + 5t - 4.9t^2 - \cancel{10} - 5(t-1) + 4.9(t-1)^2]$$

$$= (\cancel{5t} - \cancel{5t} + 5)^2 + (\cancel{5t} - \cancel{4.9t^2} - \cancel{5t} + 5 + \cancel{4.9t^2} - 9.8t + 4.9)^2$$

$$= 5^2 + (9.9 - 9.8t)^2$$

$$= 25 + 98.01 - 194.04t + 96.04t^2$$

$$= 96.04t^2 - 194.04t + 123.01$$

**iii**  $D$  is minimised when  $D^2$  is minimised.

This occurs when  $t = -\frac{-194.04}{2 \times 96.04} = \frac{99}{98} \approx 1.01$

When  $t = \frac{99}{98}$ ,  $D = \sqrt{96.04\left(\frac{99}{98}\right)^2 - 194.04\left(\frac{99}{98}\right) + 123.01}$

$$= 5$$

Now  $1 \leq 1.01 \leq 2.027$ , so both targets are in the air when the shortest distance occurs.

$\therefore$  the shortest distance between the two targets while in the air is 5 m, and this occurs after about 1.01 seconds.

**8**  $f'(x) = \sqrt{4x+5} = (4x+5)^{\frac{1}{2}}$ ,  $f(0) = -\frac{\sqrt{5}}{6}$

**a**  $f'(x)$  is defined when  $4x+5 \geq 0$

$$\therefore 4x \geq -5$$

$$\therefore x \geq -\frac{5}{4}$$

**b**  $f(x) = \int (4x+5)^{\frac{1}{2}} dx$

$$= \frac{2}{3} \times \frac{1}{4} (4x+5)^{\frac{3}{2}} + c$$

$$= \frac{1}{6} (4x+5)^{\frac{3}{2}} + c$$

Now  $f(0) = -\frac{\sqrt{5}}{6}$ ,

$$\therefore \frac{1}{6} (5)^{\frac{3}{2}} + c = -\frac{\sqrt{5}}{6}$$

$$\therefore \frac{5\sqrt{5}}{6} + c = -\frac{\sqrt{5}}{6}$$

$$\therefore c = -\frac{6\sqrt{5}}{6} = -\sqrt{5}$$

$$\therefore f(x) = \frac{1}{6} (4x+5)^{\frac{3}{2}} - \sqrt{5}$$

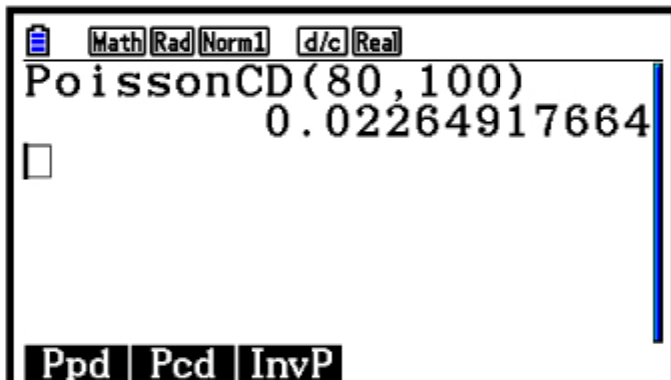
**9 a** The rate at which cars pass through the toll gate = 10 cars per minute

$$= 100 \text{ cars per 10 minutes.}$$

Let  $X$  be the number of cars passing through the toll gate over a 10 minute interval.

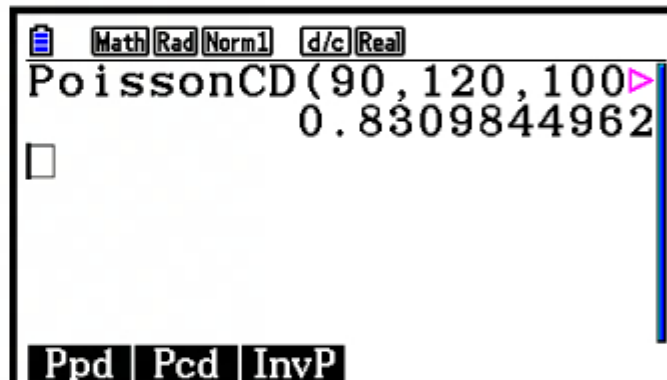
$$\therefore X \sim \text{Po}(100)$$

**i**



$$P(X \leq 80) \approx 0.0226$$

**ii**



$$P(90 \leq X \leq 120) \approx 0.831$$



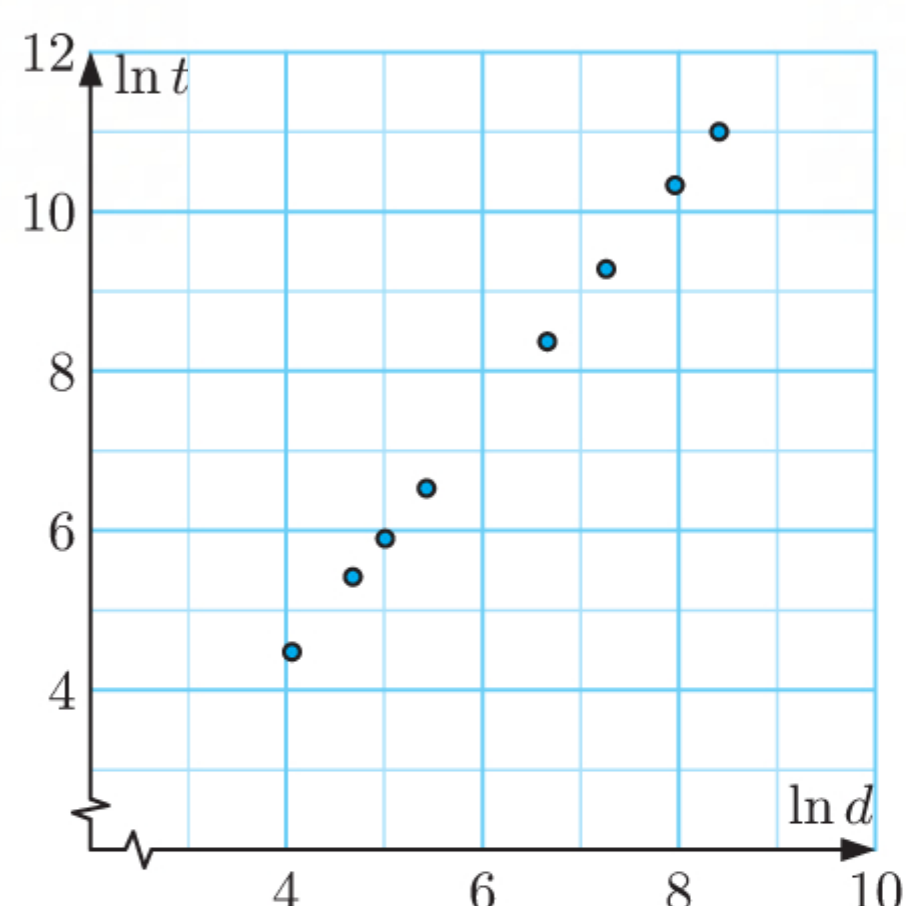
**b**  $Y = 0.75X$

**i**  $E(Y) = E(0.75X)$   
 $= 0.75 E(X)$   
 $= 0.75 \times 100$   
 $= \$75$

**ii**  $\text{Var}(Y) = \text{Var}(0.75X)$   
 $= 0.75^2 \text{Var}(X)$   
 $= 0.5625 \times 100$   
 $= 56.25$

**10 a**

$d$ (millions of km)	57.9	108.2	149.6	227.9	778.3	1427.0	2870.0	4497.0
$t$ (days)	88	225	365	687	4329	10 753	30 660	60 150
$\ln d$	4.06	4.68	5.01	5.43	6.66	7.26	7.96	8.41
$\ln t$	4.48	5.42	5.90	6.53	8.37	9.28	10.33	11.00



**b** The graph of  $\ln t$  against  $\ln d$  appears linear, so a power model is appropriate.

**c** Using technology, the linear model connecting  $\ln t$  and  $\ln d$  is

$$\ln t \approx 1.50 \ln d - 1.61$$

$$\therefore t \approx e^{1.50 \ln d - 1.61}$$

$$\therefore t \approx e^{-1.61} \times e^{1.50 \ln d}$$

$$\therefore t \approx 0.200 \times d^{1.50}$$

Rad	Norm1	d/c	Real
LinearReg(ax+b)			
a	=	1.49956632	
b	=	-1.6089251	
r	=	0.99999996	
r <sup>2</sup>	=	0.99999992	
MSe	=	4.8569E-07	
y=ax+b			
[COPY] [DRAW]			

**d** For Pluto  $d = \frac{5.907 \times 10^9}{10^6} = 5.907 \times 10^3 = 5907$

When  $d = 5907$ ,  $t \approx 0.200 \times 5907^{1.50} \approx 90\,500$

So, the length of Pluto's orbit is about 90 500 days.

## MIXED QUESTIONS SET 4

- 1 a**
- The survey is likely to under-represent full-time weekday workers.
  - The survey was taken at a suburban shopping centre, so the people surveyed are likely to prefer suburban shopping. Therefore the sample is likely to be biased toward suburban shopping.

**b** The conclusion is unreasonable since the survey is likely to contain a coverage error, as in **a**, and so the results may not accurately represent the opinions of the whole population.

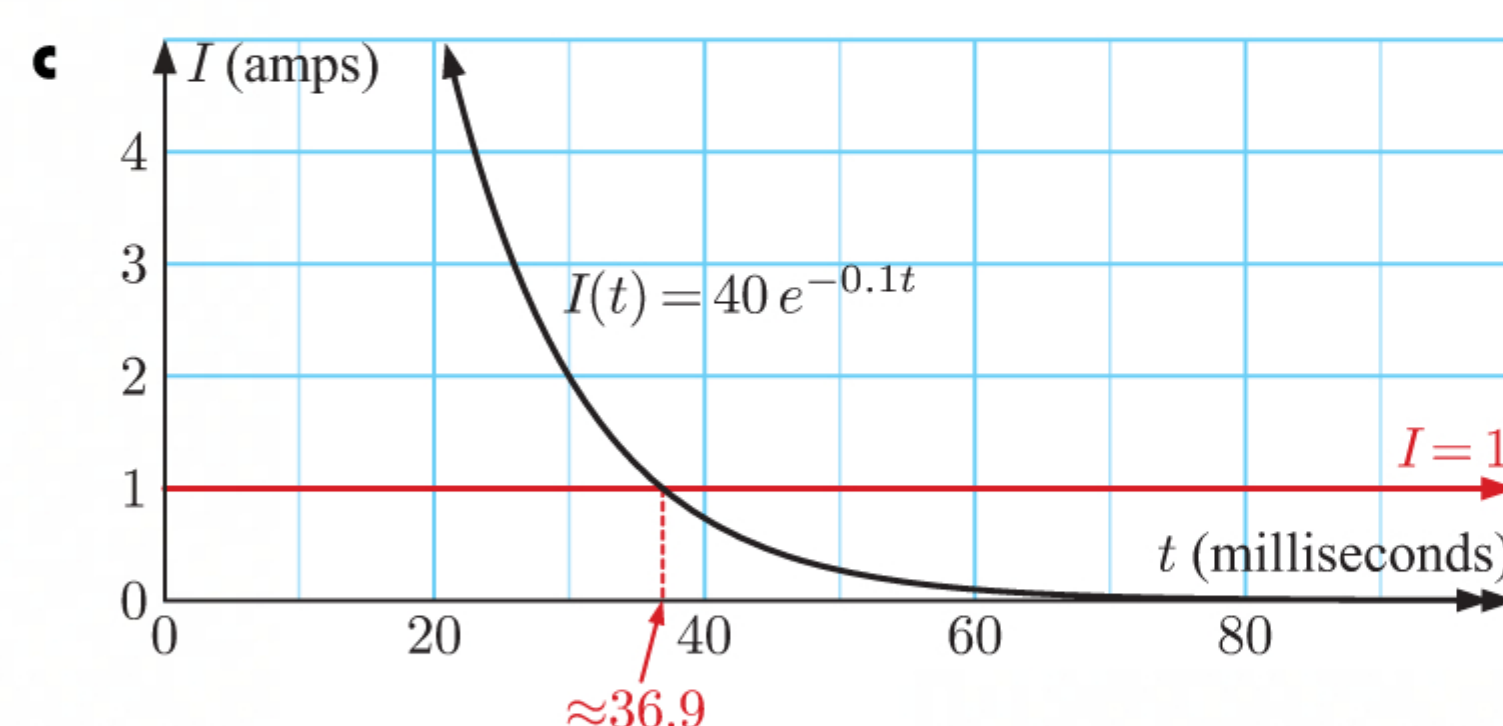
**2**  $I(t) = 40e^{-0.1t}$  amps

**a**  $I(0) = 40e^0$   
 $= 40$

$\therefore$  there was 40 amps of current flowing through the circuit initially.

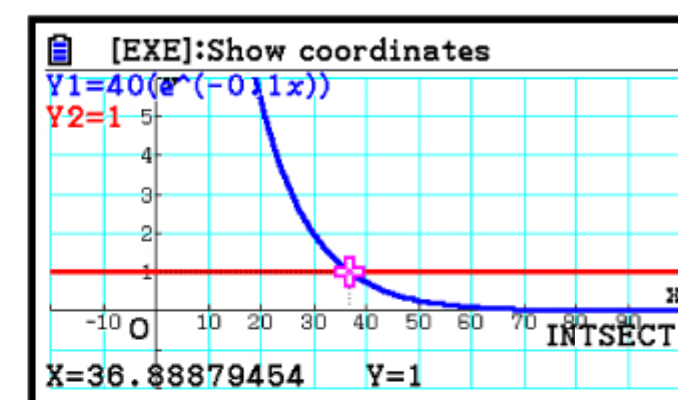
**b**  $I(100) = 40e^{-0.1 \times 100}$   
 $\approx 0.00182$

$\therefore$  after 100 milliseconds, there was about 0.001 82 amps flowing through the circuit.





- d** Using technology, it took about 36.9 milliseconds for the current to fall to 1 amp.



- 3 a**  $N = 3 \times 12 = 36$ ,  $PV = 7000$ ,  $PMT = -220$ ,  $FV = 0$ ,  $P/Y = 12$ ,  $C/Y = 12$

$$\therefore I\% \approx 8.20$$

$$\therefore r \approx 8.20$$

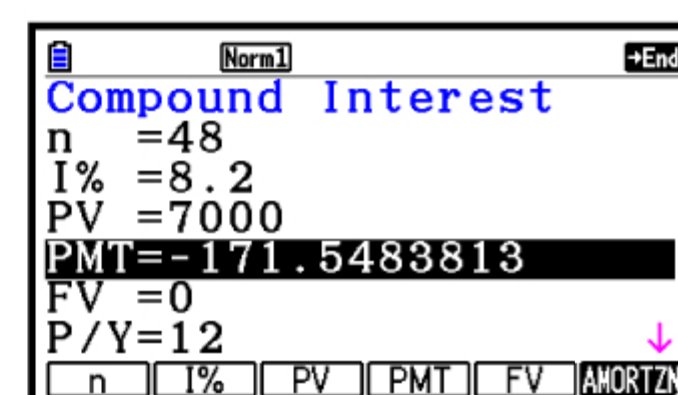
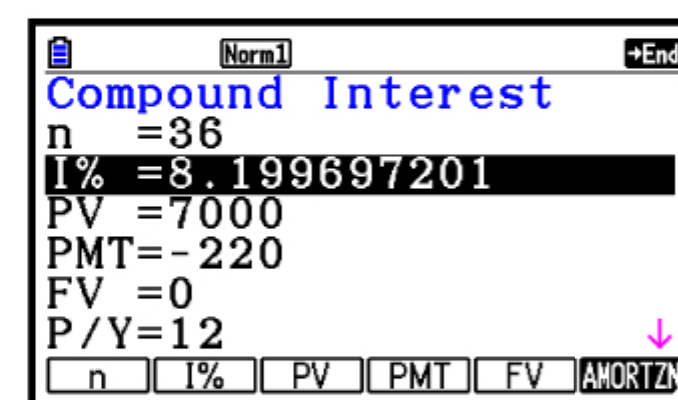
- b** Total interest = total repayment – starting principal  
 $= £220 \times 36 - £7000$   
 $= £920$

- c i**  $N = 4 \times 12 = 48$ ,  $I\% = 8.20$ ,  $PV = 7000$ ,  $FV = 0$ ,  $P/Y = 12$ ,  $C/Y = 12$

$$\therefore PMT \approx 171.55$$

Soraya's monthly repayment is £171.55.

- ii** Total interest = total repayment – starting principal  
 $= £171.55 \times 48 - £7000$   
 $= £1234.40$   
 $\therefore$  extra interest =  $£1234.40 - £920$   
 $= £314.40$



- 4**  $N(t) = at^3 + bt^2 + ct + d$

- a**  $N(0) = 2000$  {2000 people at 12 pm}  
 $N'(6) = 200$  {increasing at 200 people per hour at 6 pm}  
 $N'(8) = -500$  {decreasing at 500 people per hour at 8 pm}  
 $N(10) = 0$  {closed at 10 pm}

- b**  $N(0) = 2000 \quad \therefore a(0)^3 + b(0)^2 + c(0) + d = 2000$   
 $\therefore d = 2000$

$$N(10) = 0 \quad \therefore a(10)^3 + b(10)^2 + c(10) + 2000 = 0$$

$$\therefore 1000a + 100b + 10c = -2000$$

$$\therefore 100a + 10b + c = -200$$

$$N'(t) = 3at^2 + 2bt + c$$

$$N'(6) = 200 \quad \therefore 3a(6)^2 + 2b(6) + c = 200$$

$$\therefore 108a + 12b + c = 200$$

$$N'(8) = -500 \quad \therefore 3a(8)^2 + 2b(8) + c = -500$$

$$\therefore 192a + 16b + c = -500$$

So we have the system of equations

$$\begin{cases} 100a + 10b + c = -200 \\ 108a + 12b + c = 200 \\ 192a + 16b + c = -500 \end{cases}$$

Using technology to solve this system simultaneously gives  $a = -\frac{375}{17}$ ,  $b = \frac{4900}{17}$ , and  $c = -\frac{14900}{17}$ .

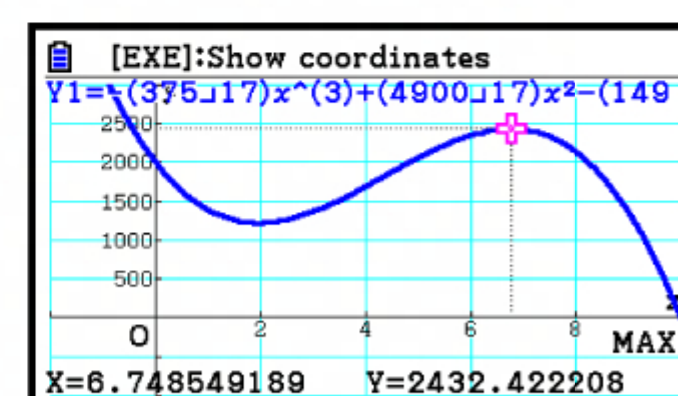
$$\therefore N(t) = -\frac{375}{17}t^3 + \frac{4900}{17}t^2 - \frac{14900}{17}t + 2000$$

	a	b	c	d
1	100	10	1	-200
2	108	12	1	200
3	192	16	1	-500
				-500

	X	Y	Z
	-22.05	288.23	-878.4
			-375/17

- c** Using technology,  $N(t)$  has a maximum of about 2432 when  $t \approx 6.75$ .

$\therefore$  the maximum number of people at the festival was about 2432 at about 6:45 pm.





- 5 We extend the table to include totals for each row and column.

	Defective	Not defective	Total
Corn	37	581	618
Pineapple	24	617	641
Total	61	1198	1259

- a There were 1259 tins included in the sample.
- b i 1198 of the 1259 tins were not defective.  
 $\therefore P(\text{is not defective}) \approx \frac{1198}{1259} \approx 0.952$
- ii 24 of the 1259 tins were defective tins of pineapple.  
 $\therefore P(\text{is a defective tin of pineapple}) \approx \frac{24}{1259} \approx 0.0191$
- iii 37 of the 618 tins of corn were defective.  
 $\therefore P(\text{is defective, given it is a tin of corn}) \approx \frac{37}{618} \approx 0.0599$
- 6 a Let  $p_1, p_2, p_3, p_4, p_5, p_6$ , and  $p_7$  be the probabilities of a break-in on Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday respectively.  
 $H_0: p_1 = \frac{1}{7}, p_2 = \frac{1}{7}, \dots, p_7 = \frac{1}{7}$   
 $H_1: \text{at least one of } p_1 \neq \frac{1}{7}, p_2 \neq \frac{1}{7}, \dots, \text{ or } p_7 \neq \frac{1}{7}$
- b Expected number of break-ins per day =  $140 \times \frac{1}{7} = 20$
- c  $df = 7 - 1 = 6$

	List 1	List 2	List 3	List 4
SUB				
1	15	20		
2	11	20		
3	17	20		
4	18	20		
				15

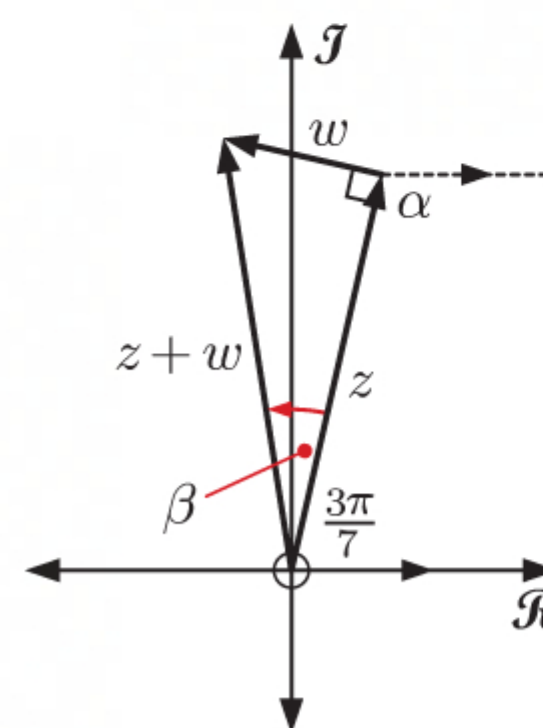
	List 1	List 2	List 3	List 4
SUB				
1	15	20		
2	11	20		
3	17	20		
4	18	20		
				15

	List 1	List 2	List 3	List 4
SUB				
1	15	20		
2	11	20		
3	17	20		
4	18	20		
				15

Using technology,  $\chi^2_{\text{calc}} = 12.9$ .

- d Since  $\chi^2_{\text{calc}} > 12.59 = \chi^2_{\text{crit}}$ , we have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% level of significance.  
 $\therefore$  break-ins are *not* equally likely to occur on each day of the week.

- 7 a  $\alpha = \pi - \frac{3\pi}{7} = \frac{4\pi}{7}$  {co-interior angles}  
 $\therefore \arg w = 2\pi - \frac{\pi}{2} - \alpha$  {angles at a point}  
 $= 2\pi - \frac{\pi}{2} - \frac{4\pi}{7}$   
 $= \frac{13\pi}{14}$



- b  $|z + w| = \sqrt{|z|^2 + |w|^2}$  {Pythagoras}  
 $= \sqrt{25 + 4}$   
 $= \sqrt{29}$

- c  $\tan \beta = \frac{2}{5}$   
 $\therefore \beta = \tan^{-1}(\frac{2}{5}) \approx 0.381$   
 $\therefore \arg(z + w) = \arg z + \beta$   
 $\approx \frac{3\pi}{7} + 0.381$   
 $\approx 1.73$

- 8  $x(t) = 20 \cos(\frac{\pi}{6}(t - 3))$ ,  $y(t) = -20 \sin(\frac{\pi}{6}(t - 3))$ ,  $0 \leq t \leq 3$ .

- a i  $x(0) = 20 \cos(\frac{\pi}{6}(-3))$   $y(0) = -20 \sin(\frac{\pi}{6}(-3))$   
 $= 20 \cos(-\frac{\pi}{2})$   $= -20 \sin(-\frac{\pi}{2})$   
 $= 0$   $= 20$

The initial position of the car is (0, 20).

- ii  $x(3) = 20 \cos 0$   $y(3) = -20 \sin 0$   
 $= 20$   $= 0$

After 3 seconds, the car is at (20, 0).

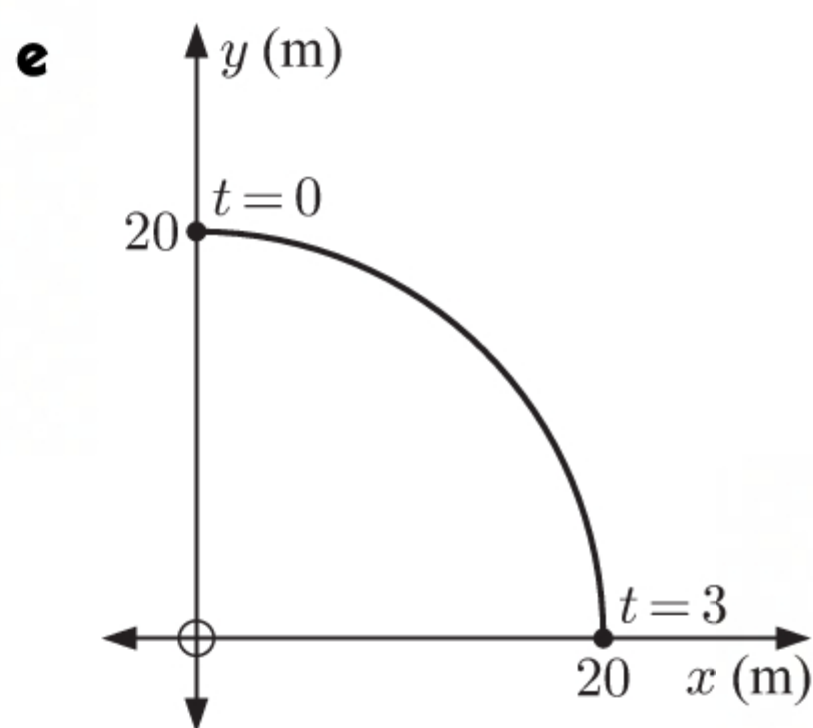


$$\begin{aligned}\mathbf{b} \quad \mathbf{v} &= \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -20 \sin\left(\frac{\pi}{6}(t-3)\right) \times \frac{\pi}{6} \\ -20 \cos\left(\frac{\pi}{6}(t-3)\right) \times \frac{\pi}{6} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{10\pi}{3} \sin\left(\frac{\pi}{6}(t-3)\right) \\ -\frac{10\pi}{3} \cos\left(\frac{\pi}{6}(t-3)\right) \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \text{Speed} = |\mathbf{v}| &= \sqrt{\left(-\frac{10\pi}{3} \sin\left(\frac{\pi}{6}(t-3)\right)\right)^2 + \left(-\frac{10\pi}{3} \cos\left(\frac{\pi}{6}(t-3)\right)\right)^2} \\ &= \sqrt{\frac{100\pi^2}{9} \sin^2\left(\frac{\pi}{6}(t-3)\right) + \frac{100\pi^2}{9} \cos^2\left(\frac{\pi}{6}(t-3)\right)} \\ &= \sqrt{\frac{100\pi^2}{9} [\sin^2\left(\frac{\pi}{6}(t-3)\right) + \cos^2\left(\frac{\pi}{6}(t-3)\right)]} \\ &= \sqrt{\frac{100\pi^2}{9}} \\ &= \frac{10\pi}{3} \approx 10.5 \text{ m s}^{-1}\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad x^2 + y^2 &= \left(20 \cos\left(\frac{\pi}{6}(t-3)\right)\right)^2 + \left(-20 \sin\left(\frac{\pi}{6}(t-3)\right)\right)^2 \\ &= 400 \cos^2\left(\frac{\pi}{6}(t-3)\right) + 400 \sin^2\left(\frac{\pi}{6}(t-3)\right) \\ &= 400 [\cos^2\left(\frac{\pi}{6}(t-3)\right) + \sin^2\left(\frac{\pi}{6}(t-3)\right)] \\ &= 400\end{aligned}$$

$\therefore$  the car travels in a circle centred at  $(0, 0)$  with radius 20 m.

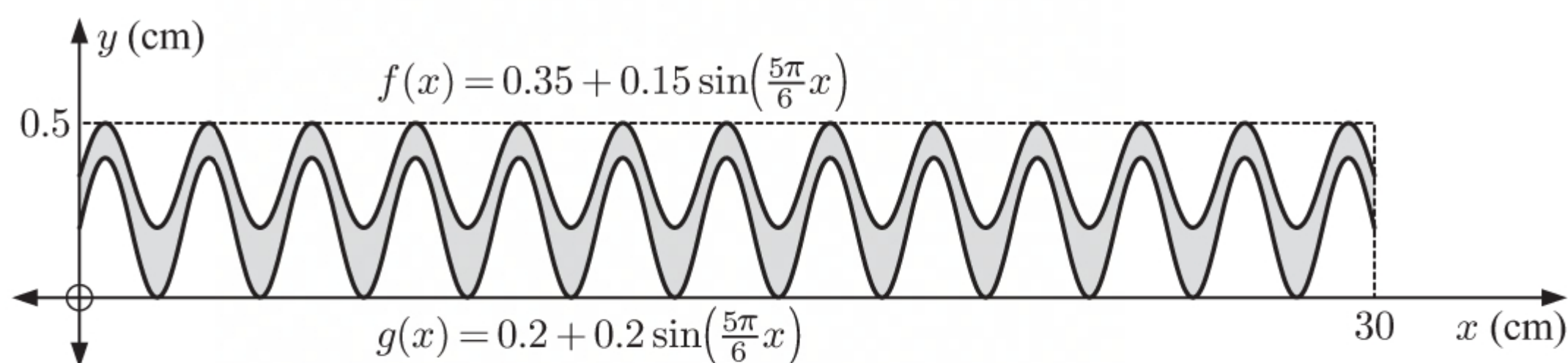


$$\begin{aligned}\mathbf{f} \quad \text{Distance travelled} &= \frac{1}{4} \times 2\pi r \\ &= \frac{1}{4} \times 2\pi \times 20 \\ &= 10\pi \approx 31.4 \text{ m}\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad \text{Speed} &= \frac{\text{distance}}{\text{time}} \\ &= \frac{10\pi}{3} \approx 10.5 \text{ m s}^{-1}\end{aligned}$$

This is the same as the speed calculated in  $\mathbf{c}$ .

**9**



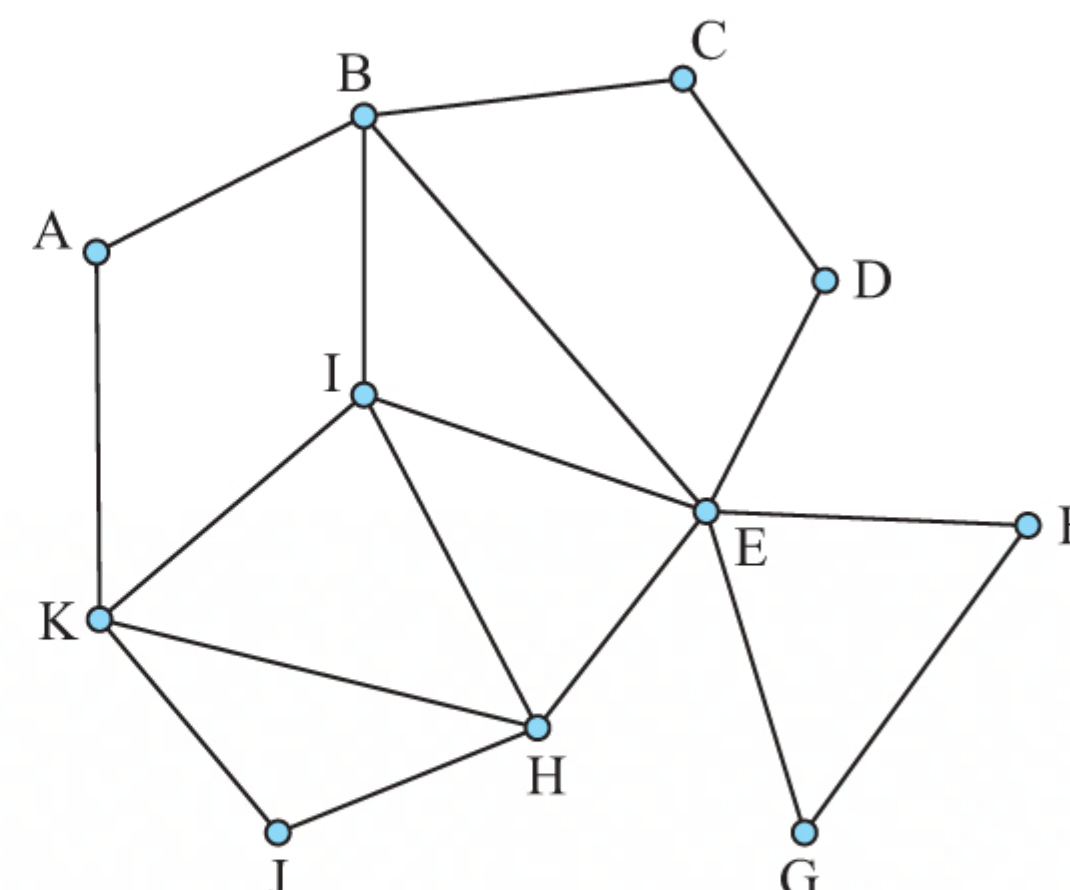
$$\begin{aligned}\text{Cross-sectional area} &= \int_0^{30} (f(x) - g(x)) dx \\ &= \int_0^{30} \left( (0.35 + 0.15 \sin(\frac{5\pi}{6}x)) - (0.2 + 0.2 \sin(\frac{5\pi}{6}x)) \right) dx \\ &= \int_0^{30} (0.15 - 0.05 \sin(\frac{5\pi}{6}x)) dx \\ &= \left[ 0.15x + 0.05\left(\frac{6}{5\pi}\right) \cos(\frac{5\pi}{6}x) \right]_0^{30} \\ &= (0.15 \times 30 + \frac{3}{50\pi} \cos(\frac{5\pi}{6} \times 30)) - (0.15 \times 0 + \frac{3}{50\pi} \cos(\frac{5\pi}{6} \times 0)) \\ &= 4.5 + \frac{3}{50\pi} \cos(25\pi) - \frac{3}{50\pi} \\ &= 4.5 + \frac{3}{50\pi} \cos \pi - \frac{3}{50\pi} \quad \{\cos \theta = \cos(\theta + 2\pi)\} \\ &= 4.5 + \frac{3}{50\pi}(-1) - \frac{3}{50\pi} \\ &= 4.5 - \frac{3}{25\pi} \text{ cm}^2\end{aligned}$$

So, volume = cross-sectional area  $\times$  length

$$\begin{aligned}&= (4.5 - \frac{3}{25\pi}) \times 100 \quad \{1 \text{ m} \equiv 100 \text{ cm}\} \\ &\approx 446 \text{ cm}^3\end{aligned}$$



- 10 a** The edges adjacent to BC are AB, BI, BE, and CD.
- b** A path of length 5 which starts at B and finishes at D is  
 $B \rightarrow A \rightarrow K \rightarrow I \rightarrow E \rightarrow D$ .
- c** All vertices have even degree, so the graph is Eulerian.
- d** An Eulerian circuit which starts and finishes at A is  
 $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow G \rightarrow E \rightarrow H \rightarrow J \rightarrow K \rightarrow H \rightarrow I \rightarrow E \rightarrow B \rightarrow A$ .



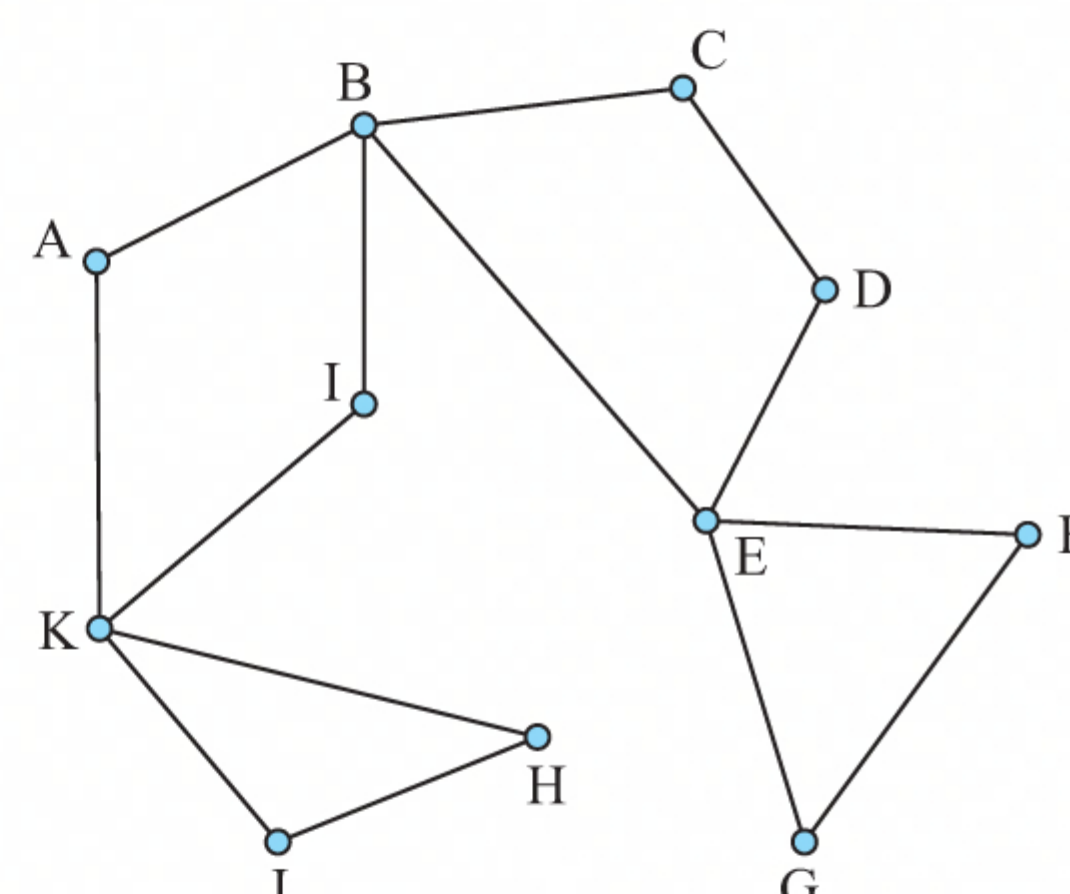
- e i** When EH is removed, vertices E and H have odd degree. The graph is still Eulerian, so we must remove at least 2 other edges: 1 connected to E, and the other connected to H.

The edges to remove must be adjacent, otherwise we will create more odd vertices.

$\therefore$  the other two removed edges are HI and EI.

$\therefore$  the other two closed tracks are HI and EI.

- ii** An Eulerian circuit which starts and finishes at A in this case is  
 $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow G \rightarrow E \rightarrow B \rightarrow I \rightarrow K \rightarrow H \rightarrow J \rightarrow K \rightarrow A$ .



## MIXED QUESTIONS SET 5

**1 a** Amount of fluoride = concentration  $\times$  volume  
 $= (3 \times 10^{-4}) \times (5.6 \times 10^8)$   
 $= 1.68 \times 10^5 \text{ g}$

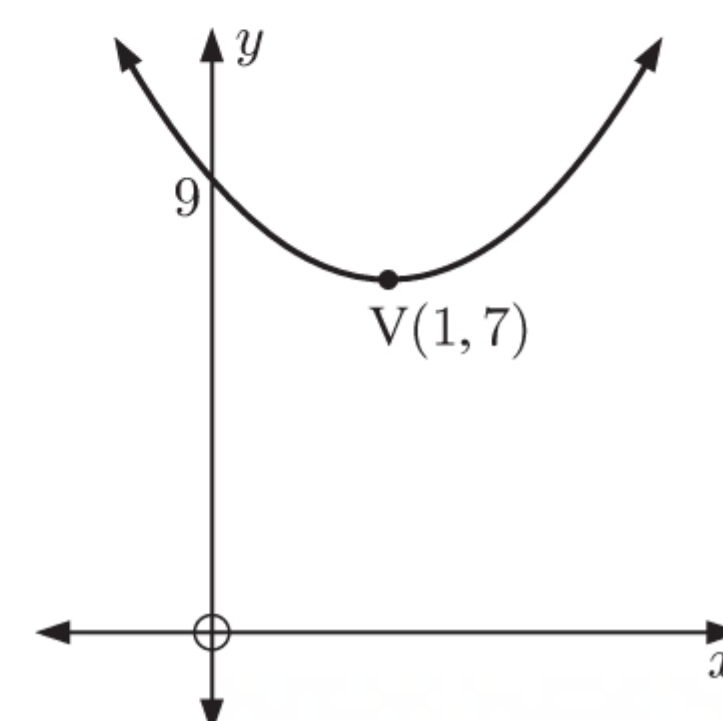
**b** Volume =  $\frac{\text{amount of fluoride}}{\text{concentration}}$   
 $= \frac{4.13 \times 10^7}{3 \times 10^{-4}}$   
 $\approx 1.38 \times 10^{11} \text{ litres}$

**2 a** The  $y$ -intercept = 9, so  $c = 9$

**b** The axis of symmetry is  $x = -\frac{b}{2a}$   
 $\therefore -\frac{b}{2a} = 1$   
 $\therefore -b = 2a$   
 $\therefore 2a + b = 0 \dots (1)$

**c** The point  $(1, 7)$  lies on the graph, so  $a(1)^2 + b(1) + 9 = 7$   
 $\therefore a + b = -2 \dots (2)$

**d** Solving (1) and (2) simultaneously gives  $a = 2$  and  $b = -4$ .



	a	b	c
1	2	1	0
2	1	1	-2

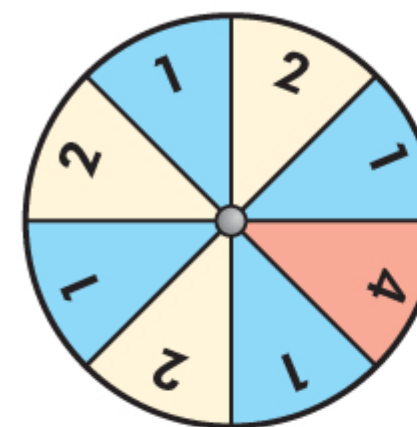
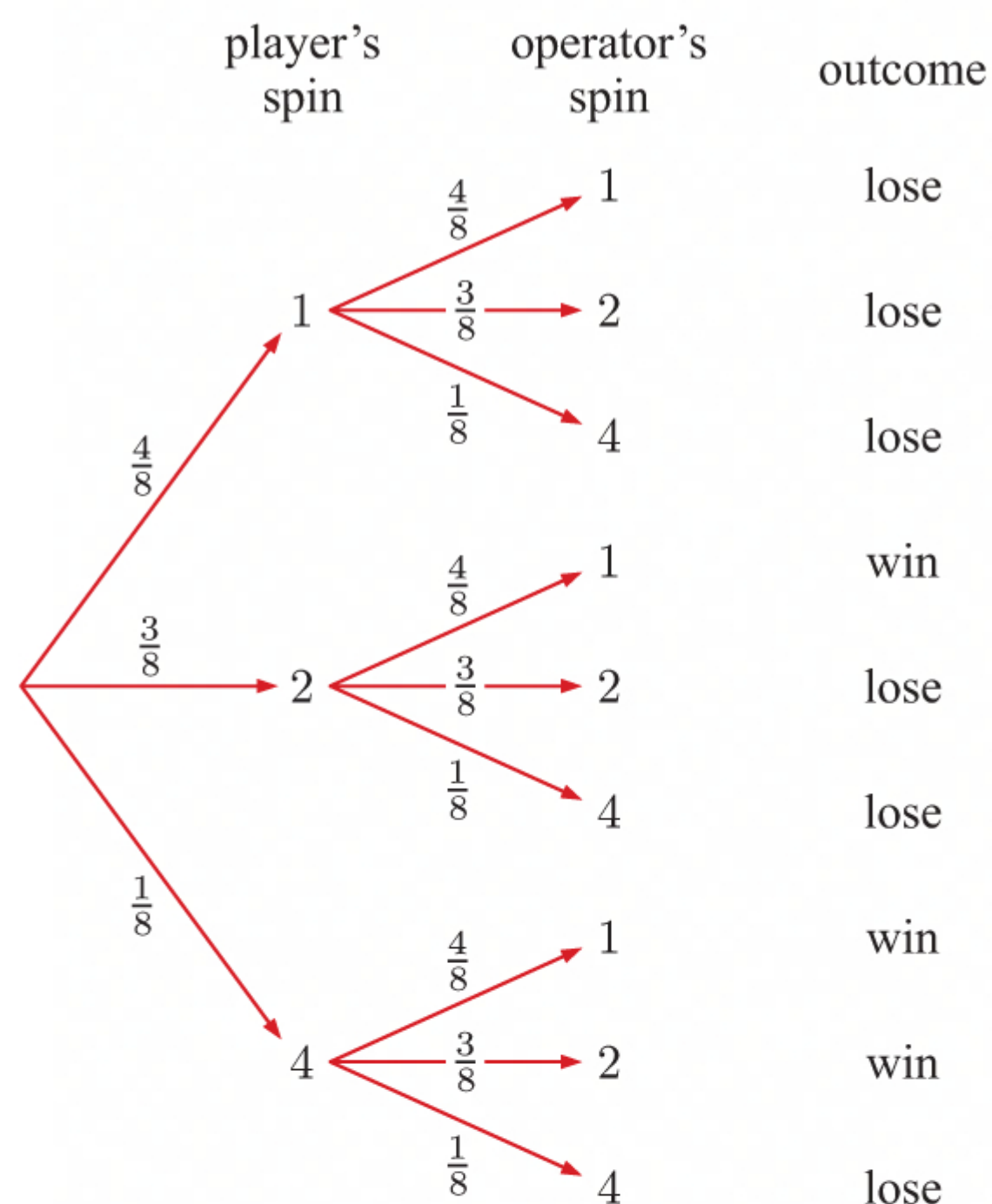
SOLVE DELETE CLEAR EDIT

	a	b	c
1	2	1	0
2	1	1	-2

REPEAT



3 We first construct a tree diagram of the possible outcomes.



The player wins if their spin is higher than the operator.

$$\begin{aligned}\therefore P(\text{win}) &= \left(\frac{3}{8} \times \frac{4}{8}\right) + \left(\frac{1}{8} \times \frac{4}{8}\right) + \left(\frac{1}{8} \times \frac{3}{8}\right) \\ &= \frac{12}{64} + \frac{4}{64} + \frac{3}{64} \\ &= \frac{19}{64}\end{aligned}$$

Outcome	Win	Lose
Winnings	\$a	\$0
Probability	$\frac{19}{64}$	$\frac{41}{64}$

Let  $X$  denote the return from one game.

$$\begin{aligned}E(X) &= \left(a \times \frac{19}{64}\right) + \left(0 \times \frac{41}{64}\right) \\ &= \frac{19a}{64} \text{ dollars}\end{aligned}$$

It costs \$ $k$  to play a game, so the expected gain =  $\frac{19a}{64} - k$  dollars.

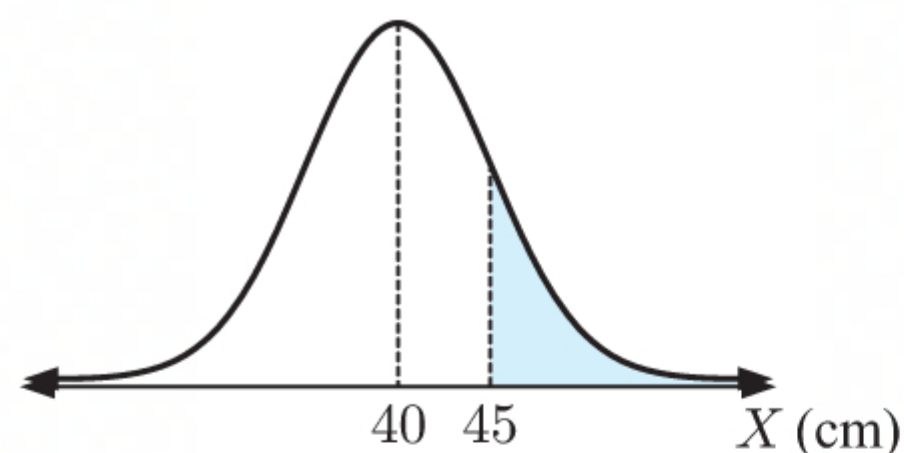
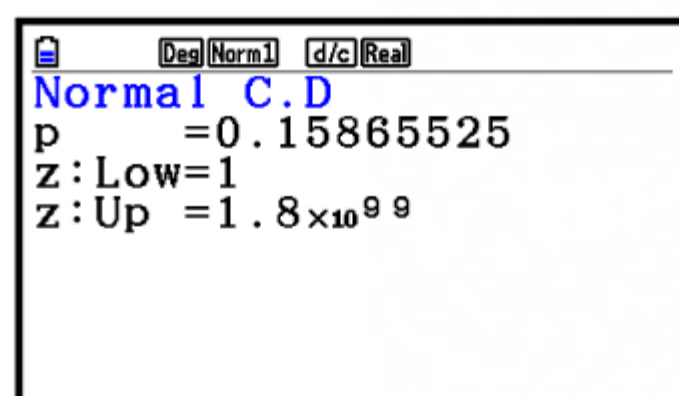
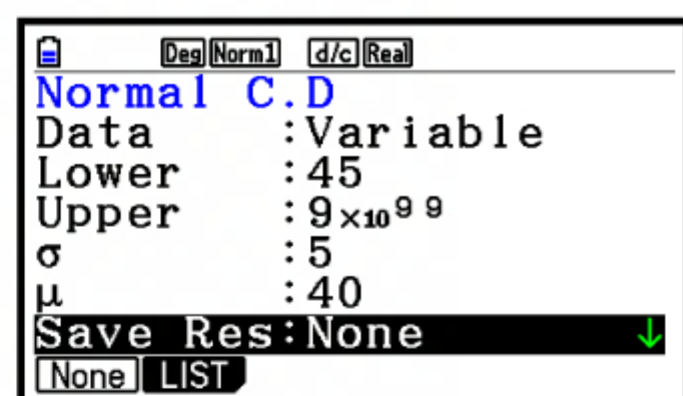
The game is fair when the expected gain is 0.

$$\therefore \frac{19a}{64} - k = 0 \quad \text{or} \quad 19a = 64k$$

4 Let the length of a randomly selected adult fish be  $X$  cm.

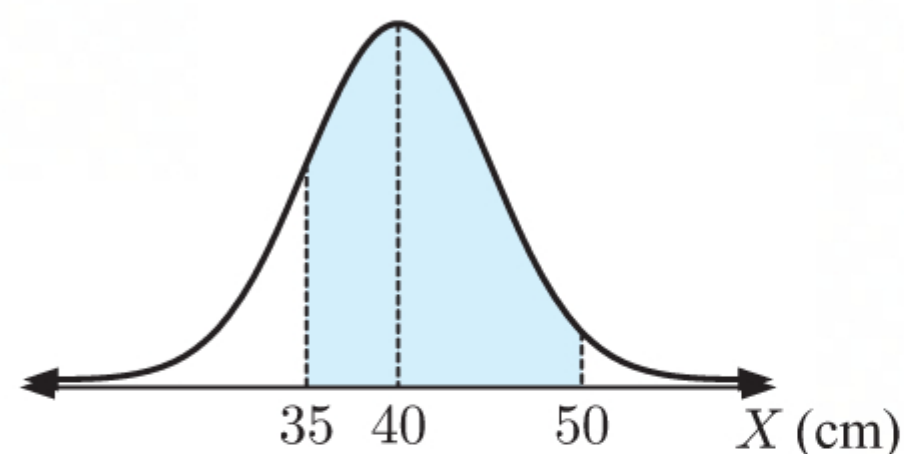
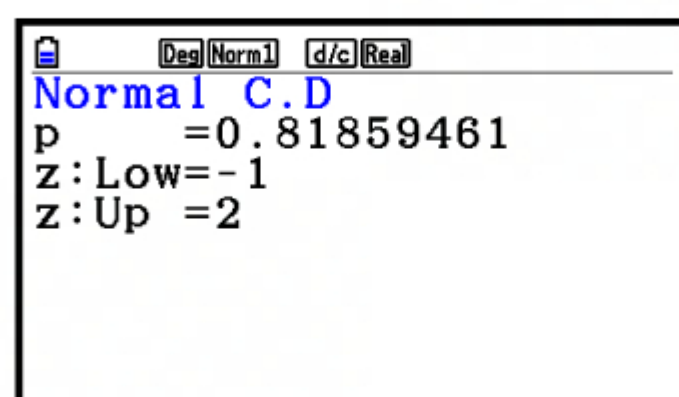
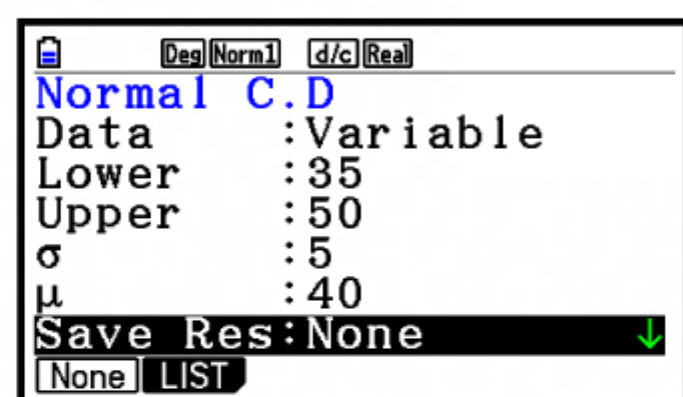
So,  $X \sim N(40, 5^2)$ .

a i



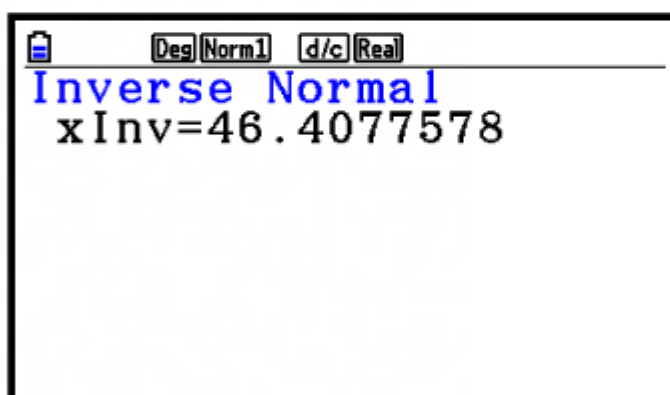
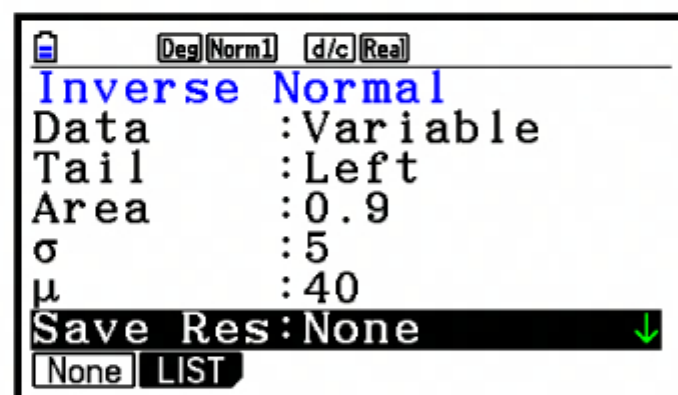
$$P(X > 45) \approx 0.159$$

ii



$$P(35 \leq X \leq 50) \approx 0.819$$

b

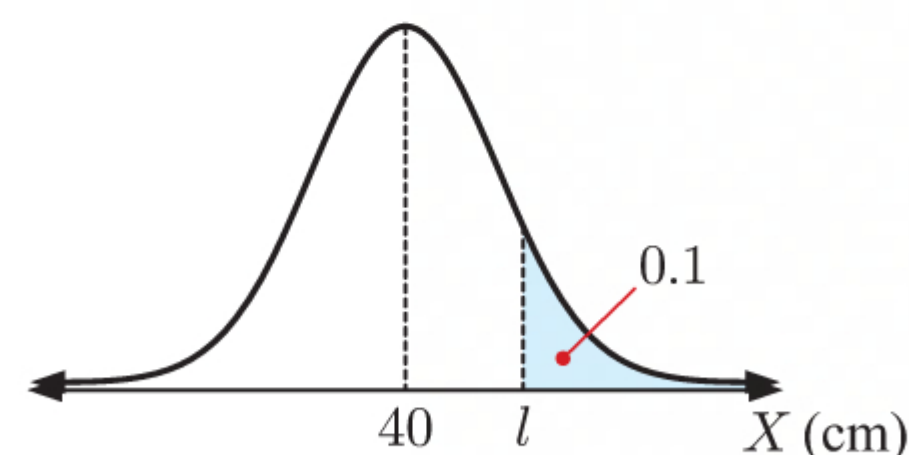


$$P(X > l) = 0.1$$

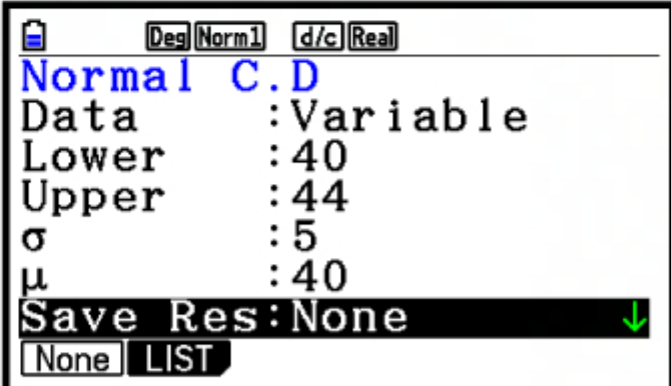
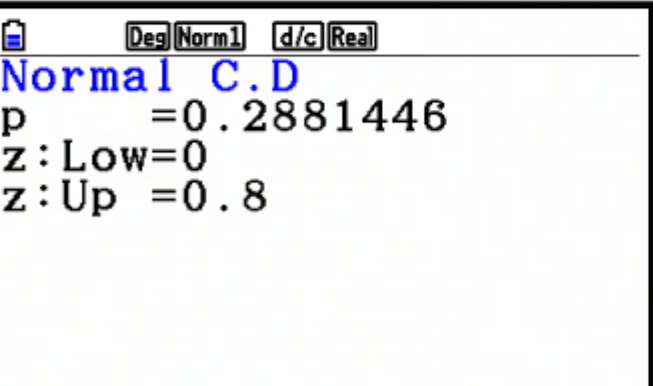
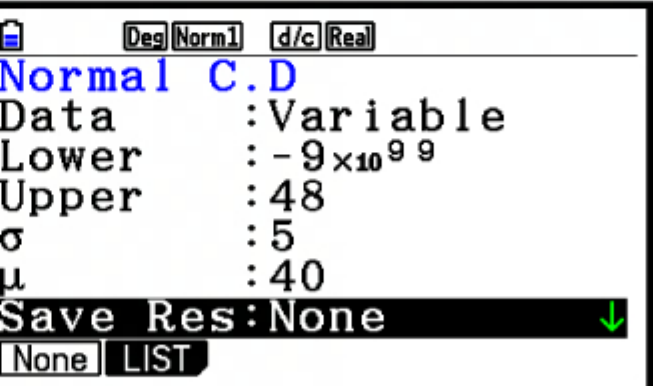
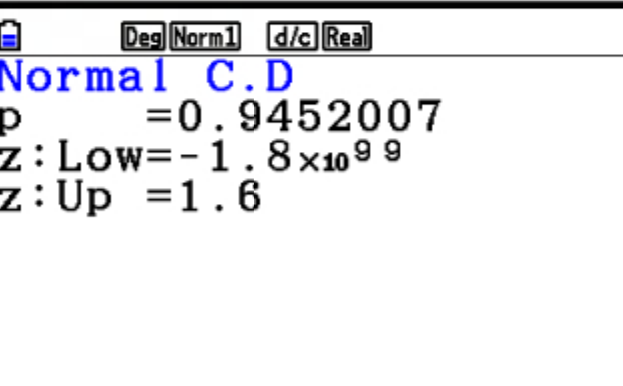
$$\therefore P(X \leq l) = 0.9$$

$$\therefore l = 46.4$$

$\therefore$  the minimum length of the longest 10% of fish is about 46.4 cm.



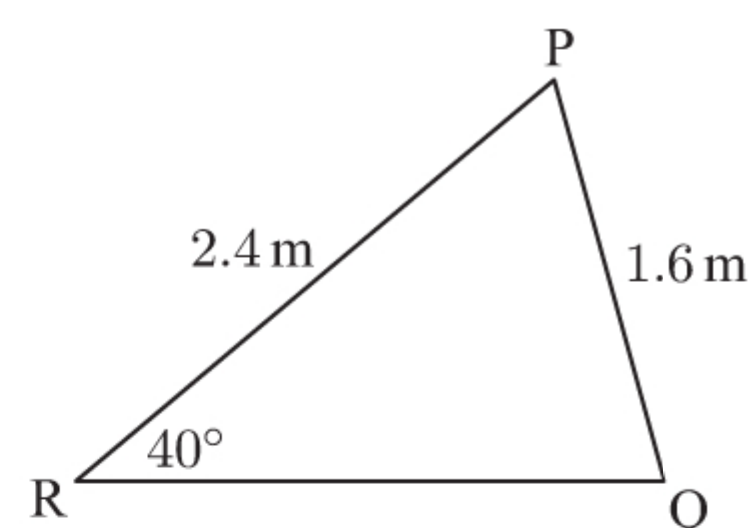


<b>c</b> 			
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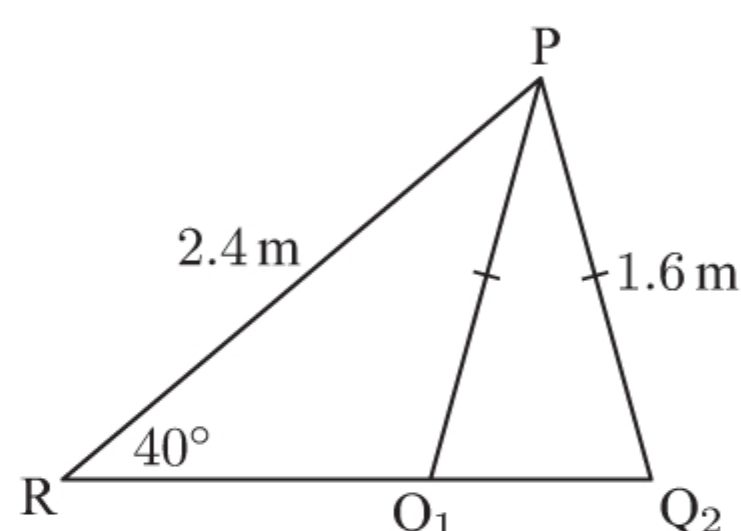
$$\begin{aligned}
 P(40 \leq X \leq 44 \mid X < 48) &= \frac{P((40 \leq X \leq 44) \cap (X < 48))}{P(X < 48)} \\
 &= \frac{P(40 \leq X \leq 44)}{P(X < 48)} \\
 &\approx \frac{0.288}{0.945} \\
 &\approx 0.305
 \end{aligned}$$

**5 a** Using the sine rule,

$$\begin{aligned}
 \frac{\sin \widehat{PQR}}{2.4} &= \frac{\sin 40^\circ}{1.6} \\
 \therefore \sin \widehat{PQR} &= \frac{2.4 \times \sin 40^\circ}{1.6} \\
 \therefore \widehat{PQR} &= \sin^{-1}\left(\frac{2.4 \times \sin 40^\circ}{1.6}\right) \text{ or its supplement} \\
 \therefore \widehat{PQR} &\approx 74.6^\circ \text{ or } 180^\circ - 74.6^\circ \\
 \therefore \widehat{PQR} &\approx 74.6^\circ \text{ or } 105.4^\circ
 \end{aligned}$$



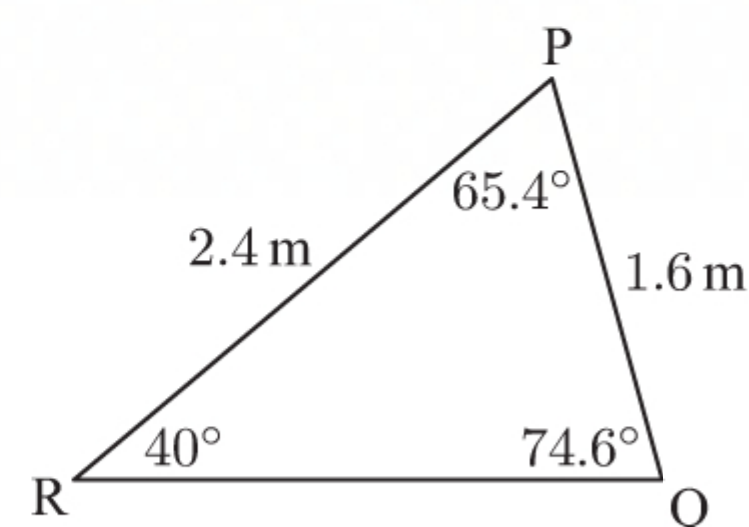
**b**



**c** For the case in which  $\widehat{PQR} \approx 74.6^\circ$ :

**i**  $\widehat{QPR} \approx 180^\circ - 40^\circ - 74.6^\circ$  {angles in a triangle}  
 $\therefore \widehat{QPR} \approx 65.4^\circ$

**ii**  $\frac{QR}{\sin \widehat{QPR}} = \frac{PQ}{\sin \widehat{PRQ}}$  {sine rule}  
 $\therefore \frac{QR}{\sin 65.4^\circ} \approx \frac{1.6}{\sin 40^\circ}$   
 $\therefore QR \approx \frac{1.6 \times \sin 65.4^\circ}{\sin 40^\circ}$   
 $\therefore QR \approx 2.26 \text{ m}$

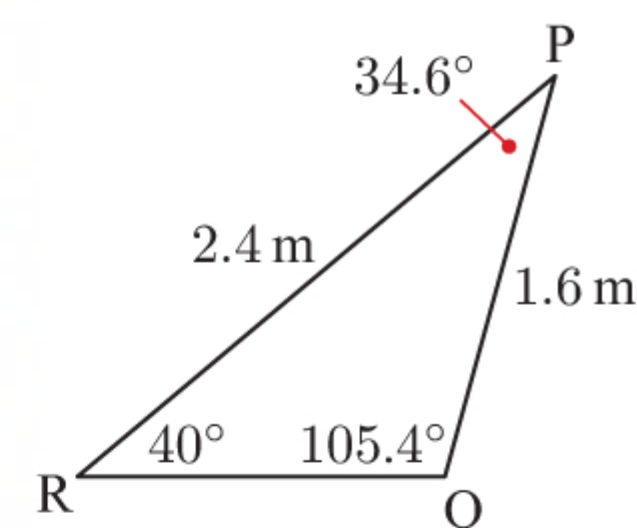


So, perimeter of the garden  $\approx (2.4 + 1.6 + 2.26) \text{ m}$   
 $\approx 6.26 \text{ m}$

For the case in which  $\widehat{PQR} \approx 105.4^\circ$ :

**i**  $\widehat{QPR} \approx 180^\circ - 40^\circ - 105.4^\circ$  {angles in a triangle}  
 $\therefore \widehat{QPR} \approx 34.6^\circ$

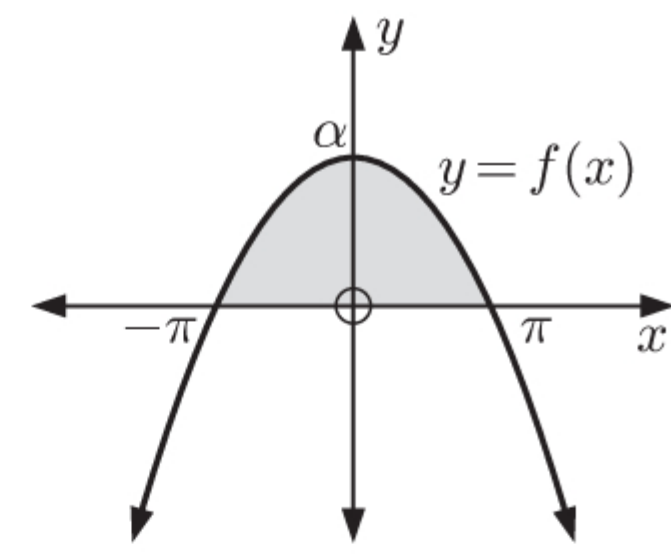
**ii**  $\frac{QR}{\sin \widehat{QPR}} = \frac{PQ}{\sin \widehat{PRQ}}$  {sine rule}  
 $\therefore \frac{QR}{\sin 34.6^\circ} \approx \frac{1.6}{\sin 40^\circ}$   
 $\therefore QR \approx \frac{1.6 \times \sin 34.6^\circ}{\sin 40^\circ}$   
 $\therefore QR \approx 1.41 \text{ m}$



So, perimeter of the garden  $\approx (2.4 + 1.6 + 1.41) \text{ m}$   
 $\approx 5.41 \text{ m}$



- 6 Suppose  $\alpha > 0$ , then the parabola has equation  $f(x) = -\frac{\alpha}{\pi^2}(x - \pi)(x + \pi)$   
 $= -\frac{\alpha}{\pi^2}(x^2 - \pi^2)$



Now area = 4 units<sup>2</sup>

$$\therefore \int_{-\pi}^{\pi} f(x) dx = 4$$

$$\therefore \int_{-\pi}^{\pi} -\frac{\alpha}{\pi^2}(x^2 - \pi^2) dx = 4$$

$$\therefore -\frac{\alpha}{\pi^2} \int_{-\pi}^{\pi} (x^2 - \pi^2) dx = 4$$

$$\therefore \int_{-\pi}^{\pi} (x^2 - \pi^2) dx = -\frac{4\pi^2}{\alpha}$$

$$\therefore \left[ \frac{1}{3}x^3 - \pi^2 x \right]_{-\pi}^{\pi} = -\frac{4\pi^2}{\alpha}$$

$$\therefore \left( \frac{1}{3}\pi^3 - \pi^2(\pi) \right) - \left( \frac{1}{3}(-\pi)^3 - \pi^2(-\pi) \right) = -\frac{4\pi^2}{\alpha}$$

$$\therefore \frac{\pi^3}{3} - \pi^3 - \left( -\frac{\pi^3}{3} + \pi^3 \right) = -\frac{4\pi^2}{\alpha}$$

$$\therefore \frac{\pi^3}{3} - \pi^3 + \frac{\pi^3}{3} - \pi^3 = -\frac{4\pi^2}{\alpha}$$

$$\therefore -\frac{4\pi^3}{3} = -\frac{4\pi^2}{\alpha}$$

$$\therefore \alpha = \frac{3}{\pi}$$

Similarly, if  $\alpha < 0$ ,  $\alpha = -\frac{3}{\pi}$ .

- 7 a Each test is identical and is done under the same conditions by the same set of students each month.

Test-retest reliability is being considered in this case.

b

Student	A	B	C	D	E	F	G	H	I	J	K	L
Test 1	25	21	18	27	19	22	21	26	14	17	20	16
Test 2	26	19	21	28	17	24	22	29	18	17	24	19

Rad(Norm1)	d/c	Real
LinearReg(ax+b)		
a = 0.91061452		
b = 3.33240223		
r = 0.87470221		
r <sup>2</sup> = 0.76510395		
MSe = 4.55698324		
y = ax + b		
COPY DRAW		

So,  $r \approx 0.875$ .

- c From **b**, there is a strong positive correlation between the students' *Test 1* and *Test 2* results. The spelling test is therefore reliable.
- d Factors which might have affected the test-retest reliability of the test include:
- the students may have remembered some of the words on the test
  - the students' spelling abilities may have improved during the month.
- e The test could be changed to consider parallel forms reliability by randomly selecting 30 words of a similar difficulty for each test.

8  $L_1: \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}, \quad t \in \mathbb{R}$

- a  $L_1$  meets the  $XY$ -plane where  $4 - 2t = 0$

$$\therefore 2t = 4$$

$$\therefore t = 2$$

$\therefore L_1$  meets the  $XY$ -plane at  $(2 + 3(2), -1 + 4(2), 0)$  which is  $(8, 7, 0)$ .



**b**  $L_2$  has direction vector  $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$  and passes through  $(3, 0, 0)$ .

$\therefore L_2$  has vector equation  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}, s \in \mathbb{R}$

$\therefore L_2$  has parametric equations  $x = 3 + 3s, y = 5s, z = 0, s \in \mathbb{R}$ .

**c**  $L_3$  has direction vector  $\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -2 \\ 3 & 5 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 4 & -2 \\ 5 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 4 \\ 3 & 5 \end{vmatrix} \mathbf{k}$$

$$= (0 + 10)\mathbf{i} - (0 + 6)\mathbf{j} + (15 - 12)\mathbf{k}$$

$$= 10\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$$

$$= \begin{pmatrix} 10 \\ -6 \\ 3 \end{pmatrix}$$

Since  $L_3$  passes through  $(-2, 5, 1)$ ,  $L_3$  has vector equation  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ -6 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$

$\therefore L_3$  has parametric equations  $x = -2 + 10\lambda, y = 5 - 6\lambda, z = 1 + 3\lambda, \lambda \in \mathbb{R}$ .

**9**  $v = \frac{3}{2}(s+1)^{\frac{1}{3}} \text{ m s}^{-1}$

**a** When  $s = 7 \text{ m}$ ,

$$v = \frac{3}{2}(7+1)^{\frac{1}{3}} = \frac{3}{2}(8)^{\frac{1}{3}} = 3 \text{ m s}^{-1}$$

**b**  $a = v \frac{dv}{ds}$

$$= \frac{3}{2}(s+1)^{\frac{1}{3}} \times \frac{1}{2}(s+1)^{-\frac{2}{3}}$$

$$= \frac{3}{4}(s+1)^{-\frac{1}{3}} \text{ m s}^{-2}$$

**c**  $v = \frac{3}{2}(s+1)^{\frac{1}{3}}, s(0) = 0$

$$\therefore \frac{ds}{dt} = \frac{3}{2}(s+1)^{\frac{1}{3}}$$

$$\therefore (s+1)^{-\frac{1}{3}} \frac{ds}{dt} = \frac{3}{2}$$

$$\therefore \int (s+1)^{-\frac{1}{3}} \frac{ds}{dt} dt = \int \frac{3}{2} dt$$

$$\therefore \int (s+1)^{-\frac{1}{3}} ds = \int \frac{3}{2} dt$$

$$\therefore \frac{3}{2}(s+1)^{\frac{2}{3}} = \frac{3}{2}t + c$$

$$\therefore (s+1)^{\frac{2}{3}} = t + c$$

$$\therefore s+1 = (t+c)^{\frac{3}{2}}$$

$$\therefore s = (t+c)^{\frac{3}{2}} - 1$$

Now  $s(0) = 0$ .  $\therefore 0 = (0+c)^{\frac{3}{2}} - 1$

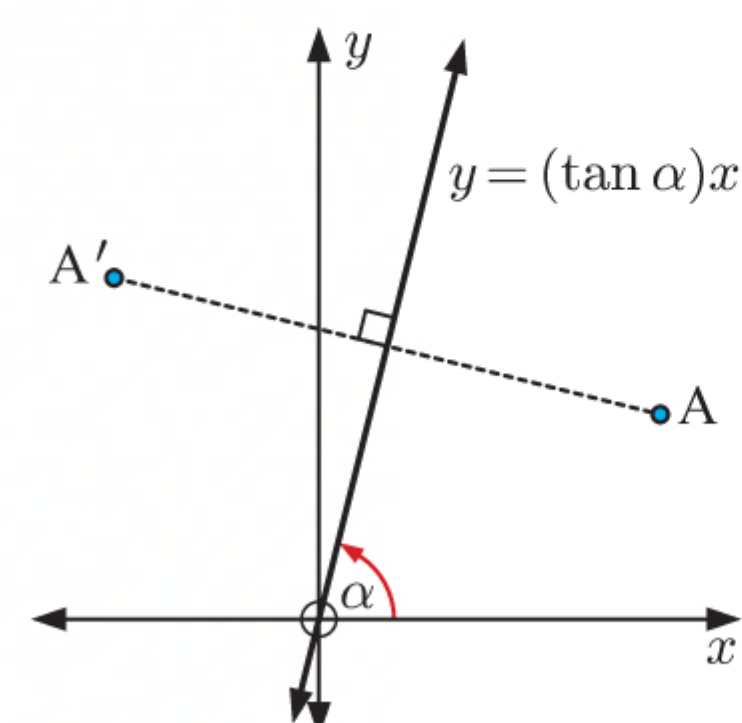
$$\therefore c^{\frac{3}{2}} = 1$$

$$\therefore c = 1$$

$$\therefore s = (t+1)^{\frac{3}{2}} - 1$$

**10 a** Gradient of  $[AA'] = \frac{5-3}{-3-5} = \frac{2}{-8} = -\frac{1}{4}$

The line  $y = (\tan \alpha)x$  is perpendicular to  $[AA']$ , so  $\tan \alpha = -\frac{1}{(-\frac{1}{4})} = 4$ .





**b**  $\alpha = \tan^{-1}(4)$

$\therefore \cos 2\alpha = -\frac{15}{17}$  and  $\sin 2\alpha = \frac{8}{17}$

$\therefore \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{pmatrix}$

**c**  $\mathbf{x}' = \mathbf{A} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$   
 $= \begin{pmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$   
 $= \begin{pmatrix} -\frac{14}{17} \\ \frac{46}{17} \end{pmatrix}$

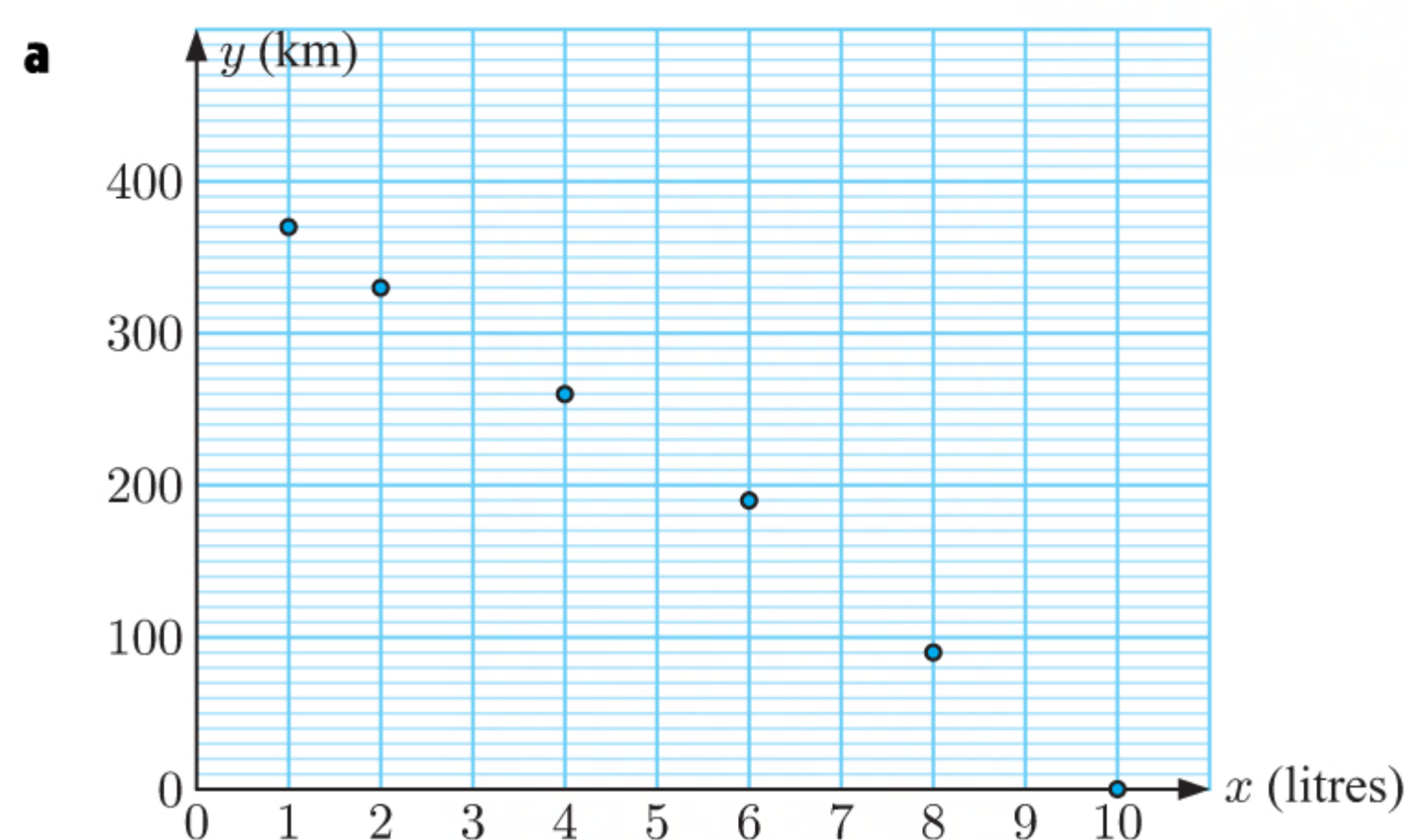
$\therefore$  the image of  $(2, 2)$  is  $(-\frac{14}{17}, \frac{46}{17})$ .

Math Rad Norm1 d/c Real  
 $\cos(2\tan^{-1} 4)$   $-\frac{15}{17}$   
 $\sin(2\tan^{-1} 4)$   $\frac{8}{17}$   
 JUMP DELETE MAT MATH

## MIXED QUESTIONS SET 6

**1**

Remaining fuel ( $x$ litres)	10	8	6	4	2	1
Distance ( $y$ km)	0	90	190	260	330	370



**b**

Deal Norm1 d/c Real  
 LinearReg(ax+b)  
 a = -40.712328  
 b = 417.013698  
 r = -0.9975176  
 r^2 = 0.99504145  
 MSe = 125.616438  
 y = ax + b  
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Using technology, the regression line is  $y \approx -40.7x + 417$ .

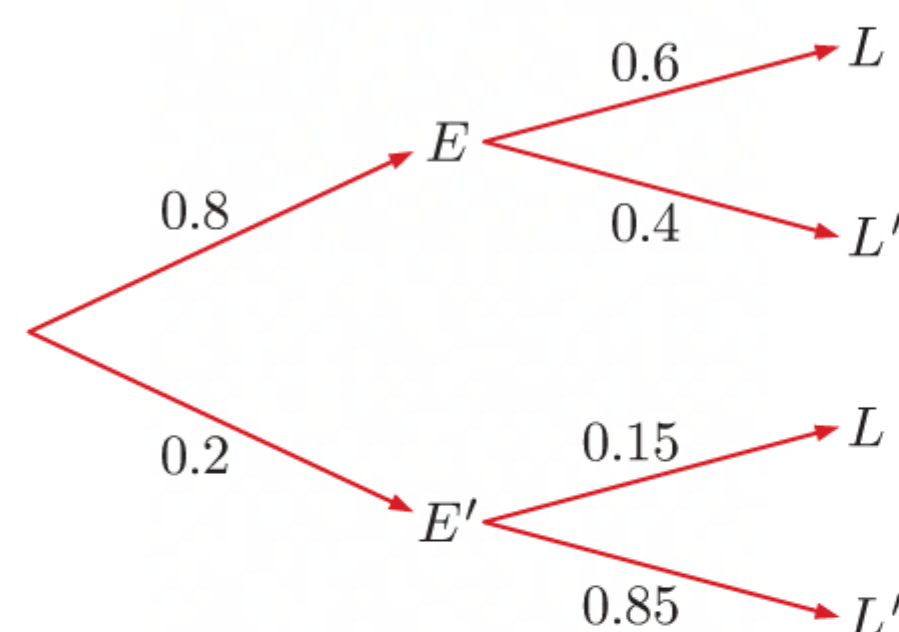
**c** The  $y$ -intercept of the regression line  $\approx 417$ . This indicates that the motorbike can travel about 417 km on a full tank of petrol.

**d i** When  $y = 220$ ,  $220 \approx -40.7x + 417$   
 $\therefore -197 \approx -40.7x$   
 $\therefore x \approx 4.84$

$\therefore$  there is about 4.84 litres of fuel left in the tank after the motorbike has travelled 220 km.

**ii** Average distance travelled per litre  $\approx \frac{220}{10 - 4.84} \approx 42.6$  km per litre.

**2 a** Let  $E$  be the event that Mark wakes up early, and  $L$  be the event that Mark packs his lunch.



**b**  $P(L) = P(E \cap L) + P(E' \cap L)$   
 $= 0.8 \times 0.6 + 0.2 \times 0.15$   
 $= 0.51$



- 3 a** In  $\triangle ABC$ , by the cosine rule:

$$\begin{aligned} BC^2 &= 65^2 + 104^2 - 2 \times 65 \times 104 \times \cos 60^\circ \\ \therefore BC &= \sqrt{65^2 + 104^2 - 2 \times 65 \times 104 \times \cos 60^\circ} \quad \{\text{as } BC > 0\} \\ \therefore BC &= 91 \text{ m} \end{aligned}$$

**b** Area of  $\triangle ABC = \frac{1}{2} \times AB \times AC \times \sin \widehat{BAC}$

$$\begin{aligned} &= \frac{1}{2} \times 65 \times 104 \times \sin 60^\circ \\ &= 65 \times 52 \times \frac{\sqrt{3}}{2} \\ &= 1690\sqrt{3} \text{ m}^2 \\ &\approx 2930 \text{ m}^2 \end{aligned}$$

$\therefore$  the total area of the field is about  $2930 \text{ m}^2$ .

**c** Area of  $A_1 = \frac{1}{2} \times AB \times AD \times \sin \widehat{BAD}$

$$\begin{aligned} &= \frac{1}{2} \times 65 \times x \times \sin 30^\circ \\ &= \frac{65x}{2} \times \frac{1}{2} \\ &= \frac{65x}{4} \text{ m}^2 \end{aligned}$$

Area of  $A_2 = \frac{1}{2} \times AC \times AD \times \sin \widehat{CAD}$

$$\begin{aligned} &= \frac{1}{2} \times 104 \times x \times \sin 30^\circ \\ &= 52x \times \frac{1}{2} \\ &= 26x \text{ m}^2 \end{aligned}$$

Now, the total area of the field  $= A_1 + A_2$

$$\therefore 1690\sqrt{3} = \frac{65x}{4} + 26x \quad \{\text{from a}\}$$

$$\therefore 1690\sqrt{3} = x\left(\frac{65}{4} + 26\right)$$

$$\therefore x = \frac{1690\sqrt{3}}{\frac{65}{4} + 26}$$

$$\therefore x \approx 69.3$$

**4 a**  $D'(t) = 0.8t - 8$

$$\begin{aligned} \therefore D(t) &= \int (0.8t - 8) dt \\ &= 0.4t^2 - 8t + c \end{aligned}$$

Now  $D(0) = 42 \quad \therefore 0.4(0)^2 - 8(0) + c = 42$

$$\therefore c = 42$$

$$\therefore D(t) = 0.4t^2 - 8t + 42$$

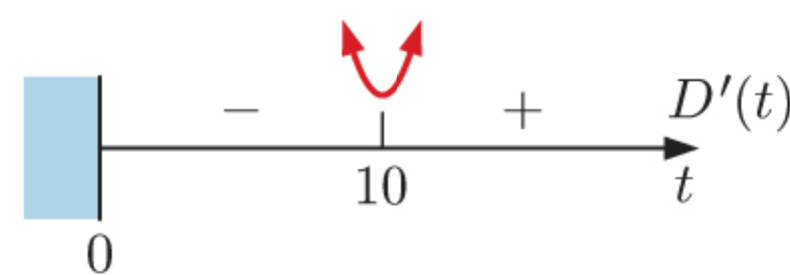
**c** The minimum distance occurs when  $D'(t) = 0$

$$\therefore 0.8t - 8 = 0$$

$$\therefore 0.8t = 8$$

$$\therefore t = 10$$

$D'(t)$  has sign diagram



$\therefore D$  is minimised when  $t = 10$ .

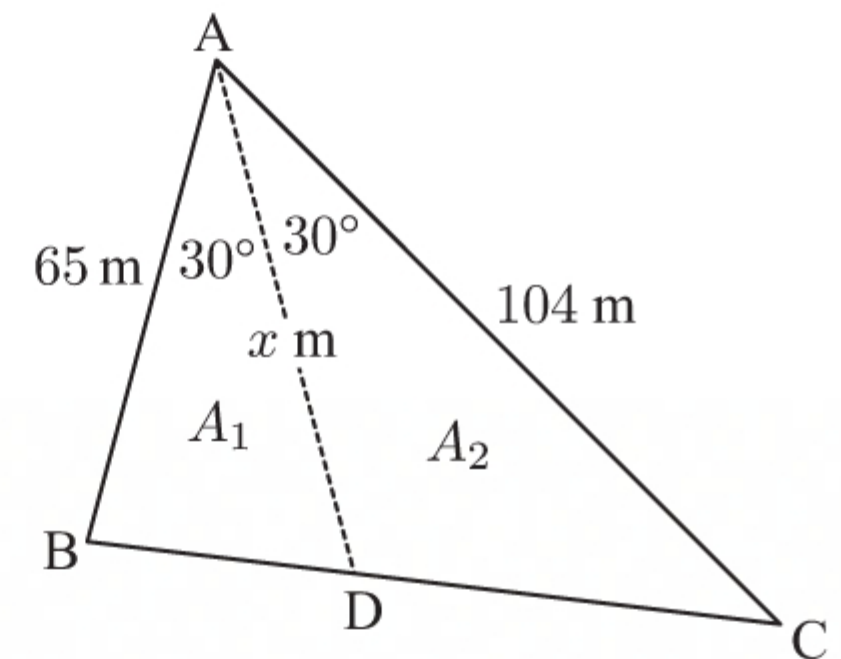
$$\begin{aligned} D(10) &= 0.4(10)^2 - 8(10) + 42 \\ &= 40 - 80 + 42 \\ &= 2 \end{aligned}$$

$\therefore$  the minimum distance between the motorcyclists is 2 m which occurs 10 seconds after they start riding.

**5 a**  $N = 7 \times 12 = 84$ ,  $I\% = 9.25$ ,  $PV = 55\,000$ ,  $FV = 0$ ,  $P/Y = 12$ ,  $C/Y = 12$

$$\therefore PMT \approx -891.90$$

The monthly repayment is £891.90



Norm1		End
Compound Interest		
n	=	84
I%	=	9.25
PV	=	55000
PMT	=	-891.8933025
FV	=	0
P/Y	=	12
n	I%	PV PMT FV AMORTZN



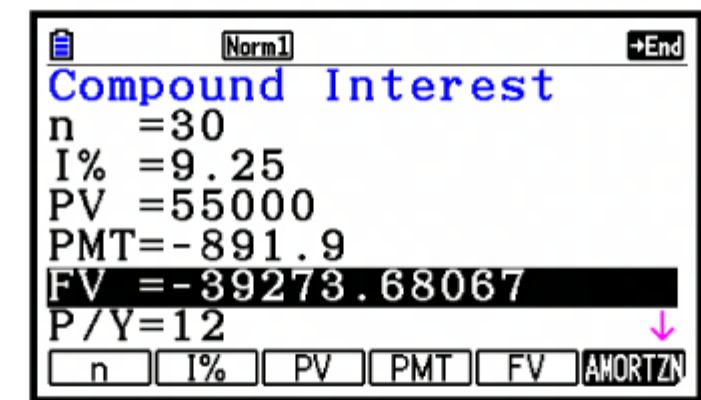
- b**  $N = 2\frac{1}{2} \times 12 = 30$ ,  $I\% = 9.25$ ,  $PV = 55\,000$ ,  $PMT = -891.90$ ,  
 $P/Y = 12$ ,  $C/Y = 12$   
 $\therefore FV \approx 39\,273.68$

The outstanding debt is £39 273.68

- c** Depreciated value after 7 years  $= £55\,000 \times (0.85)^7$   
 $= £17\,631.74$

Total repayments  $= 84 \times £891.90$   
 $= £74\,919.60$

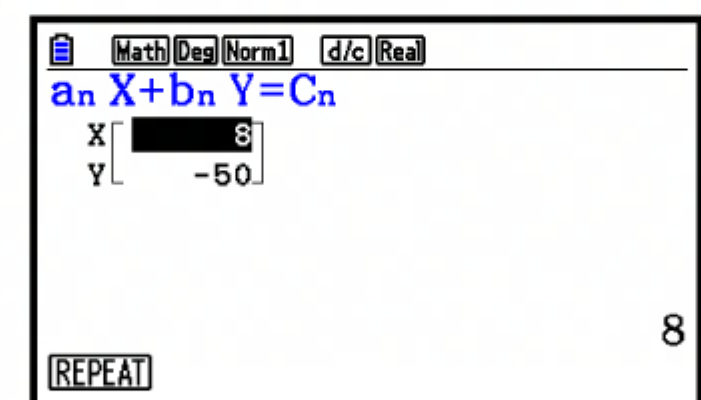
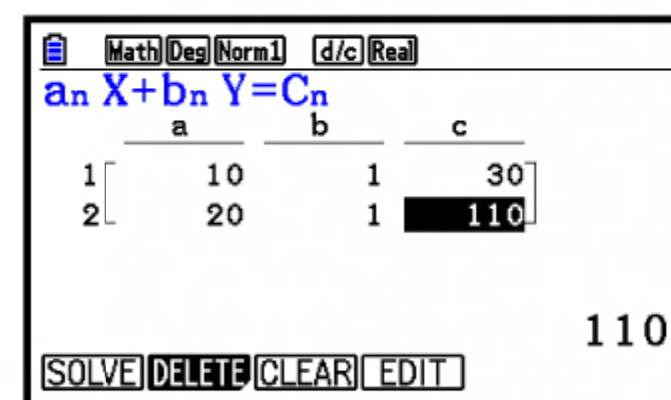
$\therefore$  cost of car to Melinda  $= £74\,919.60 - £17\,631.74$   
 $= £57\,287.86$



- 6 a** Using (1),  $10a + b = 30$

Using (2),  $20a + b = 110$

- b** Solving the system of equations  $\begin{cases} 10a + b = 30 \\ 20a + b = 110 \end{cases}$   
simultaneously gives  $a = 8$  and  $b = -50$ .



- c** From **b**, the model is  $T = 8r - 50$ .

For small values of  $r$ ,  $T \approx -50$ , which does not make any physical sense.

So, this model is not appropriate for small values of  $r$ .

- d** For Globe Park,  $r = 30$ .

$$\begin{aligned} \therefore T &= 8(30) - 50 \\ &= 240 - 50 \\ &= 190 \end{aligned}$$

$\therefore$  it will take 190 minutes to maintain Globe Park.

**e** Percentage error  $= \frac{|V_A - V_E|}{V_E} \times 100\%$   
 $= \frac{|190 - 230|}{230} \times 100\%$   
 $= \frac{40}{230} \times 100\%$   
 $\approx 17.4\%$

- f i**  $T = \text{time to mow interior} + \text{time to trim perimeter}$

Now time to mow interior  $\propto$  area of circle  $= \pi r^2$

$\therefore$  time to mow interior  $= k_1 \pi r^2$ , where  $k_1$  is a constant

and time to trim perimeter  $\propto$  circumference of circle  $= 2\pi r$

$\therefore$  time to trim perimeter  $= 2k_2 \pi r$ , where  $k_2$  is a constant

$$\therefore T = k_1 \pi r^2 + 2k_2 \pi r$$

This model has the form  $T = pr^2 + qr$  where  $p$  and  $q$  are constants.

- ii** When  $r = 10$ ,  $T = 30$

When  $r = 20$ ,  $T = 110$

$$\therefore 30 = p(10)^2 + q(10)$$

$$\therefore 110 = p(20)^2 + q(20)$$

$$\therefore 30 = 100p + 10q$$

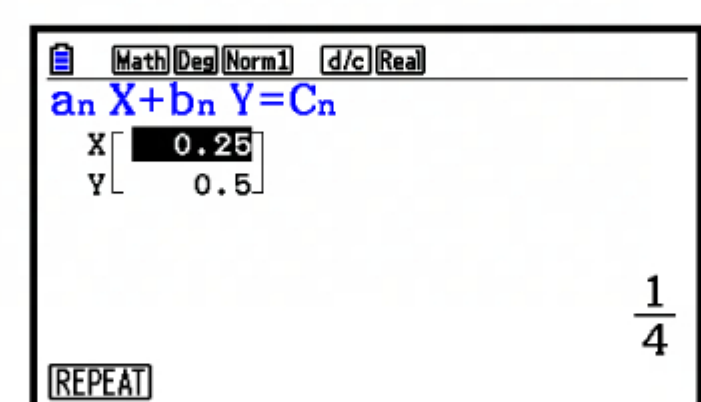
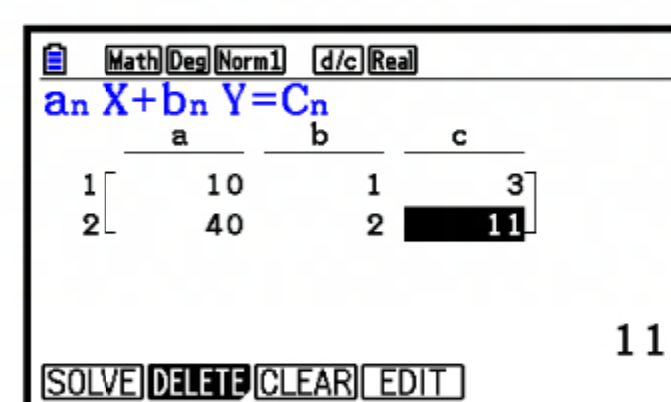
$$\therefore 110 = 400p + 20q$$

$$\therefore 10p + q = 3$$

$$\therefore 40p + 2q = 11$$

So we have the system of equations  $\begin{cases} 10p + q = 3 \\ 40p + 2q = 11 \end{cases}$ .

Using technology to solve the system simultaneously gives  $p = \frac{1}{4}$  and  $q = \frac{1}{2}$ .





iii From ii,  $T = \frac{1}{4}r^2 + \frac{1}{2}r$

For Globe Park,  $r = 30$

$$\begin{aligned}\therefore T &= \frac{1}{4}(30)^2 + \frac{1}{2}(30) \\ &= 240\end{aligned}$$

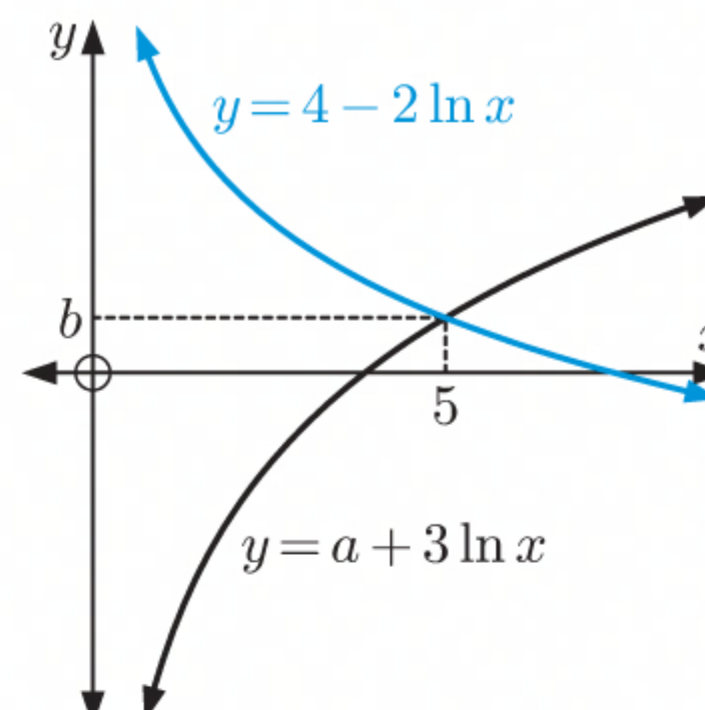
This model is better at predicting the time taken to maintain Globe Park as the predicted value is closer to the actual time than our estimate in d.

7 a When  $x = 5$ ,  $y = b$

$$\begin{aligned}\therefore b &= 4 - 2 \ln 5 \\ &= \ln(e^4) - \ln 25 \\ &= \ln\left(\frac{e^4}{25}\right)\end{aligned}$$

b When  $x = 5$ ,  $y = \ln\left(\frac{e^4}{25}\right)$  {from a}

$$\begin{aligned}\therefore a + 3 \ln 5 &= \ln\left(\frac{e^4}{25}\right) \\ \therefore a &= \ln\left(\frac{e^4}{25}\right) - \ln(5^3) \\ &= \ln\left(\frac{e^4}{3125}\right)\end{aligned}$$

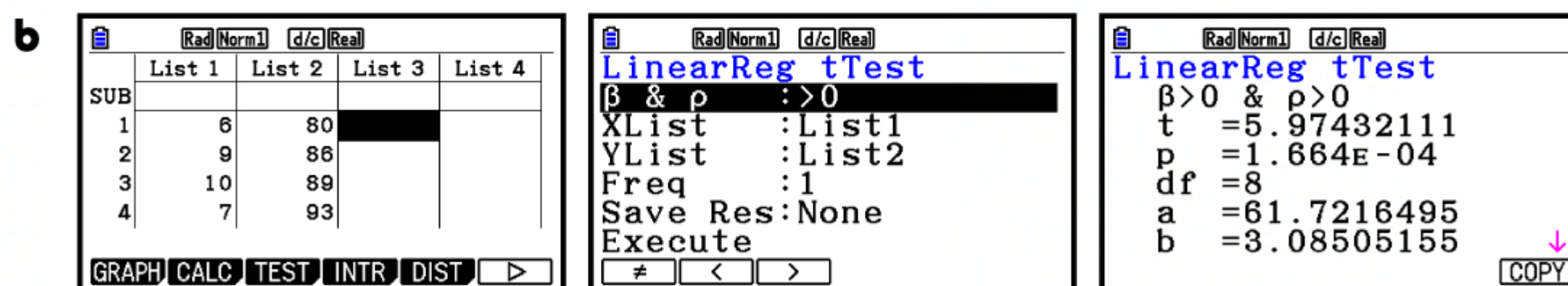


8 a Let  $\rho$  be the population product-moment correlation coefficient between the variables.

The hypotheses are:

$$H_0: \rho = 0 \quad \{\text{there is no correlation between the variables}\}$$

$$H_1: \rho > 0 \quad \{\text{the variables are positively correlated}\}$$



Using technology:

i the test statistic  $\approx 5.97$

ii  $p$ -value  $\approx 0.000166$

c The significance level is  $\alpha = 0.01$ .

Since  $p$ -value  $< 0.01 = \alpha$ , we have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 1% significance level.

We therefore accept  $H_1$ , so we conclude that there is a positive correlation between *tutorials attended* and *final mark*.

d Yes, there is a causal relationship. Attending more tutorials is likely to increase the final mark.

9 Let  $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$

$$\begin{aligned}\mathbf{a} \quad \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} &= \begin{pmatrix} 2 \times 3 + 3 \times (-4) \\ 4 \times 3 + 1 \times (-4) \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 8 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \text{From a, } \mathbf{A} \begin{pmatrix} 3 \\ -4 \end{pmatrix} &= \begin{pmatrix} -6 \\ 8 \end{pmatrix} = -2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} \\ \therefore -2 &\text{ is an eigenvalue of } \mathbf{A} \text{ with corresponding} \\ &\text{eigenvector } \begin{pmatrix} 3 \\ -4 \end{pmatrix}.\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad p(\lambda) &= \det(\lambda \mathbf{I} - \mathbf{A}) \\ &= \begin{vmatrix} \lambda - 2 & -3 \\ -4 & \lambda - 1 \end{vmatrix} \\ &= (\lambda - 2)(\lambda - 1) - 12 \\ &= \lambda^2 - 3\lambda + 2 - 12 \\ &= \lambda^2 - 3\lambda - 10\end{aligned}$$



$$\begin{aligned} \mathbf{d} \quad p(\lambda) = 0 \quad \text{where} \quad \lambda^2 - 3\lambda - 10 &= 0 \\ \therefore (\lambda - 5)(\lambda + 2) &= 0 \\ \therefore \lambda = 5 \quad \text{or} \quad -2 \end{aligned}$$

$\therefore$  the remaining eigenvalue is 5.

For  $\lambda = 5$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{aligned} \therefore \begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \therefore 3a - 3b &= 0 \\ \therefore a - b &= 0 \end{aligned}$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$  choosing  $t = 1$ ,  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue 5.

$$\mathbf{10} \quad v(t) = 30 - 20e^{-0.2t} \text{ m s}^{-1}$$

$$\begin{aligned} \mathbf{a} \quad \mathbf{i} \quad v(0) &= 30 - 20e^0 \\ &= 10 \end{aligned}$$

$\therefore$  the initial velocity of the boat is  $10 \text{ m s}^{-1}$ .

$$\begin{aligned} \mathbf{ii} \quad v(2) &= 30 - 20e^{-0.2 \times 2} \\ &\approx 16.6 \end{aligned}$$

$\therefore$  the velocity of the boat after 2 seconds is about  $16.6 \text{ m s}^{-1}$ .

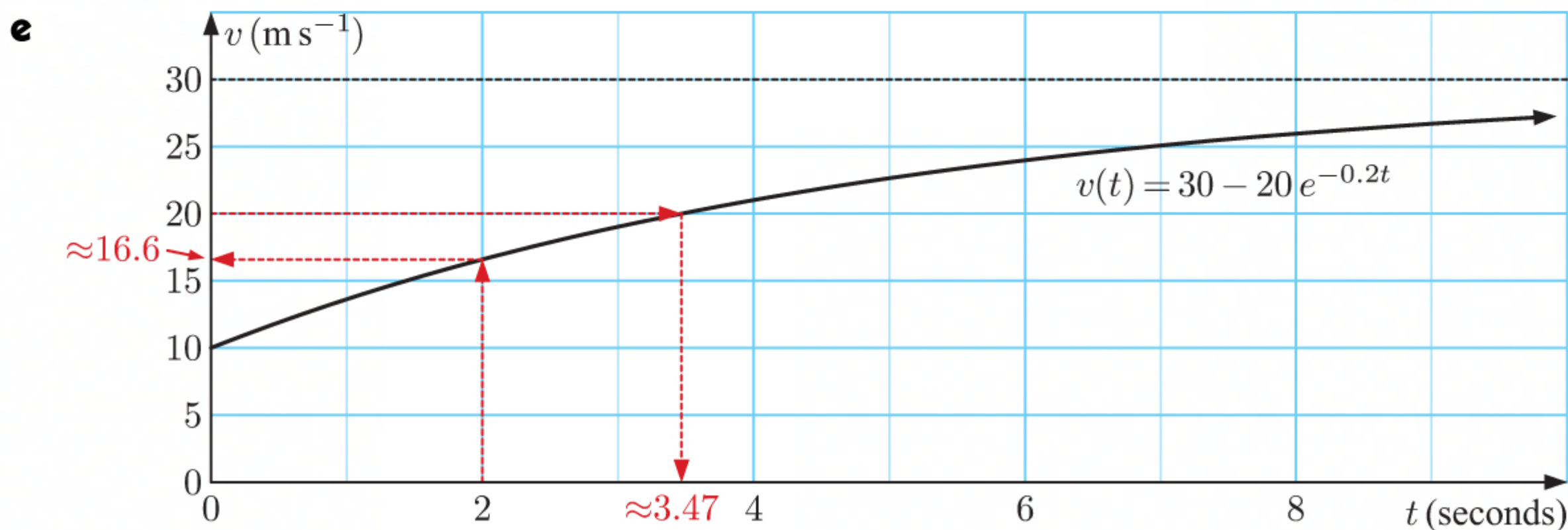
$$\begin{aligned} \mathbf{b} \quad v(t) = 20 \quad \text{when} \quad 30 - 20e^{-0.2t} &= 20 \\ \therefore 20e^{-0.2t} &= 10 \\ \therefore e^{-0.2t} &= \frac{1}{2} \\ \therefore e^{0.2t} &= 2 \\ \therefore 0.2t &= \ln 2 \\ \therefore t &= 5 \ln 2 \approx 3.47 \end{aligned}$$

It will take about 3.47 seconds for the boat's velocity to reach  $20 \text{ m s}^{-1}$ .

$$\begin{aligned} \mathbf{c} \quad \text{As } t \rightarrow \infty, \quad e^{-0.2t} &\rightarrow 0 \\ \therefore v(t) &\rightarrow 30 - 20(0) = 30 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad v(t) &= 30 - 20e^{-0.2t} \\ \therefore v'(t) &= -20e^{-0.2t}(-0.2) \quad \{\text{chain rule}\} \\ \therefore v'(t) &= 4e^{-0.2t} \\ \therefore v'(t) &> 0 \quad \{\text{as } e^{-0.2t} > 0 \text{ for all } t\} \end{aligned}$$

So, the acceleration  $v'(t)$  is always positive.





- f** The boat's velocity reached  $20 \text{ m s}^{-1}$  after  $5 \ln 2$  seconds. {from **b**}

$$\begin{aligned}
 \text{Distance travelled in first } 5 \ln 2 \text{ seconds} &= \int_0^{5 \ln 2} v(t) dt \\
 &= \int_0^{5 \ln 2} (30 - 20e^{-0.2t}) dt \\
 &= [30t + 100e^{-0.2t}]_0^{5 \ln 2} \\
 &= (150 \ln 2 + 100e^{-\ln 2}) - (0 + 100) \\
 &= 150 \ln 2 + 100 \times \frac{1}{2} - 100 \\
 &= 150 \ln 2 - 50 \\
 &\approx 54.0 \text{ m}
 \end{aligned}$$

$\therefore$  the boat travelled about 54.0 m before its velocity reached  $20 \text{ m s}^{-1}$ .

## MIXED QUESTIONS SET 7

- 1** *Neighbourhood A:* 275 281 320 265 305 258 310 430 285  
 290 297 345 195 230 269 300 258 273  
*Neighbourhood B:* 325 300 412 370 297 505 340 333 290  
 428 305 520 360 410 275 320 431 410

- a** The sale price of a house can be counted, so it is a discrete variable.

- b** *Neighbourhood A:*

1-Variable	
n	=18
minX	=195
Q1	=265
Med	=283
Q3	=305
maxX	=430

minimum = \$195 000

$Q_1$  = \$265 000

median = \$283 000

$Q_3$  = \$305 000

maximum = \$430 000

- Neighbourhood B:*

1-Variable	
n	=18
minX	=275
Q1	=305
Med	=350
Q3	=412
maxX	=520

minimum = \$275 000

$Q_1$  = \$305 000

median = \$350 000

$Q_3$  = \$412 000

maximum = \$520 000

- c** For Neighbourhood A,  $\text{IQR} = 305\,000 - 265\,000 = 40\,000$

<i>Test for outliers:</i>	upper boundary	and	lower boundary
	$= \text{upper quartile} + 1.5 \times \text{IQR}$		$= \text{lower quartile} - 1.5 \times \text{IQR}$
	$= 305\,000 + 1.5 \times 40\,000$		$= 265\,000 - 1.5 \times 40\,000$
	$= 365\,000$		$= 205\,000$

\$430 000 is above the upper boundary, so it is an outlier.

\$195 000 is below the lower boundary, so it is an outlier.

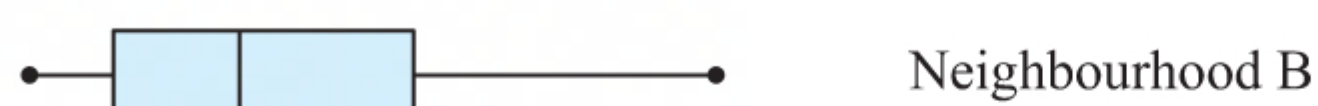
For Neighbourhood B,  $\text{IQR} = 412\,000 - 305\,000 = 107\,000$

<i>Test for outliers:</i>	upper boundary	and	lower boundary
	$= \text{upper quartile} + 1.5 \times \text{IQR}$		$= \text{lower quartile} - 1.5 \times \text{IQR}$
	$= 412\,000 + 1.5 \times 107\,000$		$= 305\,000 - 1.5 \times 107\,000$
	$= 572\,500$		$= 144\,500$

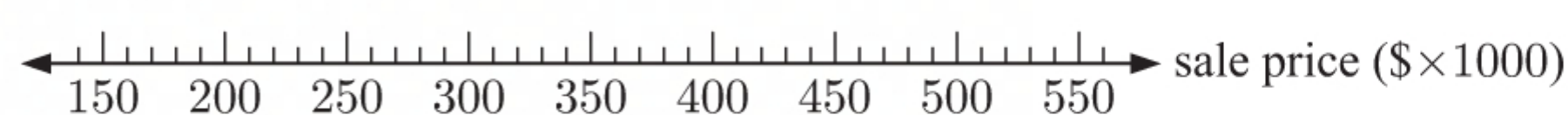
$\therefore$  there are no outliers.



Neighbourhood A



Neighbourhood B





- d** Both sets of data are positively skewed. The sale price of houses in Neighbourhood B are generally higher than those in Neighbourhood A. With the outliers removed, there is more variation in the sale price of houses in Neighbourhood B compared to Neighbourhood A.

**2 a**

$b$ (bags)	30	35	40	45	50
$P$ (rupiah)	38 000	36 000	34 000	32 000	30 000

We see that for every increase of 5 bags of rice, the price decreases by 2000 rupiah.

$\therefore P(b)$  is a linear function with gradient  $\frac{-2000}{5} = -400$

$\therefore P(b) = -400b + c$  for some constant  $c$ .

Now  $P(50) = 30\,000$

$\therefore -400 \times 50 + c = 30\,000$

$\therefore c = 50\,000$

$\therefore P(b) = -400b + 50\,000$

**b**  $P(60) = -400 \times 60 + 50\,000 = 26\,000$

$\therefore$  the total cost  $= 60 \times 26\,000 = 1\,560\,000$  rupiah

**c**  $P(150) = -400 \times 150 + 50\,000$   
 $= -10\,000$

This model should not be used to predict the cost of 150 bags of rice because it would yield a negative value.

**3 a**  $y = f(x) = a(x - p)(x - q)$  has  $x$ -intercepts  $-1$  and  $5$ .

$\therefore p = 5$  and  $q = -1$   $\{p > q\}$

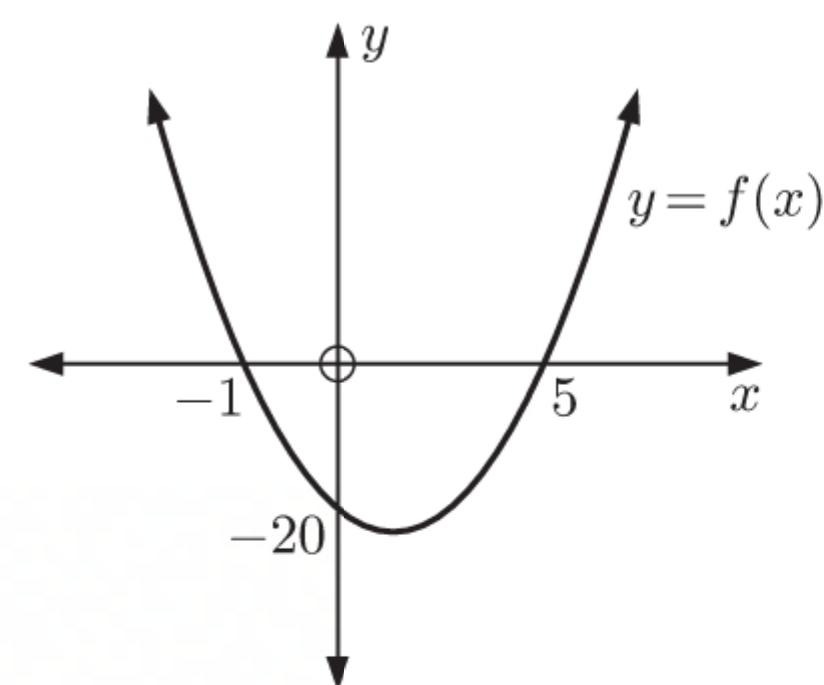
**b** From **a**,  $f(x) = a(x - 5)(x + 1)$ .

From the graph,  $f(0) = -20$

$\therefore -20 = a(-5)(1)$

$\therefore -20 = -5a$

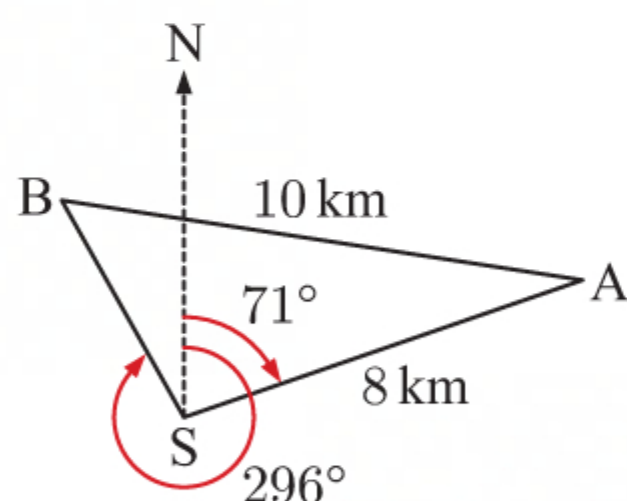
$\therefore a = 4$



**c** The axis of symmetry is midway between the  $x$ -intercepts.

$\therefore$  the axis of symmetry is  $x = \frac{-1 + 5}{2} = 2$ .

**4 a**



**b**  $N_1\hat{S}B = 360^\circ - 296^\circ$  {angles at a point}  
 $= 64^\circ$

$\therefore B\hat{S}A = 64^\circ + 71^\circ = 135^\circ$

$\therefore \frac{\sin \alpha}{8} = \frac{\sin 135^\circ}{10}$  {sine rule}

$\therefore \sin \alpha = \frac{8 \sin 135^\circ}{10}$

$\therefore \alpha = \sin^{-1}\left(\frac{8 \sin 135^\circ}{10}\right) \approx 34.4^\circ$

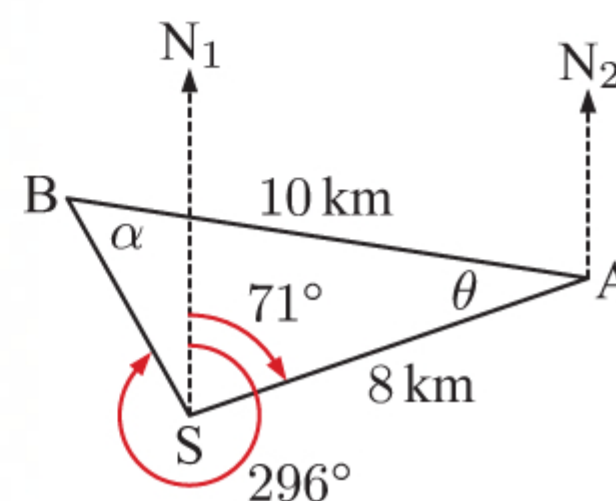
Now  $\theta = 180^\circ - 135^\circ - \alpha$  {angles in a triangle}

$\approx 180^\circ - 135^\circ - 34.4^\circ$

$\approx 10.6^\circ$

$N_2\hat{A}S = 180^\circ - 71^\circ$  {co-interior angles}  
 $= 109^\circ$

$\therefore$  bearing of B from A  $\approx 360^\circ - 109^\circ + 10.6^\circ \approx 262^\circ$





**c** By the cosine rule:

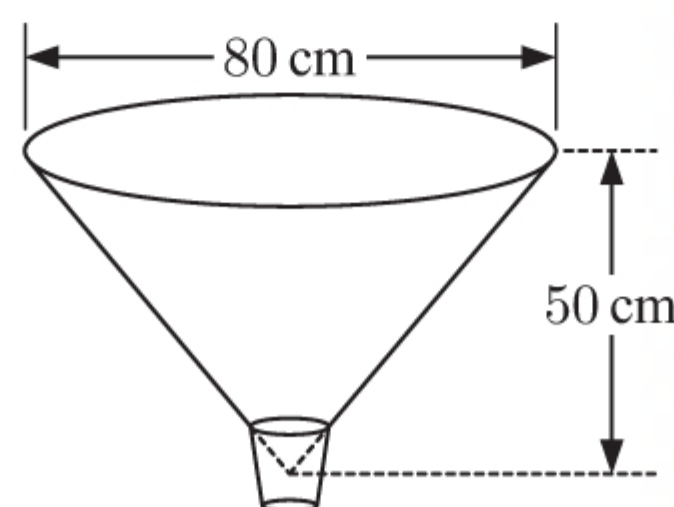
$$\begin{aligned} BS^2 &= 10^2 + 8^2 - 2(10)(8) \cos \theta \\ \therefore BS &= \sqrt{10^2 + 8^2 - 2(10)(8) \cos \theta} \\ \therefore BS &\approx \sqrt{10^2 + 8^2 - 2(10)(8) \cos 10.6^\circ} \quad \{\text{from b}\} \\ \therefore BS &\approx 2.59 \text{ km} \approx 2590 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Now time} &= \frac{\text{distance}}{\text{speed}} \\ &\approx \frac{2590}{7} \\ &\approx 370 \text{ seconds} \end{aligned}$$

It will take about 370 seconds or 6 minutes 10 seconds for train B to reach the train station.

**5 a**  $V \approx$  volume of cone

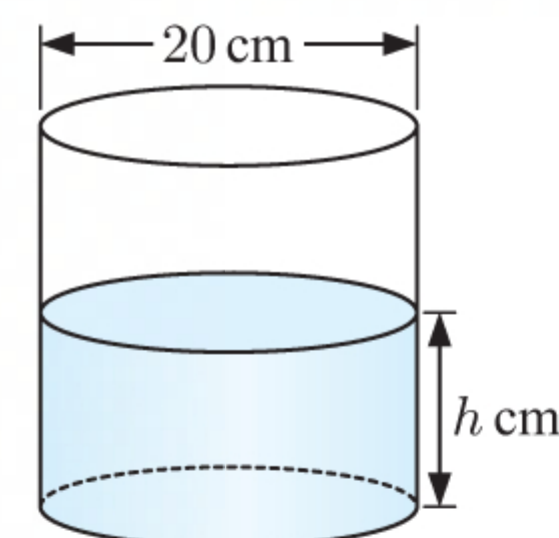
$$\begin{aligned} &\approx \frac{1}{3} \pi r^2 h \\ &\approx \frac{1}{3} \times \pi \times \left(\frac{80}{2}\right)^2 \times 50 \text{ cm}^3 \\ &\approx \frac{80\,000}{3} \pi \text{ cm}^3 \\ &\approx 83\,800 \text{ cm}^3 \end{aligned}$$



The capacity of the funnel is about 83 800 mL or  $8.38 \times 10^4$  mL.

**b** When half full, the funnel contains about  $\frac{80\,000}{3} \pi \times 0.5 \approx \frac{40\,000}{3} \pi$  mL of liquid.

$$\begin{aligned} V &\approx \frac{40\,000}{3} \pi \text{ cm}^3 \\ \therefore \pi \times \left(\frac{20}{2}\right)^2 \times h &\approx \frac{40\,000}{3} \pi \\ \therefore h &\approx \frac{40\,000}{3 \times 10^2} \\ \therefore h &\approx 133 \text{ cm} \end{aligned}$$

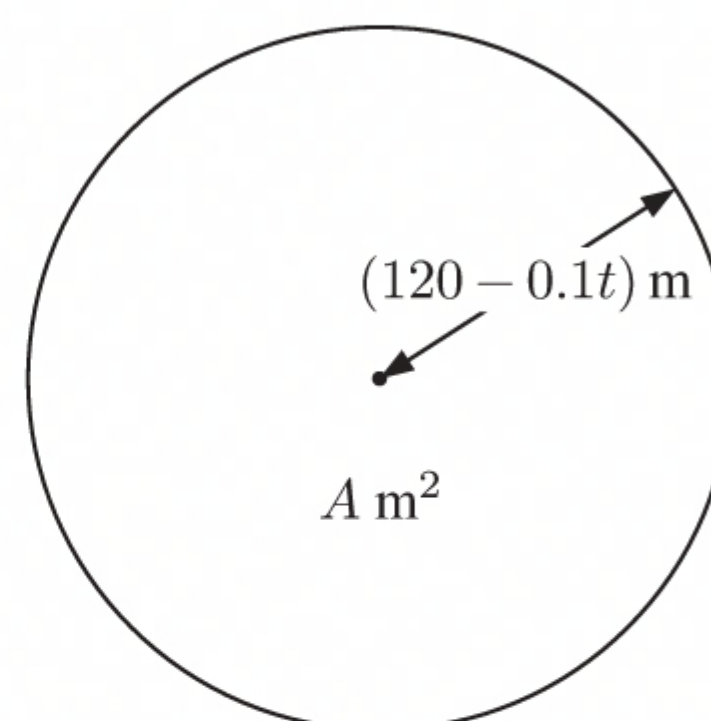


The liquid will reach about 133 cm up the tube.

**6 a** Area  $A = \pi r^2$

$$\begin{aligned} &= \pi(120 - 0.1t)^2 \\ &= \pi(14\,400 - 24t + 0.01t^2) \text{ m}^2 \end{aligned}$$

**b**  $\frac{dA}{dt} = \pi(-24 + 0.02t) \text{ m}^2 \text{ per second}$



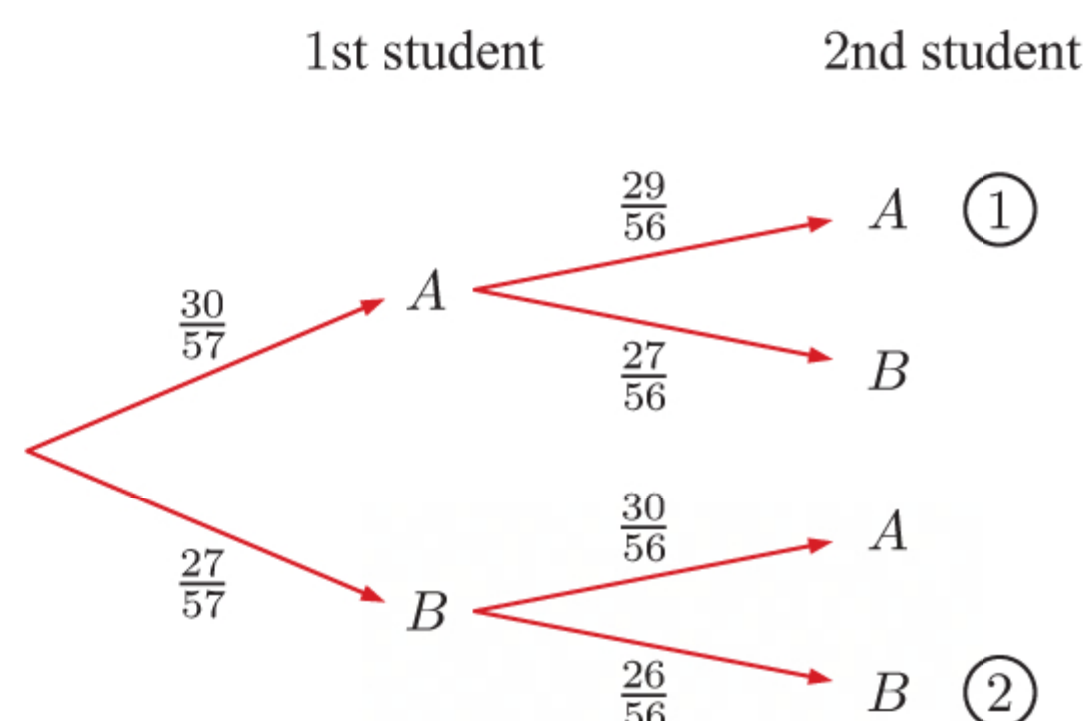
**c** The area is decreasing at  $70 \text{ m}^2$  per second when  $\frac{dA}{dt} = -70$

$$\begin{aligned} \therefore \pi(-24 + 0.02t) &= -70 \\ \therefore -24 + 0.02t &= -\frac{70}{\pi} \\ \therefore 0.02t &= 24 - \frac{70}{\pi} \\ \therefore t &= \frac{24 - \frac{70}{\pi}}{0.02} \approx 85.9 \text{ seconds.} \end{aligned}$$

**7 a** Total number of Year 7 students =  $30 + 27 = 57$

Let  $A$  represent a student selected from class A and  $B$  represent a student selected from class B.

$$\begin{aligned} P(\text{same class}) &= P(AA \text{ or } BB) \\ &= \underbrace{\frac{30}{57} \times \frac{29}{56}}_{\textcircled{1}} + \underbrace{\frac{27}{57} \times \frac{26}{56}}_{\textcircled{2}} \\ &= \frac{131}{266} \\ &\approx 0.492 \end{aligned}$$



$\therefore$  the probability that in any given week the two selected students are in the same class is  $\frac{131}{266} \approx 0.492$ .



- b** Let  $X$  be the number of weeks out of 20 that the two selected students are in the same class.

$$\therefore X \sim B\left(20, \frac{131}{266}\right) \quad \{\text{using a}\}$$

$$\begin{aligned} E(X) &= np \\ &= 20 \times \frac{131}{266} \\ &\approx 9.85 \end{aligned}$$

$\therefore$  we expect that the two selected students are in the same class about 9.85 times out of 20.

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad \text{Speed of meteor A} &= \left| \begin{pmatrix} 8 \\ 12 \\ 24 \end{pmatrix} \right| & \text{Speed of meteor B} &= \left| \begin{pmatrix} -10 \\ 20 \\ -20 \end{pmatrix} \right| \\ &= \sqrt{8^2 + 12^2 + 24^2} & &= \sqrt{(-10)^2 + 20^2 + (-20)^2} \\ &= \sqrt{784} & &= \sqrt{900} \\ &= 28 \text{ km s}^{-1} & &= 30 \text{ km s}^{-1} \end{aligned}$$

- b** The paths of the meteors intersect where  $\begin{pmatrix} 556 \\ -154 \\ -2313 \end{pmatrix} + \begin{pmatrix} 8 \\ 12 \\ 24 \end{pmatrix} t_1 = \begin{pmatrix} 3796 \\ -1594 \\ 5607 \end{pmatrix} + \begin{pmatrix} -10 \\ 20 \\ -20 \end{pmatrix} t_2$ , where  $t_1$  and  $t_2$

are the times at which meteor A and meteor B reach the point of intersection.

Equating coordinates,

$$\begin{aligned} 556 + 8t_1 &= 3796 - 10t_2, & -154 + 12t_1 &= -1594 + 20t_2, & \text{and} & -2313 + 24t_1 &= 5607 - 20t_2 \\ \therefore 8t_1 + 10t_2 &= 3240 \quad \dots (1), & 12t_1 - 20t_2 &= -1440 \quad \dots (2), & \text{and} & 24t_1 + 20t_2 &= 7920 \quad \dots (3) \end{aligned}$$

Solving (1), (2), and (3) simultaneously using technology gives  $t_1 = 180$ ,  $t_2 = 180$ .

Substituting  $t_1 = 180$  into  $\mathbf{r}_1$ , we find that the paths of the meteors intersect at  $(556 + 8(180), -154 + 12(180), -2313 + 24(180))$ , which is  $(1996, 2006, 2007)$ .

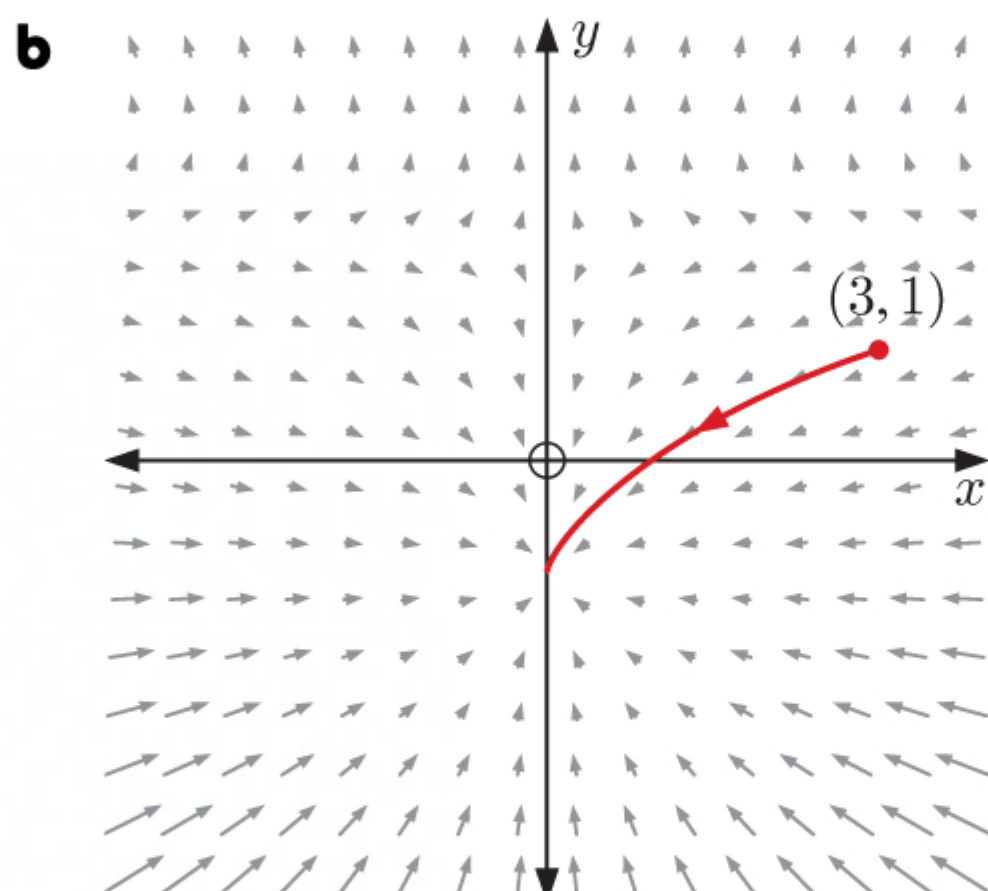
- c** The meteors will collide at the intersection point as they both pass through it after 180 seconds.

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad \text{Equilibrium points occur where } \frac{dx}{dt} &= 0 & \text{and} & \frac{dy}{dt} = 0 \\ \therefore xy - 3x &= 0 & \therefore y^2 - y - 2 &= 0 \\ \therefore x(y - 3) &= 0 & \therefore (y - 2)(y + 1) &= 0 \\ \therefore x = 0 \text{ or } y &= 3 & \therefore y = 2 \text{ or } -1 & \end{aligned}$$

If  $y = 3$ , then  $\frac{dy}{dt} \neq 0$ .

So, the equilibrium points are  $(0, 2)$  and  $(0, -1)$ .

From the phase portrait,  $(0, 2)$  is a saddle point, and  $(0, -1)$  is stable fixed point.





**10 a** Number of days =  $520 \times 7 = 3640$

$$\begin{aligned}\text{Number of days with rainfall} &= 1 \times 137 + 2 \times 169 + \dots + 7 \times 4 \\ &= 1092\end{aligned}$$

$$\begin{aligned}\therefore \text{percentage of days with rainfall} &= \frac{1092}{3640} \times 100\% \\ &= 30\%\end{aligned}$$

**b i** We want to test whether the data fits a particular probability distribution, so a  $\chi^2$  goodness of fit test is appropriate.

**ii**  $H_0$ : the data is from  $B(7, 0.3)$

$H_1$ : the data is not from  $B(7, 0.3)$

**iii**  $X \sim B(7, 0.3)$

Number of days	$f_{\text{obs}}$	$f_{\text{exp}}$
0	45	$520 \times P(X = 0) = 42.824\,236$
1	137	$520 \times P(X = 1) = 128.472\,708$
2	169	$520 \times P(X = 2) = 165.179\,196$
3	101	$520 \times P(X = 3) = 117.985\,14$
4	41	$520 \times P(X = 4) = 50.565\,06$
5	16	$520 \times P(X = 5) = 13.002\,444$
6	7	$520 \times P(X = 6) = 1.857\,492$
7	4	$520 \times P(X = 7) = 0.113\,724$

} < 5

There are expected frequencies less than 5, so we combine “categories” appropriately:

Number of days	$f_{\text{obs}}$	$f_{\text{exp}}$	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
0	45	42.824 236	$\approx 0.1105$
1	137	128.472 708	$\approx 0.5660$
2	169	165.179 196	$\approx 0.0884$
3	101	117.985 14	$\approx 2.445$
4	41	50.565 06	$\approx 1.809$
$\geq 5$	27	14.973 66	$\approx 9.659$
Total			$\approx 14.6786$

So, the test statistic  $\chi_{\text{calc}}^2 \approx 14.7$ .

Now  $\text{df} = 6 - 1 = 5$

	List 1	List 2	List 3	List 4
SUB				
1	45	42.824		
2	137	128.47		
3	169	165.17		
4	101	117.98		

	List 1	List 2	List 3	List 4
SUB				
1	45	42.824		
2	137	128.47		
3	169	165.17		
4	101	117.98		

	List 1	List 2	List 3	List 4
SUB				
1	45	42.824		
2	137	128.47		
3	169	165.17		
4	101	117.98		

Using technology,  $p\text{-value} \approx 0.0118$ .

**iv** The level of significance  $\alpha = 0.05$ .

Since  $p\text{-value} < 0.05 = \alpha$ , there is enough evidence to reject  $H_0$  in favour of  $H_1$ .

We therefore accept  $H_1$ .

So, the binomial model is not suitable for the data.



## MIXED QUESTIONS SET 8

- 1 a A deposit of  $€225\,000 \times 10\% = €22\,500$  is required.

$\therefore$  it is necessary to borrow  $€225\,000 - €22\,500 = €202\,500$

$$N = 10 \times 4 = 40, \quad I\% = 5.99, \quad PV = 202\,500, \quad FV = 0, \quad P/Y = 4, \\ C/Y = 4$$

$$\therefore PMT \approx -6765.90$$

The quarterly repayments are €6765.90.

Normal		End
Compound Interest		
n	=	40
I%	=	5.99
PV	=	202500
PMT	=	-6765.899478
FV	=	0
P/Y	=	4
n	I%	PV
PMT	FV	AMORTZ

- b Total interest = total repayment – starting principal  
 $= €6765.90 \times 10 \times 4 - €202\,500$   
 $= €68\,136$

- c The indexed value  $= €225\,000 \times (1.035)^{10}$   
 $= €317\,384.72$

- 2 a i  $P(\text{rain on both days}) = P(\text{rain} \cap \text{rain})$   
 $= 0.4 \times 0.4$   
 $= 0.16$

- ii  $P(\text{no rain on exactly one day})$   
 $= P(\text{rain} \cap \text{no rain}) + P(\text{no rain} \cap \text{rain})$   
 $= 0.4 \times 0.6 + 0.6 \times 0.4$   
 $= 0.48$

- b  $P(\text{no rain on both days}) = P(\text{no rain} \cap \text{no rain})$   
 $= 0.6 \times 0.6$   
 $= 0.36$

$$\therefore P(\text{rain on at least one day}) = 1 - 0.36$$

$$= 0.64$$

$$\text{So, } P(\text{rain on 2nd day} \mid \text{rain on at least one day}) = \frac{P(\text{rain on 2nd day} \cap \text{rain on at least one day})}{P(\text{rain on at least one day})}$$

$$= \frac{P(\text{rain on 2nd day})}{P(\text{rain on at least one day})}$$

$$= \frac{0.4}{0.64}$$

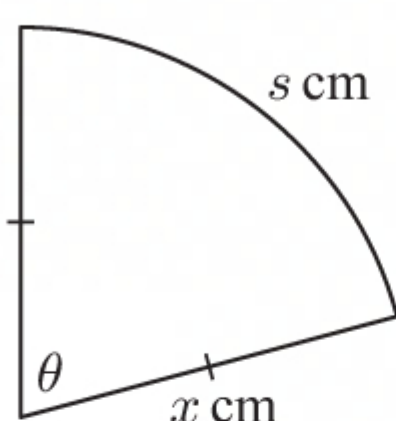
$$= 0.625$$

- 3 a Perimeter  $= 2x + s$   
 $= 2x + \theta x$

$$\therefore 2x + \theta x = 40$$

$$\therefore 2 + \theta = \frac{40}{x}$$

$$\therefore \theta = \frac{40}{x} - 2$$



- b Area  $A = \frac{\theta}{2} x^2$   
 $= \frac{(\frac{40}{x} - 2)x^2}{2}$  {using a}  
 $= 20x - x^2 \text{ cm}^2$

- c  $\frac{dA}{dx} = 20 - 2x$  which is 0 when  $x = 10$ .

The sign diagram of  $\frac{dA}{dx}$  is

$$\text{When } x = 10, \quad \theta = \frac{40}{10} - 2 = 2.$$

$\therefore A$  is a maximum when  $x = 10$  and  $\theta = 2^\circ$ .

- 4  $T(t) = A \times B^{-t} + 3$

- a i The initial internal temperature of the refrigerator was  $27^\circ\text{C}$ .

$$\text{So, } T(0) = 27$$

$$\therefore 27 = A + 3$$

$$\therefore A = 24$$

- ii After 3 hours, the internal temperature was  $6^\circ\text{C}$ .

$$\text{So, } T(3) = 6$$

$$\therefore 6 = 24 \times B^{-3} + 3 \quad \{\text{using i}\}$$

$$\therefore 24 \times B^{-3} = 3$$

$$\therefore B^{-3} = \frac{1}{8}$$

$$\therefore B^3 = 8$$

$$\therefore B = 2$$



**b**  $T(t) = 24 \times 2^{-t} + 3$  {using **a**}

$$\therefore T(5) = 24 \times 2^{-5} + 3$$

$$= 3.75$$

$\therefore$  the internal temperature is  $3.75^\circ\text{C}$  after 5 hours.

**c** As  $t \rightarrow \infty$ ,  $2^{-t} \rightarrow 0$

$$\therefore T(t) \rightarrow 24 \times 0 + 3 = 3$$

$\therefore$  the minimum temperature that the refrigerator could be expected to reach is  $3^\circ\text{C}$ .

- 5 a** Let  $\mu_1$  be the population mean battery life of *Mega* speakers, and let  $\mu_2$  be the population mean battery life of *Micro* speakers.

The hypotheses to be considered are:

$$H_0: \mu_1 = \mu_2$$

$$H_1: \mu_1 > \mu_2$$

**b**

	List 1	List 2	List 3	List 4
SUB				
1	22.4	20.8		
2	23.5	21.2		
3	24.1	22.1		
4	22.3	20.7		
				22.4

GRAPH CALC TEST INTR DIST

	List	Var
2-Sample tTest		
Data	:List	
$\mu_1$	:> $\mu_2$	
List(1)	:List1	
List(2)	:List2	
Freq(1)	:1	
Freq(2)	:1	

2-Sample tTest

$\mu_1 > \mu_2$

$t = 2.4270426$

$p = 0.01239673$

$df = 20$

$\bar{x}_1 = 22.7$

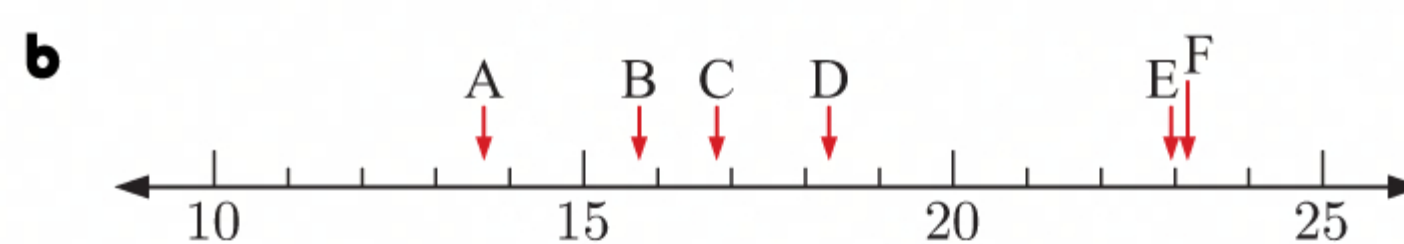
$\bar{x}_2 = 21.76$

Using technology, the test statistic  $t \approx 2.43$  and  $p$ -value  $\approx 0.0124$ .

- c** Since  $p$ -value  $< 0.1$ , we have enough evidence to reject  $H_0$  on a 10% level of significance.
- $\therefore$  we conclude that the company's claim is valid.

**6 a**

	Moon	Mass ( $x$ kg)	$\log x$
A	Carpo	$4.50 \times 10^{13}$	$\approx 13.65$
B	Leda	$5.68 \times 10^{15}$	$\approx 15.75$
C	Lysithea	$6.29 \times 10^{16}$	$\approx 16.80$
D	Amalthea	$2.08 \times 10^{18}$	$\approx 18.32$
E	Io	$8.93 \times 10^{22}$	$\approx 22.95$
F	Ganymede	$1.48 \times 10^{23}$	$\approx 23.17$



**c** For Autonoe,  $\log x = 14.0$

$$10^{\log x} = 10^{14.0}$$

$$x = 10^{14.0} = 1 \times 10^{14}$$

So, the mass of Autonoe is  $1 \times 10^{14}$  kg.

**7**

Age ( $x$ years)	28	40	21	38	30	26	18	32	25	29	20	24
Time ( $y$ min)	20	32	15	40	26	25	19	28	21	25	16	22

**a**

	List 1	List 2	List 3	List 4
SUB				
10	29	25		
11	20	16		
12	24	22		
13				

GRAPH CALC TEST INTR DIST

	List	Var
LinearReg(ax+b)		
a	=0.92969136	
b	=-1.5606535	
r	=0.89822155	
$r^2$	=0.80680195	
MSe	=10.4504043	
$y = ax + b$		

COPY

Using technology, the coefficient of determination  $r^2 \approx 0.807$ .

The dependent variable is *time*, and the independent variable is *age*.

This means that about 80.7% of the variation in *time* can be explained by the variation in *age*.

- b i** Using the screenshots in **a**, the equation of the linear regression line is  $y \approx 0.930x - 1.56$ .
- ii** The slope of the line  $m \approx 0.930$  indicates that for every additional year old a contestant is, we should expect their time taken to increase by about 0.930 minutes (or about 56 seconds).



- 8 a** The direction vector  $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$  has length  $\sqrt{(-1)^2 + 2^2 + 2^2} = 3$ .

Trisha moves with speed  $0.5 \text{ m s}^{-1}$ .

$$\therefore \text{Trisha's velocity vector is } \frac{0.5}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

- b** Trisha's position vector after  $t$  seconds is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \quad t \geq 0.$$

$$\begin{aligned} \text{When } t = 10, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + 10 \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{5}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{pmatrix} \\ &= \begin{pmatrix} 1\frac{1}{3} \\ 4\frac{1}{3} \\ 3\frac{1}{3} \end{pmatrix} \end{aligned}$$

$\therefore$  after 10 seconds, Trisha is at  $(1\frac{1}{3}, 4\frac{1}{3}, 3\frac{1}{3})$ .

- d** The escalator has direction vector  $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ .

The horizontal corresponds to the  $XY$ -plane.

The projection of  $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$  onto the  $XY$ -plane is  $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ .

If  $\theta$  is the angle the escalator makes with the horizontal,

$$\begin{aligned} \cos \theta &= \frac{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}{\left| \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \right| \left| \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right|} \\ &= \frac{1 + 4 + 0}{\sqrt{(-1)^2 + 2^2 + 2^2} \sqrt{(-1)^2 + 2^2 + 0^2}} \\ &= \frac{5}{\sqrt{9} \sqrt{5}} \\ &= \frac{\sqrt{5}}{3} \end{aligned}$$

$$\begin{aligned} \therefore \theta &= \cos^{-1} \left( \frac{\sqrt{5}}{3} \right) \\ &\approx 41.8^\circ \end{aligned}$$

$\therefore$  the escalator travels at an angle of about  $41.8^\circ$  to the horizontal.

- 9 a**  $(2, 4)$ ,  $(2, -6)$ , and  $(-1, 3)$  lie on a circle with equation  $x^2 + y^2 + ax + by + c = 0$ .

$$\therefore 2^2 + 4^2 + a(2) + b(4) + c = 0$$

$$2^2 + (-6)^2 + a(2) + b(-6) + c = 0$$

$$(-1)^2 + 3^2 + a(-1) + b(3) + c = 0$$

$$\text{which gives the system of equations } \begin{cases} 2a + 4b + c = -20 \\ 2a - 6b + c = -40 \\ -a + 3b + c = -10 \end{cases}$$

- c** After  $t$  seconds, Trisha's  $x$ -coordinate is  $x = 3 - \frac{t}{6}$ .

$$0 = 3 - \frac{t}{6}$$

$$\therefore \frac{t}{6} = 3$$

$$\therefore t = 18$$

$\therefore$  Trisha is on the escalator for 18 seconds.

$$\begin{aligned} \text{When } t = 18, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + 18 \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 6 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} \end{aligned}$$

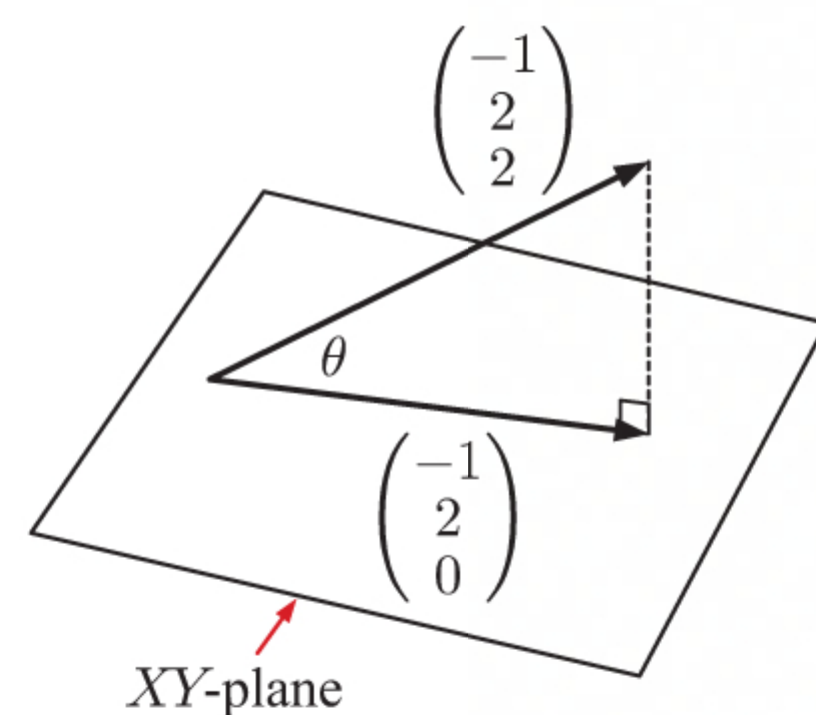
$\therefore$  the escalator ends at  $(0, 7, 6)$  and starts at  $(3, 1, 0)$ .

$\therefore$  the length of the escalator

$$= \sqrt{(0-3)^2 + (7-1)^2 + (6-0)^2}$$

$$= \sqrt{(-3)^2 + 6^2 + 6^2}$$

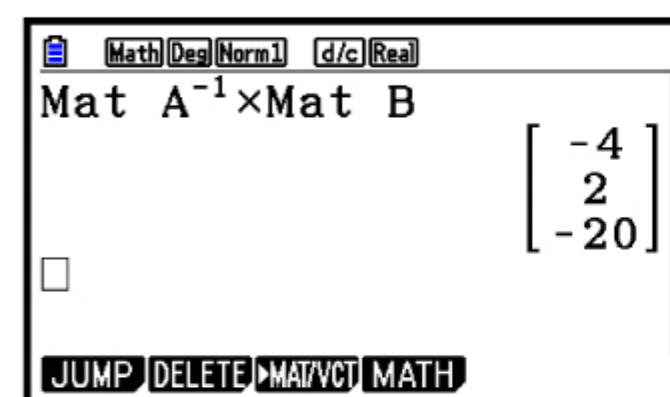
$$= 9 \text{ m}$$





**b** The system of three equations in **a** can be written in matrix form as  $\begin{pmatrix} 2 & 4 & 1 \\ 2 & -6 & 1 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -20 \\ -40 \\ -10 \end{pmatrix}$ .

$$\begin{aligned} \mathbf{c} \quad & \begin{pmatrix} 2 & 4 & 1 \\ 2 & -6 & 1 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -20 \\ -40 \\ -10 \end{pmatrix} \\ & \therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ 2 & -6 & 1 \\ -1 & 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -20 \\ -40 \\ -10 \end{pmatrix} \\ & = \begin{pmatrix} -4 \\ 2 \\ -20 \end{pmatrix} \end{aligned}$$



So  $a = -4$ ,  $b = 2$ ,  $c = -20$ .

<b>10 a</b>	$t$ (seconds)	2	4	6
	$V$ ( $\text{m s}^{-1}$ )	20	33	40
	<i>Residual for Lucinda's model</i>	$\approx 3.52$	$\approx 5.47$	$\approx 5.06$
	<i>Residual for Lewis' model</i>	$\approx -4.82$	$\approx -5.43$	$\approx -5.91$

For Lucinda's model  $SS_{\text{res}} \approx (3.52)^2 + (5.47)^2 + (5.06)^2$   
 $\approx 67.8$

For Lewis' model  $SS_{\text{res}} \approx (-4.82)^2 + (-5.43)^2 + (-5.91)^2$   
 $\approx 87.6$

**b** The sum of squared residuals is smaller for Lucinda's model than for Lewis' model.

This means that the values predicted by Lucinda's model are closer to the actual velocity, and therefore Lucinda's model is better.

## MIXED QUESTIONS SET 9

**1 a**  $f(1) = 1^2 - 2(1) = -1$

$g(1) = \frac{1}{\sqrt{1}} = 1$

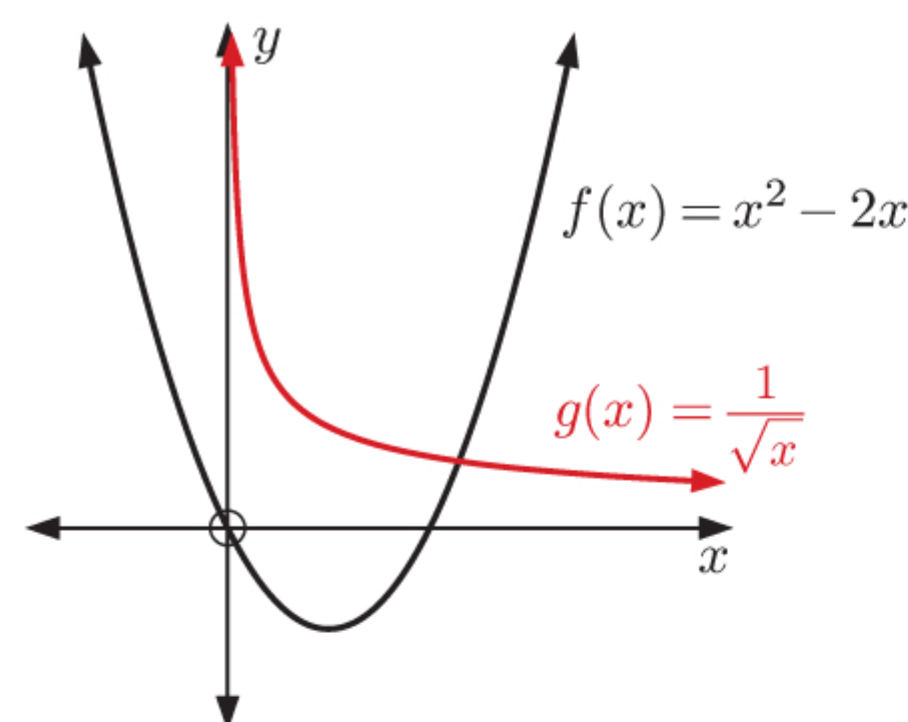
**b**  $g$  is one-to-one, so it is invertible.

$f$  is not one-to-one, so it is not invertible.

**c**  $g^{-1}(x) = 4$

$\therefore x = g(4)$

$\therefore x = \frac{1}{\sqrt{4}} = \frac{1}{2}$

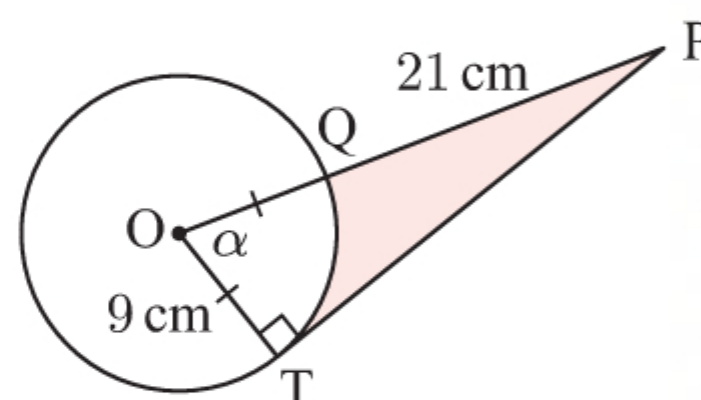


**2 a**  $\widehat{OTP} = 90^\circ$  {radius-tangent}

$\therefore \triangle OPT$  is right angled at  $T$ .

$\therefore \cos \alpha = \frac{9}{30}$

$\therefore \alpha = \cos^{-1}\left(\frac{9}{30}\right)$   
 $\approx 72.5^\circ$



**b** Area of  $\triangle OPT = \frac{1}{2} \times 9 \times 30 \times \sin \alpha$   
 $= 135 \sin \alpha \text{ cm}^2$

Area of sector OQT  $= \frac{\theta}{360} \times \pi r^2$   
 $= \frac{\alpha}{360} \times \pi \times 9^2$   
 $= \frac{9\alpha\pi}{40} \text{ cm}^2$

So, shaded area = area of  $\triangle OPT$  - area of sector OQT

$$\begin{aligned} &= 135 \sin \alpha - \frac{9\alpha\pi}{40} \\ &= 135 \sin\left(\cos^{-1}\left(\frac{9}{30}\right)\right) - \frac{9\pi}{40} \cos^{-1}\left(\frac{9}{30}\right) \quad \{\text{using a}\} \\ &\approx 77.5 \text{ cm}^2 \end{aligned}$$



- 3 a** Francesca adds \$0.50 in the first week, \$1 the next, \$1.50 the next, adding an additional \$0.50 each subsequent week.  
 $\therefore$  in the  $n$ th week, Francesca adds  $0.50n$  dollars to her money box.

Now the last week before her 11th birthday is the 51st week.

$\therefore$  in the last week before her 11th birthday, Francesca added  $\$0.50 \times 51 = \$25.50$  to her money box.

- b** Let  $P(n)$  dollars be the amount Pierre had added to his money box after  $n$  weeks, and  $F(n)$  dollars be the amount Francesca had added to her money box after  $n$  weeks.

Pierre adds \$10 each week, so after  $n$  weeks he has added  $10n$  dollars.

$$\text{So, } P(n) = 10n$$

$$\therefore P(8) = 10 \times 8 = 80$$

After 8 weeks Pierre had added \$80 to his money box.

From **a**, Francesca adds  $0.50n$  dollars in the  $n$ th week, so after  $n$  weeks she has added  $0.50 + 1 + 1.50 + \dots + 0.50n$  dollars.

Now  $0.50 + 1 + 1.50 + \dots + 0.50n$  is an arithmetic series with  $u_1 = 0.5$  and  $d = 0.5$ .

$$\begin{aligned} \therefore 0.50 + 1 + 1.50 + \dots + 0.50n &= \frac{n}{2}(2u_1 + (n-1)d) \\ &= \frac{n}{2}(2 \times 0.5 + (n-1) \times 0.5) \\ &= \frac{n}{2}(1 + 0.5n - 0.5) \\ &= \frac{n}{2}(0.5 + 0.5n) \\ &= 0.25n + 0.25n^2 \end{aligned}$$

$$\text{So, } F(n) = 0.25n + 0.25n^2$$

$$\therefore F(8) = 0.25 \times 8 + 0.25 \times 8^2 = 18$$

After 8 weeks, Francesca added \$18 to her money box.

- c** There are 52 weeks in 1 year.

$$\text{Now } P(52) = 10 \times 52 = 520$$

$$\text{and } F(52) = 0.25 \times 52 + 0.25 \times 52^2 = 689$$

$\therefore$  after 1 year, Pierre had  $\$520 + \$100 = \$620$  in his money box, and Francesca had  $\$689 + \$100 = \$789$  in her money box.

So, Francesca had more money in her money box after 1 year.

- 4 a** B is  $(6, 6, 0)$  and C is  $(0, 6, 0)$ .

$$\begin{aligned} \text{b Volume} &= \frac{1}{3} \times \text{area of base} \times \text{height} \\ &= \frac{1}{3} \times 6 \times 6 \times 7 \\ &= 84 \text{ units}^3 \end{aligned}$$

- c** The midpoint M of [BC] is  $\left(\frac{6+0}{2}, \frac{6+6}{2}, \frac{0+0}{2}\right)$  which is  $(3, 6, 0)$ .

$$\begin{aligned} \text{d MD} &= \sqrt{(3-3)^2 + (3-6)^2 + (7-0)^2} \\ &= \sqrt{0^2 + (-3)^2 + 7^2} \\ &= \sqrt{0 + 9 + 49} \\ &= \sqrt{58} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{Area of triangle BCD} &= \frac{1}{2} \times 6 \times \sqrt{58} \\ &= 3\sqrt{58} \text{ units}^2 \end{aligned}$$

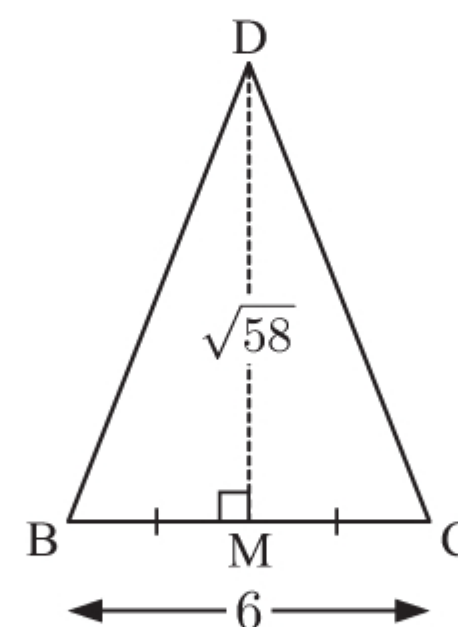
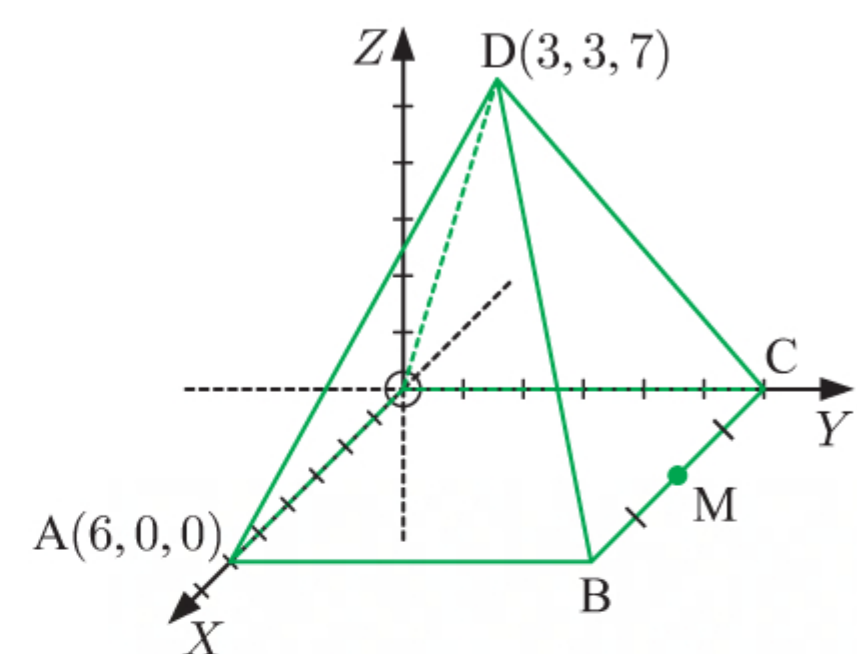
Surface area of pyramid = area of base + area of 4 triangular faces

$$= 6 \times 6 + 4 \times \text{area of } \triangle BCD$$

$$= 36 + 4 \times 3\sqrt{58}$$

$$= 36 + 12\sqrt{58} \text{ units}^2$$

$$\approx 127 \text{ units}^2$$





5

Lake	A	B	C	D	E	F	G	H
Surface area ( $x$ hectares)	25	10	35	16	19	27	14	16
Population ( $y$ )	5620	840	6125	1280	1805	3645	980	1110

**a**

Des	Norm1	d/c	Real
LinearReg(ax+b)			
a	=244.906417		
b	=-2283.7299		
r	=0.92307035		
r <sup>2</sup>	=0.85205888		
MSe	=811427.963		
y=ax+b			
[COPY][DRAW]			

So,  $r_p \approx 0.923$ .**b**

Lake	A	B	C	D	E	F	G	H
rank of $x$	6	1	8	3.5	5	7	2	3.5
rank of $y$	7	1	8	4	5	6	2	3

**c**

Des	Norm1	d/c	Real
LinearReg(ax+b)			
a	=0.97590361		
b	=0.10843373		
r	=0.97007727		
r <sup>2</sup>	=0.94104991		
MSe	=0.4126506		
y=ax+b			
[COPY][DRAW]			

So,  $r_s \approx 0.970$ .**d** There is a very strong, positive relationship between  $x$  and  $y$ .

**e**  $1180 > 1110$  and  $1180 < 1805$ , so the rank of lake D's  $y$ -value does not change.  
 $\therefore$  the value of  $r_s$  is not affected.

**6** Let  $G$  be the number of fish Gaetano catches, and  $J$  be the number of fish Jillian catches.

$\therefore G \sim \text{Po}(5)$  and  $J \sim \text{Po}(7)$

We assume that  $G$  and  $J$  are independent.

$\therefore G + J \sim \text{Po}(5 + 7) = \text{Po}(12)$

So, the number of fish they catch *together* follows a Poisson distribution with rate 12 fish per hour which is 36 fish every 3 hours.

Let  $X$  be the number of fish caught in 3 hours.

$\therefore X \sim \text{Po}(36)$

$$\begin{aligned} \text{Now } P(X \geq 40) &= 1 - P(X \leq 39) \\ &\approx 1 - 0.726 \\ &\approx 0.274 \end{aligned}$$

Math	Rad	Norm1	d/c	Real
PoissonCD(39,36)				
				0.7263036068
1-Ans				
				0.2736963932
<input type="checkbox"/>				
Ppd   Pcd   InvP				

**7 a** Let the cosine model be  $H(t) = a \cos(b(t - c)) + d$ .

Low tide =  $4.7 - 2.4 = 2.3$  m

$\therefore$  the mean height =  $\frac{2.3 + 4.7}{2} = 3.5$  m, so  $d = 3.5$ .

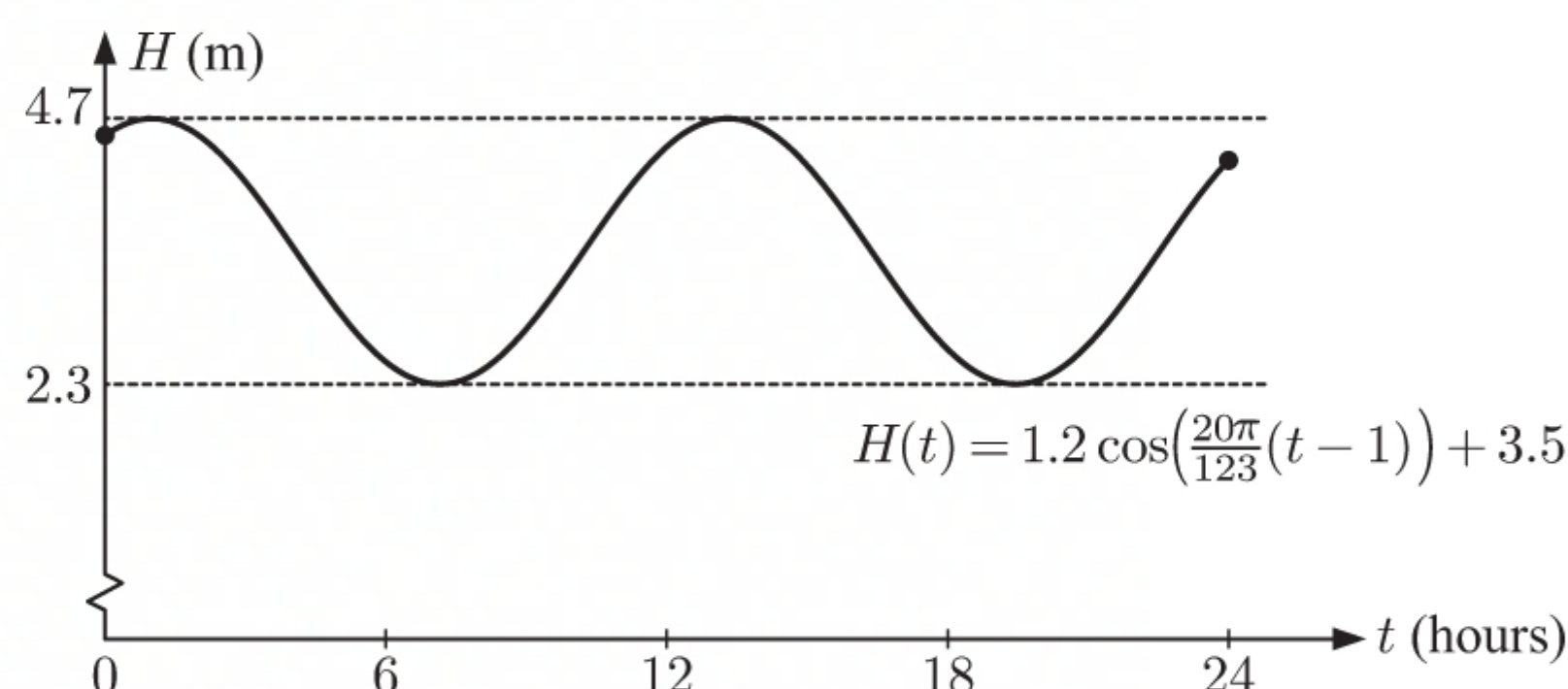
The amplitude =  $\frac{2.4}{2} = 1.2$  m, so  $a = 1.2$ .

The period = 12.3 hours, so  $b = \frac{2\pi}{12.3} = \frac{20\pi}{123}$ .

High tide occurs at 1 am, so the function is shifted 1 hour to the right, thus  $c = 1$ .

If  $t$  is the number of hours after midnight, the height  $H$  is modelled by

$$H(t) = 1.2 \cos\left(\frac{20\pi}{123}(t - 1)\right) + 3.5 \text{ m.}$$

**b**



- 8 a** Let  $p$  be the population proportion of buses which run on time.

The hypotheses to be tested are:

$$H_0: p = 0.9 \quad \{\text{90\% of buses run on time}\}$$

$$H_1: p < 0.9 \quad \{\text{less than 90\% of buses run on time}\}$$

- b** The null distribution is  $X \sim B(50, 0.9)$ .

$$\begin{aligned} P(\text{Type I error}) &= P(\text{Reject } H_0 \mid H_0 \text{ true}) \\ &= P(X \leq 40 \mid X \sim B(50, 0.9)) \\ &\approx 0.0245 \quad \{\text{technology}\} \end{aligned}$$

- c** The true distribution is  $X \sim B(50, 0.8)$ .

$$\begin{aligned} P(\text{Type II error}) &= P(\text{Retain } H_0 \mid H_0 \text{ false}) \\ &= P(X \geq 41 \mid X \sim B(50, 0.8)) \\ &\approx 0.444 \quad \{\text{technology}\} \end{aligned}$$

- 9** The system has matrix form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  where  $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 1 & -4 \end{pmatrix}$ .

$$\begin{aligned} \mathbf{a} \quad \text{If } \det(\lambda \mathbf{I} - \mathbf{A}) = 0 \text{ then } \begin{vmatrix} \lambda + 1 & 2 \\ -1 & \lambda + 4 \end{vmatrix} &= 0 \\ \therefore (\lambda + 1)(\lambda + 4) + 2 &= 0 \\ \therefore \lambda^2 + 5\lambda + 4 + 2 &= 0 \\ \therefore \lambda^2 + 5\lambda + 6 &= 0 \\ \therefore (\lambda + 2)(\lambda + 3) &= 0 \\ \therefore \lambda = -2 \text{ or } -3 \end{aligned}$$

The eigenvalues are  $-2$  and  $-3$ .

For  $\lambda_1 = -2$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{aligned} \therefore \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \therefore -a + 2b &= 0 \end{aligned}$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = 2t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$  choosing  $t = 1$ ,  $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue  $\lambda_1 = -2$ .

For  $\lambda_2 = -3$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{aligned} \therefore \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \therefore -a + b &= 0 \end{aligned}$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$  choosing  $t = 1$ ,  $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue  $\lambda_2 = -3$ .

- b** The eigenvalues  $\lambda_1 = -2$  and  $\lambda_2 = -3$  satisfy  $0 > \lambda_1 > \lambda_2$ .

$\therefore$  the equilibrium point at  $\mathbf{O}$  is a stable fixed point.

- c i** When  $t = 0$ ,  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$$\therefore \dot{\mathbf{x}} = \begin{pmatrix} -1 & -2 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$



**ii** Using the eigenvalues and eigenvectors in **a**, a general solution to the system is  $\mathbf{x} = Ae^{-2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + Be^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

When  $t = 0$ , we know  $\mathbf{x} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$$\therefore A \begin{pmatrix} 2 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

The particular solution is  $\mathbf{x} = -3e^{-2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

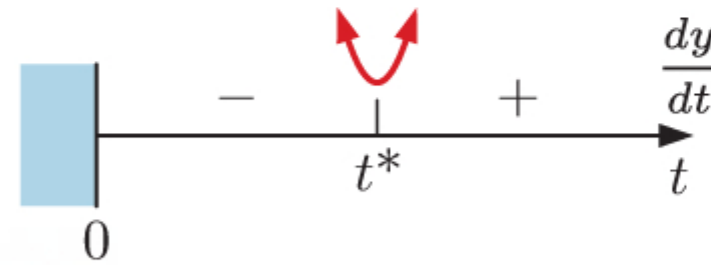
**d i** The  $y$ -value of the particular solution is  $y = -3e^{-2t} + 3e^{-3t}$

$$\begin{aligned} \therefore \frac{dy}{dt} &= 6e^{-2t} - 9e^{-3t} \\ &= 3e^{-3t}(2e^t - 3) \end{aligned}$$

Let  $t^*$  satisfy  $2e^{t^*} - 3 = 0$

$$\therefore e^{t^*} = \frac{3}{2} \quad \dots (*)$$

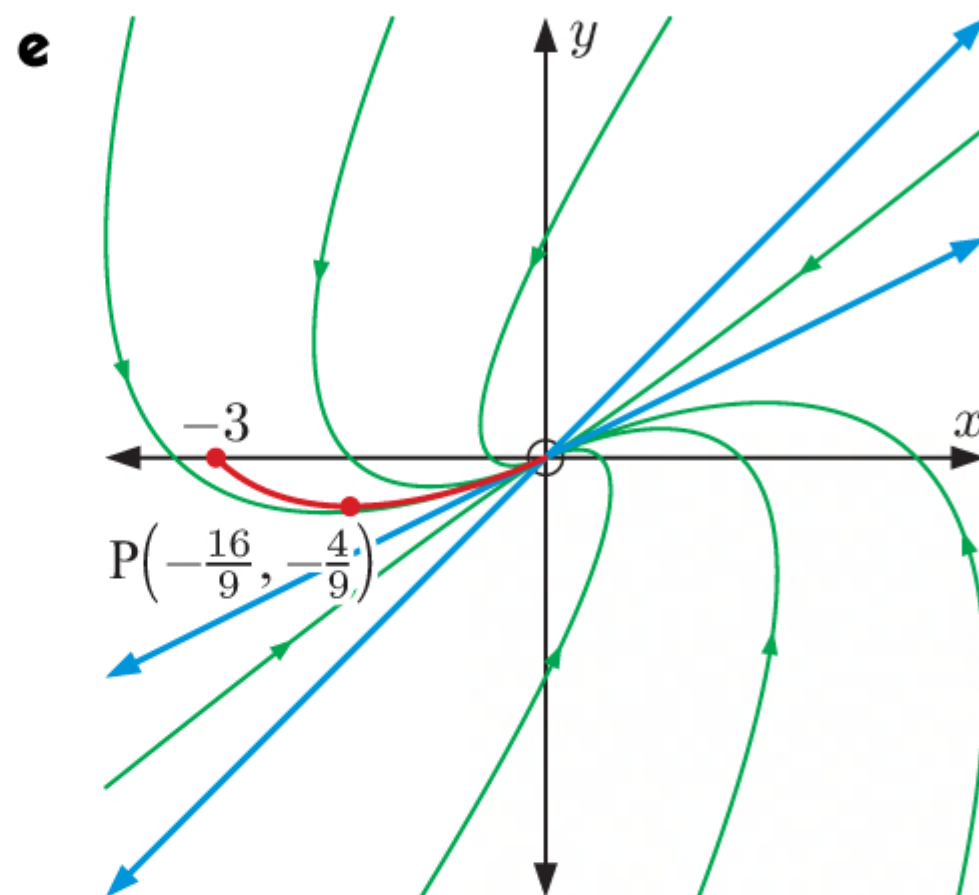
So,  $\frac{dy}{dt}$  has sign diagram:



$\therefore$  there is a minimum at  $t = t^*$ , and  $e^{t^*} = \frac{3}{2}$ .

$$\begin{aligned} \text{ii When } t = t^*, \quad \mathbf{x} &= -3e^{-2t^*} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3e^{-3t^*} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= -3(e^{t^*})^{-2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3(e^{t^*})^{-3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= -3\left(\frac{3}{2}\right)^{-2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\left(\frac{3}{2}\right)^{-3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \{\text{using } (*)\} \\ &= -\frac{4}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{8}{9} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{16}{9} \\ -\frac{4}{9} \end{pmatrix} \end{aligned}$$

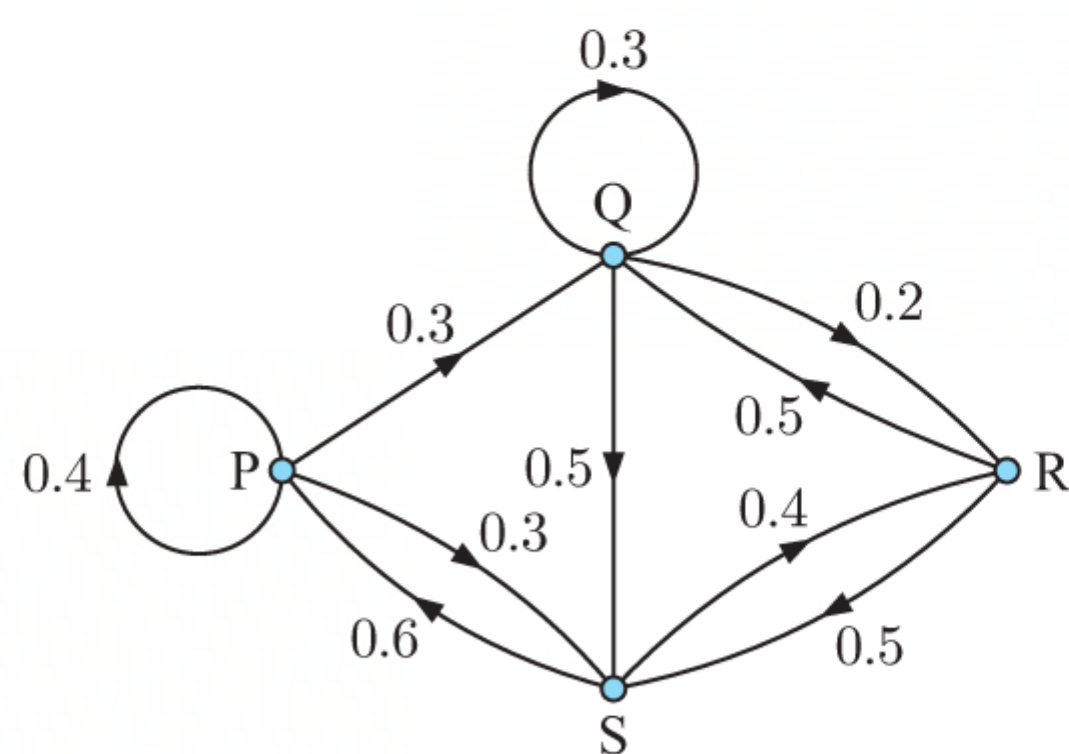
$\therefore$  P has coordinates  $\left(-\frac{16}{9}, -\frac{4}{9}\right)$ .



**f**  $0 > \lambda_1 > \lambda_2$

$\therefore$  the trajectory approaches O along  $k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ,  $k \in \mathbb{R}$ , as  $t \rightarrow \infty$ .



**10 a**

**b** The in degree of S is 3. If Clyde is at S tonight, then he could have been at one of 3 different houses (P, Q, or R) last night.

**c** 
$$\mathbf{T} = \begin{matrix} & \begin{matrix} \text{House tomorrow} \\ \text{P} & \text{Q} & \text{R} & \text{S} \end{matrix} \\ \begin{matrix} \text{House tonight} \\ \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \end{matrix} & \begin{pmatrix} 0.4 & 0.3 & 0 & 0.3 \\ 0 & 0.3 & 0.2 & 0.5 \\ 0 & 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 & 0 \end{pmatrix} \end{matrix}$$

**d** Using technology,  $\mathbf{T}^2 = \begin{pmatrix} 0.34 & 0.21 & 0.18 & 0.27 \\ 0.3 & 0.19 & 0.26 & 0.25 \\ 0.3 & 0.15 & 0.3 & 0.25 \\ 0.24 & 0.38 & 0 & 0.38 \end{pmatrix}$  and  $\mathbf{T}^3 = \begin{pmatrix} 0.298 & 0.255 & 0.15 & 0.297 \\ 0.27 & 0.277 & 0.138 & 0.315 \\ 0.27 & 0.285 & 0.13 & 0.315 \\ 0.324 & 0.186 & 0.228 & 0.262 \end{pmatrix}$ .

The values in row 3 of  $\mathbf{T}^3$  are the probabilities that Clyde is at each house in 3 nights time if he is at R tonight.

So, he is at P with probability 0.27, Q with probability 0.285, R with probability 0.13, and S with probability 0.315.

**e**  $\mathbf{s}_0 = (1 \ 0 \ 0 \ 0)$

**i** Thursday is 3 days after Monday.

$$\mathbf{s}_3 = \mathbf{s}_0 \mathbf{T}^3 = (0.298 \ 0.255 \ 0.15 \ 0.297)$$

So, the probability that Clyde will be at R on Thursday night is 0.15.

The possible routes are:  $P \rightarrow P \rightarrow Q \rightarrow R$ ,  $P \rightarrow P \rightarrow S \rightarrow R$ ,  $P \rightarrow Q \rightarrow Q \rightarrow R$ , and  $P \rightarrow Q \rightarrow S \rightarrow R$ .

**ii** Tuesday is 1 day after Monday.

$$\mathbf{s}_1 = \mathbf{s}_0 \mathbf{T} = (0.4 \ 0.3 \ 0 \ 0.3)$$

$$\therefore P(\text{at P on Tuesday}) = 0.4$$

Wednesday is 2 days after Monday.

$$\mathbf{s}_2 = \mathbf{s}_0 \mathbf{T}^2 = (0.34 \ 0.21 \ 0.18 \ 0.27)$$

$$\therefore P(\text{at P on Wednesday}) = 0.34$$

$$\begin{aligned} \text{Now } P(\text{at P on Tuesday} \mid \text{at P on Wednesday}) &= \frac{P(\text{at P on Tuesday and Wednesday})}{P(\text{at P on Wednesday})} \\ &= \frac{0.4 \times 0.4}{0.34} \\ &= \frac{0.16}{0.34} \\ &= \frac{8}{17} \approx 0.471 \end{aligned}$$

So, the probability that Clyde was at P on Tuesday night given he is at P on Wednesday night is about 0.471.

## MIXED QUESTIONS SET 10

**1**  $f'(x) = (x^2 + 2)^2 = x^4 + 4x^2 + 4$ ,  $f(1) = \frac{8}{15}$

$$\begin{aligned} \therefore f(x) &= \int (x^4 + 4x^2 + 4) dx \\ &= \frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x + c \end{aligned}$$

$$\begin{aligned} \text{Now } f(1) &= \frac{8}{15}, \quad \therefore \frac{1}{5}(1)^5 + \frac{4}{3}(1)^3 + 4(1) + c = \frac{8}{15} \\ \therefore \frac{1}{5} + \frac{4}{3} + 4 + c &= \frac{8}{15} \\ \therefore \frac{83}{15} + c &= \frac{8}{15} \\ \therefore c &= -5 \end{aligned}$$

$$\therefore f(x) = \frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x - 5$$

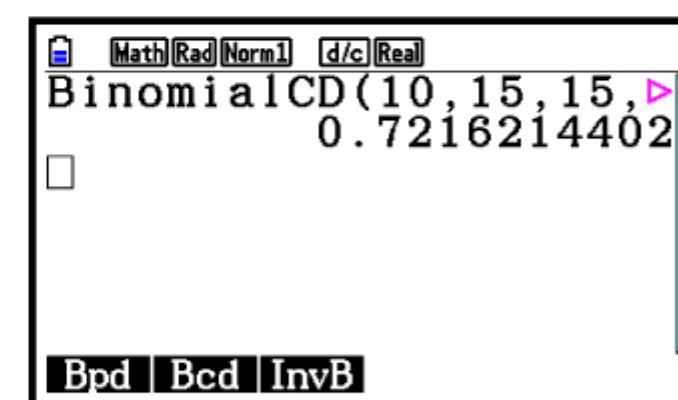


- 2 a** Let  $X$  be the number of committee members who attend a randomly selected meeting.

$$\therefore X \sim B(15, 0.7)$$

Now  $P(X \geq 10) \approx 0.722$  {using technology}

$\therefore$  approximately 72.2% of meetings will go ahead.



- b** Suppose there are  $n$  committee members, and let  $Y$  be the number of committee members who attend a randomly selected meeting.

$$\therefore Y \sim B(n, 0.7)$$

We require  $P(Y \geq 10) \geq 0.9$

If  $n = 16$ ,  $P(Y \geq 10) \approx 0.825$  ✗

If  $n = 17$ ,  $P(Y \geq 10) \approx 0.895$  ✗

If  $n = 18$ ,  $P(Y \geq 10) \approx 0.940$  ✓

So, 18 committee members are required to ensure that at least 90% of the meetings go ahead.

- 3 a** Sites  $C(6, -4)$  and  $E(-4, 2)$  are currently in the same cell, so the missing edge must be the perpendicular bisector of  $[CE]$ .

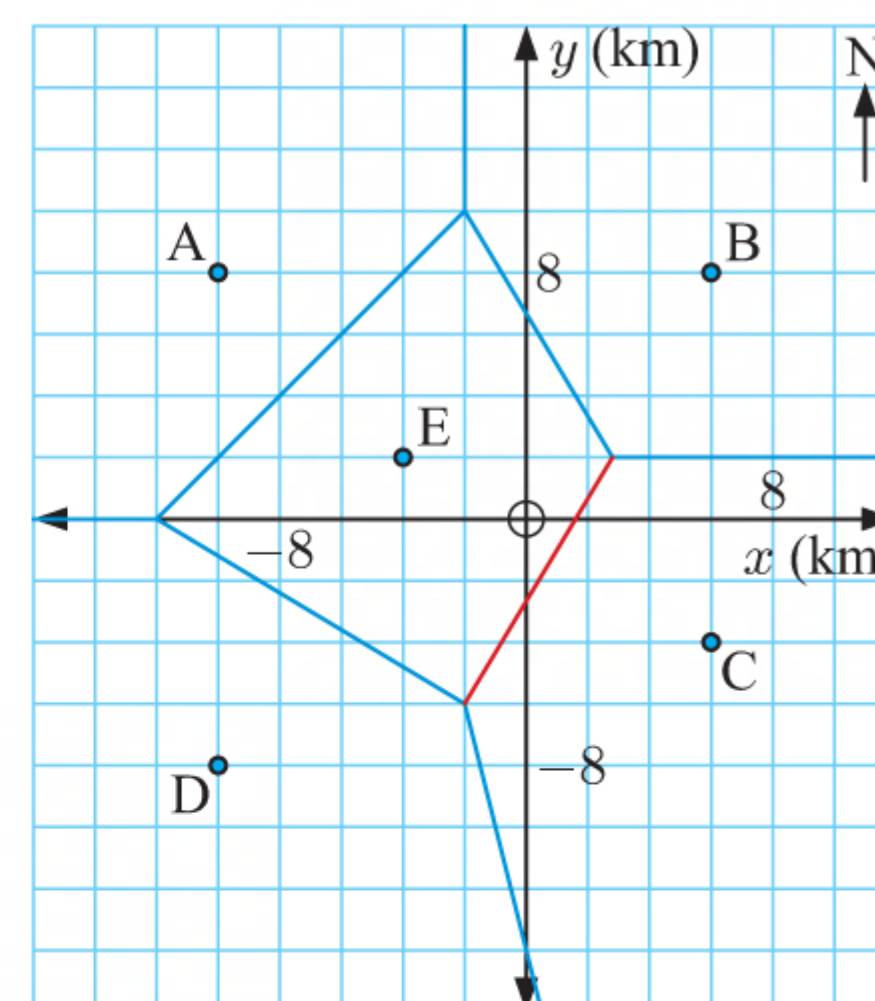
The midpoint of  $[CE]$  is  $\left(\frac{6-4}{2}, \frac{-4+2}{2}\right)$  or  $(1, -1)$ .

The gradient of  $[CE]$  is  $\frac{2-(-4)}{-4-6} = \frac{6}{-10} = -\frac{3}{5}$ .

So, the perpendicular bisector of  $[CE]$  has gradient  $\frac{5}{3}$ .

$\therefore$  its equation is  $5x - 3y = 5(1) - 3(-1)$

$$\text{or } 5x - 3y - 8 = 0$$



- b** Cell D represents all the points that are closer to petrol station D than any other petrol station.

- c i** Riley's location is the  $y$ -intercept of the edge between cell C and E.

When  $x = 0$ ,  $5(0) - 3y - 8 = 0$  {using a}

$$\therefore -3y = 8$$

$$\therefore y = -\frac{8}{3}$$

$\therefore$  Riley's location is  $R(0, -\frac{8}{3})$ .

- ii** Riley is closest to petrol stations C and E.

Distance Riley has to drive

$$= RC$$

$$= \sqrt{(6-0)^2 + (-4-(-\frac{8}{3}))^2}$$

$$= \sqrt{6^2 + (-\frac{4}{3})^2}$$

$$= \sqrt{36 + \frac{16}{9}}$$

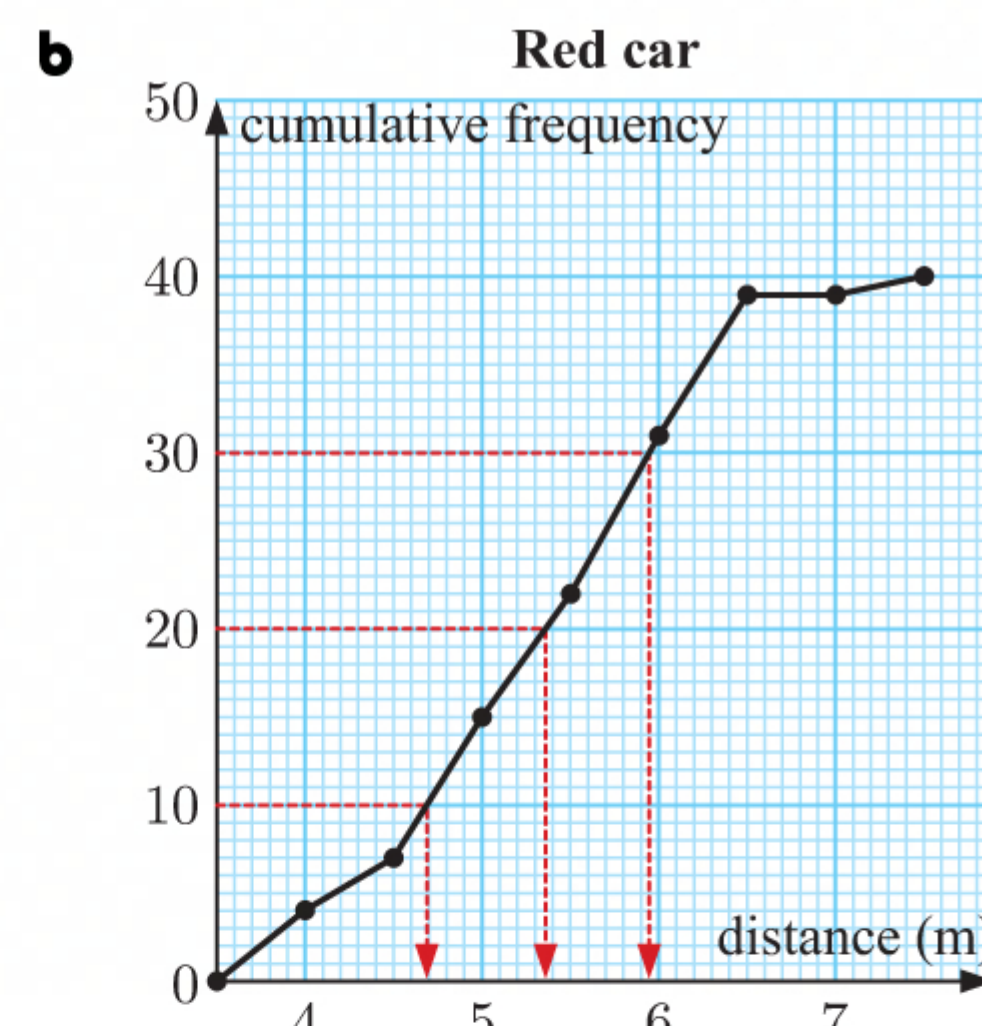
$$= \sqrt{\frac{340}{9}}$$

$$\approx 6.15 \text{ km} > 6 \text{ km}$$

$\therefore$  Riley will not be able to drive to a petrol station before his car runs out of petrol.

**4 a**

Distance (m)	Frequency	Cumulative frequency
$3.5 \leq d < 4$	4	4
$4 \leq d < 4.5$	3	7
$4.5 \leq d < 5$	8	15
$5 \leq d < 5.5$	7	22
$5.5 \leq d < 6$	9	31
$6 \leq d < 6.5$	8	39
$6.5 \leq d < 7$	0	39
$7 \leq d < 7.5$	1	40

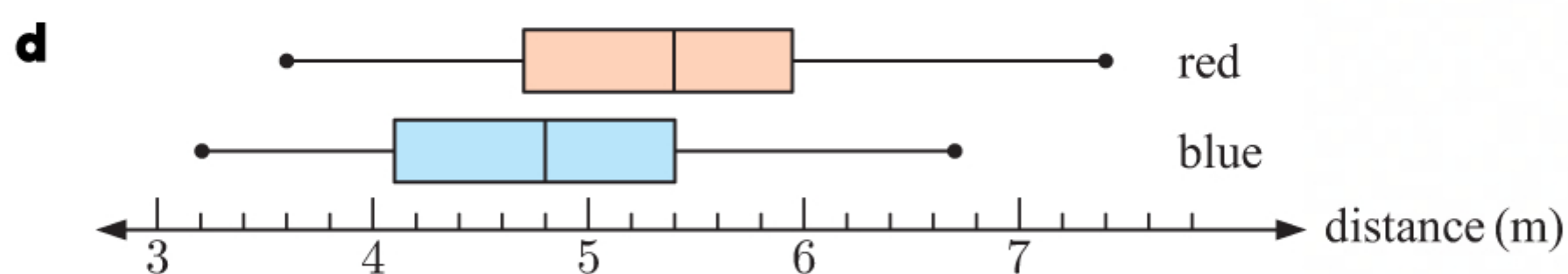


- c i** Median  $\approx 5.4$

- ii**  $Q_1 \approx 4.7$

- iii**  $Q_3 \approx 5.9$





- e** All values of the five-number summary (min,  $Q_1$ , median,  $Q_3$ , and max) for the red car are higher than those for the blue car. This evidence is strongly against the view that the cars were made by the same machine.

- 5** The sine function has the form  $y = a \sin bx + d$ , where  $a$ ,  $b$ , and  $d$  are constants.

When  $x = 0$ ,  $y = 1$

$$\therefore a \sin 0 + d = 1$$

$$\therefore 0 + d = 1$$

$$\therefore d = 1$$

The period of the graph is  $\pi$ .  $\therefore \frac{2\pi}{b} = \pi$   
 $\therefore b = 2$

When  $x = \frac{\pi}{4}$ ,  $y = 3$

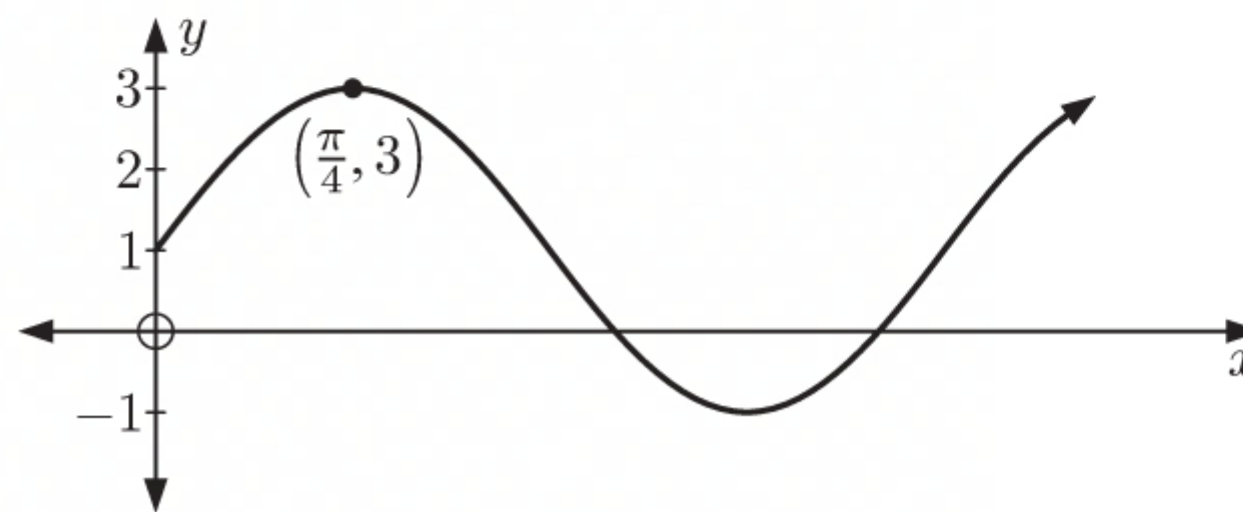
$$\therefore a \sin\left(\frac{2\pi}{4}\right) + 1 = 3$$

$$\therefore a \sin\left(\frac{\pi}{2}\right) + 1 = 3$$

$$\therefore a(1) + 1 = 3$$

$$\therefore a = 2$$

$\therefore$  the sine function has equation  $y = 2 \sin 2x + 1$ .



- 6 a** 8:30 am corresponds to time  $t = 0$  hours.

$$x = 3 - 2t, y = 3t + 1 \quad \therefore x(0) = 3, y(0) = 1$$

So, at 8:30 am the ship is at  $(3, 1)$ .

- b i** The ship's velocity vector is  $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ .

- ii** The ship's speed is  $\sqrt{4 + 9} = \sqrt{13} \text{ km h}^{-1}$ .

- c** At 10:30 am,  $t = 2$

So, the ship is at  $(3 - 2(2), 3(2) + 1)$  or  $(-1, 7)$

$$\begin{aligned} \text{and the distance to } (0, 10) &= \sqrt{(0 - (-1))^2 + (10 - 7)^2} \\ &= \sqrt{1 + 9} \\ &= \sqrt{10} \text{ km} \end{aligned}$$

- d** When the ship is directly west of the lighthouse,  $3t + 1 = 10$   
 $\therefore 3t = 9$   
 $\therefore t = 3$

$\therefore$  the time is 11:30 am.

- e**
- 
- $L(0, 10)$
- $S(3 - 2t, 3t + 1)$
- $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$$\begin{aligned} \vec{LS} &= \begin{pmatrix} 3 - 2t - 0 \\ 3t + 1 - 10 \end{pmatrix} \\ &= \begin{pmatrix} 3 - 2t \\ 3t - 9 \end{pmatrix} \end{aligned}$$

The ship is closest to the lighthouse when  $\vec{LS} \perp \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ .

$$\therefore \vec{LS} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} = 0$$

$$\therefore -2(3 - 2t) + 3(3t - 9) = 0$$

$$\therefore -6 + 4t + 9t - 27 = 0$$

$$\therefore 13t = 33$$

$$\therefore t = \frac{33}{13} \approx 2.53846 \text{ h}$$

$$\therefore t \approx 2 \text{ h } 32 \text{ min}$$

So, the ship is closest to the lighthouse at about 11:02 am.



$$\begin{aligned}\text{At time } t = \frac{33}{13}, \quad \vec{LS} &= \begin{pmatrix} 3 - 2\left(\frac{33}{13}\right) \\ 3\left(\frac{33}{13}\right) - 9 \end{pmatrix} = \begin{pmatrix} -\frac{27}{13} \\ -\frac{18}{13} \end{pmatrix} \\ \therefore |\vec{LS}| &= \sqrt{\left(-\frac{27}{13}\right)^2 + \left(-\frac{18}{13}\right)^2} \\ &= \sqrt{\frac{729}{169} + \frac{324}{169}} \\ &= \sqrt{\frac{81}{13}} \\ &= \frac{9}{\sqrt{13}} \approx 2.50 \text{ km}\end{aligned}$$

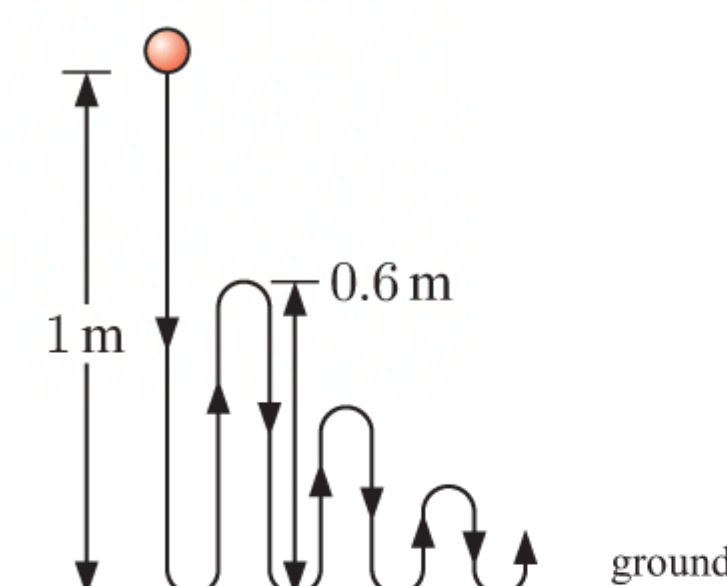
So, the distance between the ship and the lighthouse is about 2.50 km.

- 7 a**  $S = 1 + 0.6 + (0.6)^2 + (0.6)^3 + \dots$  is an infinite geometric series with  $u_1 = 1$  and  $r = 0.6$ .

$$\begin{aligned}\therefore S &= \frac{u_1}{1-r} \\ &= \frac{1}{1-0.6} \\ &= \frac{1}{0.4} \\ &= 2.5\end{aligned}$$

- b** Each time the ball bounces upward, it must travel the same distance on its way downward.

$$\begin{aligned}\therefore \text{total distance} &= 1 + 2(0.6) + 2(0.6)^2 + 2(0.6)^3 + \dots \\ &= 1 + 0.6 + (0.6)^2 + (0.6)^3 + \dots \\ &\quad + 0.6 + (0.6)^2 + (0.6)^3 + \dots \\ &= S + (S - 1) \\ &= 2S - 1 \\ &= 2(2.5) - 1 \quad \{\text{using a}\} \\ &= 5 - 1 \\ &= 4 \text{ m}\end{aligned}$$



- 8 a** When  $I = 3 \times 10^{-2}$ ,  $L = 10 \log\left(\frac{3 \times 10^{-2}}{10^{-12}}\right) \approx 105 \text{ dB}$

- b** When  $L = 85$ ,  $85 = 10 \log\left(\frac{I}{10^{-12}}\right)$   
 $\therefore$  using technology,  $I \approx 3.16 \times 10^{-4}$

<b>9</b>	France (%)	9.2	9.3	9.2	9.2	9.1	9.0	9.0	9.0	9.0	8.9	8.9	8.9
	Denmark (%)	5.1	5.2	5.0	5.3	5.2	5.1	5.1	4.9	4.9	4.9	5.3	5.1
	United Kingdom (%)	4.2	4.2	4.2	4.0	3.9	4.0	4.0	4.0	4.0	3.9	3.9	3.8

- a Step 1:** Let  $\rho$  be the population product-moment correlation coefficient between the unemployment rates of France and Denmark.

We use the hypotheses:

$H_0: \rho = 0$  {there is no correlation between the unemployment rates of France and Denmark}

$H_1: \rho \neq 0$  {the unemployment rates of France and Denmark are correlated}

**Step 2:** The significance level is  $\alpha = 0.05$ .

**Step 3:**

	List 1	List 2	List 3	List 4
SUB				
1	9.2	5.1		
2	9.3	5.2		
3	9.2	5		
4	9.2	5.3		

	Rad(Norm1)	d/c(Real)
LinearReg tTest		
$\beta \neq 0$		
XList	List1	
YList	List2	
Freq	1	
Save Res	None	
Execute		

	Rad(Norm1)	d/c(Real)
LinearReg tTest		
$\beta \neq 0$ & $\rho \neq 0$		
t	=0.99701708	
p	=0.34226948	
df	=10	
a	=2.24063745	
b	=0.31474103	

The observed value of the test statistic  $\approx 0.997$ .



Step 4:  $p\text{-value} \approx 0.342$

Step 5: Since  $p\text{-value} > 0.05 = \alpha$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% significance level. We therefore accept  $H_0$ .

Step 6: Since we have accepted  $H_0$ , we conclude that the unemployment rates of France and Denmark are not correlated.

- b** Step 1: Let  $\rho$  be the population product-moment correlation coefficient between the unemployment rates of France and the United Kingdom.

We use the hypotheses:

$H_0: \rho = 0$  {there is no correlation between the unemployment rates of France and the United Kingdom}

$H_1: \rho \neq 0$  {the unemployment rates of France and the United Kingdom are correlated}

Step 2: The significance level is  $\alpha = 0.05$ .

Step 3:

	List 1	List 2	List 3	List 4
SUB				
1	9.2	4.2		
2	9.3	4.2		
3	9.2	4.2		
4	9.2	4		

	List 1	List 2	List 3	List 4
SUB				
1	9.2	4.2		
2	9.3	4.2		
3	9.2	4.2		
4	9.2	4		

	List 1	List 2	List 3	List 4
SUB				
1	9.2	4.2		
2	9.3	4.2		
3	9.2	4.2		
4	9.2	4		

The observed value of the test statistic  $\approx 4.62$ .

Step 4:  $p\text{-value} \approx 0.000948$

Step 5: Since  $p\text{-value} < 0.05 = \alpha$ , we have enough evidence to reject  $H_0$  in favour of  $H_1$ . We therefore accept  $H_1$ .

Step 6: Since we have accepted  $H_1$ , we conclude that the unemployment rates of France and the United Kingdom are correlated.

- 10** Let  $B$  be the event that a blue ball is drawn, and  $R$  be the event that a red ball is drawn.

$$\text{Now } P(\text{both red}) = \frac{1}{3}$$

$$\therefore P(R \cap R) = \frac{1}{3}$$

$$\therefore \frac{n}{n+4} \times \frac{n-1}{n+3} = \frac{1}{3}$$

$$\therefore \frac{n(n-1)}{(n+4)(n+3)} = \frac{1}{3}$$

$$\therefore 3n(n-1) = (n+4)(n+3)$$

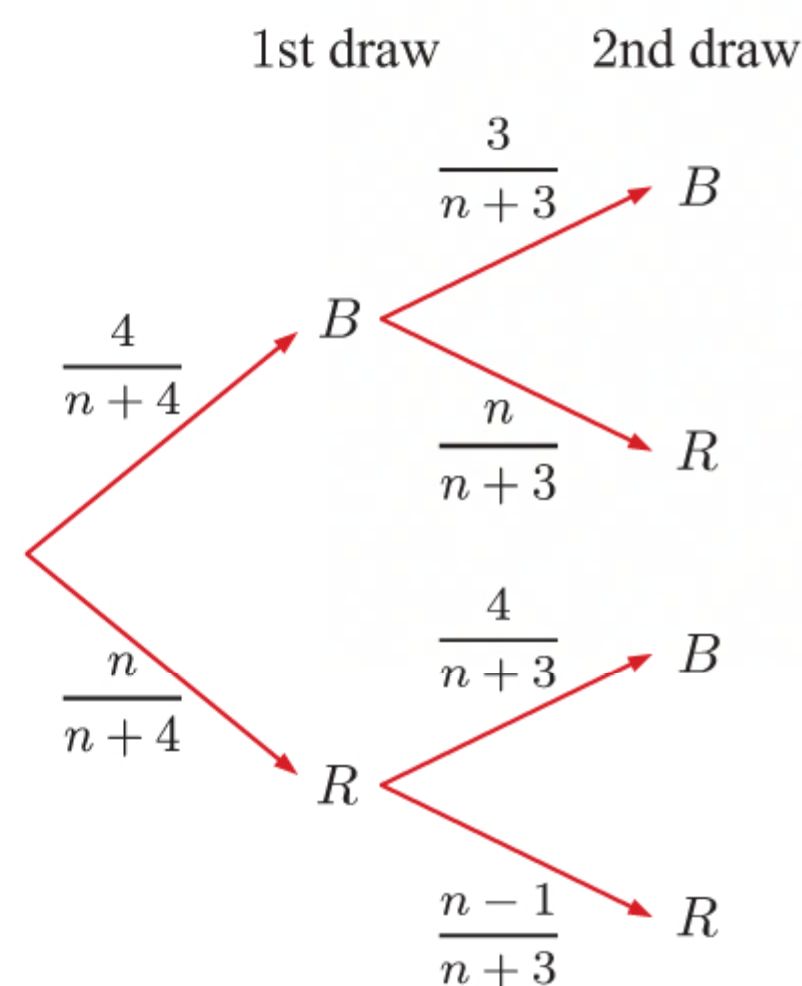
$$\therefore 3n^2 - 3n = n^2 + 7n + 12$$

$$\therefore 2n^2 - 10n - 12 = 0$$

$$\therefore n^2 - 5n - 6 = 0$$

$$\therefore (n-6)(n+1) = 0$$

$$\therefore n = 6 \quad \{n \geq 0\}$$



## MIXED QUESTIONS SET 11

**1**  $f(x) = 3 - 4^{-x}$

**a**  $f(2) = 3 - 4^{-2} = 3 - \frac{1}{16}$   
 $= 2\frac{15}{16}$

$$\therefore p = 2\frac{15}{16}$$

$$f(-2) = 3 - 4^2 = -13$$

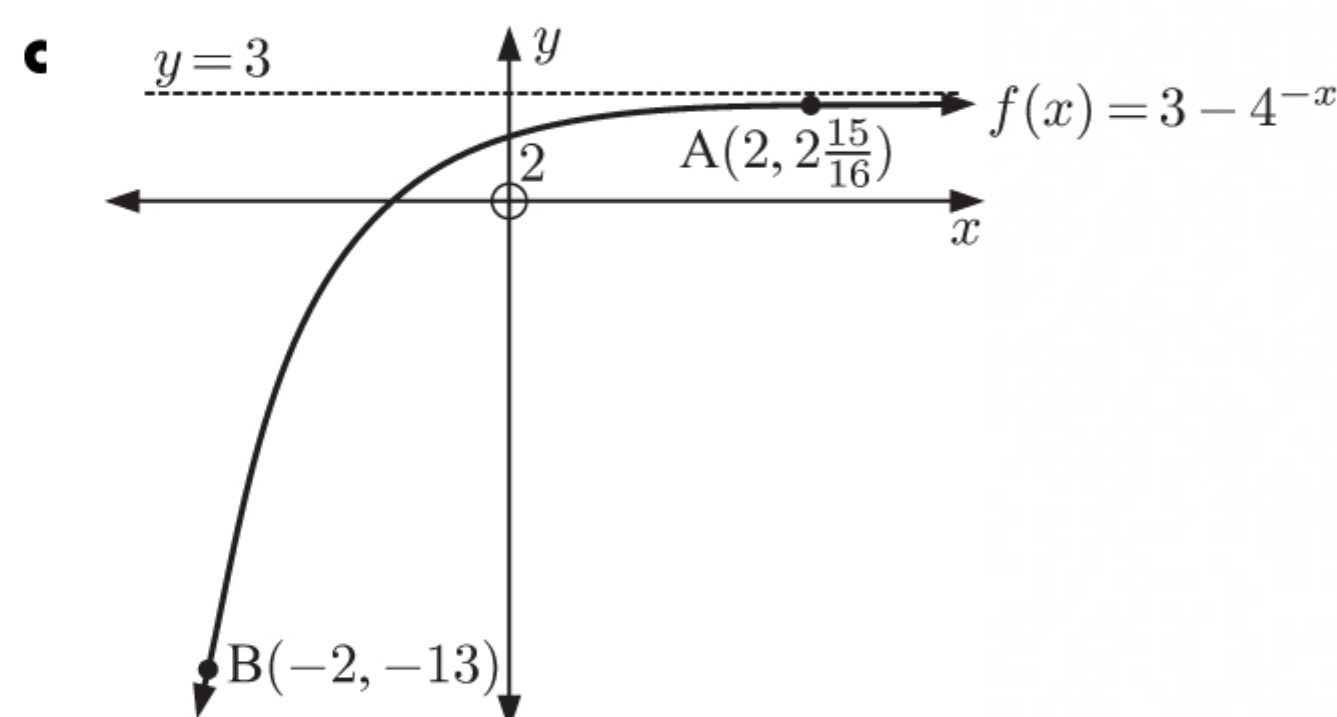
$$\therefore q = -13$$

**b i**  $f(0) = 3 - 4^0 = 2 \quad \therefore$  the  $y$ -intercept is 2.

**ii** As  $x \rightarrow \infty$ ,  $4^{-x} \rightarrow 0$  and so  $y \rightarrow 3$

$$\therefore y = 3 \text{ is the horizontal asymptote.}$$

**d** The range is  $\{y \mid y < 3\}$ .

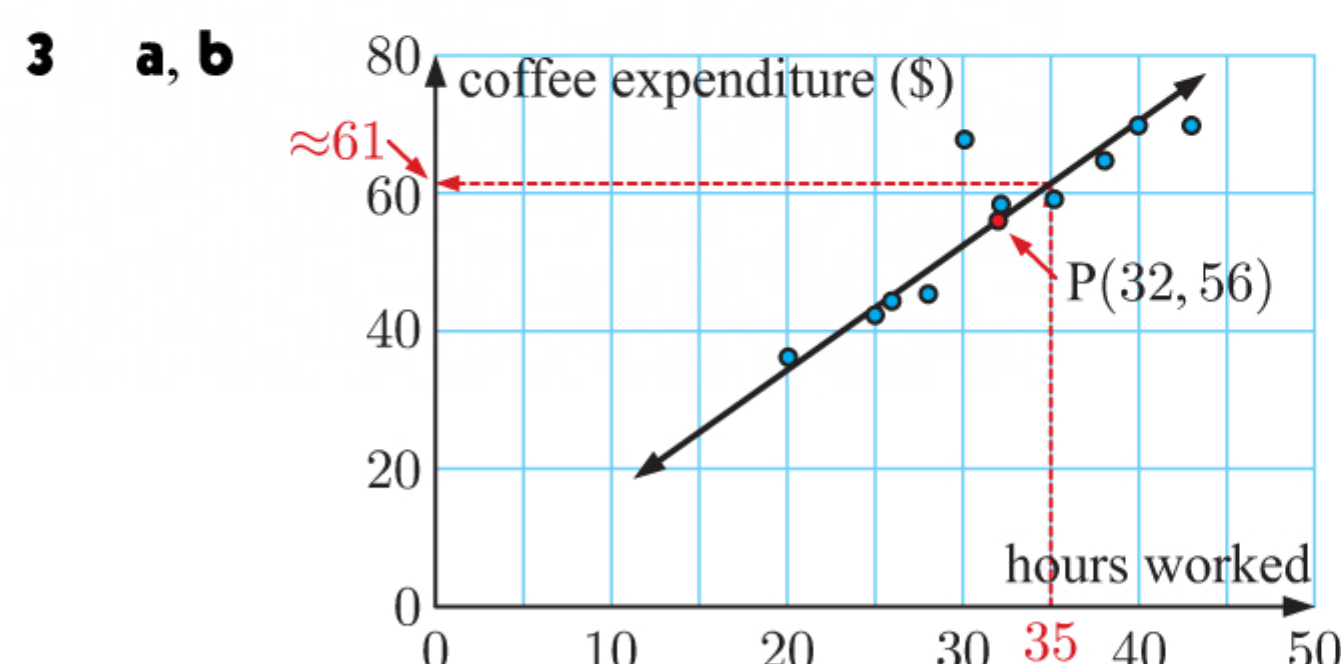
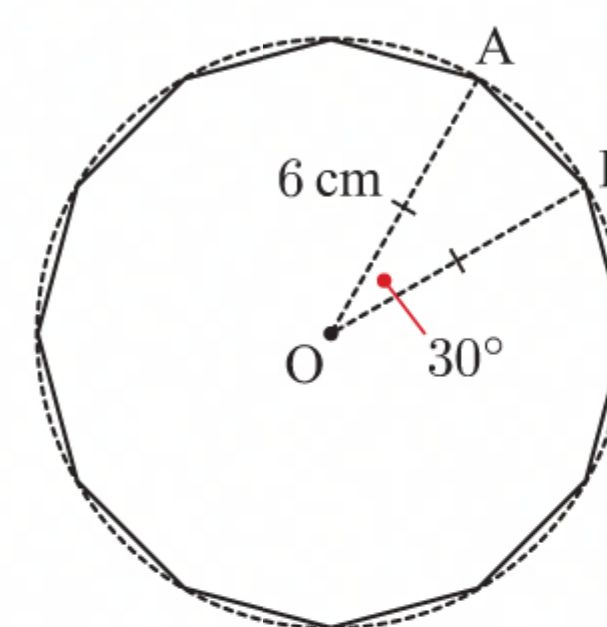




- 2 a** There are twelve equal angles at the centre of the dodecagon.

$$\therefore \widehat{AOB} = \frac{360^\circ}{12} = 30^\circ$$

$$\begin{aligned} \text{b} \quad \text{Area of } \triangle AOB &= \frac{1}{2} \times 6 \times 6 \times \sin 30^\circ \\ &= 9 \text{ cm}^2 \\ \therefore \text{area of dodecagon} &= 12 \times 9 \\ &= 108 \text{ cm}^2 \end{aligned}$$

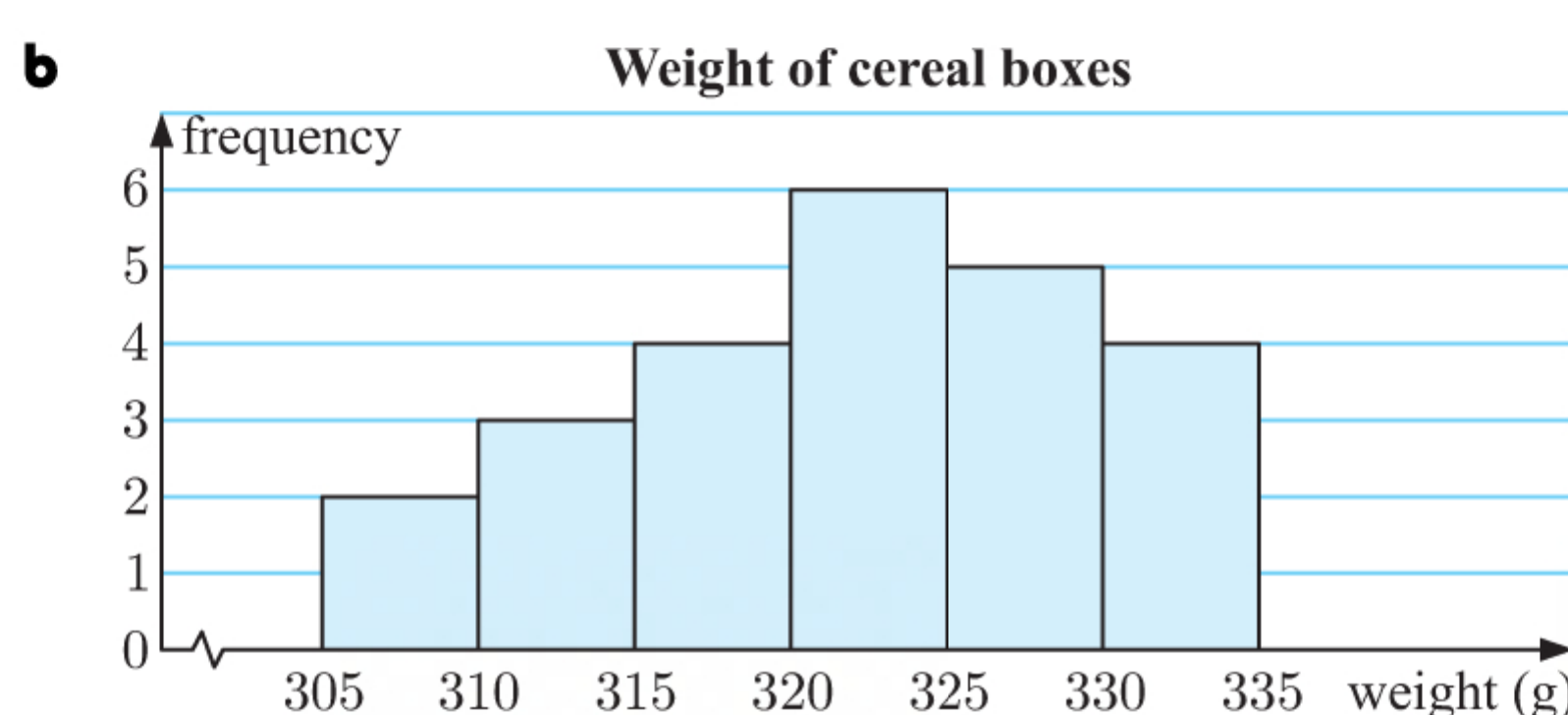


- c** From the graph, if James works a 35 hour week, he spends about \$61.

- d** There is a strong positive linear relationship between the length of time James works and the amount he spends on coffee. Since the prediction in **c** was an interpolation on strongly correlated data, it is a reliable estimate.

**4 a**

Weight ( $w$ g)	Frequency
$305 \leq w < 310$	2
$310 \leq w < 315$	3
$315 \leq w < 320$	4
$320 \leq w < 325$	6
$325 \leq w < 330$	5
$330 \leq w < 335$	4



- c** The data is slightly negatively skewed.

- d** The modal class is the interval  $320 \leq w < 325$  because it has the highest frequency.

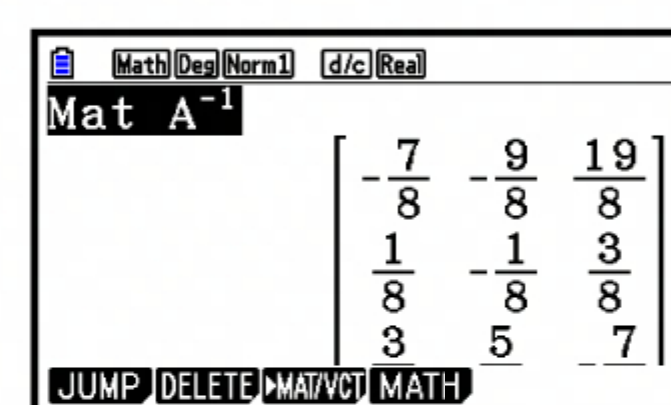
$$\begin{aligned} \text{e} \quad \text{Mean} &= \frac{312 + 320 + \dots + 324}{24} \\ &= \frac{7696}{24} \approx 320.67 \end{aligned}$$

The mean of the sample is reasonably close to the average weight that the manufacturer claims.

**5 a**

$$\mathbf{P} = \begin{pmatrix} -1 & 4 & -1 \\ 2 & -1 & 5 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\therefore \mathbf{P}^{-1} = \begin{pmatrix} -\frac{7}{8} & -\frac{9}{8} & \frac{19}{8} \\ \frac{1}{8} & -\frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{5}{8} & -\frac{7}{8} \end{pmatrix} \quad \{\text{using technology}\}$$



- b** The system of equations can be expressed in matrix form as

$$\begin{pmatrix} -1 & 4 & -1 \\ 2 & -1 & 5 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -10 \\ -5 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 4 & -1 \\ 2 & -1 & 5 \\ 1 & 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ -10 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{7}{8} & -\frac{9}{8} & \frac{19}{8} \\ \frac{1}{8} & -\frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{5}{8} & -\frac{7}{8} \end{pmatrix} \begin{pmatrix} -3 \\ -10 \\ -5 \end{pmatrix} \quad \{\text{from a}\}$$

$$= \begin{pmatrix} \frac{21}{8} + \frac{90}{8} - \frac{95}{8} \\ -\frac{3}{8} + \frac{10}{8} - \frac{15}{8} \\ -\frac{9}{8} - \frac{50}{8} + \frac{35}{8} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

So  $x = 2$ ,  $y = -1$ ,  $z = -3$ .



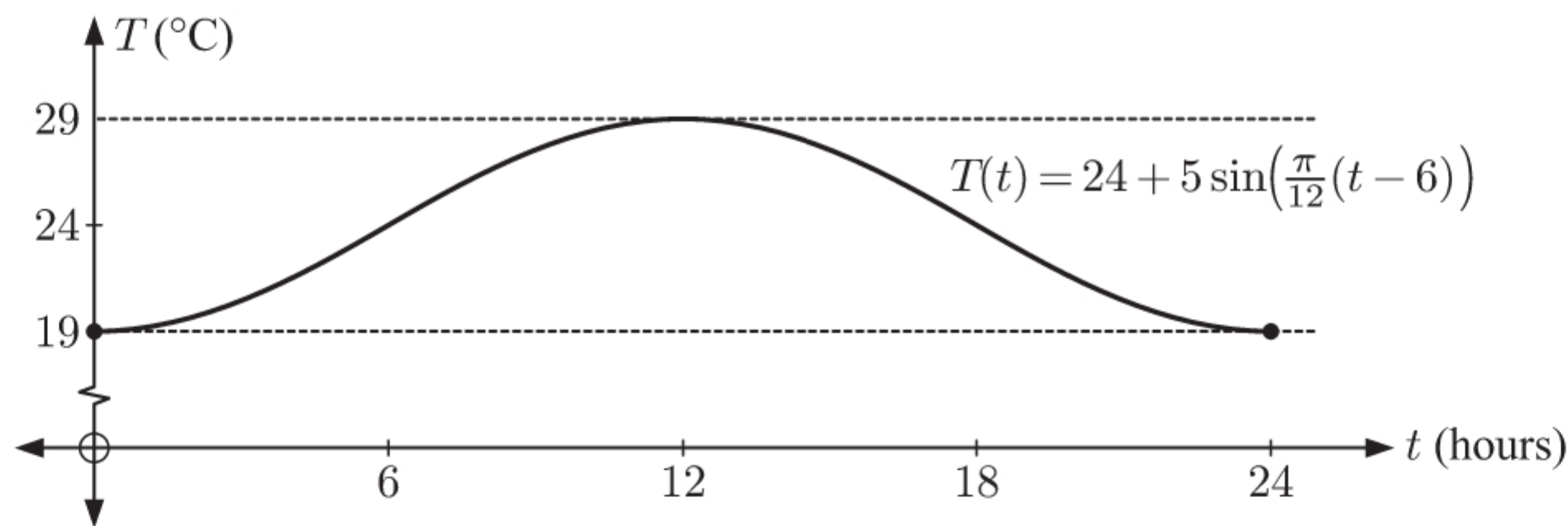
**6 a**  $Z = 3X + 2Y$

**b**  $E(Z) = E(3X + 2Y) \quad \text{Var}(Z) = \text{Var}(3X + 2Y)$   
 $= 3E(X) + 2E(Y) \quad = \text{Var}(3X) + \text{Var}(2Y) \quad \{\text{independence}\}$   
 $= 3(15) + 2(10) \quad = (3)^2 \text{Var}(X) + (2)^2 \text{Var}(Y)$   
 $= 65 \text{ minutes} \quad = 9(5^2) + 4(2^2)$   
 $= 241$

$\therefore$  standard deviation of  $Z = \sqrt{241} \approx 15.5$  minutes.

**7 a** For  $T(t) = 24 + 5 \sin\left(\frac{\pi}{12}(t - 6)\right)$ :

- the amplitude is 5
- the period is  $\frac{2\pi}{(\frac{\pi}{12})} = 24$  hours
- the horizontal translation is 6 hours to the right
- the principal axis is  $T = 24$ .



**b i** 2 pm is 8 hours after 6 am.

$$\begin{aligned} T(8) &= 24 + 5 \sin\left(\frac{\pi}{12}(8 - 6)\right) \\ &= 24 + 5 \sin\left(\frac{\pi}{12} \times 2\right) \\ &= 24 + 5 \sin \frac{\pi}{6} \\ &= 24 + 5 \times \frac{1}{2} \\ &= 26.5 \end{aligned}$$

$\therefore$  at 2 pm, the temperature inside Pam's caravan is  $26.5^\circ\text{C}$ .

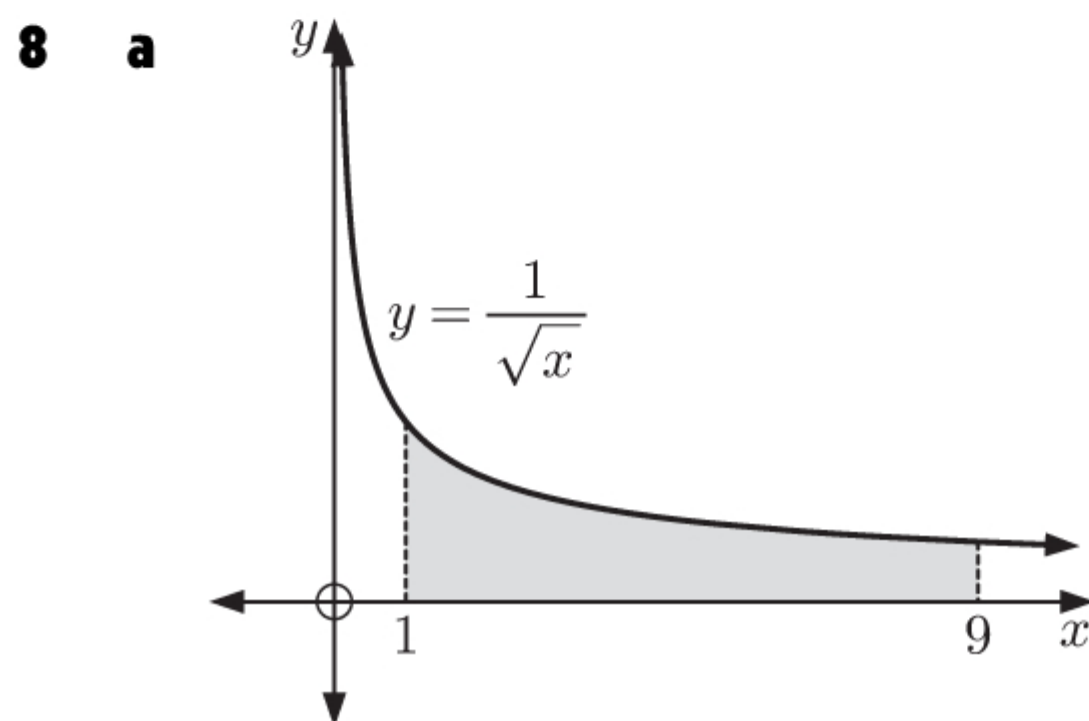
**ii** 9 pm is 15 hours after 6 am.

$$\begin{aligned} T(15) &= 24 + 5 \sin\left(\frac{\pi}{12}(15 - 6)\right) \\ &= 24 + 5 \sin\left(\frac{\pi}{12} \times 9\right) \\ &= 24 + 5 \sin \frac{3\pi}{4} \\ &= 24 + 5 \times \frac{1}{\sqrt{2}} \\ &\approx 27.5 \end{aligned}$$

$\therefore$  at 9 pm, the temperature inside Pam's caravan is about  $27.5^\circ\text{C}$ .

**c** The maximum temperature inside Pam's caravan is  $24 + 5 = 29^\circ\text{C}$ , which occurs when  $t = 12$ . 12 hours after 6 am is 6 pm.

So, the maximum temperature inside Pam's caravan occurs at 6 pm.



$$\begin{aligned} \text{Area} &= \int_1^9 x^{-\frac{1}{2}} dx \\ &= \left[ \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^9 \\ &= \left[ 2\sqrt{x} \right]_1^9 \\ &= 6 - 2 \\ &= 4 \text{ units}^2 \end{aligned}$$

**b** Volume of revolution  $= \pi \int_1^9 \left(\frac{1}{\sqrt{x}}\right)^2 dx$   
 $= \pi \int_1^9 \frac{1}{x} dx$   
 $= \pi [\ln |x|]_1^9$   
 $= \pi (\ln 9 - \ln 1)$   
 $= 2\pi \ln 3 \text{ units}^3$

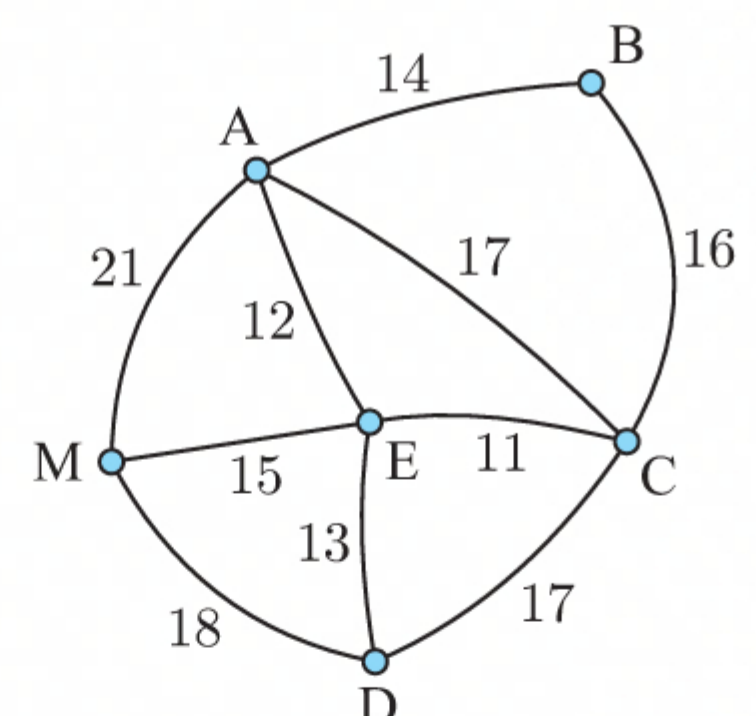
**9 a i** There are two vertices with odd degree, M and D, so the graph is semi-Eulerian.

We need to walk twice between these vertices. The most efficient way to achieve this is to traverse the edge MD twice.

A route that traverses every edge at least once will have weight equal to the sum of the weights of all the edges, plus the weight of the edge MD.

$\therefore$  the shortest distance for the course is  
 $(14 + 17 + 12 + 21 + 16 + 17 + 11 + 13 + 18 + 15) + 18 = 172 \text{ km}.$

**ii** A possible route is  $M \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow M \rightarrow E \rightarrow A \rightarrow C \rightarrow E \rightarrow D \rightarrow M$ .





- b** If the edge MD is removed, then all vertices have even degree, and the graph is Eulerian.

A route that traverses every edge once will have weight equal to the sum of the weights of all the edges.

$\therefore$  the shortest distance for the course is  $14 + 17 + 12 + 21 + 16 + 17 + 11 + 13 + 15 = 136$  km.

A possible route is  $M \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A \rightarrow C \rightarrow E \rightarrow M$ .

- 10 a** The coupled linear differential equations can be written in matrix form as  $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ .

Initially, when  $t = 0$ ,  $x = 1 \text{ cm}^2$  and  $y = 12 \text{ cm}^2$ .

$$\begin{aligned} \therefore \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} &= \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 4 + 36 \\ 2 - 12 \end{pmatrix} = \begin{pmatrix} 40 \\ -10 \end{pmatrix} \end{aligned}$$

Now  $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$  {chain rule}

$$\therefore \text{ when } t = 0, \quad -10 = \frac{dy}{dx} \times 40$$

$$\therefore \frac{dy}{dx} = -\frac{1}{4}$$

$$\begin{aligned} \mathbf{b} \quad \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= \begin{pmatrix} 12 + 3 \\ 6 - 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 + 6 \\ -2 - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \\ &= 5 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \qquad \qquad = -2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{aligned}$$

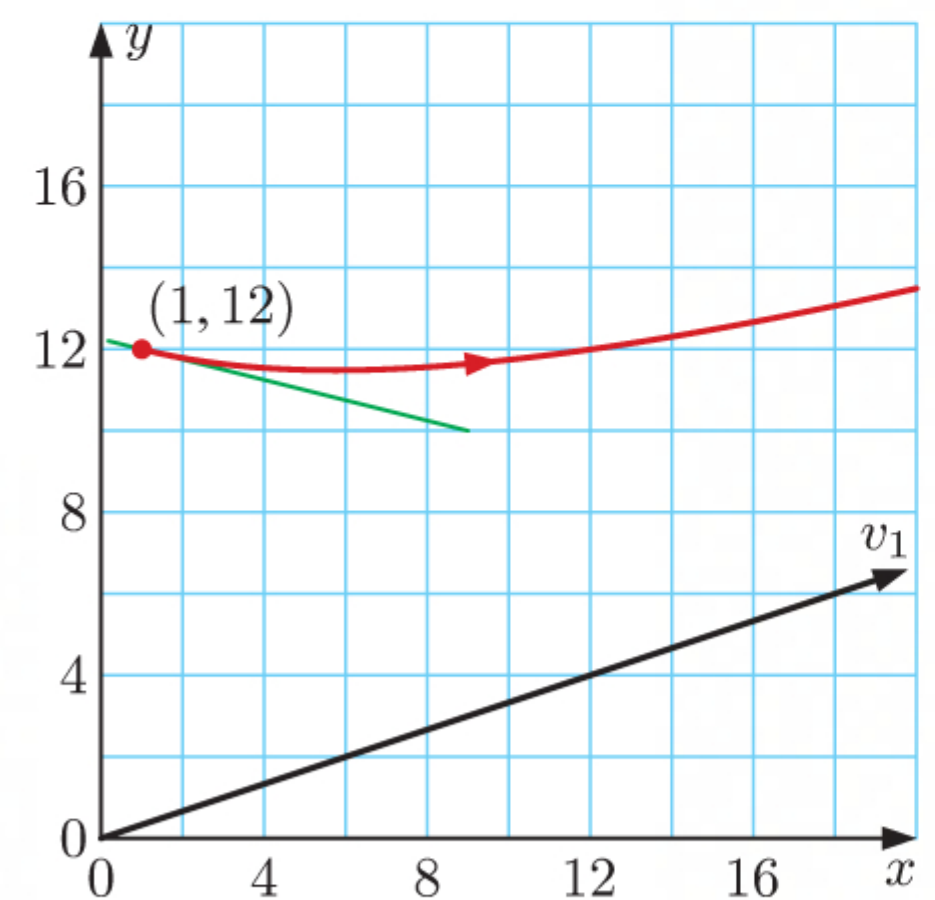
$\therefore$  the eigenvalues corresponding to  $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  and  $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$  are  $\lambda_1 = 5$  and  $\lambda_2 = -2$ , respectively.

- c** The trajectory begins at  $(1, 12)$ , with initial slope  $\frac{dy}{dx} = -\frac{1}{4}$ .

The eigenvectors  $\lambda_1$  and  $\lambda_2$  satisfy  $\lambda_1 > 0 > \lambda_2$ .

$\therefore$  the equilibrium point at O is a saddle point.

$\therefore$  as  $t \rightarrow \infty$ , the trajectory approaches  $k\mathbf{v}_1$ ,  $k \geq 0$ , which is the line  $y = \frac{1}{3}x$ ,  $x \geq 0$ .

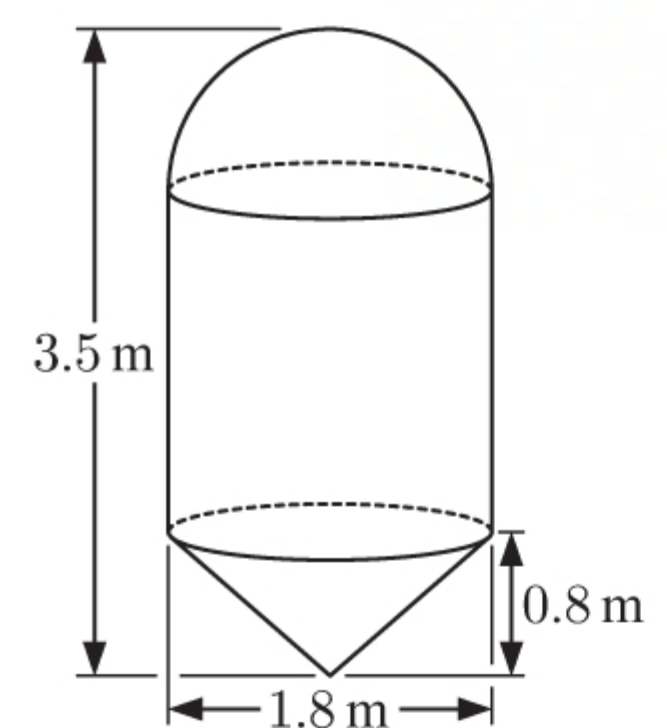


## MIXED QUESTIONS SET 12

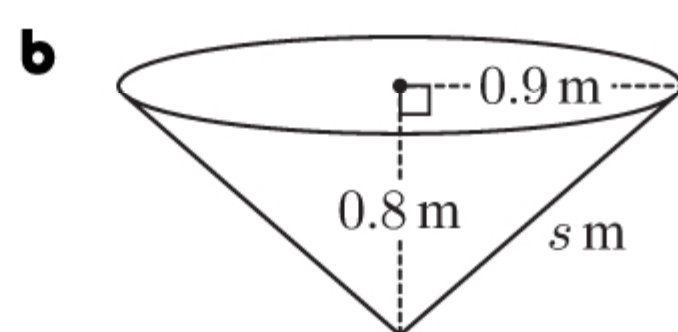
- 1 a** Height of cylinder = total height of silo – height of cone – height of hemisphere

$$= 3.5 - 0.8 - \frac{1.8}{2} \text{ m}$$

$$= 1.8 \text{ m}$$







Let the slant height of the cone be  $s$  m.

$$s^2 = (0.8)^2 + (0.9)^2$$

$$\therefore s = \sqrt{(0.8)^2 + (0.9)^2} \quad \{s > 0\}$$

$$\approx 1.204 \text{ m}$$

$$\begin{aligned} \text{Surface area of cone} &= \pi r s \\ &\approx \pi \times 0.9 \times 1.204 \\ &\approx 3.40 \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \text{Surface area of cylinder} &= 2\pi r h \\ &= 2 \times \pi \times 0.9 \times 1.8 \\ &\approx 10.2 \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \text{Surface area of hemisphere} &= 2\pi r^2 \\ &= 2 \times \pi \times (0.9)^2 \\ &\approx 5.09 \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \therefore \text{total surface area} &\approx 3.40 + 10.2 + 5.09 \text{ m}^2 \\ &\approx 18.7 \text{ m}^2 \end{aligned}$$

So, about  $18.7 \text{ m}^2$  of sheet metal was used to make the silo.

**2**  $P(x) \approx \frac{x}{\ln x}$

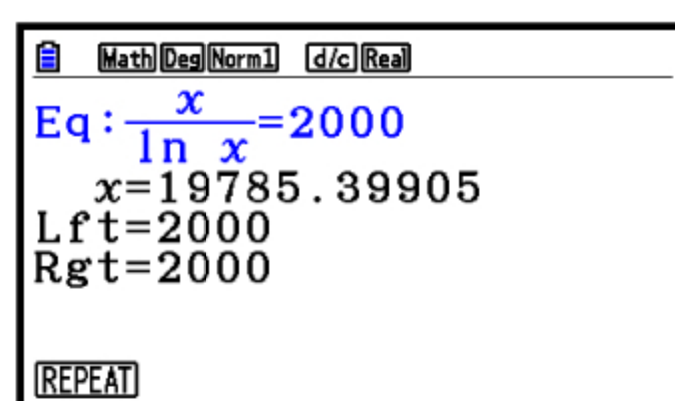
**a**  $P(1\,000\,000) \approx \frac{1\,000\,000}{\ln 1\,000\,000}$

$$\approx 72\,400$$

We estimate there are about 72 400 prime numbers less than 1 000 000.

**c**  $P(x) \approx 2000$  when  $\frac{x}{\ln x} \approx 2000$ .

Using technology,  $x \approx 19\,785$ .



So, there are about 2000 prime numbers less than or equal to 19 785.

**3** The distance travelled  $d$  is directly proportional to the square of the time taken  $t$ .

$$\therefore d = kt^2 \text{ where } k \text{ is a constant.}$$

When  $d = 19.6 \text{ m}$ ,  $t = 2 \text{ s}$ , so  $19.6 = k(2)^2$

$$\therefore k = \frac{19.6}{4} = 4.9$$

So,  $d = 4.9t^2$ .

**a** When  $t = 3 \text{ s}$ ,  $d = 4.9(3)^2$

$$= 44.1 \text{ m}$$

**c** Volume of cone  $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \pi \times 0.9^2 \times 0.8$$

$$\approx 0.679 \text{ m}^3$$

Volume of cylinder  $= \pi r^2 h$

$$= \pi \times 0.9^2 \times 1.8$$

$$\approx 4.58 \text{ m}^3$$

Volume of hemisphere  $= \frac{1}{2}(\frac{4}{3}\pi r^3)$

$$= \frac{1}{2} \times \frac{4}{3} \times \pi \times 0.9^3$$

$$\approx 1.53 \text{ m}^3$$

$$\therefore \text{total volume} \approx 0.679 + 4.58 + 1.53 \text{ m}^3$$

$$\approx 6.79 \text{ m}^3$$

Now  $6.79 \text{ m}^3 \equiv 6.79 \text{ kL}$

So, the capacity of the silo is about 6.79 kL.

**b** Percentage error  $= \frac{|V_A - V_E|}{V_E} \times 100\%$

$$\approx \frac{|72\,400 - 78\,498|}{78\,498} \times 100\%$$

$$\approx 7.8\%$$



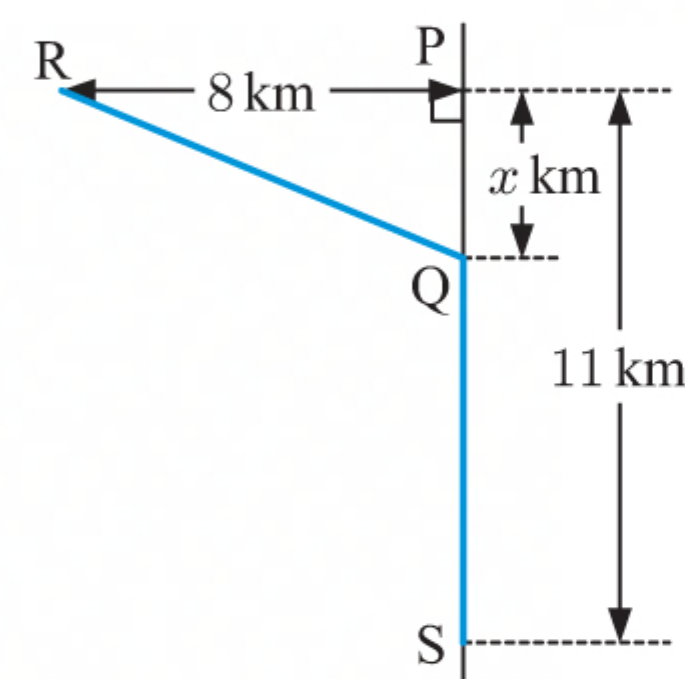
**4 a**  $QR^2 = x^2 + 8^2$  {Pythagoras}

$\therefore QR = \sqrt{x^2 + 64}$  {as  $QR > 0$ }

Also  $QS = PS - PQ$   
 $= 11 - x$

So, the length of pipeline under the sea is  $\sqrt{x^2 + 64}$  km,  
 and the length of pipeline overland is  $(11 - x)$  km.

$\therefore$  the cost  $C(x) = 5\sqrt{x^2 + 64} + 3(11 - x)$  million dollars  
 $= 5\sqrt{x^2 + 64} + 33 - 3x$  million dollars.



**b**  $C(x) = 5(x^2 + 64)^{\frac{1}{2}} + 33 - 3x$

$\therefore C'(x) = \frac{5}{2}(x^2 + 64)^{-\frac{1}{2}}(2x) - 3$   
 $= \frac{5x}{\sqrt{x^2 + 64}} - 3$

Now  $C'(x) = 0$  where  $\frac{5x}{\sqrt{x^2 + 64}} = 3$

$\therefore 5x = 3\sqrt{x^2 + 64}$

$\therefore 25x^2 = 9(x^2 + 64)$

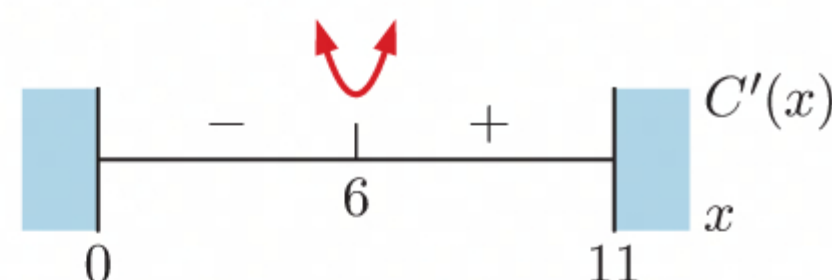
$\therefore 25x^2 = 9x^2 + 576$

$\therefore 16x^2 = 576$

$\therefore x^2 = 36$

$\therefore x = 6$   $\{x \geq 0\}$

$\therefore C'(x)$  has sign diagram



$C(6) = 5\sqrt{6^2 + 64} + 33 - 3(6)$   
 $= 65$

So, there is a local minimum at  $(6, 65)$ .

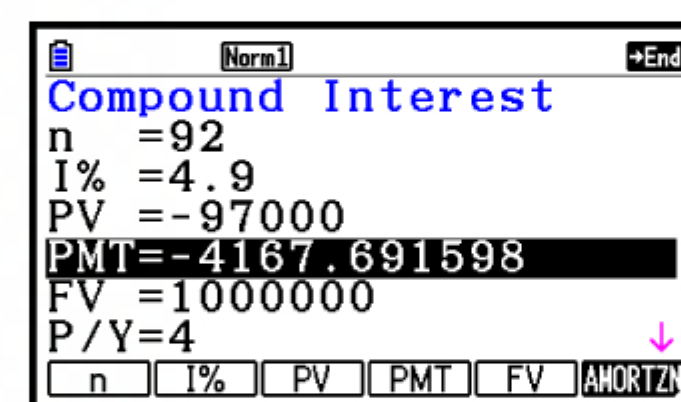
$\therefore$  the minimum cost of the pipeline is 65 million dollars, when Q is 6 km from P.

**5 a** Stig has  $55 - 32 = 23$  years before he retires.

$N = 23 \times 4 = 92$ ,  $I\% = 4.9$ ,  $PV = -97\,000$ ,  $FV = 1\,000\,000$ ,  $P/Y = 4$ ,  
 $C/Y = 4$

$\therefore PMT \approx -4167.70$

Stig should contribute \$4167.70 to his fund each quarter.

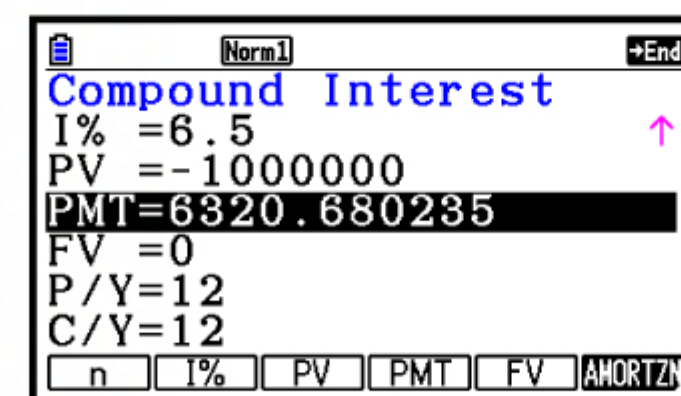


**b** Interest generated  $\approx \$1\,000\,000 - \$97\,000 - \$4167.70 \times 92$   
 $\approx \$519\,571.60$

**c**  $N = 30 \times 12 = 360$ ,  $I\% = 6.5$ ,  $PV = -1\,000\,000$ ,  $FV = 0$ ,  $P/Y = 12$ ,  
 $C/Y = 12$

$\therefore PMT \approx 6320.68$

Stig can afford to withdraw \$6320.68 each month.



**d** To index Stig's standard of living for inflation, we increase it by 3.7% each year for 23 years.

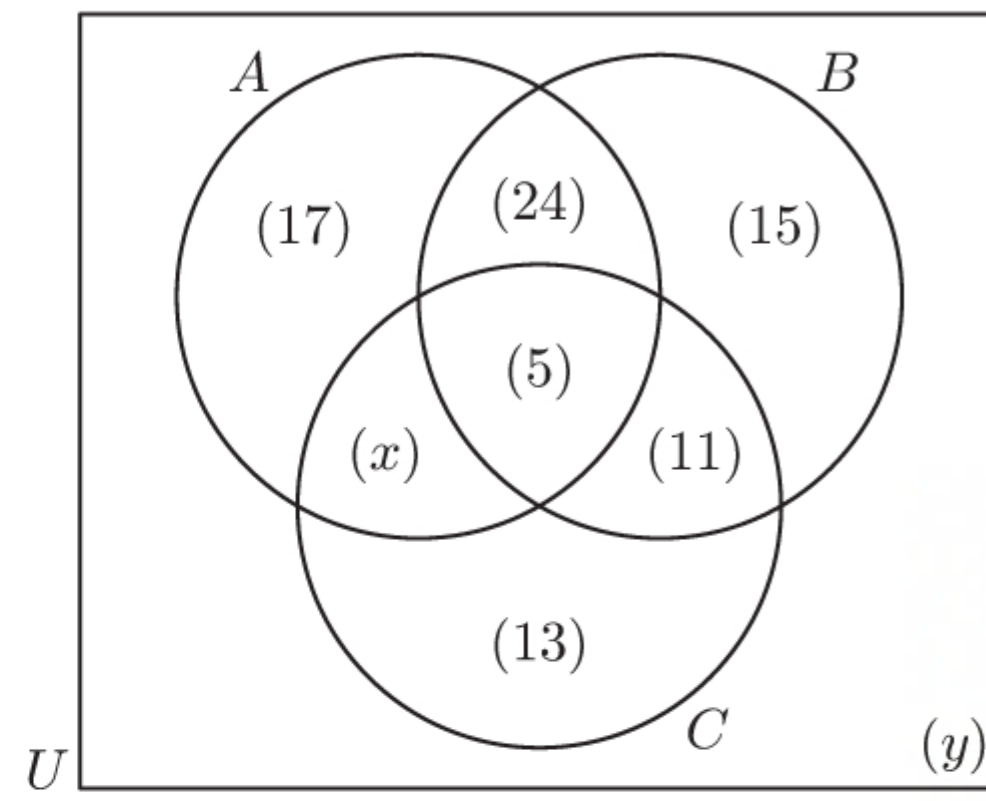
$\therefore$  indexed value  $= \$2700 \times (1.037)^{23}$   
 $= \$6226.95$

This is less than our answer in **c**, so Stig can maintain his standard of living at the time of his retirement.



$$\begin{aligned}
 \text{6 a} \quad n(A) &= 48 \\
 \therefore 17 + 24 + 5 + x &= 48 \\
 \therefore 46 + x &= 48 \\
 \therefore x &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } n(U) &= 100 \\
 \therefore 17 + 24 + 5 + 2 + 15 + 11 + 13 + y &= 100 \\
 \therefore 87 + y &= 100 \\
 \therefore y &= 13
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad n(A) &= 48, \quad n(B) = 24 + 5 + 15 + 11 = 55, \quad n(C) = 2 + 5 + 11 + 13 = 31 \\
 \therefore \text{course } B &\text{ was the most popular.}
 \end{aligned}$$

$$\begin{aligned}
 \text{c i} \quad P(\text{liked all the courses}) &= \frac{5}{100} \\
 &= \frac{1}{20}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad P(B \cap C') &= \frac{24 + 15}{100} \\
 &= \frac{39}{100}
 \end{aligned}$$

$$\begin{aligned}
 \text{iii} \quad P(\text{liked exactly 2 courses} \mid C) &= \frac{n(\text{liked } C \text{ and exactly one other course})}{n(C)} \\
 &= \frac{2 + 11}{31} \\
 &= \frac{13}{31}
 \end{aligned}$$

$$\begin{aligned}
 \text{iv} \quad P(\text{liked none} \mid B') &= \frac{n(\text{liked none})}{n(B')} \\
 &= \frac{13}{17 + 2 + 13 + 13} \\
 &= \frac{13}{45}
 \end{aligned}$$

$$\text{7 a} \quad \mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \text{ where } |\mathbf{A}| = \left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2 = 1.$$

Since  $\mathbf{A}$  has the form  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$  and  $|\mathbf{A}| = 1$ ,  $\mathbf{A}$  is a rotation matrix.

$\therefore$  the transformation is a rotation.

$$\text{b} \quad \text{If the angle of rotation is } \theta, \cos \theta = \frac{1}{\sqrt{5}} \text{ and } \sin \theta = \frac{2}{\sqrt{5}}, \text{ so } \theta \text{ is in quadrant 1.}$$

$$\begin{aligned}
 \therefore \theta &= \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \\
 &\approx 1.11^\circ
 \end{aligned}$$

$\therefore$  the transformation is an anticlockwise rotation about  $O(0, 0)$  through about  $1.11^\circ$ .

$$\text{c} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{A} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{5}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{5} \end{pmatrix}$$

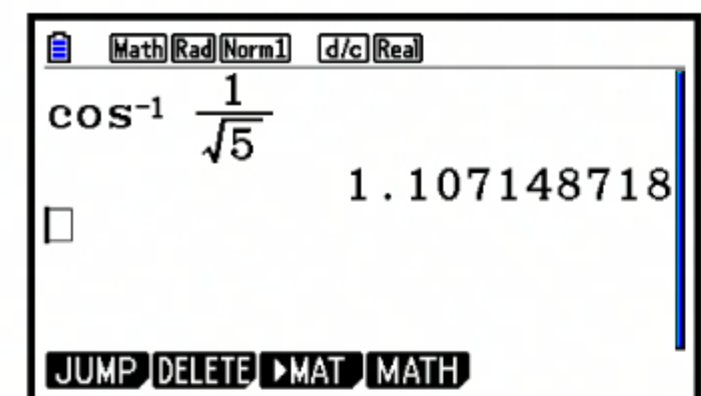
$\therefore$  the image of  $(2, 1)$  is  $(0, \sqrt{5})$ .

$$\begin{aligned}
 \text{8 a} \quad \text{If } y &= x \ln x, \text{ then } \frac{dy}{dx} = (1) \ln x + x \left(\frac{1}{x}\right) \\
 &= \ln x + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad T' &= \frac{1}{20}(1 + \ln N) \\
 \therefore T &= \int \frac{1}{20}(1 + \ln N) dN \\
 &= \frac{1}{20}N \ln N + c \quad \{\text{using a}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now when } N &= 50, T = 10, \text{ so } 10 = \frac{50}{20} \ln 50 + c \\
 \therefore c &= 10 - \frac{5}{2} \ln 50
 \end{aligned}$$

$$\begin{aligned}
 \text{So if } N &= 100, T = 5 \ln 100 + 10 - \frac{5}{2} \ln 50 \\
 &\approx 23 \text{ seconds}
 \end{aligned}$$





- 9 Under  $H_0$ , the  $z$ -score of 24 is  $\frac{24 - 25}{\frac{3}{\sqrt{15}}} = -\frac{\sqrt{15}}{3}$ .

So, the decision rule can be equivalently written as: accept  $H_0$  if  $z > -\frac{\sqrt{15}}{3}$

reject  $H_0$  if  $z \leq -\frac{\sqrt{15}}{3}$

$$\begin{aligned} \mathbf{a} \quad \alpha &= P(\text{Type I error}) \\ &= P(\text{Reject } H_0 \mid H_0 \text{ true}) \\ &= P\left(Z \leq -\frac{\sqrt{15}}{3} \mid Z \sim N(0, 1^2)\right) \\ &\approx 0.0984 \quad \{\text{technology}\} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad &\text{The true distribution of } Z \text{ is } N\left(\frac{24.5 - 25}{\frac{3}{\sqrt{15}}}, 1^2\right) \equiv N\left(-\frac{\sqrt{15}}{6}, 1^2\right). \\ \therefore \beta &= P(\text{Type II error}) \\ &= P(\text{Retain } H_0 \mid H_0 \text{ false}) \\ &= P\left(Z > -\frac{\sqrt{15}}{3} \mid Z \sim N\left(-\frac{\sqrt{15}}{6}, 1^2\right)\right) \\ &\approx 0.741 \quad \{\text{technology}\} \end{aligned}$$

10 **a**  $\tau = \mathbf{r}^* \times \mathbf{F}^*$

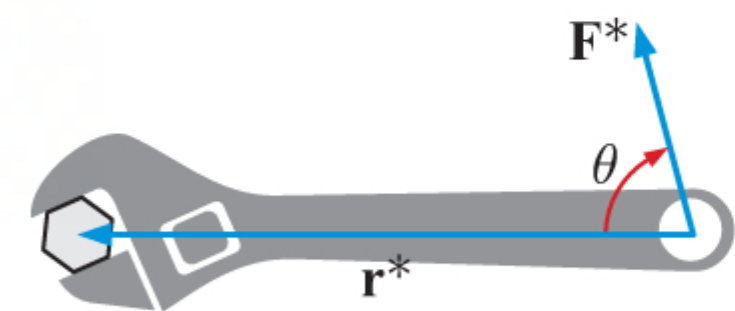
$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.2 & -0.15 & 0 \\ 2 & -1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -0.15 & 0 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0.2 & 0 \\ 2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0.2 & -0.15 \\ 2 & -1 \end{vmatrix} \mathbf{k} \\ &= (-0.3\mathbf{i} - 0.4\mathbf{j} + 0.1\mathbf{k}) \text{ Nm} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{Magnitude} &= |\tau| \\ &= \sqrt{(-0.3)^2 + (-0.4)^2 + (0.1)^2} \\ &= \sqrt{0.26} \\ &\approx 0.510 \text{ Nm} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \tau &= \mathbf{r}^* \times \mathbf{F}^* \\ \therefore |\tau| &= |\mathbf{r}^* \times \mathbf{F}^*| \\ &= |\mathbf{r}^*| |\mathbf{F}^*| \sin \theta \quad \{\theta \text{ is the angle between } \mathbf{r}^* \text{ and } \mathbf{F}^*\} \end{aligned}$$

Now  $\sin \theta$  has maximum 1 when  $\theta = 90^\circ$

$$\begin{aligned} \therefore |\tau| \text{ has maximum} &= |\mathbf{r}^*| |\mathbf{F}^*| \\ &= \sqrt{(0.2)^2 + (-0.15)^2 + 0^2} \times \sqrt{2^2 + (-1)^2 + 2^2} \\ &= 0.75 \text{ Nm} \end{aligned}$$



## MIXED QUESTIONS SET 13

- 1 **a** For £1000 in sales, the salesperson makes  $\text{£}1000 \times 0.05 = \text{£}50$  commission.

$$\therefore k = 50$$

- b** For sales above £1000, the gradient of the line is  $\frac{90 - k}{1500 - 1000} = \frac{90 - 50}{1500 - 1000} = 0.08$ .

$\therefore$  the salesperson makes 8% commission on sales above £1000.

- c** For £1800 in sales, the salesperson makes  $\text{£}50 + \text{£}800 \times 0.08 = \text{£}114$  commission.

- d** Let  $\text{£}x$  be the value in sales needed to earn £150 commission.

$$\therefore 50 + (x - 1000) \times 0.08 = 150$$

$$\therefore (x - 1000) \times 0.08 = 100$$

$$\therefore x - 1000 = 1250$$

$$\therefore x = 2250$$

$\therefore$  £2250 in sales are needed to earn £150 commission.

- 2 **a** There are  $n = 5 \times 4 = 20$  time periods.

Each time period the investment increases by  $i = \frac{4.4\%}{4} = 1.1\%$ .

$$\begin{aligned} \therefore \text{the amount after 5 years is } u_{20} &= u_0 \times (1 + i)^{20} \\ &= 2000 \times (1.011)^{20} \quad \{1.1\% = 0.011\} \\ &\approx 2489.16 \end{aligned}$$

The final value of the investment is \$2489.16.



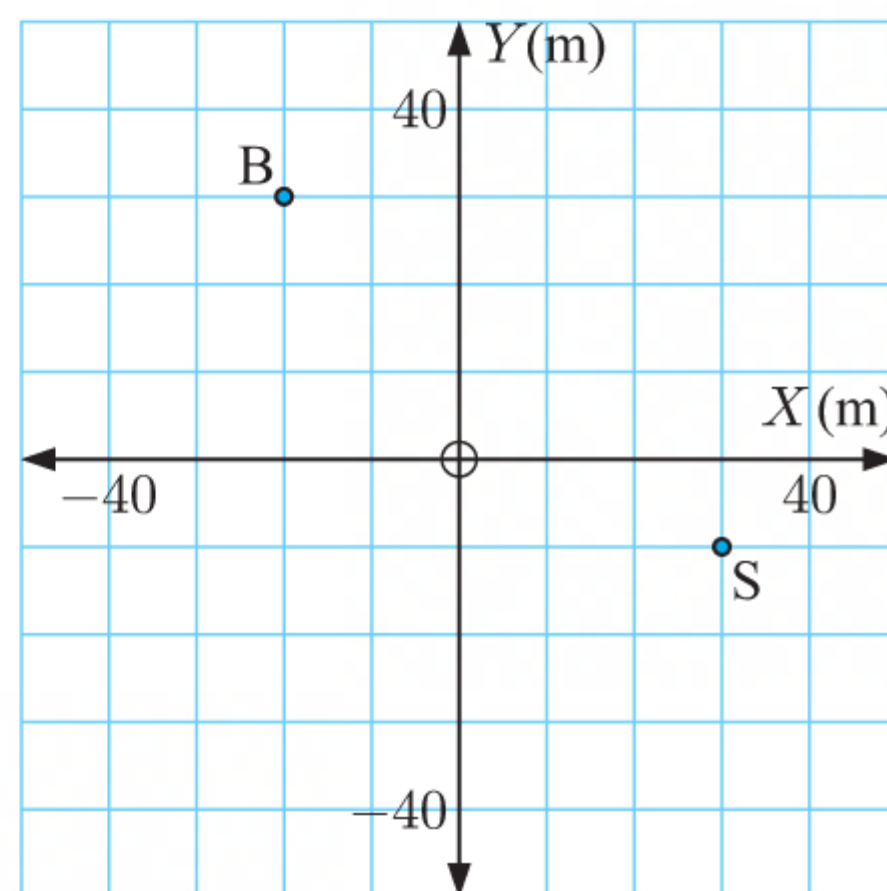
$$\begin{aligned}\text{b Interest} &= \$2489.16 - \$2000 \\ &= \$489.16\end{aligned}$$

$$\begin{aligned}\text{c real value} \times (1.025)^5 &= \$2489.16 \\ \therefore \text{real value} &= \frac{\$2489.16}{(1.025)^5} \\ &= \$2200.05\end{aligned}$$

- 3 a i The anchor has coordinates  $A(-20, 30, -50)$ .  
 ii The shipwreck has coordinates  $S(30, -10, -40)$ .

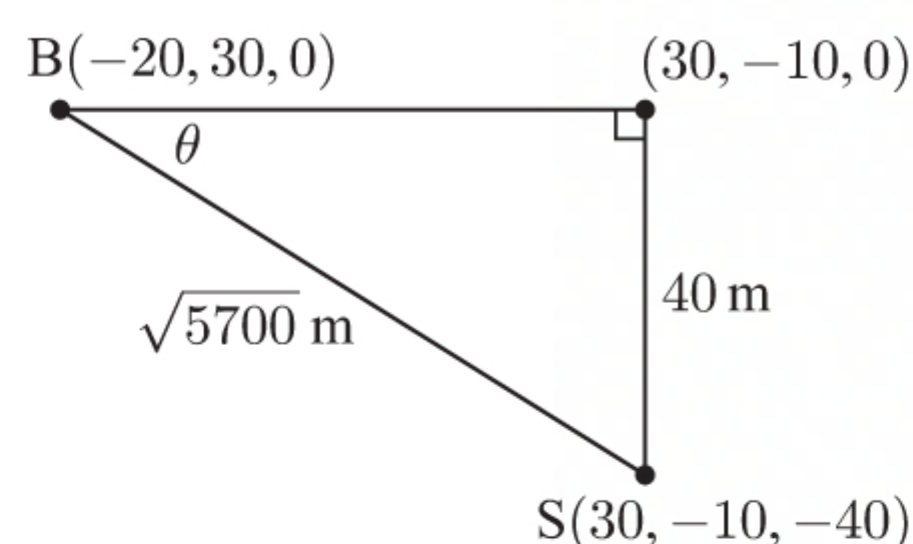
$$\begin{aligned}\text{b BS} &= \sqrt{(30 - (-20))^2 + (-10 - 30)^2 + (-40 - 0)^2} \\ &= \sqrt{50^2 + (-40)^2 + (-40)^2} \\ &= \sqrt{5700} \\ &\approx 75.5 \text{ m}\end{aligned}$$

$\therefore$  the diver has to swim about 75.5 m.



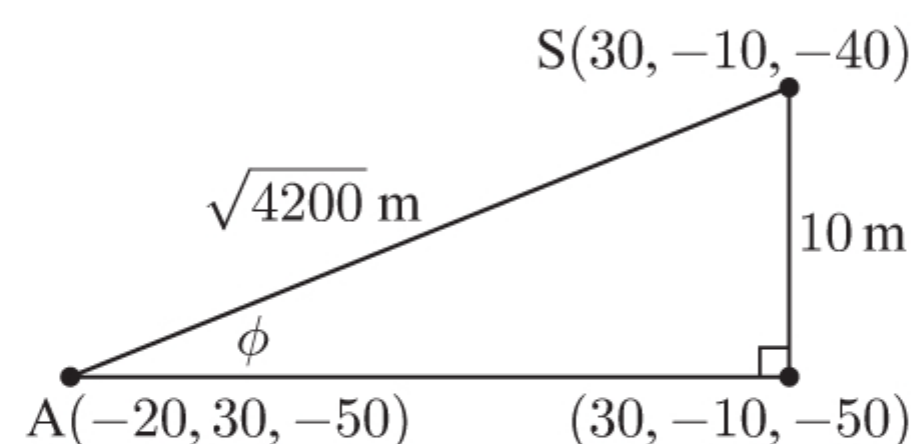
$$\begin{aligned}\text{c i } \sin \theta &= \frac{40}{\sqrt{5700}} \\ \therefore \theta &= \sin^{-1}\left(\frac{40}{\sqrt{5700}}\right) \\ \therefore \theta &\approx 32.0^\circ\end{aligned}$$

The angle of depression from the boat to the shipwreck is about  $32.0^\circ$ .



$$\begin{aligned}\text{ii AS} &= \sqrt{(-20 - 30)^2 + (30 - (-10))^2 + (-50 - (-40))^2} \\ &= \sqrt{(-50)^2 + 40^2 + (-10)^2} \\ &= \sqrt{4200} \text{ m} \\ \therefore \sin \phi &= \frac{10}{\sqrt{4200}} \\ \therefore \phi &= \sin^{-1}\left(\frac{10}{\sqrt{4200}}\right) \\ \therefore \phi &\approx 8.88^\circ\end{aligned}$$

The angle of elevation from the anchor to the shipwreck is about  $8.88^\circ$ .



$$4 \text{ a } f(x) \xrightarrow[\text{scale factor 2}]{\text{horizontal stretch}} f\left(\frac{1}{2}x\right) \xrightarrow[\text{scale factor 3}]{\text{vertical stretch}} 3f\left(\frac{1}{2}x\right) = g(x)$$

A horizontal stretch with scale factor 2, then a vertical stretch with scale factor 3 maps  $y = f(x)$  onto  $y = g(x)$ .

- b Each point on  $y = g(x)$  is 2 times their distance that  $y = f(x)$  is from the  $y$ -axis, and 3 times their distance that  $y = f(x)$  is from the  $x$ -axis.

The point  $(-6, 3)$  on  $y = f(x)$  is 6 units from the  $y$ -axis, and 3 units from the  $x$ -axis. The corresponding point on  $y = g(x)$ , which is  $2 \times 6 = 12$  units from the  $y$ -axis and  $3 \times 3 = 9$  units from the  $x$ -axis, is  $(-12, 9)$ .

- c Each point on  $y = f(x)$  is  $\frac{1}{2}$  times their distance that  $y = g(x)$  is from the  $y$ -axis, and  $\frac{1}{3}$  times their distance that  $y = g(x)$  is from the  $x$ -axis.

The point  $(4, -9)$  on  $y = g(x)$  is 4 units from the  $y$ -axis, and 9 units from the  $x$ -axis. The corresponding point on  $y = f(x)$ , which is  $\frac{1}{2} \times 4 = 2$  units from the  $y$ -axis and  $\frac{1}{3} \times 9 = 3$  units from the  $x$ -axis, is  $(2, -3)$ .

$$5 \text{ a } N = (8 - t)e^{t-6}, \quad 0 \leq t \leq 8$$

$$\begin{aligned}\frac{dN}{dt} &= (-1)e^{t-6} + (8 - t)e^{t-6} \quad \{\text{product rule}\} \\ &= e^{t-6}(-1 + 8 - t) \\ &= (7 - t)e^{t-6}\end{aligned}$$



- b i** The turning point occurs when

$$\frac{dN}{dt} = 0$$

$$\therefore (7-t)e^{t-6} = 0$$

$$\therefore 7-t = 0 \quad \{e^{t-6} > 0 \text{ for all } t\}$$

$$\therefore t = 7$$

$$\text{When } t = 7, \quad N = (8-7)e^{7-6}$$

$$= (1)e^1$$

$$= e$$

$\therefore$  the turning point has coordinates  $(7, e)$ .

**iii**  $N = 0$

$$\therefore (8-t)e^{t-6} = 0$$

$$\therefore 8-t = 0 \quad \{e^{t-6} > 0 \text{ for all } t\}$$

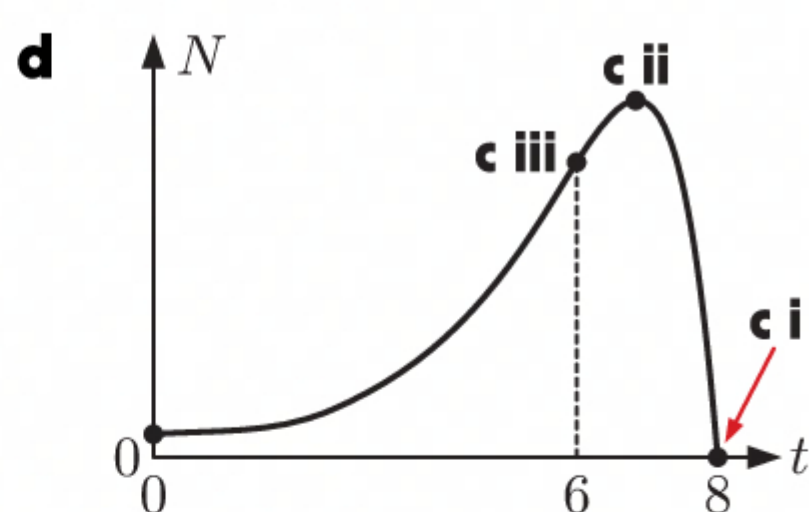
$$\therefore t = 8$$

$\therefore$  the  $t$ -intercept has coordinates  $(8, 0)$ .

- c i** Using the  $t$ -intercept, the time when all the bacteria are dead is  $t = 8$  hours.

- ii** Using the turning point, the maximum number of bacteria reached in the sample was  $N = e \approx 2.71$  million bacteria.

- iii** Using the point of inflection, the time at which the rate of increase of the bacteria is a maximum is  $t = 6$  hours.



- 6 a** When  $t = 0$ ,  $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$ .

$\therefore$  ship A is initially at  $(-1, 3)$  and ship B is initially at  $(7, 4)$ .

- b** Ship A has velocity vector  $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$\therefore$  its speed is  $\sqrt{4^2 + (-1)^2} = \sqrt{17} \text{ km h}^{-1}$ .

Ship B has velocity vector  $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$\therefore$  its speed is  $\sqrt{(-2)^2 + (-1)^2} = \sqrt{5} \text{ km h}^{-1}$ .

- c** Let  $\theta$  be the acute angle between the paths of the ships.

$$\begin{aligned} \therefore \cos \theta &= \frac{\left| \begin{pmatrix} 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 4 \\ -1 \end{pmatrix} \right| \left| \begin{pmatrix} -2 \\ -1 \end{pmatrix} \right|} \\ &= \frac{|-8+1|}{\sqrt{4^2 + (-1)^2} \sqrt{(-2)^2 + (-1)^2}} \\ &= \frac{7}{\sqrt{17}\sqrt{5}} \end{aligned}$$

$$\begin{aligned} \therefore \theta &= \cos^{-1} \left( \frac{7}{\sqrt{17}\sqrt{5}} \right) \\ &\approx 40.6^\circ \end{aligned}$$

$\therefore$  the acute angle between the paths of the ships is about  $40.6^\circ$ .

**ii**  $\frac{d^2N}{dt^2} = (-1)e^{t-6} + (7-t)e^{t-6} \quad \{\text{product rule}\}$

$$= e^{t-6}(-1+7-t)$$

$$= (6-t)e^{t-6}$$

The point of inflection occurs when

$$\frac{d^2N}{dt^2} = 0$$

$$\therefore (6-t)e^{t-6} = 0$$

$$\therefore 6-t = 0 \quad \{e^{t-6} > 0 \text{ for all } t\}$$

$$\therefore t = 6$$

$$\text{When } t = 6, \quad N = (8-6)e^{6-6}$$

$$= 2e^0$$

$$= 2$$

$\therefore$  the point of inflection has coordinates  $(6, 2)$ .



- d** Suppose the ships pass through the same point, and that ship A is there at time  $t_A$  and ship B is there at time  $t_B$ .

$$\begin{aligned} x_A = x_B &\Rightarrow -1 + 4t_A = 7 - 2t_B & y_A = y_B &\Rightarrow 3 - t_A = 4 - t_B \\ \therefore 4t_A + 2t_B &= 8 & \therefore t_A - t_B &= -1 \quad \dots (2) \\ \therefore 2t_A + t_B &= 4 \quad \dots (1) \end{aligned}$$

$$\begin{array}{rcl} 2t_A + t_B &= 4 & \{(1)\} \\ t_A - t_B &= -1 & \{(2)\} \\ \hline \text{Adding, } 3t_A &= 3 & \\ \therefore t_A &= 1 & \end{array}$$

$$\begin{aligned} \text{Substituting into (1) gives } 2 + t_B &= 4 \\ \therefore t_B &= 2 \end{aligned}$$

When  $t_A = 1$ ,  $x_A = -1 + 4 = 3$  and  $y_A = 3 - 1 = 2$

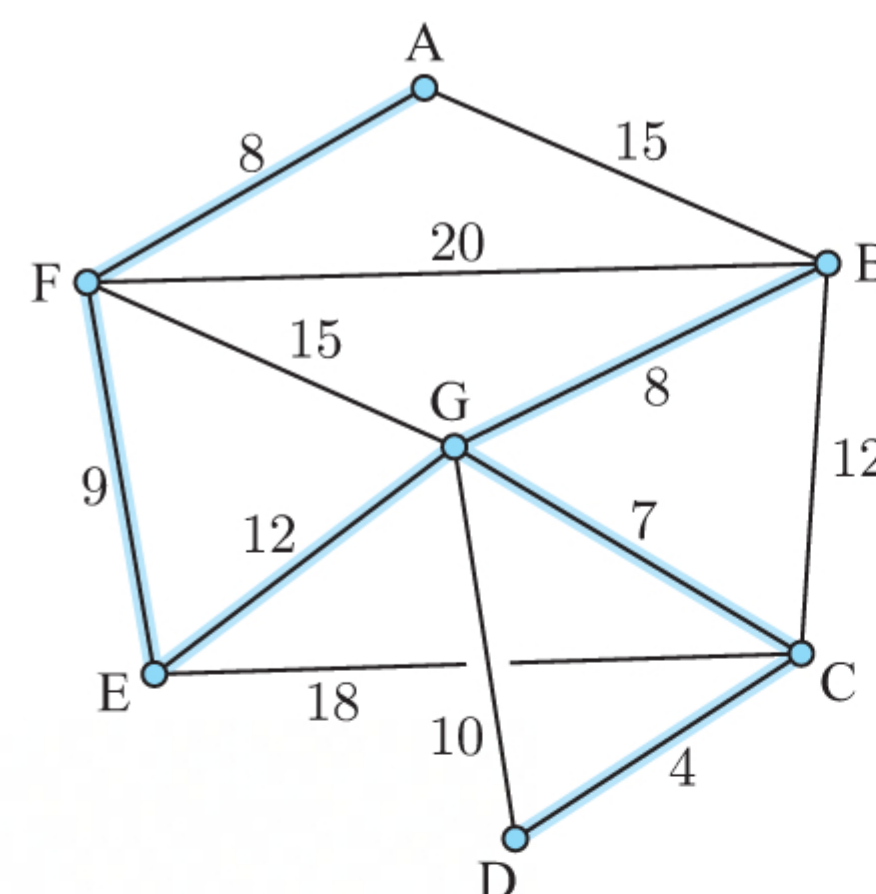
So, the two ships both pass through  $(3, 2)$ . Ship A is there after 1 hour, and ship B is there after 2 hours.

- 7 a** We choose a vertex at random, say vertex A.

The vertex nearest A is F, so we choose edge AF.

We then choose FE, EG, GC, CD, and GB.

$$\begin{aligned} \text{Minimum length of road} &= 8 + 9 + 12 + 7 + 4 + 8 \\ &= 48 \text{ km} \end{aligned}$$

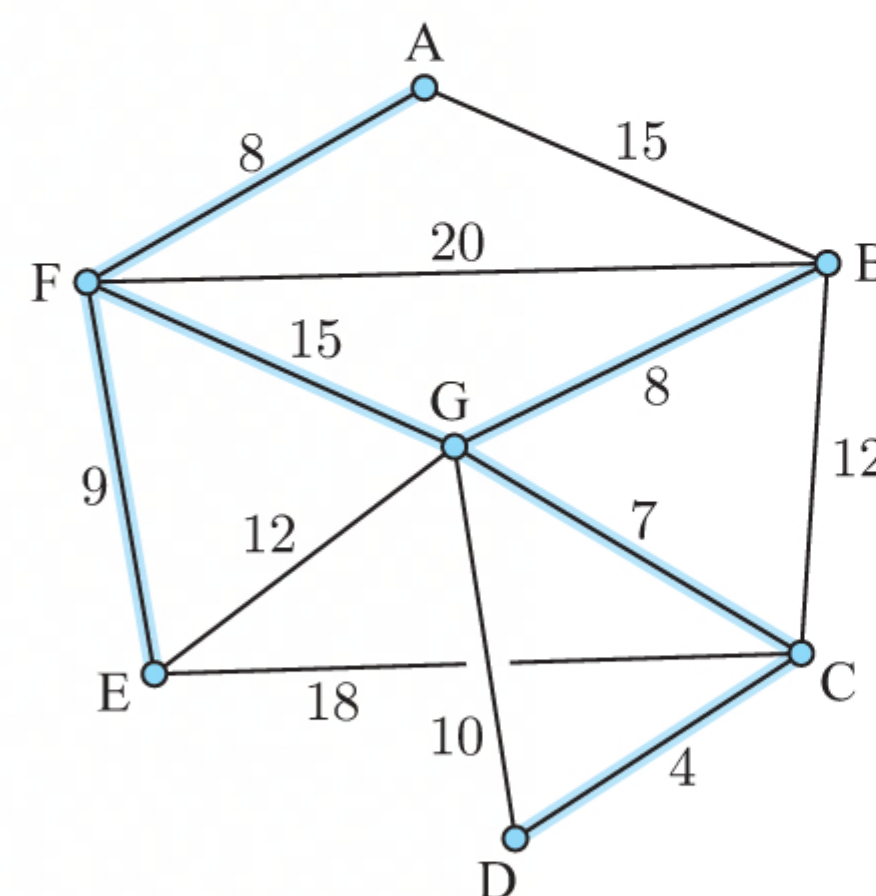


- b** We choose FG first.

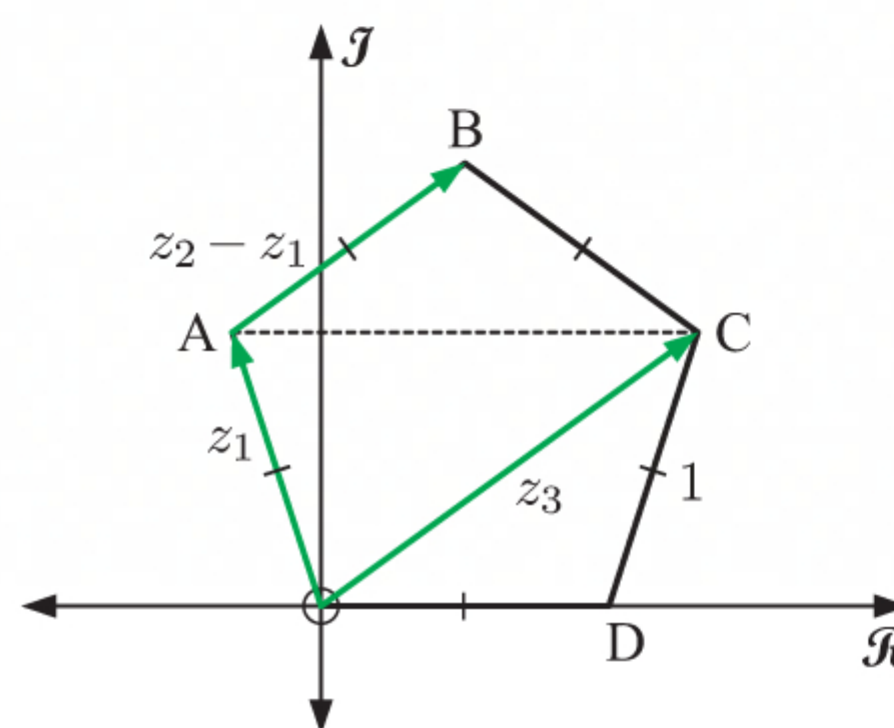
We then use Kruskal's algorithm to select the edges of least weight without creating a cycle.

We choose DC, CG, BG, FA, and FE.

$$\begin{aligned} \text{Minimum length of road} &= 15 + 4 + 7 + 8 + 8 + 9 \\ &= 51 \text{ km} \end{aligned}$$



- 8 a i**  $\widehat{AOD} = \frac{3\pi}{5}$  {OABCD is a regular pentagon}
- $$\therefore z_1 = \text{cis } \frac{3\pi}{5}$$
- ii**  $z_2 - z_1$  represents  $\overrightarrow{AB}$
- Now  $\widehat{OAC} = \pi - \widehat{AOD}$  {supplementary angles}
- $$\begin{aligned} &= \pi - \frac{3\pi}{5} \\ &= \frac{2\pi}{5} \end{aligned}$$
- and  $\widehat{BAC} = \widehat{OAB} - \widehat{OAC}$
- $$\begin{aligned} &= \frac{3\pi}{5} - \frac{2\pi}{5} \\ &= \frac{\pi}{5} \end{aligned}$$
- $$\therefore z_2 - z_1 = \text{cis } \frac{\pi}{5}$$





$$\text{iii} \quad \widehat{COD} = \widehat{OCD} \quad \{\text{base angles of isosceles } \triangle OCD\}$$

$$\text{Now } \widehat{COD} = \pi - \widehat{ODC} - \widehat{OCD}$$

$$\therefore \widehat{COD} = \pi - \frac{3\pi}{5} - \widehat{COD}$$

$$\therefore 2\widehat{COD} = \frac{2\pi}{5}$$

$$\therefore \widehat{COD} = \frac{\pi}{5}$$

$$\text{So, } \arg(z_3) = \frac{\pi}{5}$$

$$\begin{aligned} \text{Using the cosine rule in } \triangle OCD, \quad OC^2 &= 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos \frac{3\pi}{5} \\ &= 2 - 2 \cos \frac{3\pi}{5} \end{aligned}$$

$$\therefore OC = \sqrt{2 - 2 \cos \frac{3\pi}{5}} \quad \{OC > 0\}$$

$$\text{So, } |z_3| = \sqrt{2 - 2 \cos \frac{3\pi}{5}}$$

$$\therefore z_3 = \sqrt{2 - 2 \cos \frac{3\pi}{5}} \operatorname{cis} \frac{\pi}{5}$$

$$\text{b} \quad z_3^2 = \left( \sqrt{2 - 2 \cos \frac{3\pi}{5}} \operatorname{cis} \frac{\pi}{5} \right)^2 \quad \text{and} \quad \sqrt{3} - i = 2 \operatorname{cis} \frac{5\pi}{6} = 2e^{\frac{5\pi i}{6}}$$

$$= \left( \sqrt{2 - 2 \cos \frac{3\pi}{5}} e^{i\frac{\pi}{5}} \right)^2$$

$$= (2 - 2 \cos \frac{3\pi}{5}) e^{\frac{2\pi i}{5}}$$

$$\begin{aligned} \therefore w &= \frac{z_3^2}{k(\sqrt{3} - i)} \\ &= \frac{(2 - 2 \cos \frac{3\pi}{5}) e^{\frac{2\pi i}{5}}}{2ke^{\frac{5\pi i}{6}}} \\ &= \frac{1 - \cos \frac{3\pi}{5}}{k} e^{-\frac{13\pi i}{30}} \end{aligned}$$

Now successive powers of  $w$  lie on a circle if  $|w| = 1$

$$\therefore \frac{1 - \cos \frac{3\pi}{5}}{k} = 1$$

$$\therefore k = 1 - \cos \frac{3\pi}{5}$$

9 a Let  $\lambda$  be the true injury rate.

The hypotheses to be considered are:

$$H_0: \lambda = 0.8 \quad \{\text{injury rate has stayed the same}\}$$

$$H_1: \lambda < 0.8 \quad \{\text{injury rate has improved (decreased)}\}$$

$$\text{b} \quad n \times \lambda_0 = 30 \times 0.8 = 24$$

So the null distribution is  $T \sim \text{Po}(24)$ .

c The test statistic is  $t = 20$ .

$$\begin{aligned} p\text{-value} &= P(T \leq t) \\ &= P(T \leq 20) \\ &\approx 0.243 \end{aligned}$$

Now the significance level  $\alpha = 0.1$ .

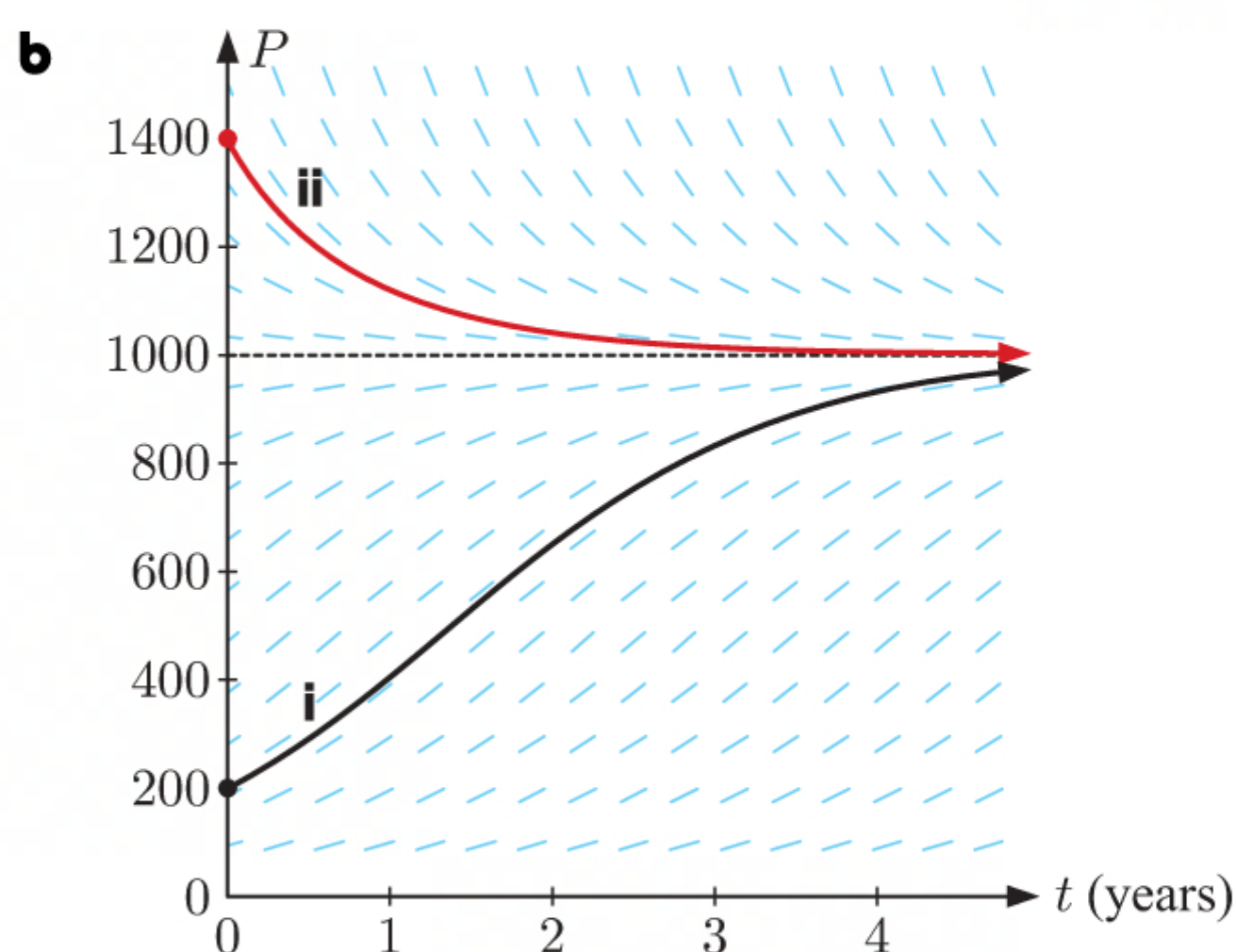
Since  $p\text{-value} > 0.1 = \alpha$ , we do not have enough evidence to reject  $H_0$  on a 10% level of significance.

So, we cannot conclude that the injury rate has improved.

10 a At the point  $(2, 400)$  on the slope field, the line segment has a *positive* gradient.

$\therefore$  the population is increasing after 2 years.



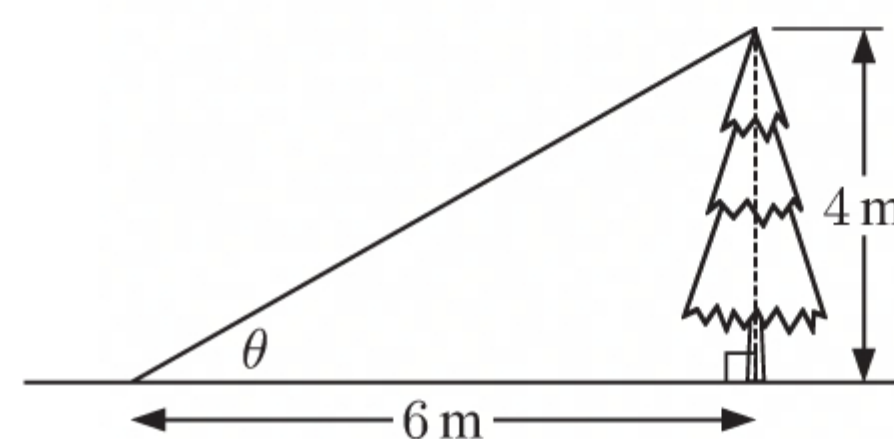


- c** If the initial population was 200, the population initially increases exponentially. The growth of the population slows until it reaches a maximum of 1000, at which it remains steady.

If the initial population was 1400, the population decreases exponentially until it reaches a minimum of 1000, at which it remains steady.

## MIXED QUESTIONS SET 14

**1 a**  $\sin \theta \approx \frac{4}{6}$   
 $\therefore \theta \approx \sin^{-1}\left(\frac{4}{6}\right)$   
 $\approx 41.8^\circ$



- b** The height of the tree could be from  $3\frac{1}{2}$  m to  $4\frac{1}{2}$  m.  
 The distance to the tree could be from  $5\frac{1}{2}$  m to  $6\frac{1}{2}$  m.  
 As  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ,  $\sin \theta$  increases.

$\therefore$  the lower boundary for  $\theta$  is  $\sin^{-1}\left(\frac{3\frac{1}{2}}{6\frac{1}{2}}\right) \approx 32.6^\circ$

and the upper boundary for  $\theta$  is  $\sin^{-1}\left(\frac{4\frac{1}{2}}{5\frac{1}{2}}\right) \approx 54.9^\circ$ .

**c** Percentage error =  $\frac{|\theta_A - \theta_E|}{\theta_E} \times 100\%$

If the exact angle  $\theta_E \approx 32.6^\circ$ , the percentage error  $\approx \frac{|41.8^\circ - 32.6^\circ|}{32.6^\circ} \times 100\%$   
 $\approx 28.2\%$

If the exact angle  $\theta_E \approx 54.9^\circ$ , the percentage error  $\approx \frac{|41.8^\circ - 54.9^\circ|}{54.9^\circ} \times 100\%$   
 $\approx 23.9\%$

$\therefore$  the maximum percentage error in our estimate in **a** is about 28.2%.

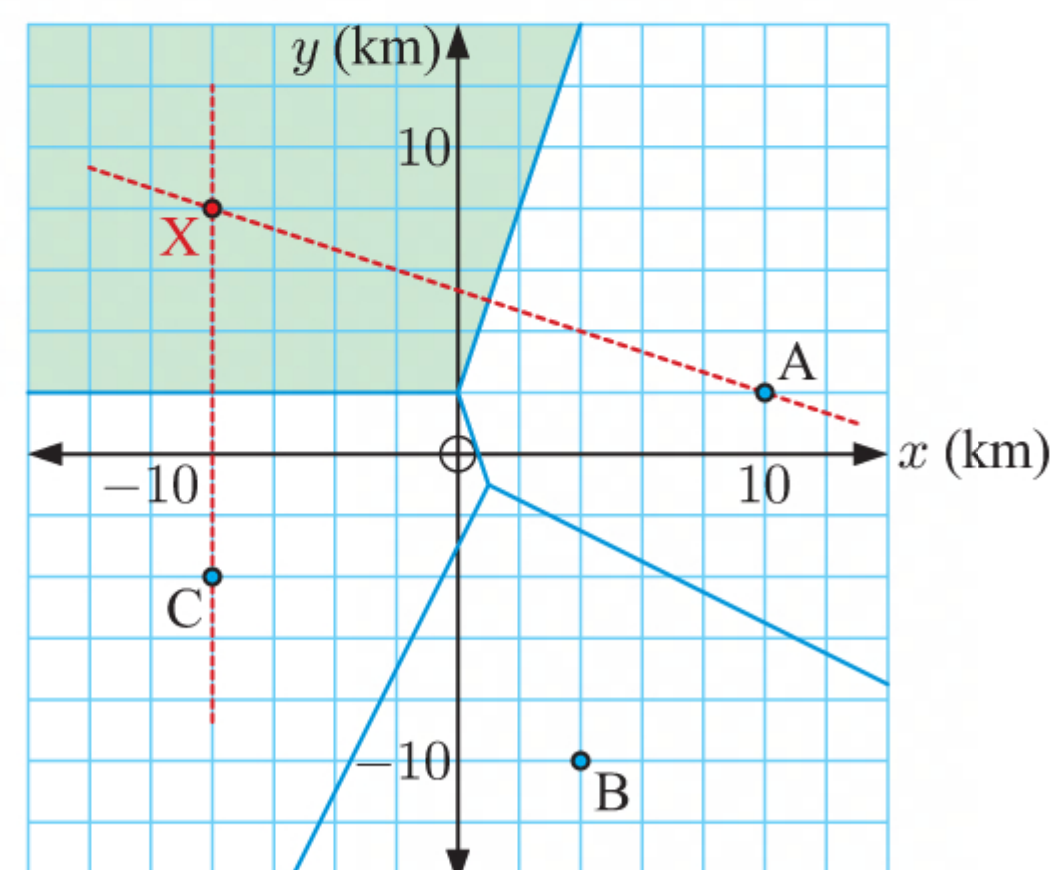
- 2 a, b** The missing hospital X must lie in the shaded cell, as this cell currently has no hospital.

The perpendicular bisector of [AX] has gradient 3, so [AX] has gradient  $-\frac{1}{3}$ .

The perpendicular bisector of [CX] is horizontal, so [CX] is vertical.

We draw (AX) and (CX) through A and C respectively. Their intersection is point X.

We observe that X has coordinates  $(-8, 8)$ .



- c i**  $(-1, 2)$  lies on the edge adjacent to cells C and X, so it is equally closest to hospitals C and X.  
**ii**  $(4, -1)$  lies in cell A, so it is closest to hospital A.



**d** Let  $H(a, b)$  be the location of the patient's house.

Since  $H$  is equally closest to hospitals  $A$  and  $X$ ,  $H$  must lie on the perpendicular bisector of  $[AX]$ .

Now the perpendicular bisector of  $[AX]$  has gradient 3 and passes through  $(0, 2)$ .

$\therefore$  its equation is  $y - 2 = 3(x - 0)$

which is  $y = 3x + 2$ .

So,  $H$  has coordinates  $(a, 3a + 2)$ .

$$\begin{aligned} \text{We have } AH &= \sqrt{(a - 10)^2 + (3a + 2 - 2)^2} \\ &= \sqrt{a^2 - 20a + 100 + 9a^2} \\ &= \sqrt{10a^2 - 20a + 100} \\ &= \sqrt{10(a^2 - 2a + 10)} \text{ km} \end{aligned}$$

Since  $H$  lies on the perpendicular bisector,  $HX = AH = \sqrt{10(a^2 - 2a + 10)}$  km.

The ambulance travels in a straight line from  $A$  to  $H$ , and from  $H$  to  $X$ , with a total distance of  $12\sqrt{5}$  km.

$$\therefore AH + HX = 12\sqrt{5}$$

$$\therefore 2\sqrt{10(a^2 - 2a + 10)} = 12\sqrt{5}$$

$$\therefore 10(a^2 - 2a + 10) = 180$$

$$\therefore a^2 - 2a + 10 = 18$$

$$\therefore a^2 - 2a - 8 = 0$$

$$\therefore (a - 4)(a + 2) = 0$$

$$\therefore a = 4 \text{ or } -2$$

If  $a = -2$ , then  $H$  is at  $(-2, 3(-2) + 2)$  which is  $(-2, -4)$ .

But  $(-2, -4)$  is closest to hospital  $C$ .

So,  $a = 4$  and  $H$  is at  $(4, 3(4) + 2)$  which is  $(4, 14)$ .

$\therefore$  the patient's house is at  $(4, 14)$ .

**3 a**  $P(\text{win}) = P(\text{1st ball red} \cap \text{2nd ball red} \cap \text{3rd ball red})$

$$\begin{aligned} &= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \\ &= \frac{1}{6} \end{aligned}$$

**b** If  $X$  is the number of wins when the game is played 60 times, then  $X \sim B(60, \frac{1}{6})$ .

$$\begin{aligned} \text{i } \mu &= np & \sigma &= \sqrt{np(1-p)} \\ &= 60(\frac{1}{6}) & &= \sqrt{60(\frac{1}{6})(\frac{5}{6})} \\ &= 10 & &= \sqrt{\frac{25}{3}} \\ & & &\approx 2.89 \end{aligned}$$

$$\begin{aligned} \text{ii } P(X = \mu) &= P(X = 10) \\ &= \binom{60}{10} (\frac{1}{6})^{10} (\frac{5}{6})^{50} \\ &\approx 0.137 \end{aligned}$$

$$\begin{aligned} \text{iii } P(\mu - \sigma \leq X \leq \mu + \sigma) &= P\left(10 - \sqrt{\frac{25}{3}} \leq X \leq 10 + \sqrt{\frac{25}{3}}\right) \\ &= P(7.11 \leq X \leq 12.9) \\ &= P(8 \leq X \leq 12) \\ &\approx 0.614 \quad \{\text{using technology}\} \end{aligned}$$

**4 a**  $f(x)$  is defined when  $x \geq 0$  and  $x \neq 3$ .

$\therefore$  the domain is  $\{x \mid x \geq 0, x \neq 3\}$ .

$$\begin{aligned} \text{b } f(x) &= \frac{\sqrt{x}}{x-3} = \frac{x^{\frac{1}{2}}}{x-3} \\ \therefore f'(x) &= \frac{\frac{1}{2}x^{-\frac{1}{2}}(x-3) - (1)x^{\frac{1}{2}}}{(x-3)^2} \quad \{\text{quotient rule}\} \\ &= \frac{x-3-2x}{2\sqrt{x}(x-3)^2} \\ &= \frac{-3-x}{2\sqrt{x}(x-3)^2} \end{aligned}$$



**c i**  $f'(x) = 0$   
 $\therefore -3 - x = 0$   
 $\therefore -x = 3$   
 $\therefore x = -3$

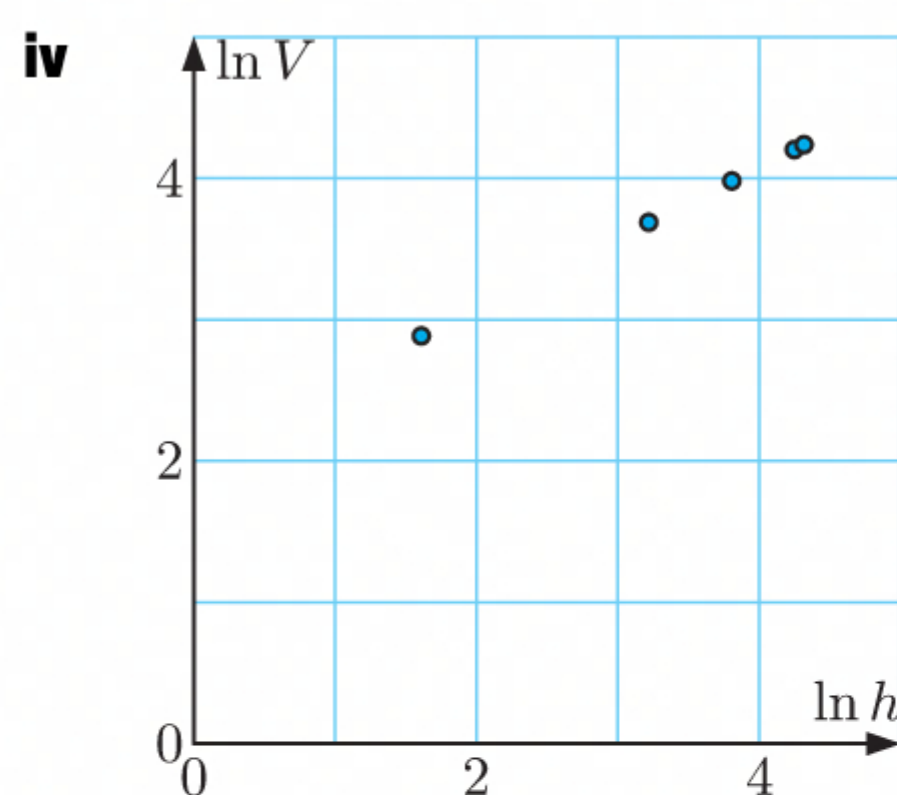
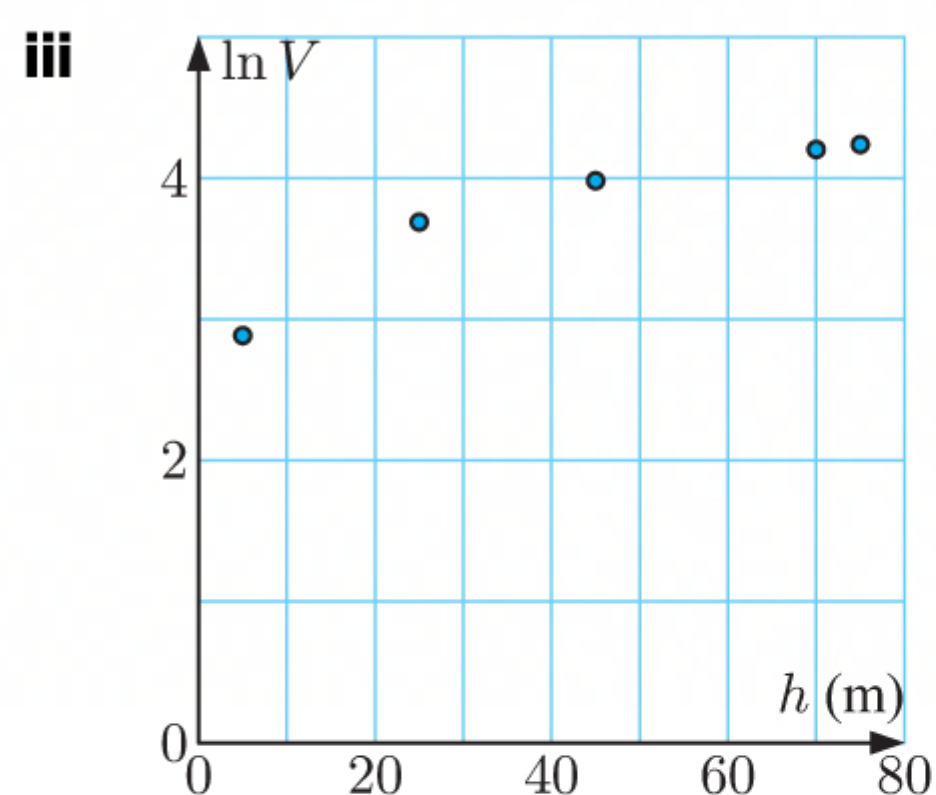
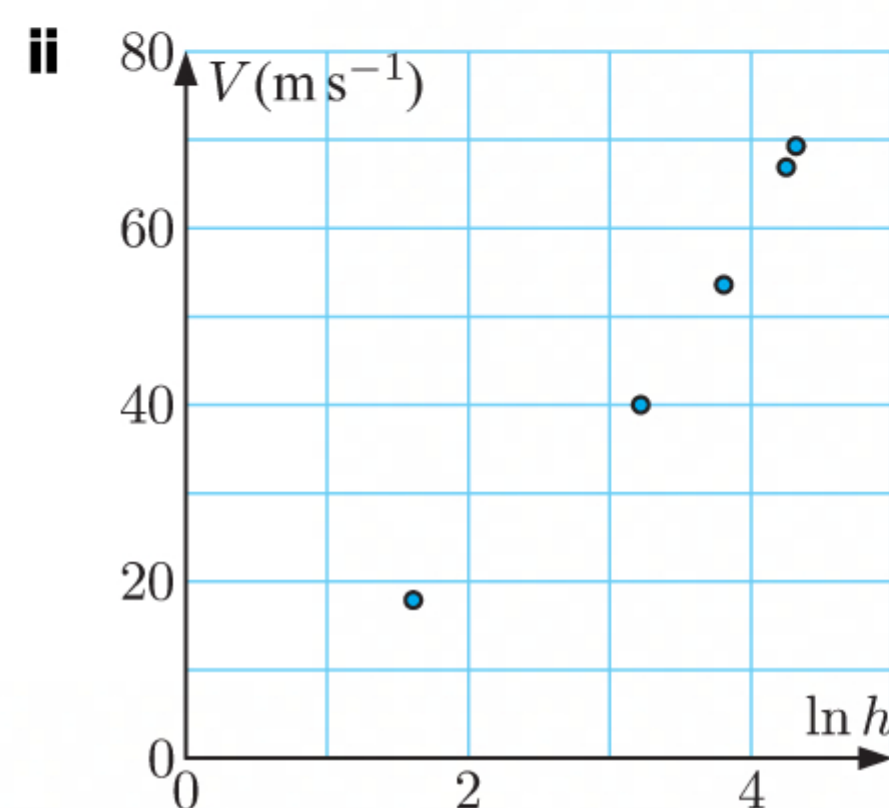
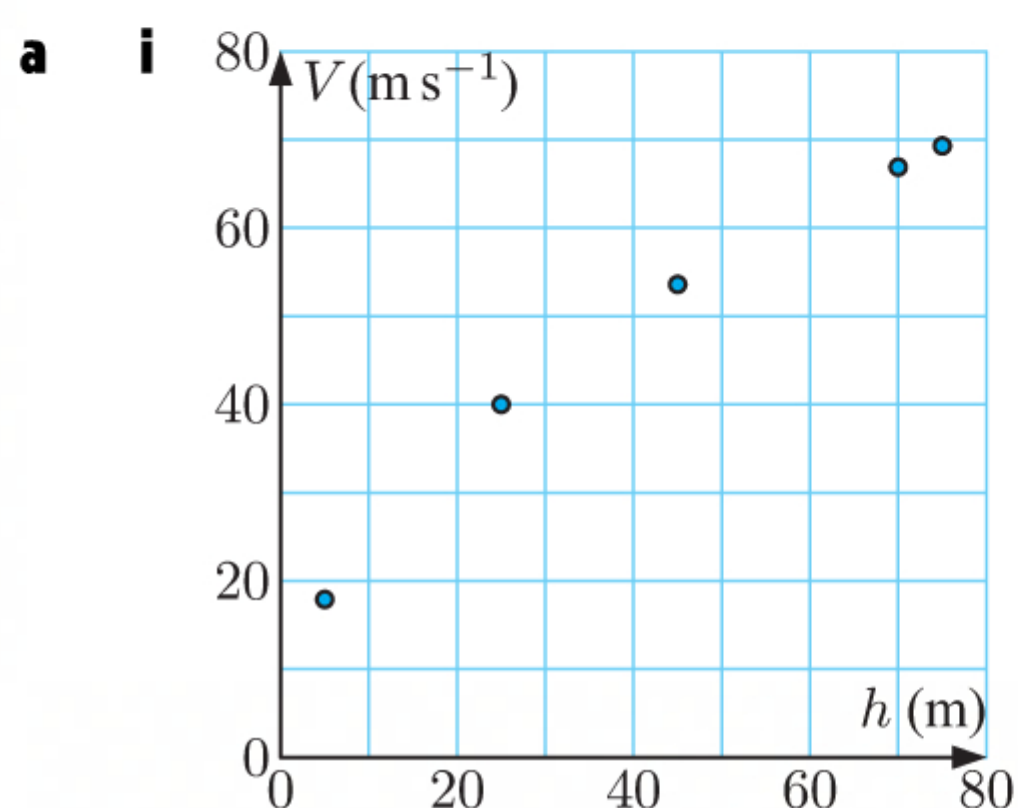
But  $f'(-3)$  is undefined, so  $f'(x)$  is never zero.

**ii**  $f'(x)$  is undefined when  $x \leq 0$ , and when  
 $2\sqrt{x}(x-3)^2 = 0$   
 $\therefore \sqrt{x} = 0$  or  $(x-3)^2 = 0$   
 $\therefore x = 0$  or  $x = 3$

That is, when  $x \leq 0$  or  $x = 3$ .

**5**

$h$	5	25	45	70	75
$V$	17.9	40	53.6	66.9	69.3
$\ln h$	1.61	3.22	3.81	4.25	4.32
$\ln V$	2.88	3.69	3.98	4.20	4.24



**b** The graph of  $\ln V$  against  $\ln h$  appears linear, so a power model is appropriate.

**c**

	List 1	List 2	List 3	List 4
SUB				
1	5	17.9	1.6094	2.8848
2	25	40	3.2188	3.6888
3	45	53.6	3.8066	3.9815
4	70	66.9	4.2484	4.2031
				2.884800713

StatGraph1  
 Graph Type : Scatter  
 XList : List3  
 YList : List4  
 Frequency : 1  
 Mark Type : ☐  
 Color Link : Off

LinearReg(ax+b)  
 a = 0.49963911  
 b = 2.08052019  
 r = 0.99999941  
 r² = 0.99999883  
 MSe = 4.7883E-07  
 y = ax + b

Using technology, the linear model connecting  $\ln V$  and  $\ln h$  is  $\ln V \approx 0.500 \ln h + 2.081$

$$\therefore V \approx e^{0.500 \ln h + 2.081}$$

$$\therefore V \approx e^{2.081} \times e^{0.500 \ln h}$$

$$\therefore V \approx 8.01 \times h^{0.500}$$

**d** When  $h = 50$  m,  $V \approx 8.01 \times 50^{0.500}$   
 $\approx 56.5 \text{ m s}^{-1}$

**e** The model may not be appropriate for heights greater than 90 m as this is outside the range of the given data. We would be extrapolating, which does not give reliable results.

**6 a**  $s_{n-1} = \sqrt{\frac{n}{n-1}} s_n$   
 $\therefore s_{n-1} = \sqrt{\frac{15}{14}} \times 3.5$   
 $\approx 3.62$

**b**

	1-Sample tInterval
Lower	= 46.1524452
Upper	= 49.4475548
$\bar{x}$	= 47.8
sx	= 3.62284419
n	= 15

Using technology, a 90% confidence interval for  $\mu$  is  $46.2 \leq \mu \leq 49.4$ .



**c** 45 minutes is below the confidence interval.

So, the company's claim does not appear to be justified.

$$\begin{aligned} 7 \quad \mathbf{a} \quad (g \circ f)(x) &= g(f(x)) \\ &= g(25 - x^2) \\ &= \frac{2}{\sqrt{25 - x^2}} \end{aligned}$$

The domain is  $\{x \mid -5 < x < 5\}$ .

$$\begin{aligned} \mathbf{b} \quad (g \circ f)(x) &= 1 \\ \therefore \frac{2}{\sqrt{25 - x^2}} &= 1 \quad \{\text{using } \mathbf{a}\} \\ \therefore 2 &= \sqrt{25 - x^2} \\ \therefore 4 &= 25 - x^2 \\ \therefore x^2 &= 21 \\ \therefore x &= \pm\sqrt{21} \end{aligned}$$

**c**  $\sqrt{25 - x^2} = 0$  when  $x = \pm 5$ , so  $x = -5$  and  $x = 5$  are the vertical asymptotes.

**8** Let the distance travelled by the ski jumper for each jump be the independent random variables  $X_1$ ,  $X_2$ , and  $X_3$ .

Let the combined distance be  $Y_3 = \sum_{k=1}^3 X_k$ , where  $X_k \sim N(170, 15^2)$ ,  $k = 1, 2, 3$ .

$$\begin{aligned} \text{Now } E(Y_3) &= \sum_{k=1}^3 E(X_k) \quad \text{and} \quad \text{Var}(Y_3) = \sum_{k=1}^3 \text{Var}(X_k) \\ &= 3 \times 170 &= 3 \times 15^2 \\ &= 510 \text{ m} &= 675 \text{ m}^2 \end{aligned}$$

$$\therefore Y_3 \sim N(510, 675)$$

$$\text{Now } P(Y_3 \geq 500) \approx 0.650$$

$\therefore$  there is a 65.0% chance that the ski jumper will travel a combined distance of at least 500 m.

**9 a** Gemma's drone has speed  $0.9 \text{ m s}^{-1}$ .

$$\begin{aligned} \therefore |0.1\mathbf{i} + 0.8\mathbf{j} + a\mathbf{k}| &= 0.9 \\ \therefore \sqrt{(0.1)^2 + (0.8)^2 + a^2} &= 0.9 \\ \therefore 0.01 + 0.64 + a^2 &= 0.81 \\ \therefore a^2 &= 0.16 \\ \therefore a &= 0.4 \quad \{a > 0\} \end{aligned}$$

**b** Let  $\theta$  be the acute angle between the paths of the two drones.

Nassa's drone has direction  $\mathbf{b}_1 = 0.5\mathbf{i} - 0.2\mathbf{j} + 0.6\mathbf{k}$ .

Gemma's drone has direction  $\mathbf{b}_2 = 0.1\mathbf{i} + 0.8\mathbf{j} + 0.4\mathbf{k}$ .

$$\begin{aligned} \therefore \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{|0.5 \times 0.1 + (-0.2) \times 0.8 + 0.6 \times 0.4|}{\sqrt{(0.5)^2 + (-0.2)^2 + (0.6)^2} \sqrt{(0.1)^2 + (0.8)^2 + (0.4)^2}} \\ &\approx 0.1792 \\ \therefore \theta &\approx 79.7^\circ \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \text{When the drones take off, the distance between them is } &\sqrt{(15 - 9)^2 + (8 - 20)^2 + (0 - 0)^2} = \sqrt{6^2 + (-12)^2} \\ &= \sqrt{180} \\ &= 6\sqrt{5} \approx 13.4 \text{ m} \end{aligned}$$

$$\mathbf{d} \quad \mathbf{i} \quad \text{Nassa's drone has vector equation } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 20 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.5 \\ -0.2 \\ 0.6 \end{pmatrix} t.$$

$$\text{Gemma's drone has vector equation } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 8 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.1 \\ 0.8 \\ 0.4 \end{pmatrix} t.$$

At time  $t$ , Nassa's drone is at  $(9 + 0.5t, 20 - 0.2t, 0.6t)$ , and Gemma's drone is at  $(15 + 0.1t, 8 + 0.8t, 0.4t)$ .

The distance between them is

$$\begin{aligned} D &= \sqrt{[(9 + 0.5t) - (15 + 0.1t)]^2 + [(20 - 0.2t) - (8 + 0.8t)]^2 + [0.6t - 0.4t]^2} \\ &= \sqrt{(-6 + 0.4t)^2 + (12 - t)^2 + (0.2t)^2} \\ &= \sqrt{36 - 4.8t + 0.16t^2 + 144 - 24t + t^2 + 0.04t^2} \\ &= \sqrt{1.2t^2 - 28.8t + 180} \text{ m} \end{aligned}$$

Now  $D$  is minimised when  $D^2 = 1.2t^2 - 28.8t + 180$  is minimised.

$$\text{This occurs when } t = -\frac{-28.8}{2 \times 1.2} = 12.$$

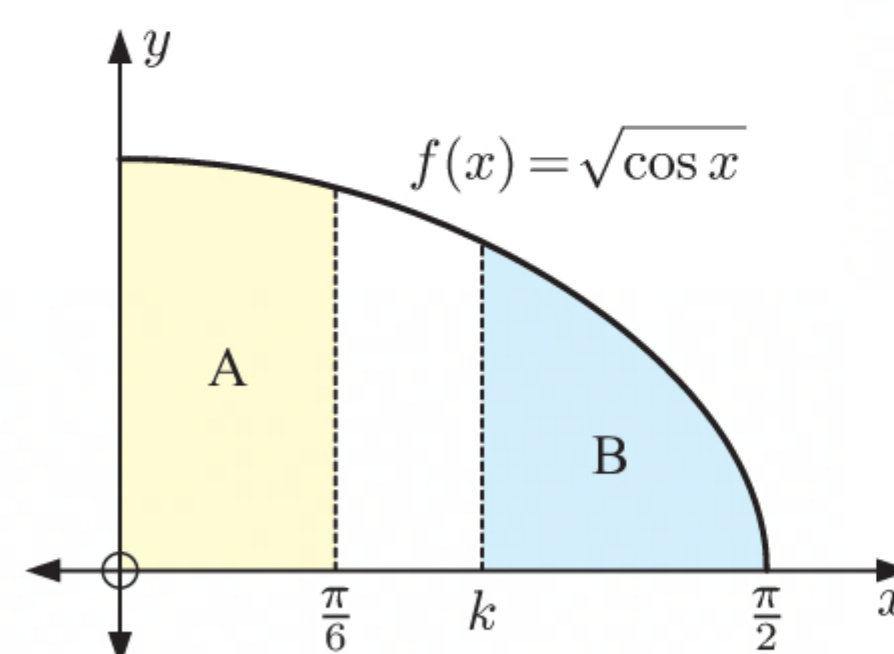
So, the drones are closest to each other after 12 seconds.



$$\begin{aligned}
 \text{ii When } t = 12, \quad D &= \sqrt{1.2(12)^2 - 28.8(12) + 180} \\
 &= \sqrt{7.2} \\
 &\approx 2.68
 \end{aligned}$$

$\therefore$  to the nearest cm, the shortest distance between the drones is about 2.68 m.

$$\begin{aligned}
 \text{10 a For region A, volume} &= \pi \int_0^{\frac{\pi}{6}} [f(x)]^2 dx \\
 &= \pi \int_0^{\frac{\pi}{6}} \cos x dx \\
 &= \pi \left[ \sin x \right]_0^{\frac{\pi}{6}} \\
 &= \pi \left( \sin \frac{\pi}{6} - \sin 0 \right) \\
 &= \frac{\pi}{2} \text{ units}^3
 \end{aligned}$$



$$\begin{aligned}
 \text{b For region B, volume} &= \pi \int_k^{\frac{\pi}{2}} [f(x)]^2 dx \\
 \therefore \frac{1}{2} \left( \frac{\pi}{2} \right) &= \pi \int_k^{\frac{\pi}{2}} \cos x dx \\
 \therefore \frac{\pi}{4} &= \pi \left[ \sin x \right]_k^{\frac{\pi}{2}} \\
 \therefore \frac{1}{4} &= \sin \frac{\pi}{2} - \sin k \\
 \therefore \sin k &= 1 - \frac{1}{4} = \frac{3}{4} \\
 \therefore k &= \sin^{-1} \left( \frac{3}{4} \right) \\
 &\approx 0.848
 \end{aligned}$$

## MIXED QUESTIONS SET 15

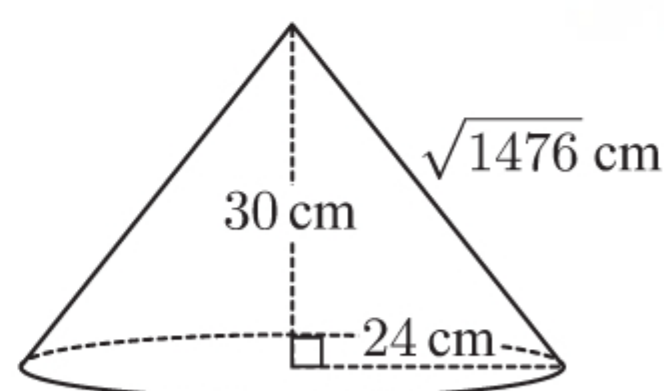
$$\text{1 a } r = \frac{0.25}{0.125} = 2$$

$$\begin{aligned}
 \text{b Using } u_n &= u_1 \times r^{n-1}, \\
 u_{20} &= 0.125 \times 2^{19} \\
 &= 65\,536
 \end{aligned}$$

$$\begin{aligned}
 \text{c Using } S_n &= \frac{u_1(r^n - 1)}{r - 1}, \\
 S_{10} &= \frac{0.125(2^{10} - 1)}{2 - 1} \\
 &= 127.875
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad & \begin{array}{c} \text{Diagram: A right-angled triangle with a vertical side of 30 cm, a horizontal side of 24 cm, and a hypotenuse of length } s. \end{array} \\
 & s^2 = 24^2 + 30^2 \quad \{\text{Pythagoras}\} \\
 & \therefore s^2 = 1476 \\
 & \therefore s = \sqrt{1476} \quad \{s > 0\} \\
 & \therefore s \approx 38.4 \text{ cm}
 \end{aligned}$$

$$\begin{aligned}
 \text{b Surface area} &= \pi r s + \pi r^2 \\
 &= \pi(24)\sqrt{1476} + \pi(24)^2 \\
 &\approx 4710 \text{ cm}^2 \\
 &\approx 4.71 \times 10^3 \text{ cm}^2
 \end{aligned}$$



Number of weeds	Frequency
0 - 4	9
5 - 9	15
10 - 14	10
15 - 19	$p$
20 - 24	5
25 - 29	2
Total	50

$$\begin{aligned}
 \text{Total number of sample spots} &= 50 \\
 \therefore 9 + 15 + 10 + p + 5 + 2 &= 50 \\
 \therefore p + 41 &= 50 \\
 \therefore p &= 9
 \end{aligned}$$



<b>b</b>	Number of weeds	Midpoint ( $x$ )	Frequency ( $f$ )	$xf$
	0 - 4	2	9	18
	5 - 9	7	15	105
	10 - 14	12	10	120
	15 - 19	17	9	153
	20 - 24	22	5	110
	25 - 29	27	2	54
	Total		$\sum f = 50$	$\sum xf = 560$

$$\begin{aligned}\bar{x} &= \frac{\sum xf}{\sum f} \\ &= \frac{560}{50} \\ &= 11.2\end{aligned}$$

We estimate the mean number of weeds per spot to be 11.2.

$$\begin{aligned}\text{c Percentage fewer than 10 weeds} &= \frac{9 + 15}{50} \times 100\% \\ &= \frac{24}{50} \times 100\% \\ &= 48\%\end{aligned}$$

$$4 \quad \text{a } N = 3 \times 12 = 36, \quad I\% = 8.5, \quad PV = 7000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$

$$\therefore PMT \approx -220.98$$

The repayments are £220.98 per month.

$$\begin{aligned}\text{b Interest} &= \text{total repayment} - \text{amount borrowed} \\ &= £220.98 \times 36 - £7000 \\ &= £955.28\end{aligned}$$

$$\begin{aligned}5 \quad \text{a } E(X - Y) &= E(X) - E(Y) & \text{Var}(X - Y) &= \text{Var}(X) + (-1)^2 \text{Var}(Y) \quad \{\text{independence}\} \\ &= 140 - 136 & &= 5^2 + 4^2 \\ &= 4 \text{ km h}^{-1} & &= 41\end{aligned}$$

$$\therefore \sigma(X - Y) = \sqrt{41} \approx 6.40 \text{ km h}^{-1}$$

So,  $X - Y$  is normally distributed with mean  $4 \text{ km h}^{-1}$  and standard deviation  $\sqrt{41} \approx 6.40 \text{ km h}^{-1}$ .

$$\begin{aligned}\text{b i } P(\text{Jerry's pitch is faster than } 143 \text{ km h}^{-1}) &= P(X > 143) \\ &\approx 0.274\end{aligned}$$

$$\begin{aligned}\text{ii } P(\text{Eric's pitch is faster than Jerry's}) &= P(Y > X) \\ &= P(X - Y < 0) \\ &\approx 0.266\end{aligned}$$

Norm1		End
Compound Interest		
n	=36	
I%	=8.5	
PV	=7000	
PMT	=-220.972762	
FV	=0	
P/Y	=12	
n	I%	PV
PMT	FV	AMORTZN

Math/Des/Normal		d/c/Real
NormCD(143, 9, 5, 1)		
0.2742531178		
Npd   Ncd   InvN		

Math/Des/Normal		d/c/Real
NormCD(-9, 0, 41, 1)		
0.2660856121		
Npd   Ncd   InvN		

$$6 \quad \text{a The concentration was initially 5 units.}$$

$$\therefore \text{when } t = 0, \quad C = 5$$

$$\therefore 5 = \frac{L}{1 + 9}$$

$$\therefore L = 50$$

The limiting concentration of the substance is 50 units.

$$\text{b The concentration was 20 units after 6 seconds.}$$

$$\therefore \text{when } t = 6, \quad C = 20$$

$$\therefore 20 = \frac{50}{1 + 9e^{-6k}}$$

$$\therefore 1 + 9e^{-6k} = \frac{5}{2}$$

$$\therefore 9e^{-6k} = \frac{3}{2}$$

$$\therefore e^{-6k} = \frac{1}{6}$$

$$\therefore e^{-k} = 6^{-\frac{1}{6}} \quad \dots (*)$$

$$\therefore -k = -\frac{1}{6} \ln 6$$

$$\therefore k = \frac{1}{6} \ln 6 \approx 0.299$$

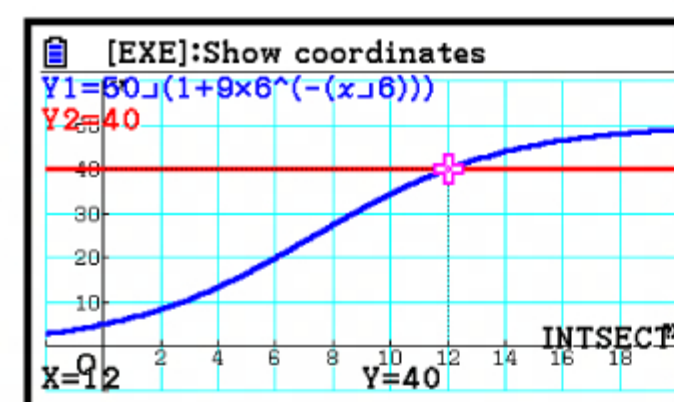
$$\text{c Using } (*), \quad C = \frac{50}{1 + 9 \times 6^{-\frac{t}{6}}}$$

$$\text{When } t = 10, \quad C = \frac{50}{1 + 9 \times 6^{-\frac{10}{6}}} \approx 34.4$$

$\therefore$  after 10 seconds, the concentration was about 34.4 units.



- d** The concentration was 40 units when  $40 = \frac{50}{1 + 9 \times 6^{-\frac{t}{6}}}$   
 $\therefore t = 12$  seconds



- 7 a** A vertical stretch with scale factor  $\frac{3}{5}$  has the transformation matrix  $\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{pmatrix}$ .

For a clockwise rotation through  $\frac{\pi}{4}$  about O, we have  $\theta = -\frac{\pi}{4}$ ,  $\sin \theta = -\frac{1}{\sqrt{2}}$ ,  $\cos \theta = \frac{1}{\sqrt{2}}$ .

$\therefore$  the rotation has transformation matrix  $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ .

$$\text{So, } \mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

**b**  $\mathbf{x}' = \mathbf{A} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{7}{5\sqrt{2}} \\ -\frac{13}{5\sqrt{2}} \end{pmatrix}$

$\therefore (2, -1)$  has image  $\left(\frac{7}{5\sqrt{2}}, -\frac{13}{5\sqrt{2}}\right)$ .

- c**  $(1, k)$  has image  $(2\sqrt{2}, \sqrt{2})$

$$\therefore \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{3k}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} + \frac{3k}{5\sqrt{2}} \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \left(1 + \frac{3k}{5}\right) \\ \frac{1}{\sqrt{2}} \left(-1 + \frac{3k}{5}\right) \end{pmatrix}$$

$$\therefore 2\sqrt{2} = \frac{1}{\sqrt{2}} \left(1 + \frac{3k}{5}\right) \quad \dots (1)$$

$$\text{and } \sqrt{2} = \frac{1}{\sqrt{2}} \left(-1 + \frac{3k}{5}\right) \quad \dots (2)$$

$$\text{From (1), } 2\sqrt{2} = \frac{1}{\sqrt{2}} \left(1 + \frac{3k}{5}\right)$$

$$\therefore 4 = 1 + \frac{3k}{5}$$

$$\therefore \frac{3k}{5} = 3$$

$$\therefore k = 5$$

$$\text{Check: in (2), } \frac{1}{\sqrt{2}} \left(-1 + \frac{3(5)}{5}\right) = \frac{2}{\sqrt{2}} = \sqrt{2} \quad \checkmark$$

**8 a**  $y = 20 \left(\frac{x}{10} - 1\right)^3 + 10$

$$\therefore y - 10 = 20 \left(\frac{x}{10} - 1\right)^3$$

$$\therefore \frac{y - 10}{20} = \left(\frac{x}{10} - 1\right)^3$$

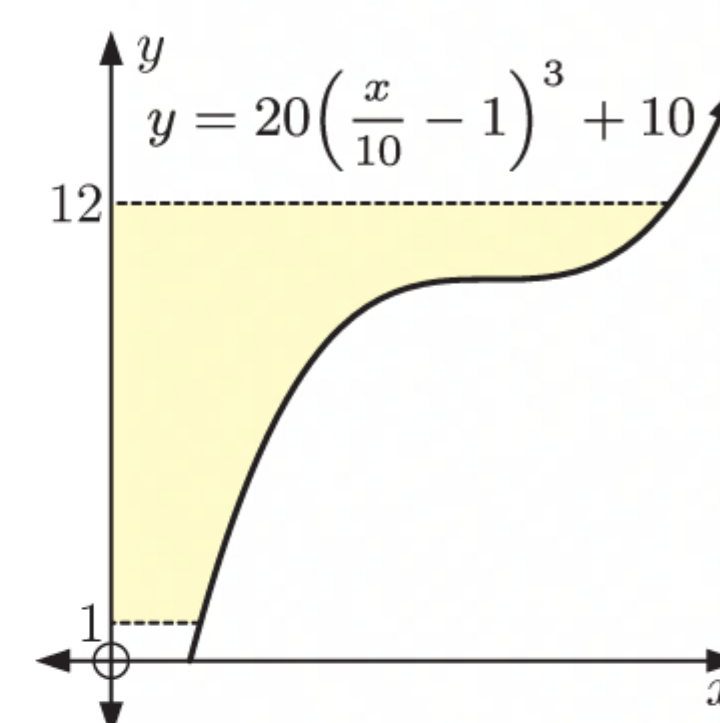
$$\therefore \frac{x}{10} - 1 = \left(\frac{y - 10}{20}\right)^{\frac{1}{3}}$$

$$\therefore \frac{x}{10} = \left(\frac{y - 10}{20}\right)^{\frac{1}{3}} + 1$$

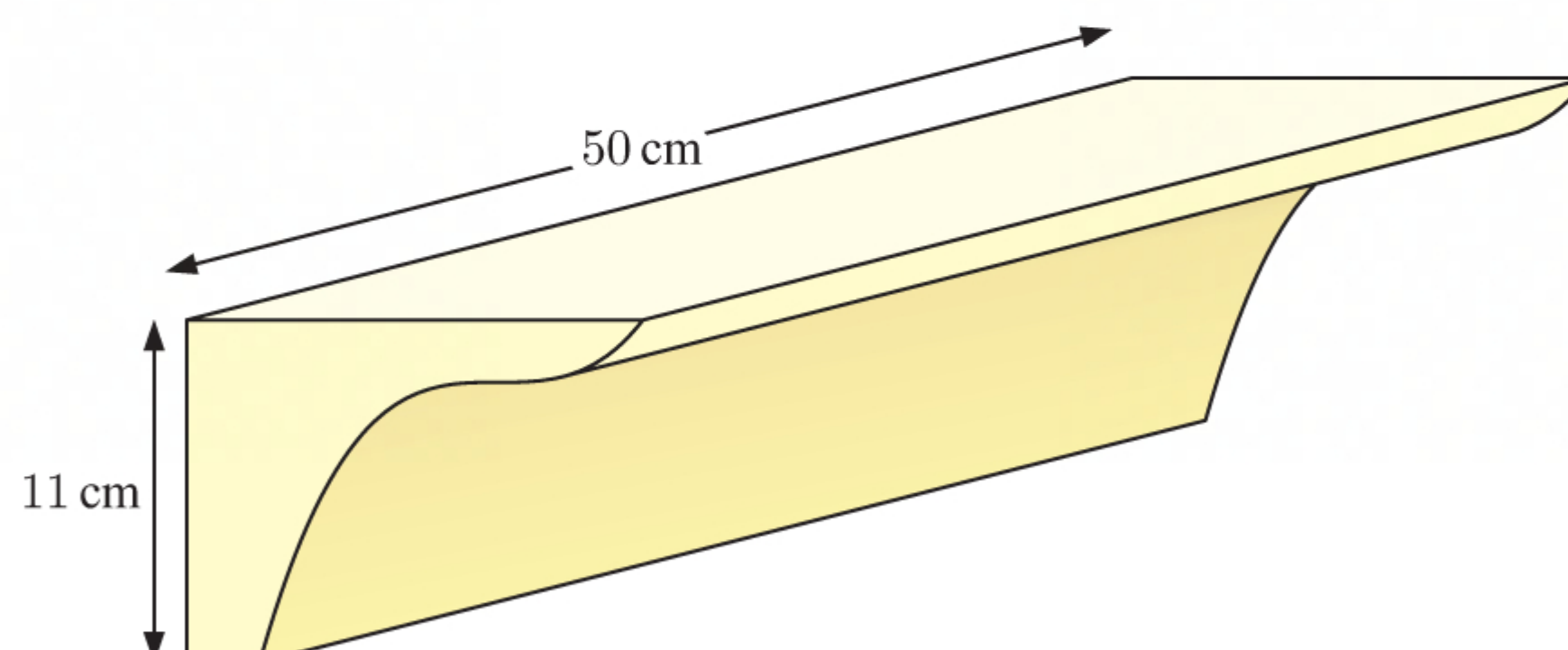
$$\therefore x = 10 \left(\frac{1}{20}y - \frac{1}{2}\right)^{\frac{1}{3}} + 10$$



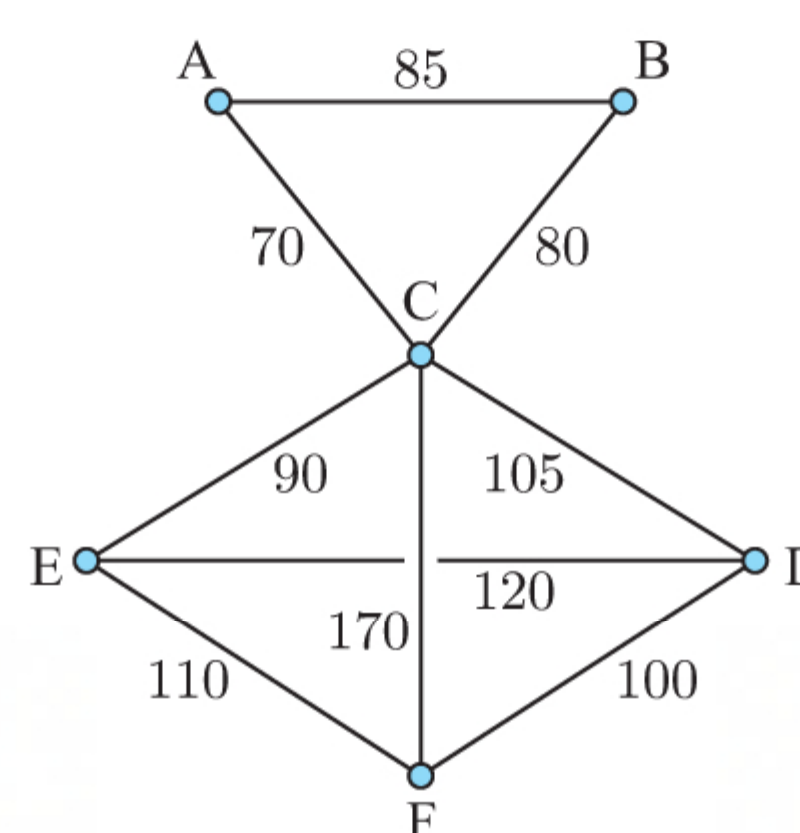
$$\begin{aligned}
 \text{b Area of cross-section} &= \int_1^{12} \left( 10 \left( \frac{1}{20}y - \frac{1}{2} \right)^{\frac{1}{3}} + 10 \right) dy \\
 &= \left[ 10 \times \frac{3}{4} \left( \frac{1}{20}y - \frac{1}{2} \right)^{\frac{4}{3}} \times 20 + 10y \right]_1^{12} \\
 &= \left[ 150 \left( \frac{1}{20}y - \frac{1}{2} \right)^{\frac{4}{3}} + 10y \right]_1^{12} \\
 &= 150 \left( \frac{3}{5} - \frac{1}{2} \right)^{\frac{4}{3}} + 120 - \left( 150 \left( \frac{1}{20} - \frac{1}{2} \right)^{\frac{4}{3}} + 10 \right) \\
 &\approx 65.2 \text{ cm}^2
 \end{aligned}$$



$$\begin{aligned}
 \text{c Volume of the shelf} &\approx 65.2 \times 50 \\
 &\approx 3260 \text{ cm}^3
 \end{aligned}$$



$$\text{9 a } A = \begin{matrix} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$



$$\text{b Using technology, } A^4 = \begin{pmatrix} 9 & 8 & 14 & 12 & 12 & 12 \\ 8 & 9 & 14 & 12 & 12 & 12 \\ 14 & 14 & 39 & 26 & 26 & 26 \\ 12 & 12 & 26 & 23 & 22 & 22 \\ 12 & 12 & 26 & 22 & 23 & 22 \\ 12 & 12 & 26 & 22 & 22 & 23 \end{pmatrix}$$

The value in row 1, column 4 indicates that there are 12 ways to travel from A to D in exactly 4 trips.

c Vertices C, D, E, and F have odd degree.

$\therefore$  the graph is not Eulerian.

d There are 3 possible pairings of the vertices of odd degree: CD and EF, CE and DF, CF and DE.

Pairing	Shortest path	Distance	Total distance
CD	$C \rightarrow D$	105	215
EF	$E \rightarrow F$	110	
CE	$C \rightarrow E$	90	190
DF	$D \rightarrow F$	100	
CF	$C \rightarrow F$	170	290
DE	$D \rightarrow E$	120	

The combination with the lowest total distance is CE and DF. The shortest route traverses routes  $C \rightarrow E$  and  $D \rightarrow F$  twice each.

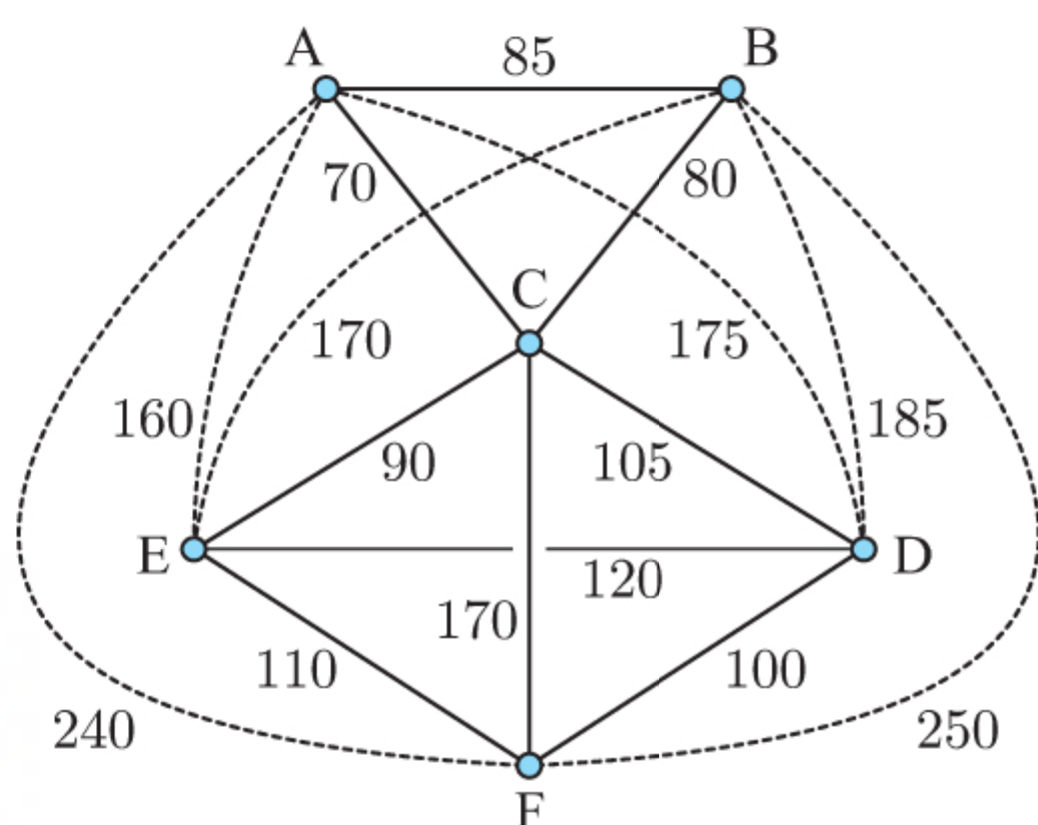
A possible route is  $A \rightarrow B \rightarrow C \rightarrow E \rightarrow C \rightarrow D \rightarrow F \rightarrow D \rightarrow E \rightarrow F \rightarrow C \rightarrow A$  with total distance 1120 km.

e Any path which passes through each vertex must pass through C at least twice.

So, the graph does not contain a Hamiltonian cycle.

$\therefore$  the graph is not Hamiltonian.



**f**

- g i** From B, the nearest vertex is C, so we choose edge BC. We then choose CA, AE, EF, then FD.

All vertices have now been visited, so we add DB to complete the Hamiltonian cycle  $B \rightarrow C \rightarrow A \rightarrow E \rightarrow F \rightarrow D \rightarrow B$ .

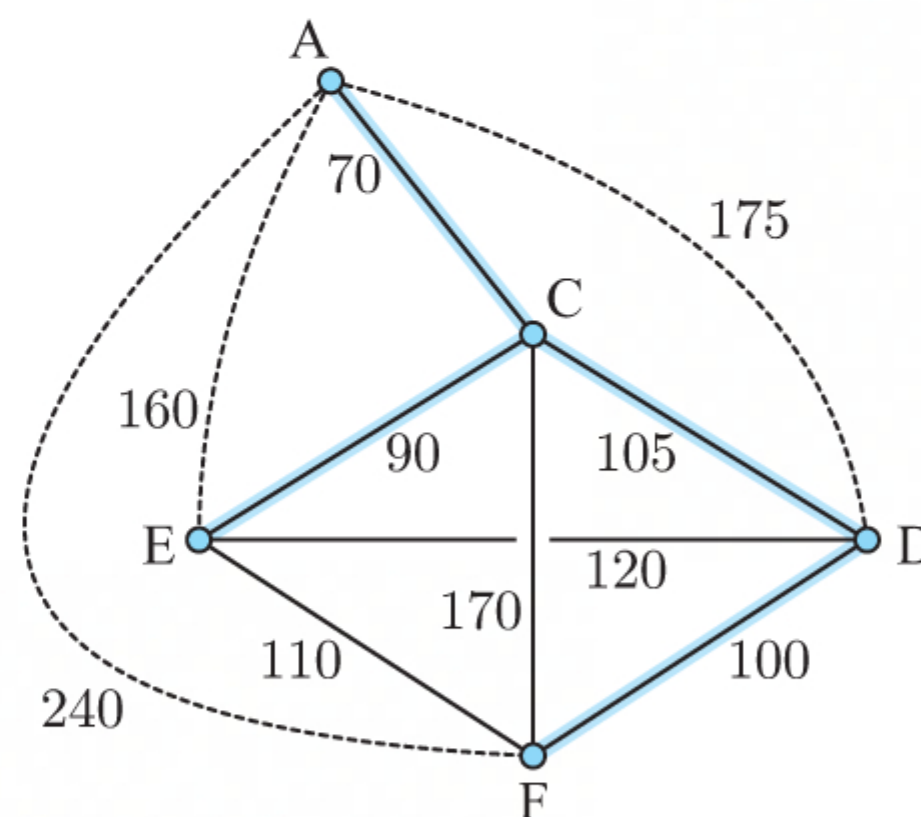
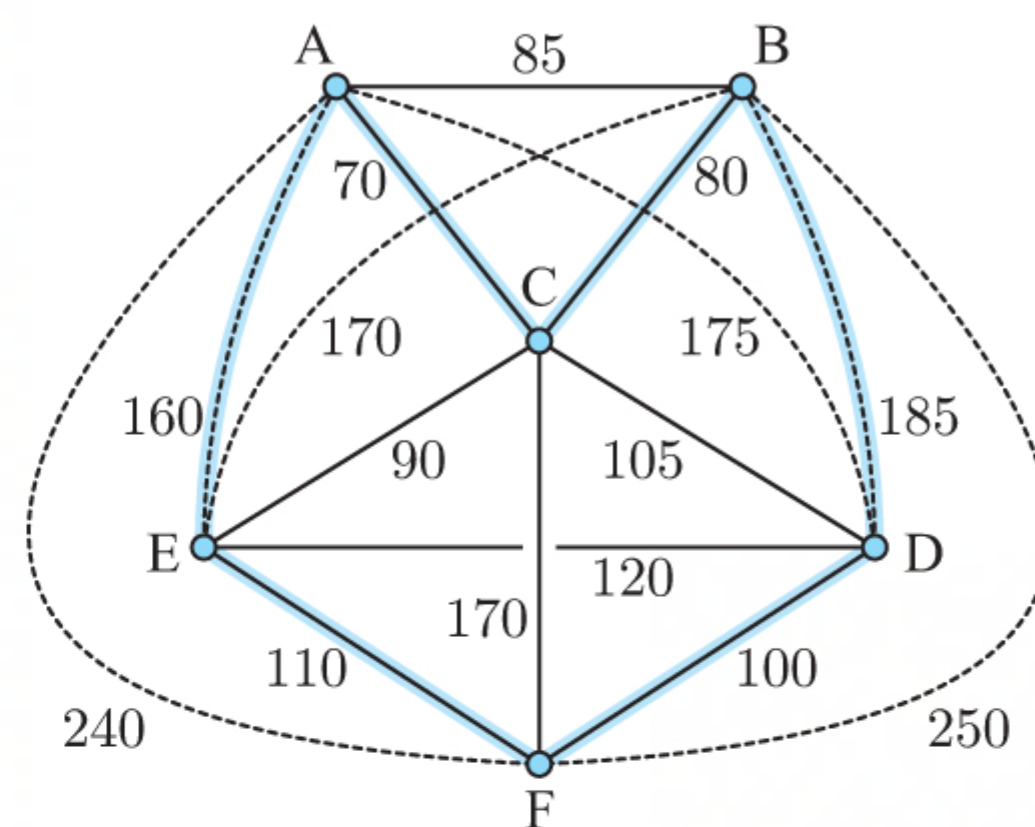
So, the upper bound for the total distance of the trip is  $80 + 70 + 160 + 110 + 100 + 185 = 705$  km.

- ii** We find a minimum spanning tree with vertex B and all edges connected to it deleted.

The minimum spanning tree has weight  $70 + 90 + 100 + 105 = 365$ .

The two shortest deleted edges have weight 80 and 85.

So, the lower bound for the total distance of the trip is  $365 + 80 + 85 = 530$  km.



- 10 a** We are given that

$$\begin{aligned}\frac{dV}{dt} &\propto \sqrt{V} \\ \therefore \frac{dV}{dt} &= -k\sqrt{V} \quad \text{where } k \text{ is a positive constant} \\ \therefore \frac{1}{\sqrt{V}} \frac{dV}{dt} &= -k \\ \therefore \int \frac{1}{\sqrt{V}} \frac{dV}{dt} dt &= \int -k dt \\ \therefore \int V^{-\frac{1}{2}} dV &= \int -k dt \\ \therefore 2V^{\frac{1}{2}} &= -kt + c \\ \therefore V &= \left(\frac{c-kt}{2}\right)^2\end{aligned}$$

Now when  $t = 0$ ,  $V = 100$

$$\begin{aligned}\therefore 100 &= \left(\frac{c}{2}\right)^2 \\ \therefore \frac{c}{2} &= 10 \\ \therefore c &= 20 \\ \therefore V &= \left(\frac{20-kt}{2}\right)^2\end{aligned}$$

- b** When  $t = 4$ ,  $V = 100 - 19 = 81$

$$\begin{aligned}\therefore 81 &= \left(\frac{20-4k}{2}\right)^2 \\ \therefore 9 &= 10 - 2k \\ \therefore 2k &= 1 \\ \therefore k &= \frac{1}{2}\end{aligned}$$

$$\text{So, } V = \left(\frac{20 - \frac{1}{2}t}{2}\right)^2 = \left(\frac{40-t}{4}\right)^2$$

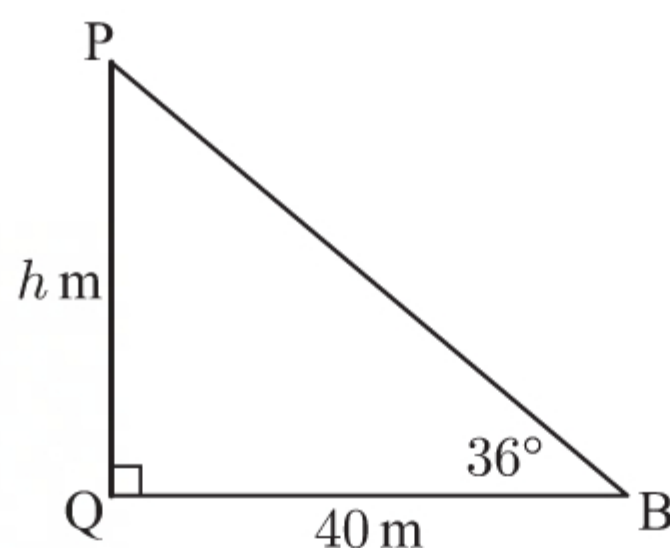
$$\begin{aligned}\text{Now } V = 0 \text{ when } \left(\frac{40-t}{4}\right)^2 &= 0 \\ \therefore 40 - t &= 0 \\ \therefore t &= 40\end{aligned}$$

$\therefore$  the vessel will be empty after 40 hours.



## MIXED QUESTIONS SET 16

1 a

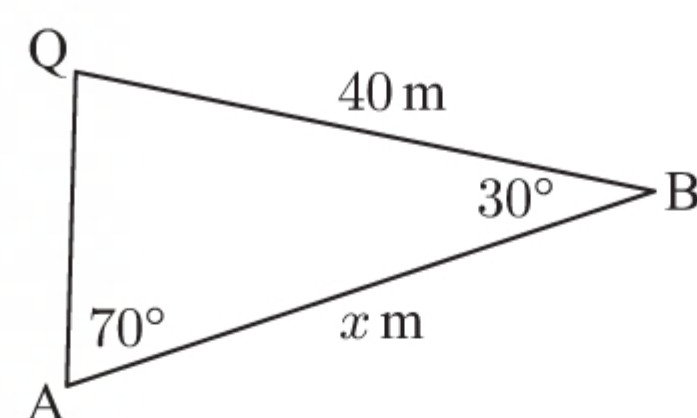

 Let PQ be  $h$  m.

$$\therefore \tan 36^\circ = \frac{h}{40}$$

$$\therefore h = 40 \tan 36^\circ \approx 29.1$$

The height of the pole is about 29.1 m.

b


 Let AB be  $x$  m.

$$\begin{aligned} \angle AQB &= 180^\circ - 70^\circ - 30^\circ \quad \{\text{angles in a triangle}\} \\ &= 80^\circ \end{aligned}$$

$$\therefore \frac{x}{\sin 80^\circ} = \frac{40}{\sin 70^\circ} \quad \{\text{sine rule}\}$$

$$\therefore x = \frac{40 \sin 80^\circ}{\sin 70^\circ}$$

$$\therefore x \approx 41.9$$

The distance between A and B is about 41.9 m.

 2 a Let  $\mu$  be the population mean flight time of Robin's arrows with the longer bow.

The hypotheses to be tested are:

$$H_0: \mu = 1.523 \quad \{\text{the mean flight time is 1.523 seconds}\}$$

$$H_1: \mu < 1.523 \quad \{\text{the mean flight time is less than 1.523 seconds}\}$$

 b i  $\bar{x} = 1.5094$  seconds,  $s \approx 0.0203$  seconds,  $n = 10$ 

$$\text{The value of the test statistic is } t \approx \frac{1.5094 - 1.523}{\frac{0.0203}{\sqrt{10}}} \approx -2.12$$

$$\begin{aligned} \text{ii Since } H_1: \mu < 1.523, \quad p\text{-value} &\approx P(T < -2.12) \quad \text{where } T \sim t_9 \\ &\approx 0.0315 \quad \{\text{using technology}\} \end{aligned}$$

 c Since  $p\text{-value} > 0.01 = \alpha$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  at the 1% significance level. We therefore accept  $H_0$ .

 Since we have accepted  $H_0$ , we cannot conclude that the longer bow has decreased the flight time of Robin's arrows.

 3 a The bin has capacity 500 litres  $\equiv 500\,000 \text{ cm}^3$ 

$$\therefore \pi r^2 h = 500\,000$$

$$\therefore h = \frac{500\,000}{\pi r^2}$$

 b Surface area  $A = 2\pi r h + \pi r^2$ 

$$= 2\pi r \left( \frac{500\,000}{\pi r^2} \right) + \pi r^2$$

$$= 1\,000\,000 r^{-1} + \pi r^2$$

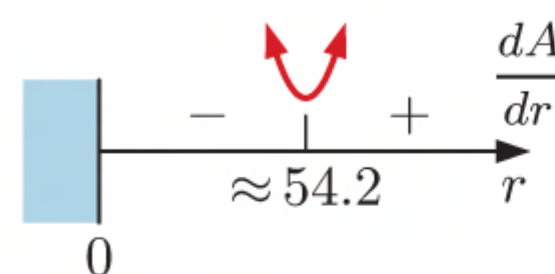
$$\text{c } \frac{dA}{dr} = -\frac{1\,000\,000}{r^2} + 2\pi r$$

$$\text{Now } \frac{dA}{dr} = 0 \quad \text{when} \quad 2\pi r = \frac{1\,000\,000}{r^2}$$

$$\therefore 2\pi r^3 = 1\,000\,000$$

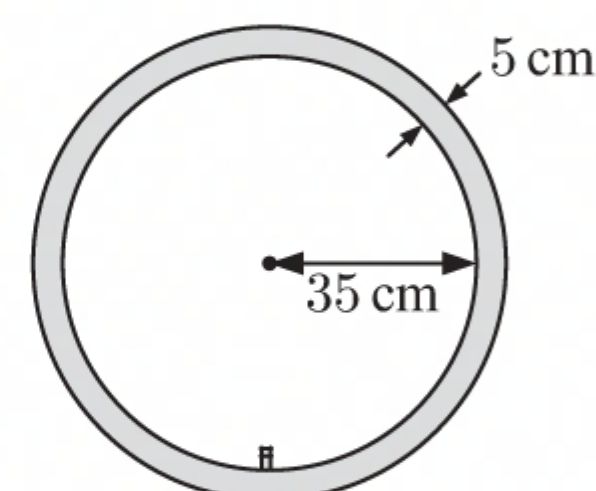
$$\therefore r = \sqrt[3]{\frac{1\,000\,000}{2\pi}} \approx 54.2$$

$$\text{and } h \approx \frac{500\,000}{\pi(54.2)^2} \approx 54.2$$



So, the surface area of the bin is minimised when the bin has a base radius and height of about 54.2 cm.

4 a i At 0 seconds, the valve is at its lowest position, closest to the road, so the height of the valve above the road is 5 cm.





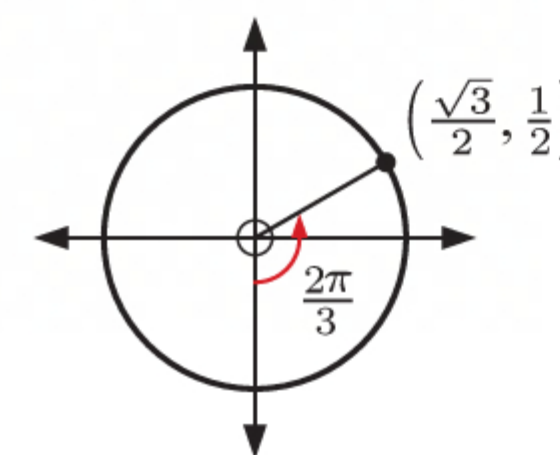
ii The wheel rotates at a constant speed of 4 revolutions per second.

$\therefore$  after  $\frac{1}{12}$  second, the wheel has rotated  $\frac{1}{12} \times 4 = \frac{1}{3}$  revolution.

$\therefore$  the valve will have moved through an angle of  $\frac{2\pi}{3}$ .

So, the valve will be at a height of  $1\frac{1}{2}$  times the radius of the wheel, plus 5 cm.

$\therefore$  the height of the valve above the road  $= 1.5 \times 35 + 5$   
 $= 57.5$  cm



b  $H(t) = a \sin(b(t - c)) + d$  cm

i Amplitude = radius of the wheel

$$= 35 \text{ cm}$$

$$\therefore a = 35$$

iii There are 4 revolutions per second, so the period is  $\frac{1}{4}$  second.

$$\text{The period} = \frac{2\pi}{b}$$

$$\therefore \frac{1}{4} = \frac{2\pi}{b}$$

$$\therefore b = 8\pi$$

ii The centre of the wheel is 40 cm above the ground, so the principal axis is at  $H = 40$  cm.

$$\therefore d = 40$$

iv  $H(t) = 35 \sin(8\pi(t - c)) + 40$

Now from a i,  $H(0) = 5$

$$\therefore 5 = 35 \sin(8\pi(-c)) + 40$$

$$\therefore -1 = \sin(8\pi(-c))$$

$$\therefore -\frac{\pi}{2} = 8\pi(-c)$$

$$\therefore c = \frac{1}{16}$$

c From b,  $H(t) = 35 \sin(8\pi(t - \frac{1}{16})) + 40$

Now  $H(t) = 60$  when  $35 \sin(8\pi(t - \frac{1}{16})) + 40 = 60$

$$\therefore \sin(8\pi(t - \frac{1}{16})) = \frac{20}{35}$$

$$\therefore t \approx 0.0867 \quad \{\text{using technology}\}$$

It takes approximately 0.0867 seconds for the valve to rise to 60 cm above the road.

5 a i If  $A \propto \theta$  then  $A = k\theta$  where  $k$  is a constant.

Using the first point,  $10.48 = 3 \times k$

$$\therefore k = \frac{10.48}{3}$$

$\therefore$  the model is  $A = \frac{10.48}{3} \theta$

ii Substituting  $\theta = 22^\circ$ ,  $A = \frac{10.48}{3} \times 22 \approx 76.85$

Substituting  $\theta = 37^\circ$ ,  $A = \frac{10.48}{3} \times 37 \approx 129.25$

These values are significantly different from the areas observed, so we conclude that Robert's suggestion is incorrect.

b i

$\theta$	$3^\circ$	$22^\circ$	$37^\circ$	$52^\circ$	$74^\circ$
$\tan \theta$	0.0524	0.4040	0.7536	1.2799	3.4874
$A \text{ (m}^2\text{)}$	10.48	80.81	150.71	255.99	697.48

ii The correlation coefficient is very close to 1, so the fit is excellent.

The power is very close to 1, so it is reasonable to conclude that  $A$  is directly proportional to  $\tan \theta$ .

The model is  $A \approx 200 \tan \theta$ .

iii The power model suggests  $A \propto \tan \theta$ , and so supports David's claim.

iv When  $\theta = 43^\circ$ ,  $A \approx 200 \times \tan 43^\circ$

$$\approx 187 \text{ m}^2$$

6 a People who have driven over the speed limit may not have been caught speeding.

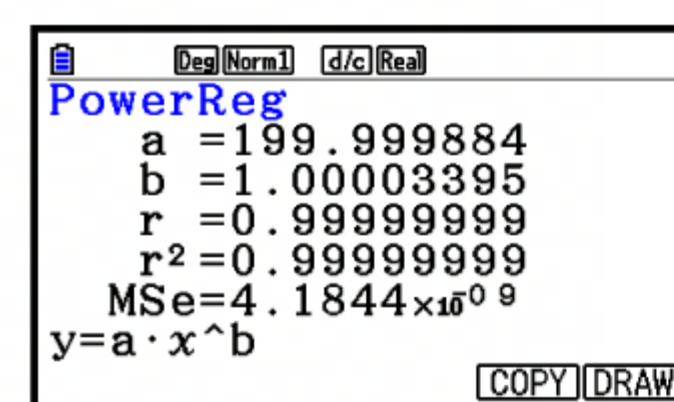
$\therefore$  the question is likely to produce a measurement error.

b The question can be reworded as "Have you ever been caught speeding?"

7  $v = t^3 - 9t^2 + 24t \text{ m s}^{-1}$ ,  $0 \leq t \leq 6$

a  $a = \frac{dv}{dt}$

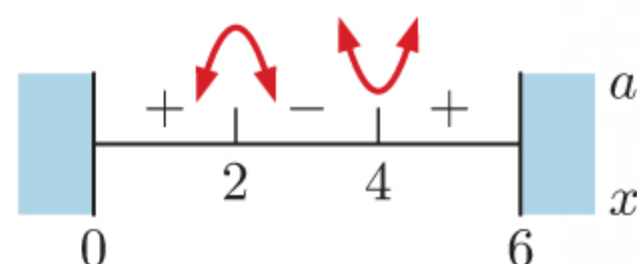
$$= 3t^2 - 18t + 24 \text{ m s}^{-2}$$





- b** The greatest velocity of the particle occurs when  $\frac{dv}{dt} = a = 0$
- $$\therefore 3t^2 - 18t + 24 = 0$$
- $$\therefore t^2 - 6t + 8 = 0$$
- $$\therefore (t - 2)(t - 4) = 0$$
- $$\therefore t = 2 \text{ or } 4$$

The sign diagram of  $a$  is



$\therefore$  there is a local maximum at  $t = 2$ .

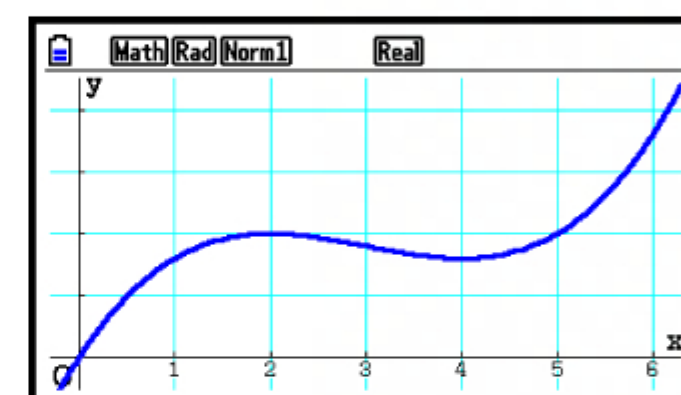
Critical value ( $t$ )	$v$ ( $\text{m s}^{-1}$ )
0 (end point)	0
2 (local maximum)	20
6 (end point)	36

$\therefore$  the greatest velocity of the particle is  $36 \text{ m s}^{-1}$  which occurs at  $t = 6$  seconds.

- c** The speed of the particle is decreasing when  $v$  and  $a$  have opposite signs.

Now  $v \geq 0$  for all  $0 \leq t \leq 6$ .

$\therefore$  the speed of the particle is decreasing when  $a < 0$ , which is when  $2 < t < 4$ .



- 8 a** The matrix  $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$  diagonalises  $\mathbf{A}$  with  $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ .

**b**  $\mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix} \frac{1}{-1} \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -2^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 - 2^n \\ 0 & 2^n \end{pmatrix}$$

**c**

$$\det(\mathbf{A}^k) = 512$$

$$\therefore \begin{vmatrix} 1 & 1 - 2^k \\ 0 & 2^k \end{vmatrix} = 512 \quad \{\text{using b}\}$$

$$\therefore 2^k = 512$$

$$\therefore k = 9 \quad \{\text{technology}\}$$

$$\text{So, } \mathbf{A}^k = \mathbf{A}^9$$

$$= \begin{pmatrix} 1 & 1 - 2^9 \\ 0 & 2^9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -511 \\ 0 & 512 \end{pmatrix}$$

**9 a**  $\mathbf{p} = \left(1 - \frac{t}{12}\right)\mathbf{a} + \frac{t}{12}\mathbf{b}$

At time  $t = 0$ ,  $\mathbf{p} = (1 - 0)\mathbf{a} + 0\mathbf{b}$

$$\therefore \mathbf{p} = \mathbf{a}$$

So, the train is at point A at time  $t = 0$ .

**b** The train is at point B when  $\mathbf{p} = 0\mathbf{a} + 1\mathbf{b} = \mathbf{b}$

$$\therefore t = 12$$

It takes 12 minutes for the train to reach B.

**c i** Distance from A to B  $= \sqrt{(2 - 1)^2 + (2 - 3)^2 + (1 - 0)^2}$

$$= \sqrt{1 + 1 + 1}$$

$$= \sqrt{3} \text{ km}$$

**ii** 12 minutes  $= \frac{1}{5}$  hour

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{\sqrt{3} \text{ km}}{\frac{1}{5} \text{ hour}}$$

$$= 5\sqrt{3} \text{ km h}^{-1}$$

**10 a** Let  $Q = \frac{dP}{dt}$

$$\therefore \frac{dQ}{dt} = \frac{d^2P}{dt^2}$$

$$\therefore \text{the system is } \begin{cases} \frac{dP}{dt} = Q \\ \frac{dQ}{dt} = Q - \frac{1}{40}PQ. \end{cases}$$



- b** At time  $t_0 = 0$ ,  $P_0 = 20$  and  $Q_0 = 15$ .

Using step size  $h = 0.5$ ,  $t_i = t_{i-1} + 0.5$ ,

$$P_i = P_{i-1} + 0.5Q_{i-1},$$

$$\text{and } Q_i = Q_{i-1} + 0.5\left(Q_{i-1} - \frac{1}{40}P_{i-1}Q_{i-1}\right).$$

- i** At time  $t_1 = 0.5$ ,  $P_1 = 20 + 0.5 \times 15 = 27.5$

$$\begin{aligned}\text{and } Q_1 &= 15 + 0.5\left(15 - \frac{1}{40} \times 20 \times 15\right) \\ &= 18.75\end{aligned}$$

$$\text{At time } t_2 = 1, \quad P_2 = 27.5 + 0.5 \times 18.75 = 36.875$$

So, after 1 month, there are about 37 frogs.

- ii** Using step size  $h = 0.5$ ,  $P_i \approx 86.8$  for large values of  $i$ .

$\therefore$  the limiting population is about 87 frogs.

**iii**  $P = \frac{80}{1 + 3e^{-t}}$

$$\text{As } t \rightarrow \infty, \quad e^{-t} \rightarrow 0^+$$

$$\therefore P \rightarrow \frac{80}{1 + 0^+} = 80^-$$

$\therefore$  the actual limiting population is 80 frogs.

The Euler approximation in **ii** is an over estimate, but is still a reasonable approximation of the correct limiting population.

## MIXED QUESTIONS SET 17

- 1 a** Let  $u_n$  km be the distance Hayley cycled in the  $n$ th week, and  $v_n$  km be the distance Patrick cycled in the  $n$ th week.

Hayley cycled an additional 20 km each week.

$$\therefore u_n = 60 + 20(n - 1)$$

$$\therefore u_5 = 60 + 20 \times 4 = 140$$

So, Hayley cycled 140 km in the 5th week of training.

Patrick increased his distance by 20% each week.

$$\begin{aligned}\therefore v_n &= 60(1 + 0.2)^{n-1} \quad \{20\% = 0.2\} \\ &= 60(1.2)^{n-1}\end{aligned}$$

$$\therefore v_5 = 60(1.2)^4 \approx 124$$

So, Patrick cycled about 124 km in the 5th week of training.

- b**  $u_n = 210$  where  $60 + 20(n - 1) = 210$

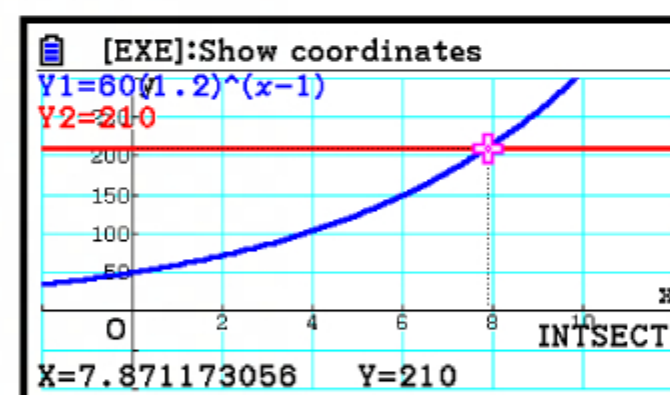
$$\therefore 20(n - 1) = 150$$

$$\therefore n - 1 = 7.5$$

$$\therefore n = 8.5$$

$$v_n = 210 \text{ where } 60(1.2)^{n-1} = 210$$

Using technology,  $n \approx 7.87$ .



So, Hayley first cycled 210 km in the 9th week, and Patrick first cycled 210 km in the 8th week.

$\therefore$  Patrick was the first to cycle 210 km in one week.

- c**  $u_n$  is an arithmetic sequence with  $u_1 = 60$  and  $d = 20$ .

$$\begin{aligned}\therefore \text{the total distance Hayley cycled in the first 12 weeks} &= \frac{n}{2}(2u_1 + (n - 1)d) \\ &= \frac{12}{2}(2 \times 60 + 11 \times 20) \\ &= 2040 \text{ km}\end{aligned}$$



$v_n$  is a geometric sequence with  $v_1 = 60$  and  $r = 1.2$ .

$$\begin{aligned}\therefore \text{the total distance Patrick cycled in the first 12 weeks} &= \frac{v_1(1-r^n)}{1-r} \\ &= \frac{60(1-(1.2)^{12})}{1-1.2} \\ &\approx 2375 \text{ km}\end{aligned}$$

So, Patrick cycled a greater total distance in the first 12 weeks.

- 2 a** The circle centred at X has radius [AX].

The length of the large arc [AB] is half of the circumference.

$$\begin{aligned}\therefore \frac{1}{2}(2 \times \pi \times AX) &= 95 \\ \therefore \pi \times AX &= 95 \\ \therefore AX &= \frac{95}{\pi} \text{ units} \\ &\approx 30.2 \text{ units}\end{aligned}$$

**b** In  $\triangle AXY$ ,  $\sin \frac{\theta}{2} = \frac{AX}{AY}$

$$\begin{aligned}&= \frac{\frac{95}{\pi}}{50} \\ &= \frac{19}{10\pi} \\ \therefore \frac{\theta}{2} &= \sin^{-1}\left(\frac{19}{10\pi}\right) \\ \therefore \theta &= 2 \sin^{-1}\left(\frac{19}{10\pi}\right) \\ &\approx 74.4^\circ\end{aligned}$$

- c** We first divide the figure into areas  $A_1$  and  $A_2$ .

Now  $A_1$  = area of semi-circle centred at X

$$\begin{aligned}&= \frac{1}{2} \times \pi \times \left(\frac{95}{\pi}\right)^2 \quad \{\text{from a}\} \\ &\approx 1436 \text{ units}^2\end{aligned}$$

and  $A_2$  = area of  $\triangle ABY$

$$\begin{aligned}&= \frac{1}{2} \times 50 \times 50 \times \sin \theta \\ &\approx 1250 \times \sin 74.4^\circ \quad \{\text{from b}\} \\ &\approx 1204 \text{ units}^2\end{aligned}$$

So, total area of figure =  $A_1 + A_2$

$$\approx 1436 + 1204$$

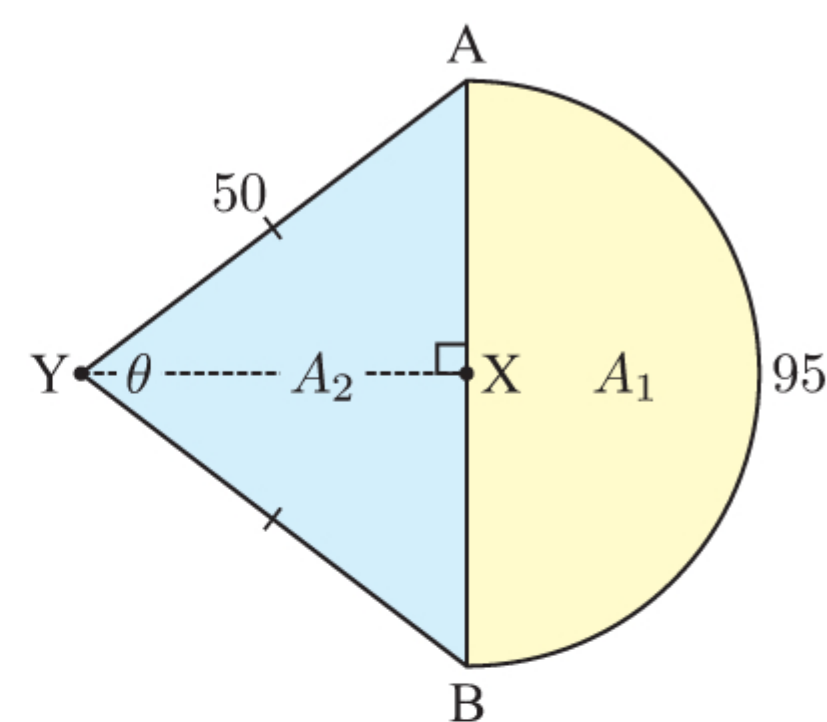
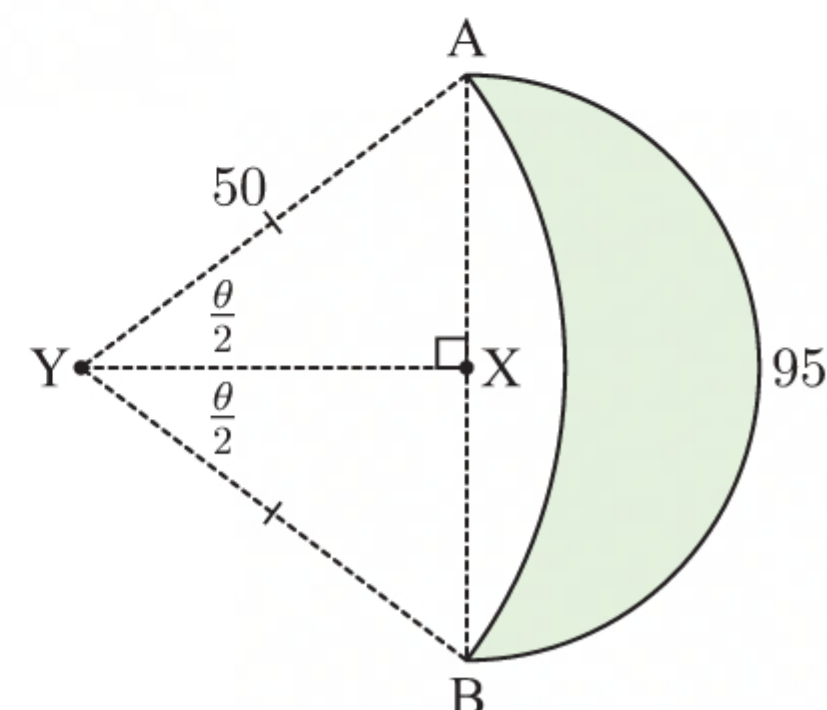
$$\approx 2640 \text{ units}^2$$

Now, shaded area = total area of figure – area of sector ABY

$$\approx 2640 - \frac{\theta}{360^\circ} \pi r^2$$

$$\approx 2640 - \frac{74.4}{360} \pi \times 50^2$$

$$\approx 1020 \text{ units}^2 \quad \{\text{to 3 significant figures}\}$$



- 3 a**  $f(x) = xe^x$

**i**  $n = 2$ ,  $a = 0$ ,  $b = 1$

$$h = \frac{b-a}{n} = \frac{1}{2}$$

$$x_i = \frac{1}{2}i$$

$i$	$x_i$	$f(x_i)$
0	0	0
1	$\frac{1}{2}$	0.824 361
2	1	2.718 282

Using the trapezoidal rule, the area  $\approx \frac{h}{2}(f(x_0) + 2f(x_1) + f(x_2))$

$$\approx 1.0918 \text{ units}^2$$



ii  $n = 6, a = 0, b = 1$

$$h = \frac{b-a}{n} = \frac{1}{6}$$

$$x_i = \frac{1}{6}i$$

$i$	$x_i$	$f(x_i)$
0	0	0
1	$\frac{1}{6}$	0.196 893
2	$\frac{1}{3}$	0.465 204
3	$\frac{1}{2}$	0.824 361
4	$\frac{2}{3}$	1.298 489
5	$\frac{5}{6}$	1.917 480
6	1	2.718 282

Using the trapezoidal rule, the area  $\approx \frac{h}{2}(f(x_0) + 2f(x_1) + \dots + 2f(x_5) + f(x_6))$   
 $\approx 1.0103 \text{ units}^2$

**b** Given the exact area is  $1 \text{ unit}^2$ , the estimate in **a ii** is more accurate than our estimate in **a i**.

Increasing the number of subintervals increases the accuracy of our estimate.

**4** Let  $X$  kg be the mass of a randomly selected sea lion.

$$\therefore X \sim N(700, 80^2)$$

**a**

Inverse Normal

Data : Variable

Tail : Left

Area : 0.65

$\sigma$  : 80

$\mu$  : 700

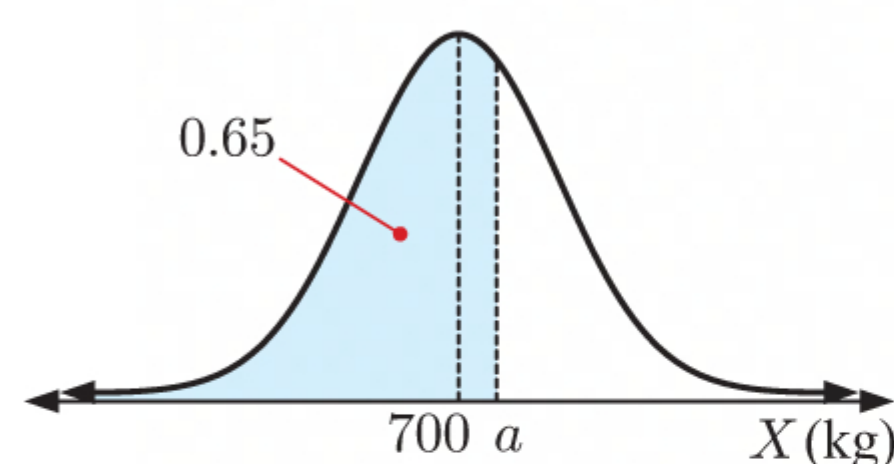
Save Res: None

None LIST

Inverse Normal

xInv=730.825637

If  $P(X < a) = 0.65$   
then  $a \approx 731$



**b**

Normal C.D

Data : Variable

Lower : 600

Upper :  $9 \times 10^9$

$\sigma$  : 80

$\mu$  : 700

Save Res: None

None LIST

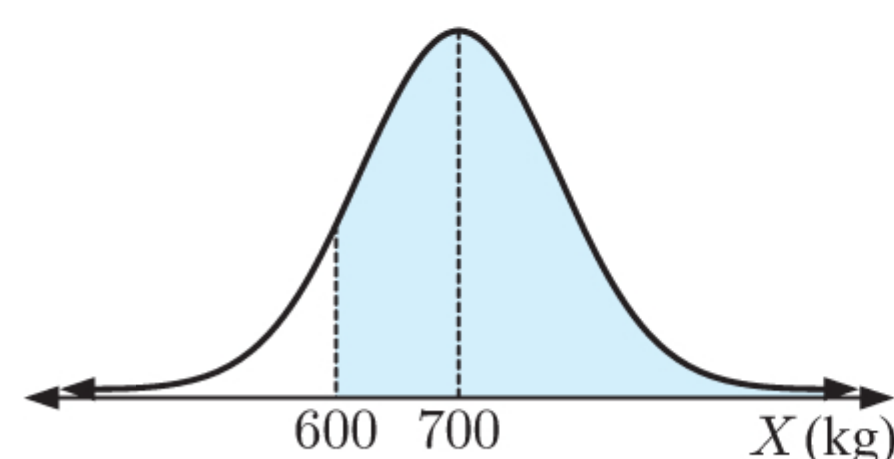
Normal C.D

p = 0.89435022

z: Low = -1.25

z: Up =  $1.125 \times 10^9$

$P(X > 600) \approx 0.894 350$   
 $\approx 0.894$



**c**  $P(X < 600) = 1 - P(X \geq 600)$   
 $\approx 1 - 0.894 350$  {using **b**}  
 $\approx 0.105 650$

$$\therefore Y \sim B(20, 0.105 650)$$

**i**

$\mu_Y = np$ $\approx 20 \times 0.105 650$ $\approx 2.11$	$\sigma_Y = \sqrt{np(1-p)}$ $\approx \sqrt{20 \times 0.105 650 \times 0.894 350}$ $\approx 1.37$
---	--

**ii** Using technology,  $P(Y > 3) = P(Y \geq 4)$   
 $\approx 0.154$

Math Deg Norm1 d/c Real

BinomialCD(4, 20, 20, 0)

0.1539249412

JUMP DELETE MATH VCT MATH



5 The ordered data set is:

132 140 149 155 159 160 161 163 164 165 (20 data values)  
 169 171 173 181 185 191 200 207 212 303

a Since  $n = 20$ ,  $\frac{n+1}{2} = 10.5$   $\therefore$  the median is the average of the 10th and 11th value.

~~132 140 149 155 159 160 161 163 164 165~~  
~~169 171 173 181 185 191 200 207 212 303~~

$$\begin{aligned}\therefore \text{median} &= \frac{10\text{th value} + 11\text{th value}}{2} \\ &= \frac{\$165 + \$169}{2} \\ &= \$167\end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set in two.

lower half

132	140	149	155	159	160	161	163	164	165
169	171	173	181	185	191	200	207	212	303

upper half

$$Q_1 = \text{median of lower half} = \frac{\$159 + \$160}{2} = \$159.50$$

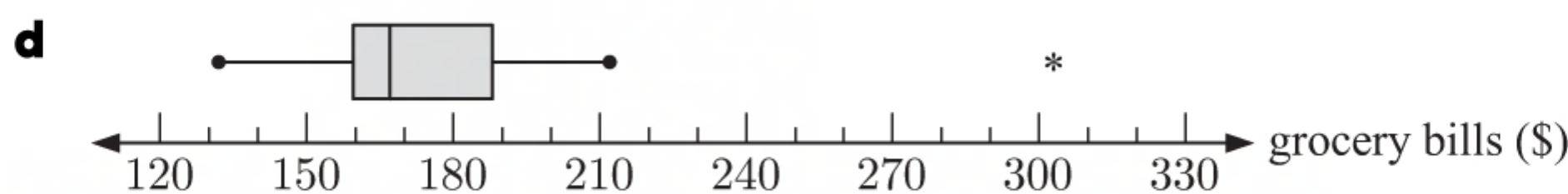
$$Q_3 = \text{median of upper half} = \frac{\$185 + \$191}{2} = \$188$$

b  $IQR = Q_3 - Q_1$   
 $= \$188 - \$159.50$   
 $= \$28.50$

c Test for outliers:      upper boundary                      and                      lower boundary

$= \text{upper quartile} + 1.5 \times IQR$ $= \$188 + 1.5 \times 28.50$ $= \$230.75$	$= \text{lower quartile} - 1.5 \times IQR$ $= \$159.50 - 1.5 \times 28.50$ $= \$116.75$
--	---

\$303 is above the upper boundary, so it is an outlier.



6  $y = xe^{2x}$

a  $\frac{dy}{dx} = e^{2x} + 2xe^{2x}$   
 $= e^{2x}(1 + 2x)$

The tangent to the curve is horizontal where  $\frac{dy}{dx} = 0$   
 $\therefore e^{2x}(1 + 2x) = 0$   
 $\therefore x = -\frac{1}{2} \quad \{e^{2x} > 0\}$

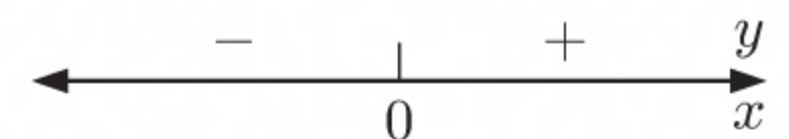
When  $x = -\frac{1}{2}$ ,  $y = (-\frac{1}{2})e^{2(-\frac{1}{2})} = -\frac{1}{2e}$ .

$\therefore y = k$  is a horizontal tangent to the curve when  $k = -\frac{1}{2e}$ .

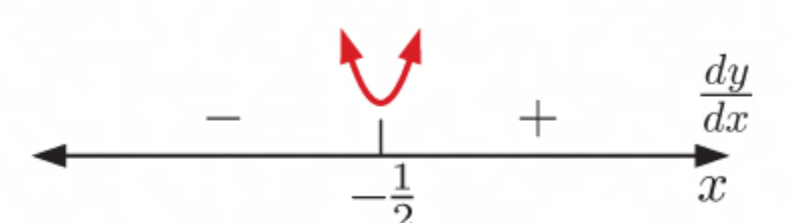
b When  $y = 0$ ,  $xe^{2x} = 0$   
 $\therefore x = 0 \quad \{e^{2x} > 0\}$

When  $\frac{dy}{dx} = 0$ ,  $x = -\frac{1}{2}$  {from a}

So,  $y$  has sign diagram:



So,  $\frac{dy}{dx}$  has sign diagram:

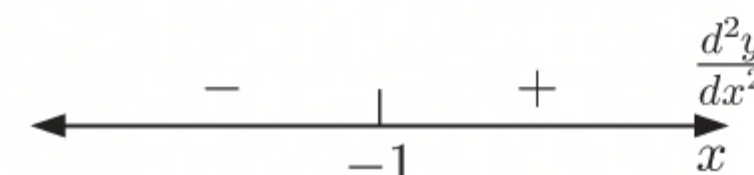




$$\begin{aligned}\text{Now } \frac{d^2y}{dx^2} &= 2e^{2x}(1+2x) + 2e^{2x} \\ &= 2e^{2x}(2+2x) \\ &= 4e^{2x}(1+x)\end{aligned}$$

$$\begin{aligned}\text{When } \frac{d^2y}{dx^2} &= 0, \quad 4e^{2x}(1+x) = 0 \\ \therefore x &= -1 \quad \{e^{2x} > 0\}\end{aligned}$$

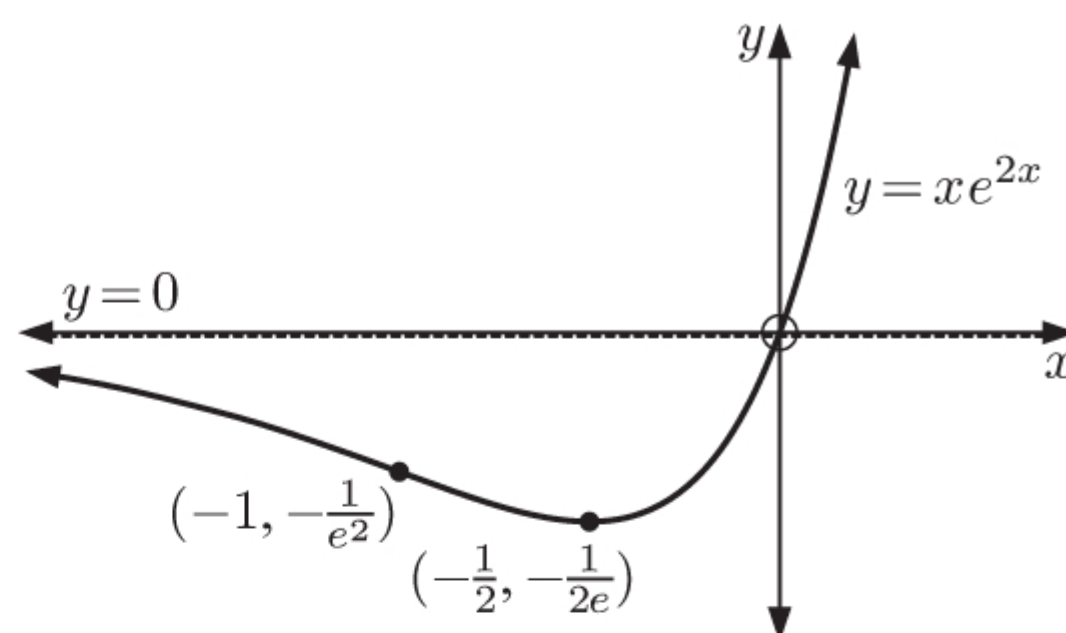
So,  $\frac{d^2y}{dx^2}$  has sign diagram:



$$\text{When } x = -1, \quad y = (-1)e^{2(-1)} = -\frac{1}{e^2}.$$

$\therefore$  there is a non-stationary point of inflection at  $\left(-1, -\frac{1}{e^2}\right)$ .

So, as  $x \rightarrow \infty$ ,  $y = xe^{2x} \rightarrow \infty$   
and as  $x \rightarrow -\infty$ ,  $y = xe^{2x} \rightarrow 0^-$ .



- i  $y = k$  meets the curve at exactly one point for  $k = -\frac{1}{2e}$  or  $k \geq 0$ .
  - ii  $y = k$  meets the curve at two distinct points for  $-\frac{1}{2e} < k < 0$ .
  - iii  $y = k$  meets the curve at no points for  $k < -\frac{1}{2e}$ .
- c  $y = xe^{ax}$ ,  $a \in \mathbb{R}$ ,  $a > 0$

$$\begin{aligned}\text{i } y = x \text{ meets the curve } y &= xe^{ax} \text{ where } xe^{ax} = x \\ \therefore xe^{ax} - x &= 0 \\ \therefore x(e^{ax} - 1) &= 0 \\ \therefore x = 0 \text{ or } e^{ax} &= 1 \\ \therefore x = 0 \text{ or } ax &= 0 \\ \therefore x = 0 \quad \{a > 0\}\end{aligned}$$

$$\text{Now } y = xe^{ax}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= e^{ax} + axe^{ax} \\ &= e^{ax}(1+ax)\end{aligned}$$

$$\text{When } x = 0, \quad y = 0e^0 = 0$$

$$\text{and } \frac{dy}{dx} = e^0(1+0) = 1$$

$\therefore$  the tangent to the curve at  $x = 0$  has gradient 1, and the point of contact is  $(0, 0)$ .

$\therefore$  the equation of the tangent is  $y - 0 = 1(x - 0)$   
 $\therefore y = x$  as required.

ii From i, the tangent to the curve at  $y = xe^{ax}$  when  $x = 0$  is  $y = x$ .

$\therefore$  the normal to the curve at  $x = 0$  has gradient  $-1$ , and the point of contact is  $(0, 0)$ .

$\therefore$  the equation of the normal is  $y - 0 = -1(x - 0)$   
 $\therefore y = -x$ .

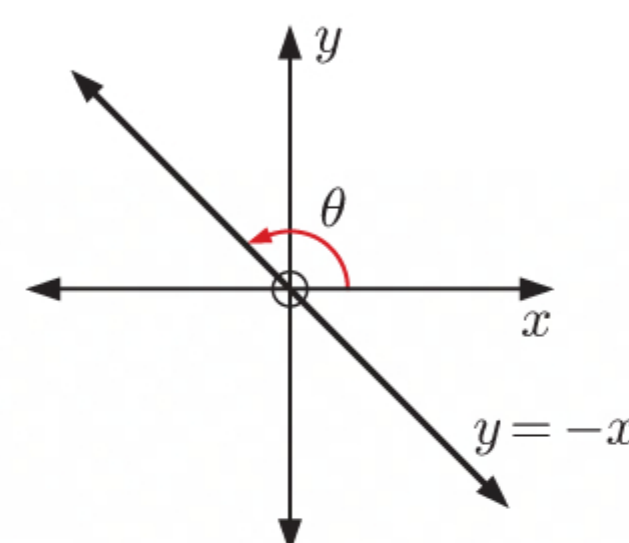
Let  $\theta$  be the angle the normal makes with the positive  $x$ -axis.

The normal has gradient  $-1$ .

$$\therefore \tan \theta = -1$$

$$\therefore \theta = \frac{3\pi}{4} \quad \{0 < \theta < \pi\}$$

So, the acute angle the normal makes with the  $x$ -axis is  $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$ .





- 7 a** The test statistic is the number of sixes that appear in 10 rolls of the die,  $X$ .

$\therefore$  the null distribution is  $B(10, \frac{1}{6})$ .

$x$	$p\text{-value} = P(X \geq x)$	
0	1	$p\text{-value} > \alpha$
1	$\approx 0.838$	
2	$\approx 0.515$	
3	$\approx 0.225$	
4	$\approx 0.0697$	
5	$\approx 0.0155$	$p\text{-value} \leq \alpha$
6	$\approx 0.00244$	
7	$\approx 2.68 \times 10^{-4}$	
8	$\approx 1.94 \times 10^{-5}$	
9	$\approx 8.43 \times 10^{-7}$	
10	$\approx 1.65 \times 10^{-8}$	

**i**  $C = \{5, 6, 7, 8, 9, 10\}$

**ii**  $A = \{0, 1, 2, 3, 4\}$

**iii** The value in  $C$  with the largest  $p$ -value is 5.

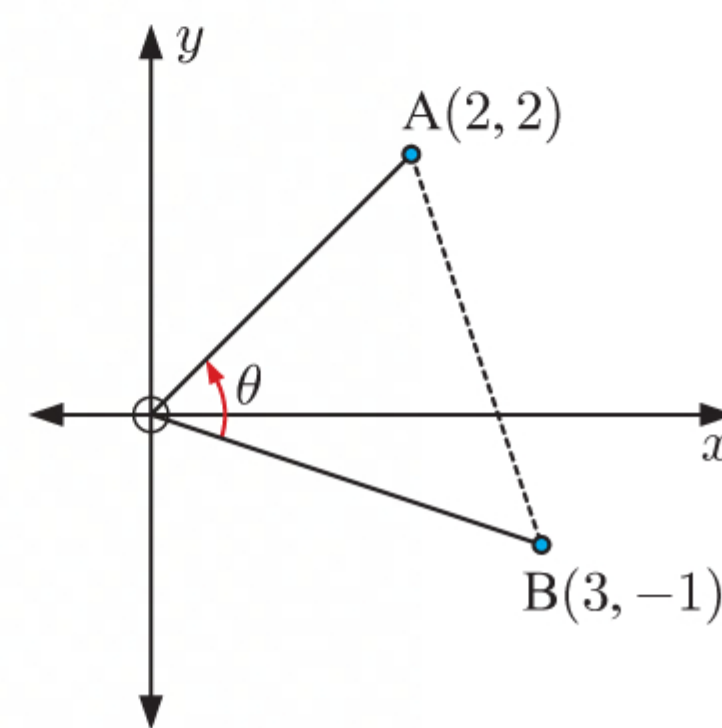
$\therefore$  the critical value  $c = 5$ .

**b**  $3 \notin C$ , so we do not have sufficient evidence to reject  $H_0$ .

**8 a**  $\vec{OA} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$  and  $\vec{OB} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Now  $\theta$  is the acute angle between  $\vec{OA}$  and  $\vec{OB}$ .

$$\begin{aligned} \therefore \cos \theta &= \frac{|\vec{OA} \cdot \vec{OB}|}{|\vec{OA}| |\vec{OB}|} \\ &= \frac{|2 \times 3 + 2 \times (-1)|}{\sqrt{2^2 + 2^2} \sqrt{3^2 + (-1)^2}} \\ &= \frac{4}{\sqrt{8} \sqrt{10}} \\ &= \frac{4}{2\sqrt{2} \times \sqrt{2} \times \sqrt{5}} \\ &= \frac{1}{\sqrt{5}} \end{aligned}$$



**b i**  $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \left(\frac{1}{\sqrt{5}}\right)^2 + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\therefore \sin \theta = \frac{2}{\sqrt{5}} \quad \{\theta \text{ is acute}\}$$

**ii** From **a**,  $|\vec{OA}| = \sqrt{8}$  and  $|\vec{OB}| = \sqrt{10}$

$$\begin{aligned} \therefore \text{area of triangle OAB} &= \frac{1}{2} \times \sqrt{8} \times \sqrt{10} \times \sin \theta \\ &= \frac{\sqrt{80}}{2} \times \frac{2}{\sqrt{5}} \\ &= 4 \text{ units}^2 \end{aligned}$$

**c i** If  $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$  then  $\begin{vmatrix} \lambda - 2 & -3 \\ -1 & \lambda - 4 \end{vmatrix} = 0$

$$\therefore (\lambda - 2)(\lambda - 4) - 3 = 0$$

$$\therefore \lambda^2 - 6\lambda + 8 - 3 = 0$$

$$\therefore \lambda^2 - 6\lambda + 5 = 0$$

$$\therefore (\lambda - 1)(\lambda - 5) = 0$$

$$\therefore \lambda = 1 \text{ or } 5$$

The eigenvalues are 1 and 5.



For  $\lambda = 1$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} -1 & -3 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -a - 3b = 0$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = -3t$ .

$$\therefore \mathbf{x} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} t, t \neq 0$$

Any vector of the form  $\begin{pmatrix} -3 \\ 1 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue 1.

For  $\lambda = 5$ , consider  $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -a + b = 0$$

Letting  $a = t$ ,  $t \neq 0$ , then  $b = t$ .

$$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t, t \neq 0$$

Any vector of the form  $\begin{pmatrix} 1 \\ 1 \end{pmatrix} t$ ,  $t \neq 0$  is an eigenvector corresponding to the eigenvalue 5.

**ii**  $\overrightarrow{\text{OB}} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} = -\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\therefore \overrightarrow{\text{OB}}$  is an eigenvector corresponding to the eigenvalue 1.

$$\therefore \mathbf{A}(\overrightarrow{\text{OB}}) = \overrightarrow{\text{OB}}$$

So, points on  $[\text{OB}]$  do not move under this transformation.

**iii**  $\overrightarrow{\text{OA}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore \overrightarrow{\text{OA}}$  is an eigenvector corresponding to the eigenvalue 5.

$$\therefore \mathbf{A}(\overrightarrow{\text{OA}}) = 5(\overrightarrow{\text{OA}})$$

Under this transformation, the points on  $[\text{OA}]$  are enlarged with scale factor 5.

**iv** Let triangle  $\text{O}'\text{A}'\text{B}'$  be the image of triangle  $\text{OAB}$  under this transformation.

From **ii**,  $|\overrightarrow{\text{O}'\text{B}'}| = |\overrightarrow{\text{OB}}| = \sqrt{10}$ .

From **iii**,  $|\overrightarrow{\text{O}'\text{A}'}| = 5|\overrightarrow{\text{OA}}| = 5\sqrt{8}$ .

$$\begin{aligned} \therefore \text{area of triangle O}'\text{A}'\text{B}' &= \frac{1}{2} \times \sqrt{10} \times 5\sqrt{8} \times \sin \theta \\ &= \frac{5\sqrt{80}}{2} \times \frac{2}{\sqrt{5}} \\ &= 20 \text{ units}^2 \end{aligned}$$

**v** Area of image  $= |\det \mathbf{A}| \times \text{area of object}$   
 $= |8 - 3| \times 4 \quad \{\text{using b ii}\}$   
 $= 5 \times 4$   
 $= 20 \text{ units}^2 \quad \checkmark$

<b>9 a</b>	$t$ (minutes)	0	10	20	30	40	50	60
	$P$ (%)	100	52.3	24.1	14	5.9	3.0	1.8
	$\ln P$	4.605	3.957	3.182	2.639	1.775	1.099	0.588

**b** The relationship between  $\ln P$  and  $t$  being approximately linear suggests that  $P$  and  $t$  are related by an exponential model.



- c** Using the given gradient and vertical intercept:

$$\ln P = -0.0685t + 4.604$$

$$\begin{aligned}\therefore P &= e^{-0.0685t+4.604} \\ &= (e^{4.604}) \times e^{-0.0685t} \\ &\approx 99.88e^{-0.0685t}\end{aligned}$$

- d i** 8 hours  $\equiv$  480 minutes

$$\begin{aligned}\text{When } t = 480, \quad P &\approx 99.88e^{-0.0685 \times 480} \\ &\approx 5.25 \times 10^{-13}\end{aligned}$$

$\therefore$  after 8 hours, about  $5.25 \times 10^{-11}\%$  of nitrogen remains in type A tissue.

- ii** The prediction in **i** is an extrapolation well beyond the given data points.

$\therefore$  it is unlikely to be an accurate prediction.

- 10 a** From the graph, when  $x = 2$ ,  $kx = \frac{1}{4}(x-2)^2 + 6$

$$\therefore 2k = 6$$

$$\therefore k = 3$$

$$\begin{aligned}\mathbf{b} \quad f(6) &= \frac{1}{4}(6-2)^2 + 6 \quad \{\text{as } 2 < x \leq 6\} \\ &= \frac{1}{4}(4)^2 + 6 \\ &= 4 + 6 \\ &= 10\end{aligned}$$

$\therefore f$  has range  $\{y \mid 0 \leq y \leq 10\}$ .

- c** For  $0 \leq x \leq 2$ ,  $y = 3x$

$$\therefore x = \frac{1}{3}y, \quad 0 \leq y \leq 6$$

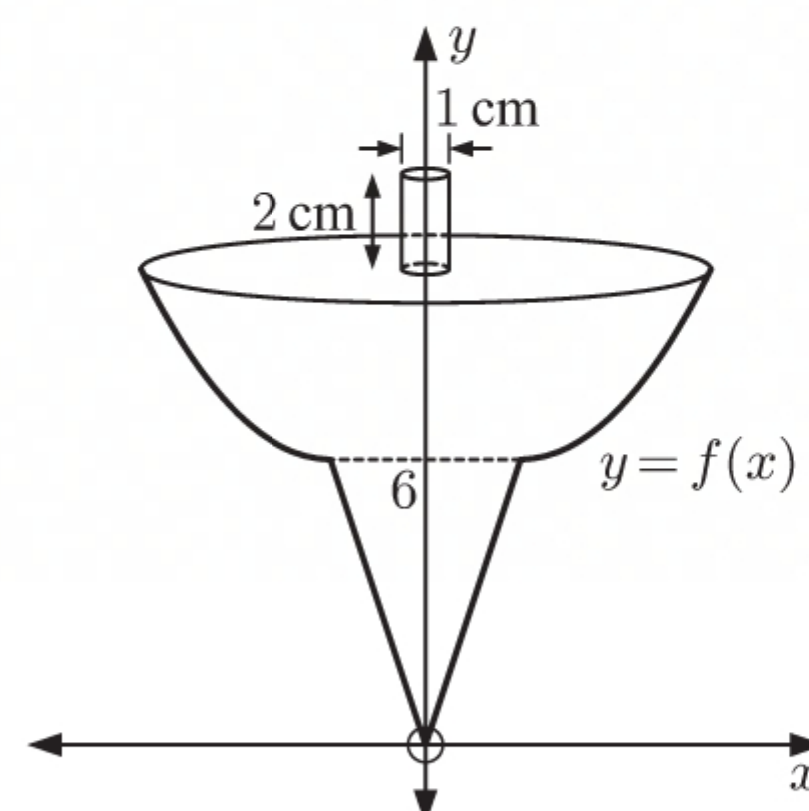
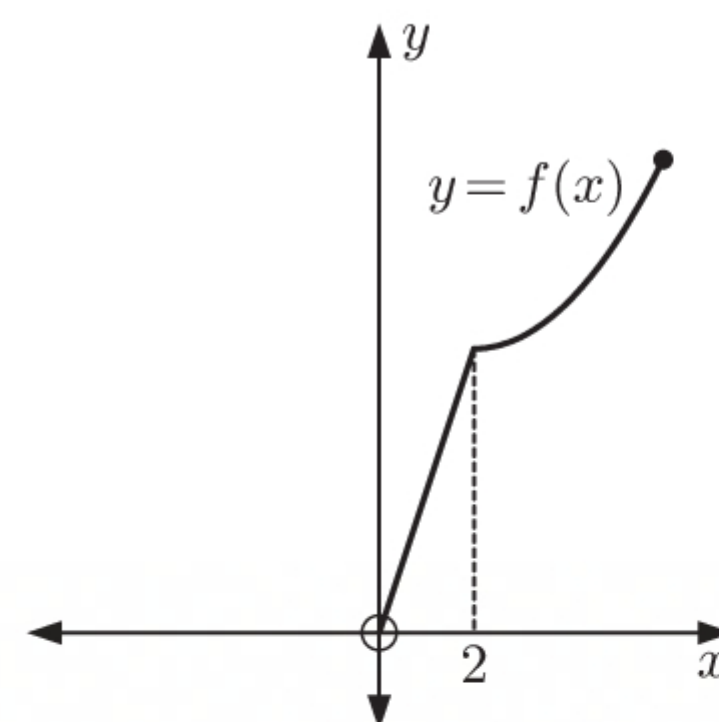
$$\text{For } 2 < x \leq 6, \quad y = \frac{1}{4}(x-2)^2 + 6$$

$$\therefore y - 6 = \frac{1}{4}(x-2)^2$$

$$\therefore 4(y-6) = (x-2)^2$$

$$\therefore x - 2 = 2\sqrt{y-6} \quad \{2 < x \leq 6\}$$

$$\therefore x = 2 + 2\sqrt{y-6}, \quad 6 \leq y \leq 10$$



$$\therefore \text{volume of spinning top} = \pi \int_0^{10} x^2 dy + \text{volume of cylindrical handle}$$

$$= \pi \left( \int_0^6 x^2 dy + \int_6^{10} x^2 dy \right) + \pi(0.5)^2(2)$$

$$= \pi \left( \int_0^6 \frac{1}{9}y^2 dy + \int_6^{10} (2 + 2\sqrt{y-6})^2 dy \right) + \frac{\pi}{2}$$

$$= \pi \left( \int_0^6 \frac{1}{9}y^2 dy + \int_6^{10} (4 + 8\sqrt{y-6} + 4(y-6)) dy \right) + \frac{\pi}{2}$$

$$= \pi \left( \int_0^6 \frac{1}{9}y^2 dy + \int_6^{10} (8(y-6)^{\frac{1}{2}} + 4y - 20) dy \right) + \frac{\pi}{2}$$

$$= \pi \left( \left[ \frac{1}{27}y^3 \right]_0^6 + \left[ \frac{16}{3}(y-6)^{\frac{3}{2}} + 2y^2 - 20y \right]_6^{10} \right) + \frac{\pi}{2}$$

$$= \pi \left( (8 - 0) + \left( \frac{128}{3} + 200 - 200 - (0 + 72 - 120) \right) \right) + \frac{\pi}{2}$$

$$= \pi \left( 8 + \frac{128}{3} + 48 \right) + \frac{\pi}{2}$$

$$= \frac{595}{6} \pi \text{ cm}^3$$

$$\approx 312 \text{ cm}^3$$



## MIXED QUESTIONS SET 18

1  $W(t) = 5 \times (0.965)^t$  grams,  $t \geq 0$

- a The weight of the radioactive substance at the *end* of each year forms a geometric sequence with common ratio  $r = 0.965$ .

$$\begin{aligned}\text{So, percentage decrease} &= (1 - r) \times 100\% \\ &= 0.035 \times 100\% \\ &= 3.5\%\end{aligned}$$

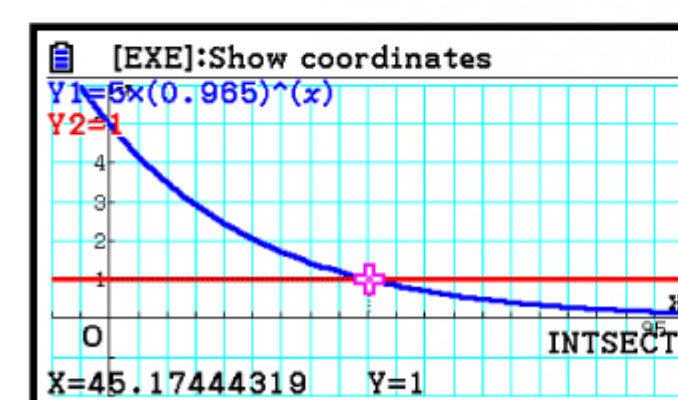
b  $W(300) = 5 \times (0.965)^{300}$   
 $\approx 0.000114$   
 $\approx 1.14 \times 10^{-4}$

The weight of the substance after 300 years is about  $1.14 \times 10^{-4}$  grams.

c We need to solve  $W(t) = 1$   
 $\therefore 5 \times (0.965)^t = 1$

Using technology,  $t \approx 45.2$

$\therefore$  it will take about 45.2 years for the weight of the substance to fall below 1 g.



- 2 a i  $(2, 4)$  is closest to Q, so we estimate a lead concentration of 47 ppm at  $(2, 4)$ .  
 ii  $(-2, -3)$  is closest to R, so we estimate a lead concentration of 62 ppm at  $(-2, -3)$ .  
 iii  $(-4, -1)$  is equally closest to P and R, so we estimate a lead concentration of  $\frac{28 + 62}{2} = 45$  ppm at  $(-4, -1)$ .

- b i S lies in the original cell R, so we construct the perpendicular bisector of [RS] within this cell.

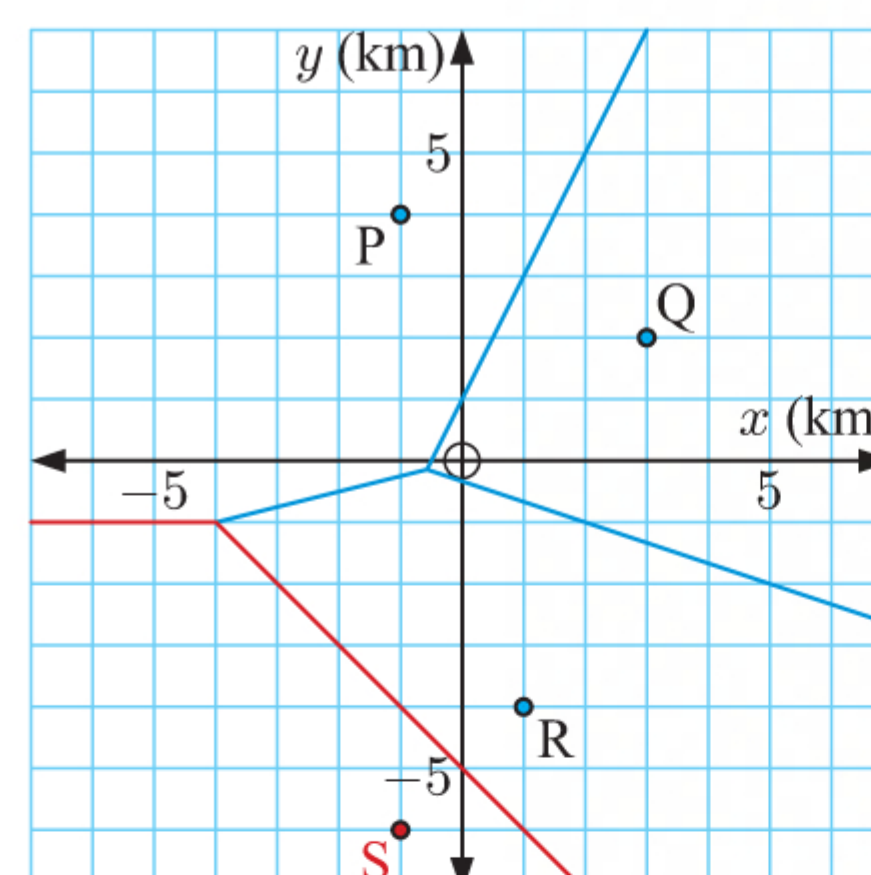
We then construct the perpendicular bisector of [PS] within the original cell P.

Finally, we remove the segment of the perpendicular bisector of [PR] which now lies within cell S.

- ii  $(2, 4)$  is still closest to Q, so its estimate is unchanged.

$(-2, -3)$  is now equally closest to R and S, so we estimate a lead concentration of  $\frac{62 + 55}{2} = 58.5$  ppm at  $(-2, -3)$ .

$(-4, -1)$  is now equally closest to P, R, and S, so we estimate a lead concentration of  $\frac{28 + 62 + 55}{3} \approx 48.3$  ppm at  $(-4, -1)$ .

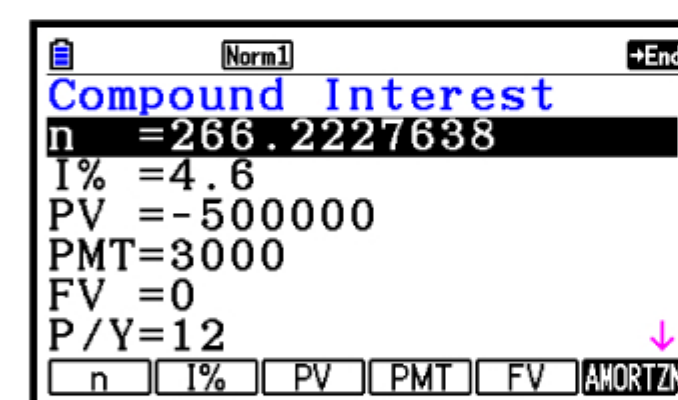


- 3 a We need to find how long it will take for the future value to fall to €0.

$$I\% = 4.6, \quad PV = -500\,000, \quad PMT = 3000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$

$$\therefore N \approx 266$$

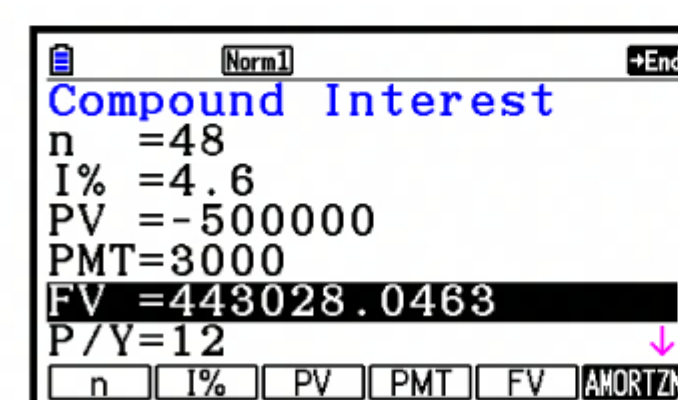
Raman will be able to withdraw €3000 for 266 months, and then less in the 267th month.



- b  $N = 4 \times 12 = 48$ ,  $I\% = 4.6$ ,  $PV = -500\,000$ ,  $PMT = 3000$ ,  $P/Y = 12$ ,  $C/Y = 12$

$$\therefore FV \approx 443\,028.05$$

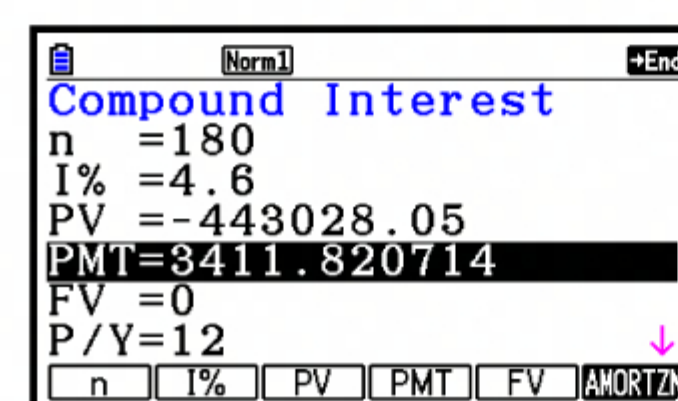
After 4 years, the balance is €443 028.05.



- c  $N = 15 \times 12 = 180$ ,  $I\% = 4.6$ ,  $PV = -443\,028.05$ ,  $FV = 0$ ,  $P/Y = 12$ ,  $C/Y = 12$

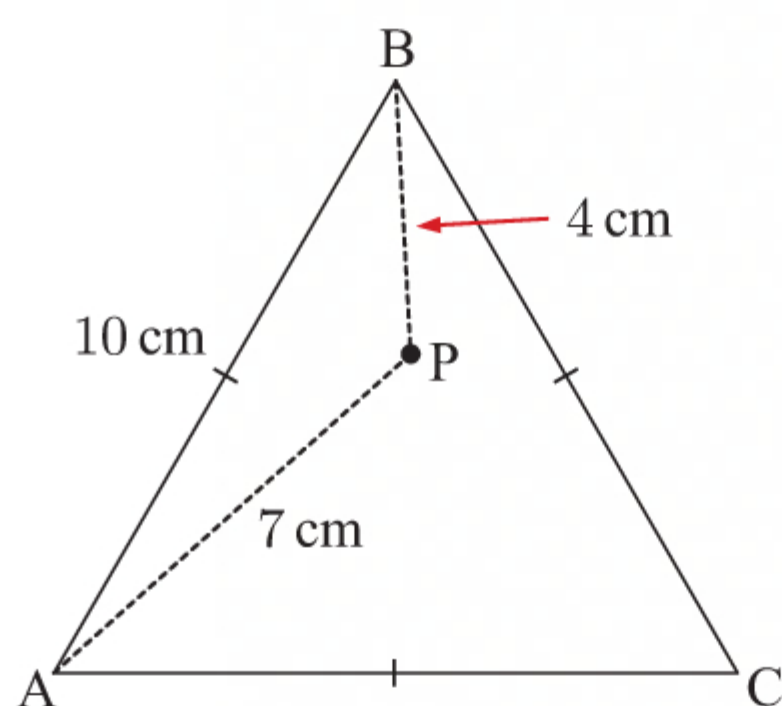
$$\therefore PMT \approx 3411.82$$

Raman can afford to withdraw €3411.82 each month.





4 a


 b i By the cosine rule in  $\triangle BAP$ :

$$\cos \widehat{BAP} = \frac{10^2 + 7^2 - 4^2}{2 \times 10 \times 7}$$

$$\therefore \cos \widehat{BAP} = \frac{133}{140}$$

$$\therefore \widehat{BAP} = \cos^{-1}\left(\frac{133}{140}\right) \approx 18.2^\circ$$

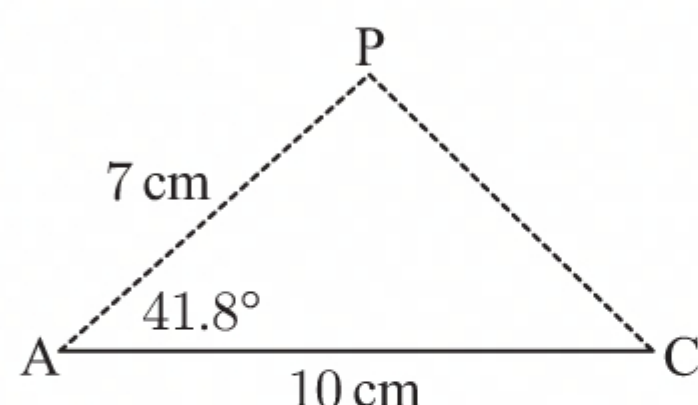
 ii  $\widehat{BAC} = 60^\circ$  {angles in an equilateral triangle}

$$\therefore \widehat{CAP} = 60^\circ - \widehat{BAP}$$

$$\approx 60^\circ - 18.2^\circ$$

$$\approx 41.8^\circ$$

c


 By the cosine rule in  $\triangle APC$ :

$$CP^2 \approx 10^2 + 7^2 - 2(10)(7) \cos 41.8^\circ$$

$$\therefore CP \approx \sqrt{10^2 + 7^2 - 2(10)(7) \cos 41.8^\circ}$$

$$\therefore CP \approx 6.68 \text{ cm}$$

5 a

	Painting	Sketching	Sculpting	Sum
Male	30	35	15	80
Female	20	15	25	60
Sum	50	50	40	140

The expected number of male sculptors is

$$\frac{80 \times 40}{140} \approx 23.$$

 b Using technology,  $\chi^2_{\text{calc}} \approx 9.84$ .

	1	2	3
1	30	35	15
2	20	15	25

	1	2	3
1	30	35	15
2	20	15	25

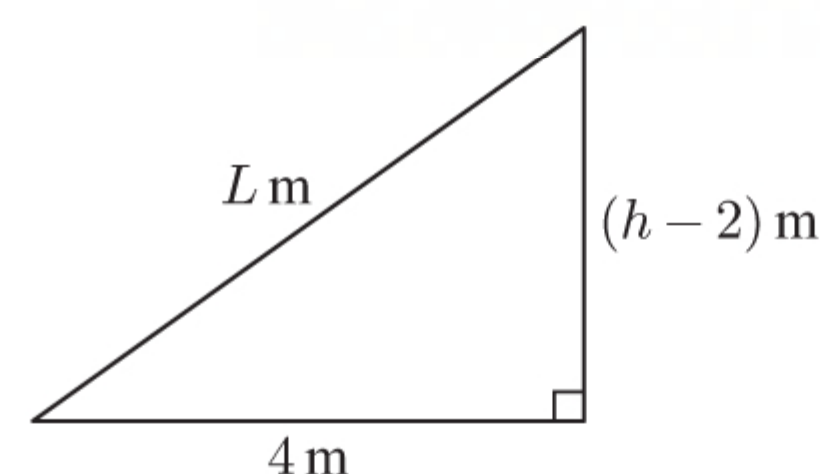
c Since  $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$ , we reject the null hypothesis that *gender* and *choice of art speciality* are independent. We therefore conclude at the 1% significance level that *gender* and *choice of art speciality* are dependent.

 6 a Using Pythagoras,  $L^2 = (h - 2)^2 + 4^2$ 

$$= h^2 - 4h + 4 + 16$$

$$= h^2 - 4h + 20$$

$$\therefore L = \sqrt{h^2 - 4h + 20} \quad \{L > 0\}$$


 b The ladder is extended at  $0.1 \text{ m s}^{-1} \therefore \frac{dL}{dt} = 0.1$ 

$$\text{Now } \frac{dL}{dt} = \frac{dL}{dh} \times \frac{dh}{dt} \quad \{\text{chain rule}\}$$

$$\therefore 0.1 = \frac{1}{2}(h^2 - 4h + 20)^{-\frac{1}{2}}(2h - 4) \times \frac{dh}{dt}$$

$$\therefore 0.1 = \frac{h - 2}{\sqrt{h^2 - 4h + 20}} \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{\sqrt{h^2 - 4h + 20}}{10(h - 2)}$$

i When  $h = 5$ ,  $\frac{dh}{dt} = \frac{\sqrt{5^2 - 4(5) + 20}}{10(5 - 2)}$

$$= \frac{5}{10 \times 3}$$

$$= \frac{1}{6} \approx 0.167$$

$\therefore$  the height is increasing by about  $0.167 \text{ m s}^{-1}$  when the height is 5 m.



ii When  $L = 6$ ,  $6 = \sqrt{h^2 - 4h + 20}$

$$\therefore 36 = h^2 - 4h + 20$$

$$\therefore h^2 - 4h - 16 = 0$$

$$\therefore h = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-16)}}{2}$$

$$= \frac{4 \pm \sqrt{80}}{2}$$

$$= 2 \pm 2\sqrt{5}$$

But  $h > 0 \therefore h = 2 + 2\sqrt{5}$

When  $h = 2 + 2\sqrt{5}$ ,  $\frac{dh}{dt} = \frac{\sqrt{(2 + 2\sqrt{5})^2 - 4(2 + 2\sqrt{5}) + 20}}{10(2 + 2\sqrt{5} - 2)}$

$$= \frac{\sqrt{4 + 8\sqrt{5} + 20 - 8 - 8\sqrt{5} + 20}}{10 \times 2\sqrt{5}}$$

$$= \frac{\sqrt{36}}{20\sqrt{5}}$$

$$= \frac{6}{20\sqrt{5}}$$

$$= \frac{3}{10\sqrt{5}} \approx 0.134$$

$\therefore$  the height is increasing by about  $0.134 \text{ m s}^{-1}$  when the length of the ladder is 6 m.

<b>7</b>	Number of games ( $x$ )	10	4	7	15	3	11	9	15	12	10
	Number of runs ( $y$ )	107	82	150	212	40	165	185	241	101	183

a Using technology,  $r \approx 0.789$ .

There is a moderate positive correlation between  $x$  and  $y$ .

Des	Norm1	d/c	Real
LinearReg(ax+b)			
a = 12.2061994			
b = 29.4204851			
r = 0.78894415			
r <sup>2</sup> = 0.62243287			
MSe = 1676.51128			
y = ax + b			
[COPY] [DRAW]			

b From the screenshot in a, the equation of the least squares regression line is  $y \approx 12.2x + 29.4$ .

c The gradient of the regression line  $\approx 12.2$ .

This means that for every additional game played, the number of runs scored will increase by an average of about 12.2 runs.

d When  $x = 6$ ,  $y \approx 12.2 \times 6 + 29.4$   
 $\approx 103$

$\therefore$  a player who played 6 games is expected to score about 103 runs.

8 a i Area of base plane  $= \frac{1}{2} |\mathbf{b} \times \mathbf{c}| \text{ units}^2$

ii Let  $h$  be the perpendicular height.

$$\therefore \sin \theta = \frac{h}{|\mathbf{a}|}$$

$$\therefore h = |\mathbf{a}| \sin \theta \text{ units}$$

b Volume = area of base  $\times$  perpendicular height

$$= \frac{1}{2} |\mathbf{b} \times \mathbf{c}| \times |\mathbf{a}| \sin \theta \quad \{\text{using a i and a ii}\}$$

$$= \frac{1}{2} |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \sin \theta \text{ units}^3$$

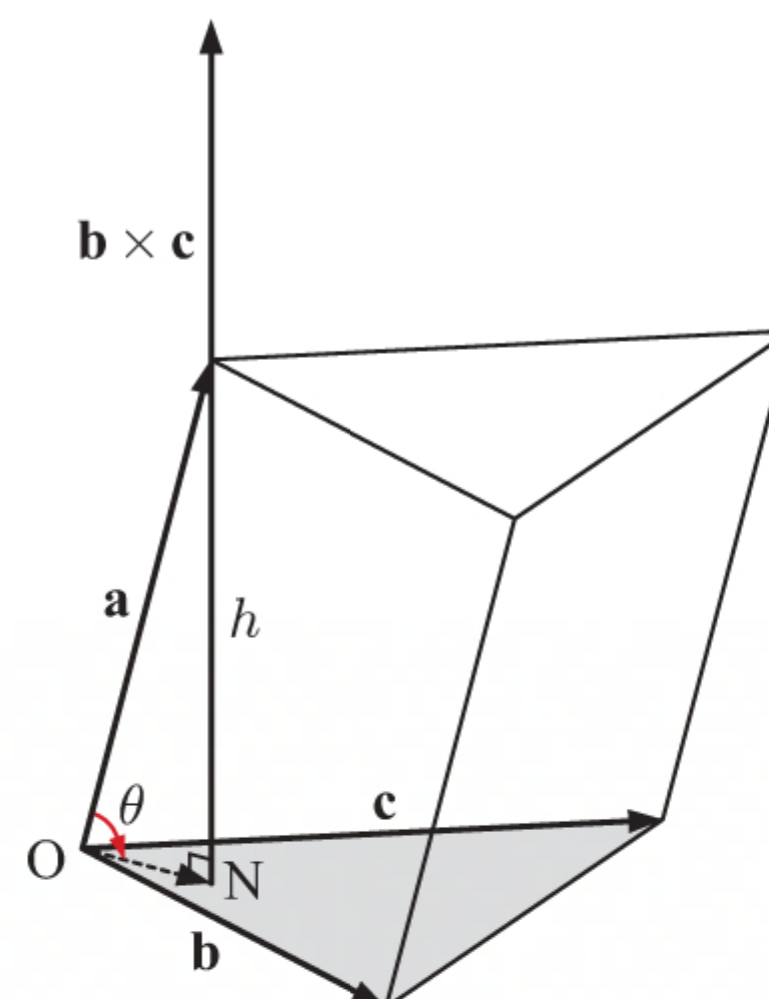
Now  $\theta$  is the angle between  $\mathbf{a}$  and the base plane.

$$\therefore \theta = \sin^{-1} \left( \frac{|(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}|}{|\mathbf{b} \times \mathbf{c}| |\mathbf{a}|} \right)$$

$$\therefore \sin \theta = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a}| |\mathbf{b} \times \mathbf{c}|}$$

$$\therefore |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \sin \theta$$

$$\text{So, volume} = \frac{1}{2} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \text{ units}^3.$$

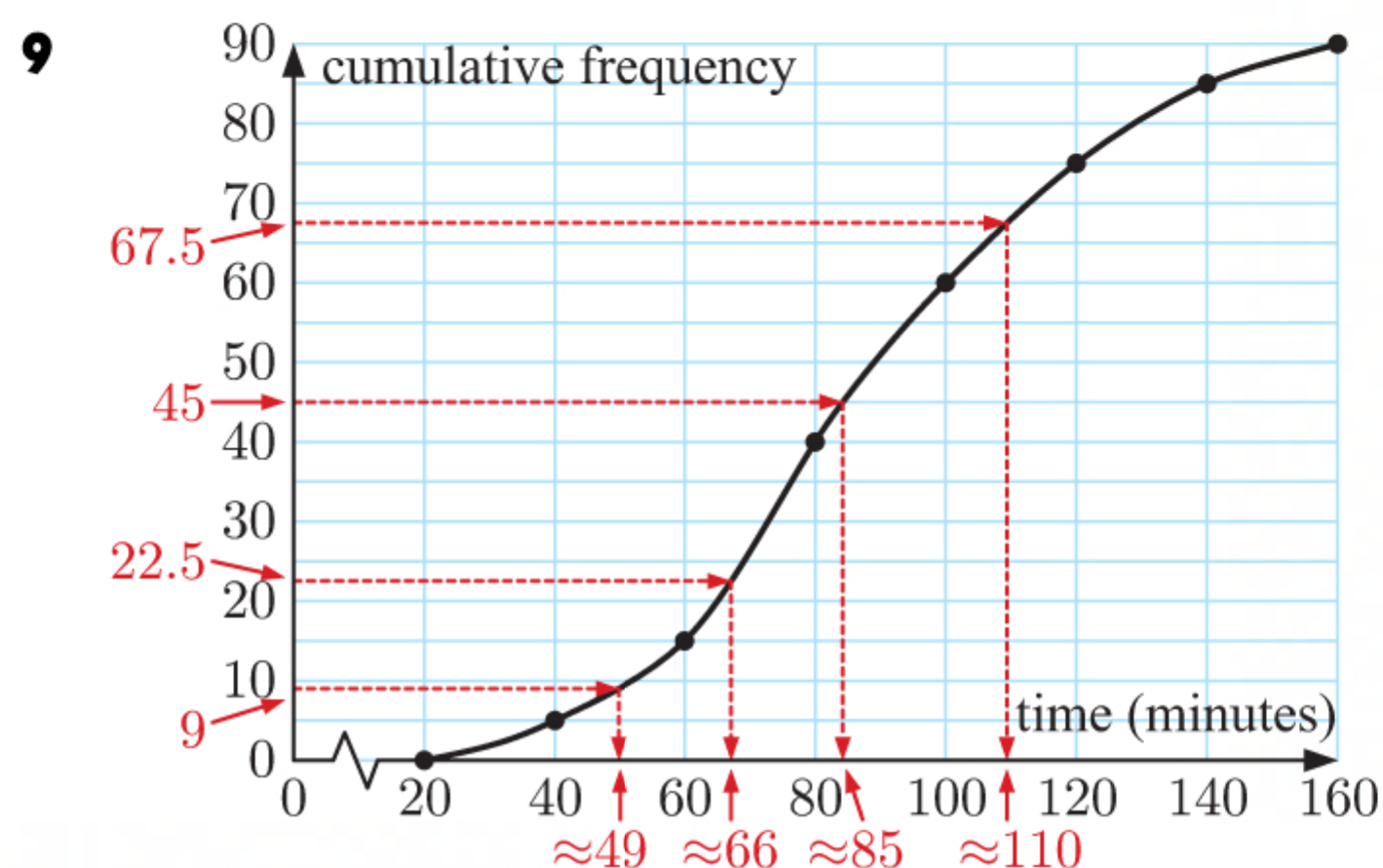




$$\text{c } \vec{OA} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{OB} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{OC} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \vec{OB} \times \vec{OC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k} \\ &= 4\mathbf{i} - 6\mathbf{k} \\ &= \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{Volume of shape} &= \frac{1}{2} \left| \vec{OA} \bullet (\vec{OB} \times \vec{OC}) \right| \\ &= \frac{1}{2} \left| \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} \right| \\ &= \frac{1}{2} |8 - 6| \\ &= 1 \text{ unit}^3 \end{aligned}$$



a From the graph, 90 games were played.

b The median corresponds to cumulative frequency = 45.

Hence, the median game length is about 85 minutes.

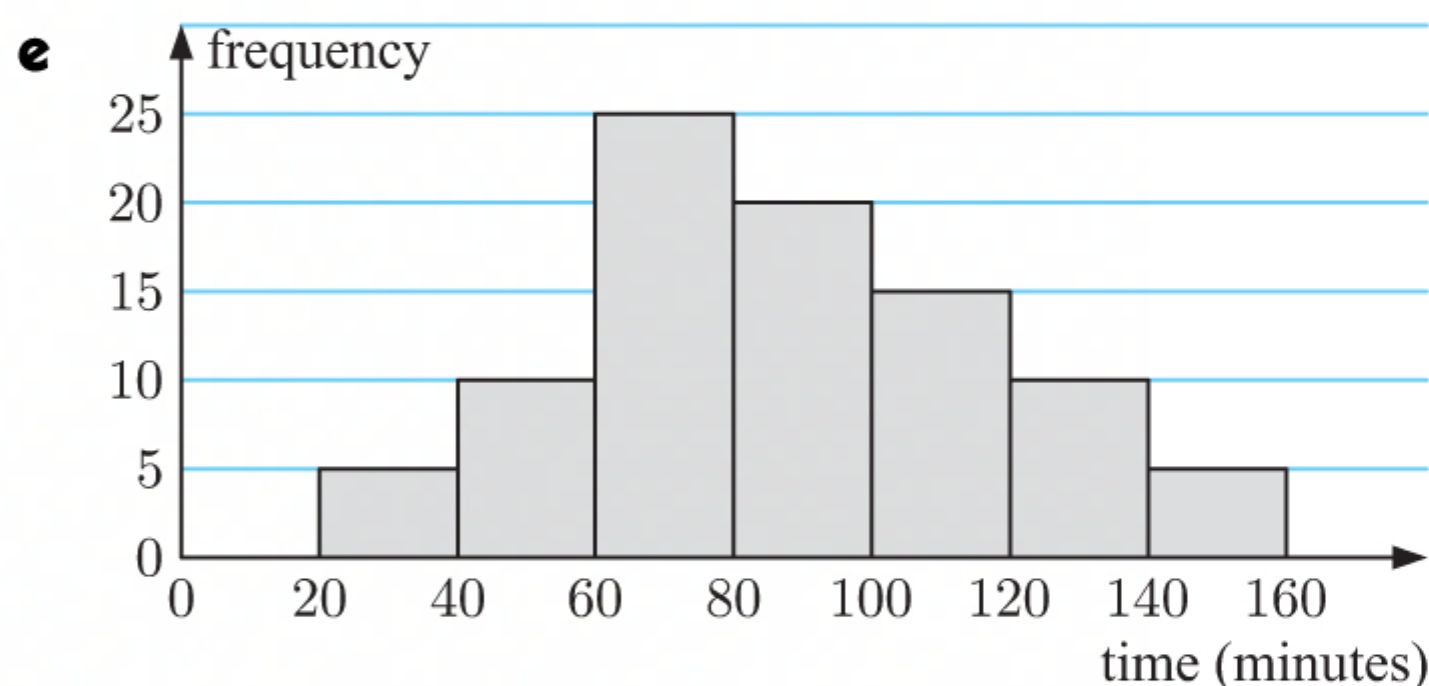
c  $IQR = Q_3 - Q_1$

$Q_3$  corresponds to cumulative frequency = 67.5 which is ≈ 110 minutes.

$Q_1$  corresponds to cumulative frequency = 22.5 which is ≈ 66 minutes.

∴  $IQR \approx 110 - 66 = 44$  minutes.

d The 10th percentile corresponds to cumulative frequency = 9 which is ≈ 49 minutes.





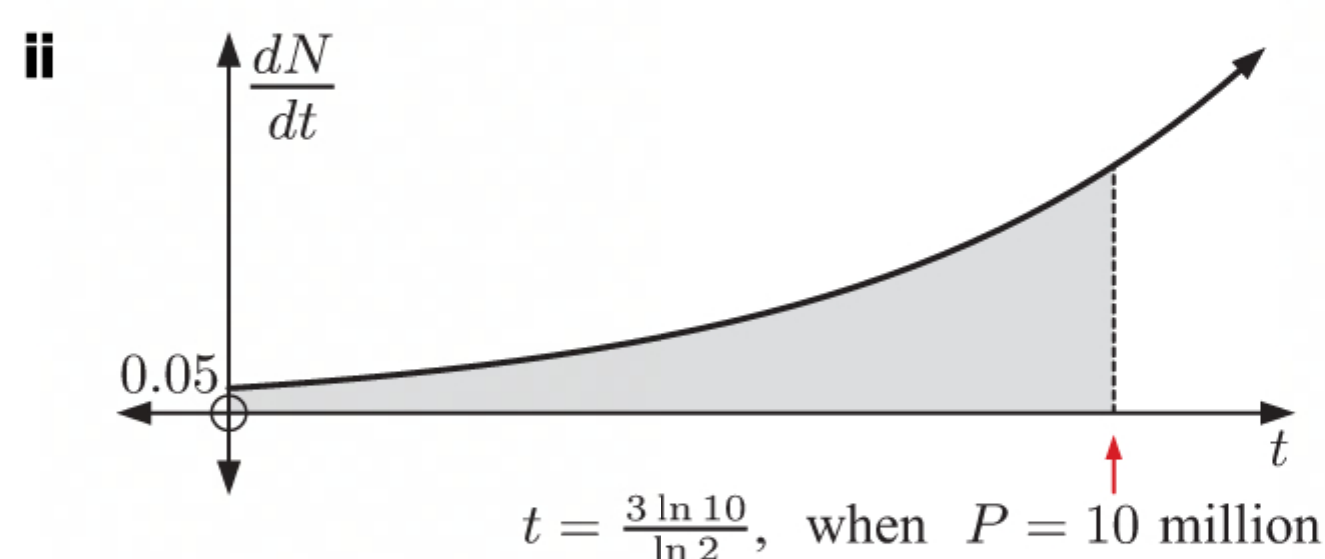
**10 a**  $\frac{dP}{dt} \propto P$   
 $\therefore \frac{dP}{dt} = kP$ , where  $k$  is a constant  
 $\therefore \frac{1}{P} \frac{dP}{dt} = k$   
 $\therefore \int \frac{1}{P} \frac{dP}{dt} dt = \int k dt$   
 $\therefore \int \frac{1}{P} dP = kt + c$   
 $\therefore \ln|P| = kt + c$   
 $\therefore \ln P = kt + c$  {as  $P > 0$ }  
 $\therefore P = e^{kt+c}$   
 $\therefore P = Ae^{kt}$  { $A = e^c$ }

**d i**  $\frac{dN}{dt} \propto P$   
 $\therefore \frac{dN}{dt} = dP$  where  $d$  is a constant  
 $\therefore \frac{dN}{dt} = de^{\frac{t}{3} \ln 2}$   
 But when  $t = 0$ ,  $\frac{dN}{dt} = 0.05$   
 $\therefore 0.05 = de^0 = d$   
 So,  $\frac{dN}{dt} = 0.05e^{\frac{t}{3} \ln 2}$  units  $\text{h}^{-1}$

**b** When  $t = 0$ ,  $P = 1$  million,  $\therefore 1 = Ae^0 = A$   
 $\therefore P = e^{kt}$  million

When  $t = 3$ ,  $P = 2$  million  
 $\therefore 2 = e^{3k}$   
 $\therefore 3k = \ln 2$   
 $\therefore k = \frac{1}{3} \ln 2$  ( $\approx 0.2310$ )

**c**  $P = 10$  million when  $10 = e^{\frac{t}{3} \ln 2}$   
 $\frac{t}{3} \ln 2 = \ln 10$   
 $t = \frac{3 \ln 10}{\ln 2} \approx 9.966$  hours



Amount of nutrient consumed

$$\begin{aligned}
 &= \int_0^{\frac{3 \ln 10}{\ln 2}} 0.05e^{\frac{t}{3} \ln 2} dt \\
 &= 0.05 \left[ \frac{1}{\frac{1}{3} \ln 2} e^{\frac{t}{3} \ln 2} \right]_0^{\frac{3 \ln 10}{\ln 2}} \\
 &= \frac{0.05}{\frac{1}{3} \ln 2} \left[ e^{\frac{t}{3} \ln 2} \right]_0^{\frac{3 \ln 10}{\ln 2}} \\
 &= \frac{0.15}{\ln 2} [10 - 1] \\
 &= \frac{1.35}{\ln 2} \\
 &\approx 1.95 \text{ g}
 \end{aligned}$$

## MIXED QUESTIONS SET 19

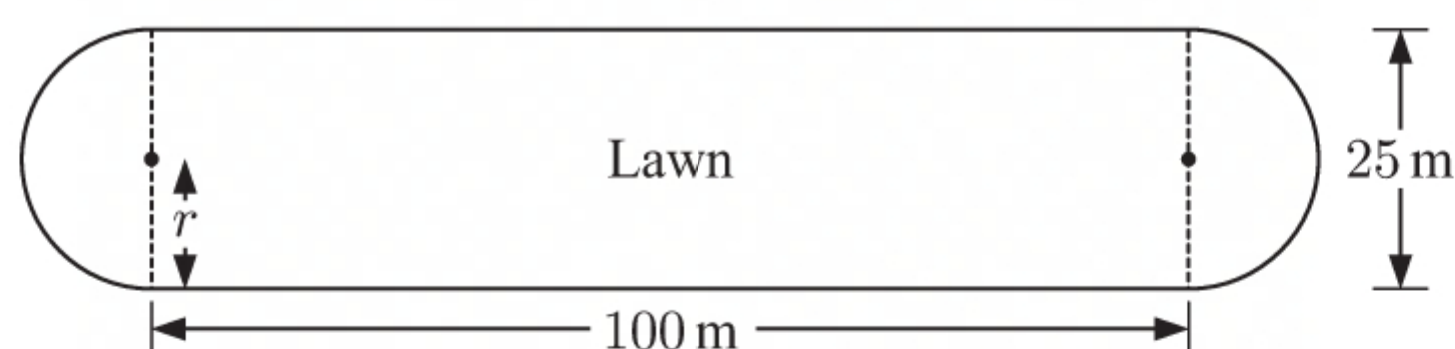
**1 a** Area = area of two semi-circles + area of rectangle  
 $= \pi r^2 + 100 \times 2r$   
 $= \pi \times 12.5^2 + 100 \times 25$   
 $= (156.25\pi + 2500) \text{ m}^2$

**b** Using  $\pi \approx 3$ , area  $\approx 156.25 \times 3 + 2500$   
 $\approx 2969 \text{ m}^2$

**2**  $V(t) = 10t^2 - \frac{1}{3}t^3$ ,  $0 \leq t \leq 30$

**a**  $V(5) = 10(5)^2 - \frac{1}{3}(5)^3$   
 $= 250 - \frac{1}{3}(125)$   
 $= 208\frac{1}{3}$  litres

After 5 minutes there is  $208\frac{1}{3}$  litres of water in the tank.



**c** Percentage error  
 $= \frac{|V_A - V_E|}{V_E} \times 100\%$   
 $= \frac{|2969 - (156.25\pi + 2500)|}{156.25\pi + 2500} \times 100\%$   
 $\approx 0.73\%$  {2 significant figures}

**b**  $V'(t) = 20t - t^2$  litres per minute



$$\begin{aligned} \text{c} \quad V'(t) &= 0 \\ \therefore 20t - t^2 &= 0 \\ \therefore t(20 - t) &= 0 \\ \therefore t &= 0 \text{ or } 20 \end{aligned}$$

$$\begin{aligned} \text{e} \quad V'(t) &= 75 \\ \therefore 20t - t^2 &= 75 \\ \therefore t^2 - 20t + 75 &= 0 \\ \therefore (t - 5)(t - 15) &= 0 \\ \therefore t &= 5 \text{ or } 15 \end{aligned}$$

So, the volume is increasing by 75 litres per minute at times 5 minutes and 15 minutes.

- 3 a** Let Maggie's eye level be at M, the car be at C, and the base of the building be at B.

$$\begin{aligned} MB &= \text{Maggie's height} + \text{building height} \\ &= 51.55 \text{ m} \end{aligned}$$

Now  $\widehat{MCB} = 67^\circ$  {alternate angles}

$$\begin{aligned} \therefore \tan 67^\circ &= \frac{51.55}{d} \\ \therefore d &= \frac{51.55}{\tan 67^\circ} \approx 21.9 \end{aligned}$$

So the car is about 21.9 m away from the base of the building.

- b** Let S be Sven's location.

$$\begin{aligned} \text{i} \quad \text{In } \triangle MBC, \quad \sin 67^\circ &= \frac{51.55}{MC} \\ \therefore MC &= \frac{51.55}{\sin 67^\circ} \\ &\approx 56.0 \text{ m} \end{aligned}$$

Since the car is directly opposite to Maggie,  $\triangle MCS$  is right angled at C.

$$\therefore MS^2 \approx 10^2 + 56.0^2 \quad \{\text{Pythagoras}\}$$

$$\therefore MS \approx \sqrt{3236} \approx 56.9 \text{ m}$$

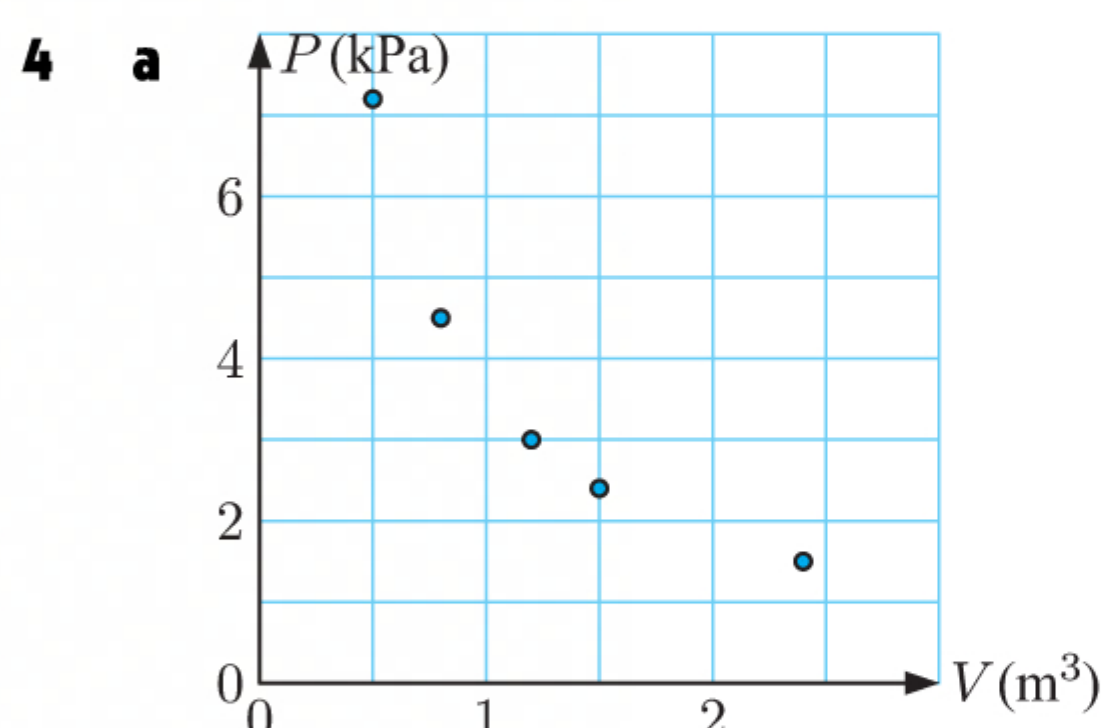
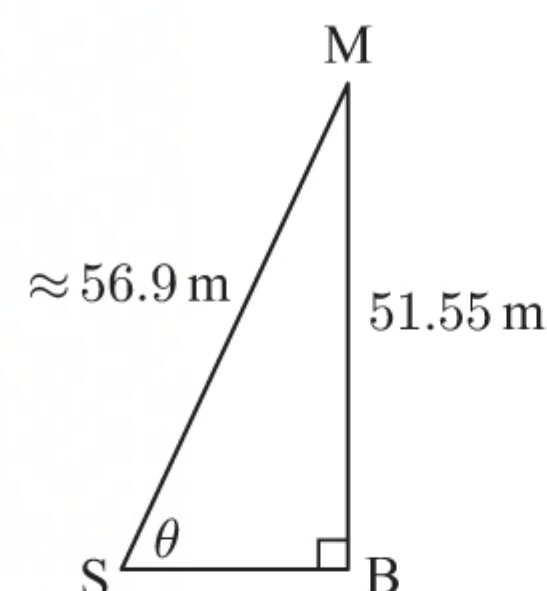
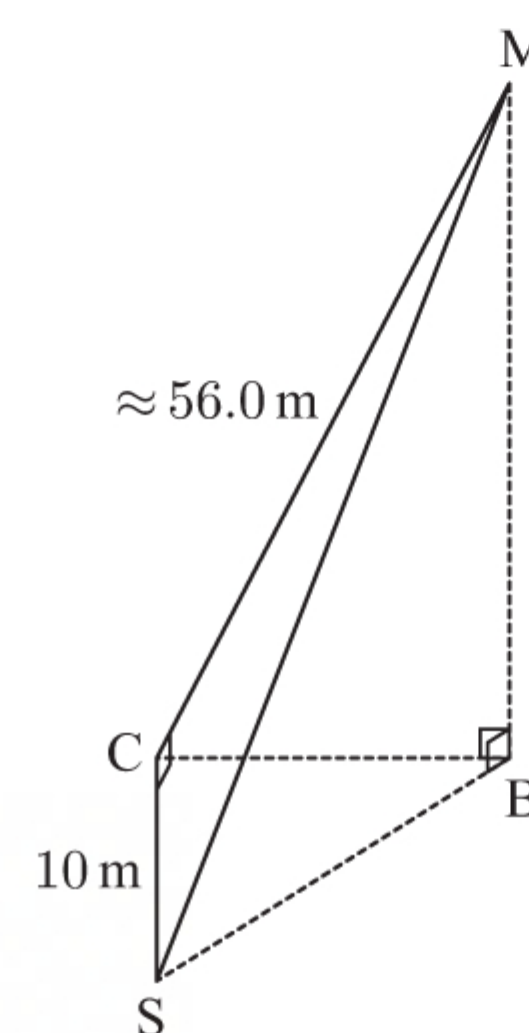
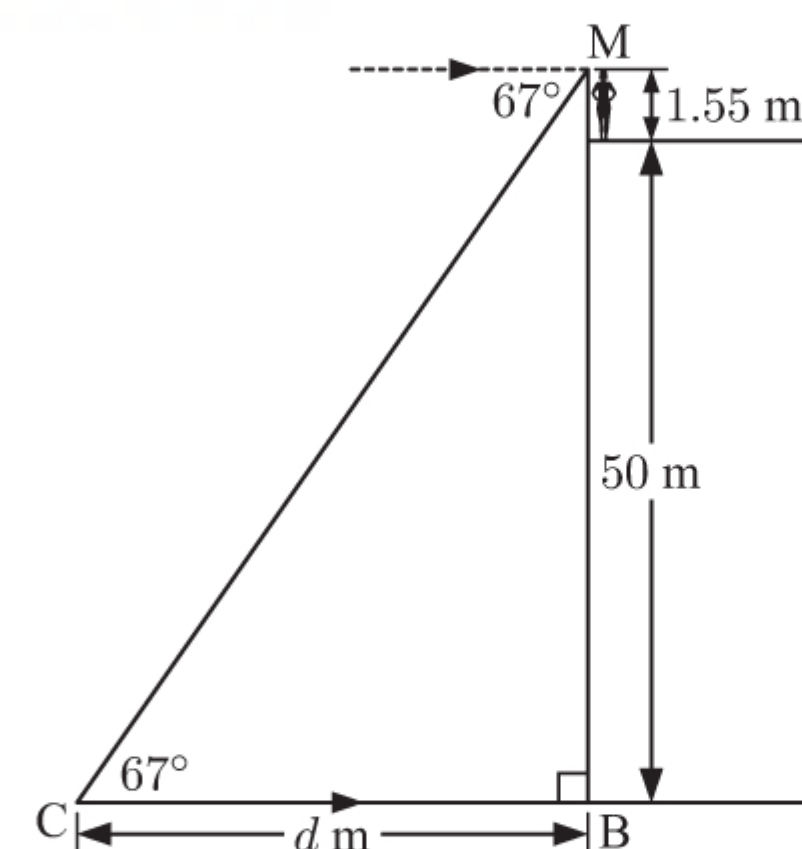
The distance between Maggie and Sven is about 56.9 m.

- ii** Let  $\theta$  be the angle of elevation from Sven to Maggie.

Now  $\triangle MBS$  is right angled at B.

$$\begin{aligned} \therefore \sin \theta &\approx \frac{51.55}{56.9} \\ \therefore \theta &\approx \sin^{-1}\left(\frac{51.55}{56.9}\right) \approx 65.0^\circ \end{aligned}$$

Sven needs to look up at an angle of about  $65.0^\circ$  to see Maggie.



The points appear to lie on a curve for which the axes are both asymptotes. This suggests that an inverse variation model is appropriate.



- b** If  $V$  and  $P$  are inversely proportional, then  $P = \frac{k}{V}$  for some constant  $k$ .

When  $V = 0.5$ ,  $P = 7.2$ , so  $7.2 = \frac{k}{0.5}$   
 $\therefore k = 7.2 \times 0.5 = 3.6$

Check: When  $V = 0.8$ ,  $P = \frac{3.6}{0.8} = 4.5$  ✓

$V = 1.2$ ,  $P = \frac{3.6}{1.2} = 3$  ✓

$V = 1.5$ ,  $P = \frac{3.6}{1.5} = 2.4$  ✓

$V = 2.4$ ,  $P = \frac{3.6}{2.4} = 1.5$  ✓

$\therefore$  the model connecting  $V$  and  $P$  is  $P = \frac{3.6}{V}$ .

- c** When  $V = 3$ ,  $P = \frac{3.6}{3} = 1.2$  kPa

- 5 a** The interest is calculated annually, so  $n = 7$  time periods.

$$\begin{aligned} u_7 &= u_0 \times (1 + i)^7 \\ &= 2000 \times (1.0825)^7 \quad \{8.25\% = 0.0825\} \\ &\approx 3484 \end{aligned}$$

The total value of Kapil's investment on January 1st 2019 is 3484 rupees.

- b** There are  $n = 7 \times 12 = 84$  time periods.

Each time period the investment increases by

$$i = \frac{8\%}{12} \approx 0.6667\%$$

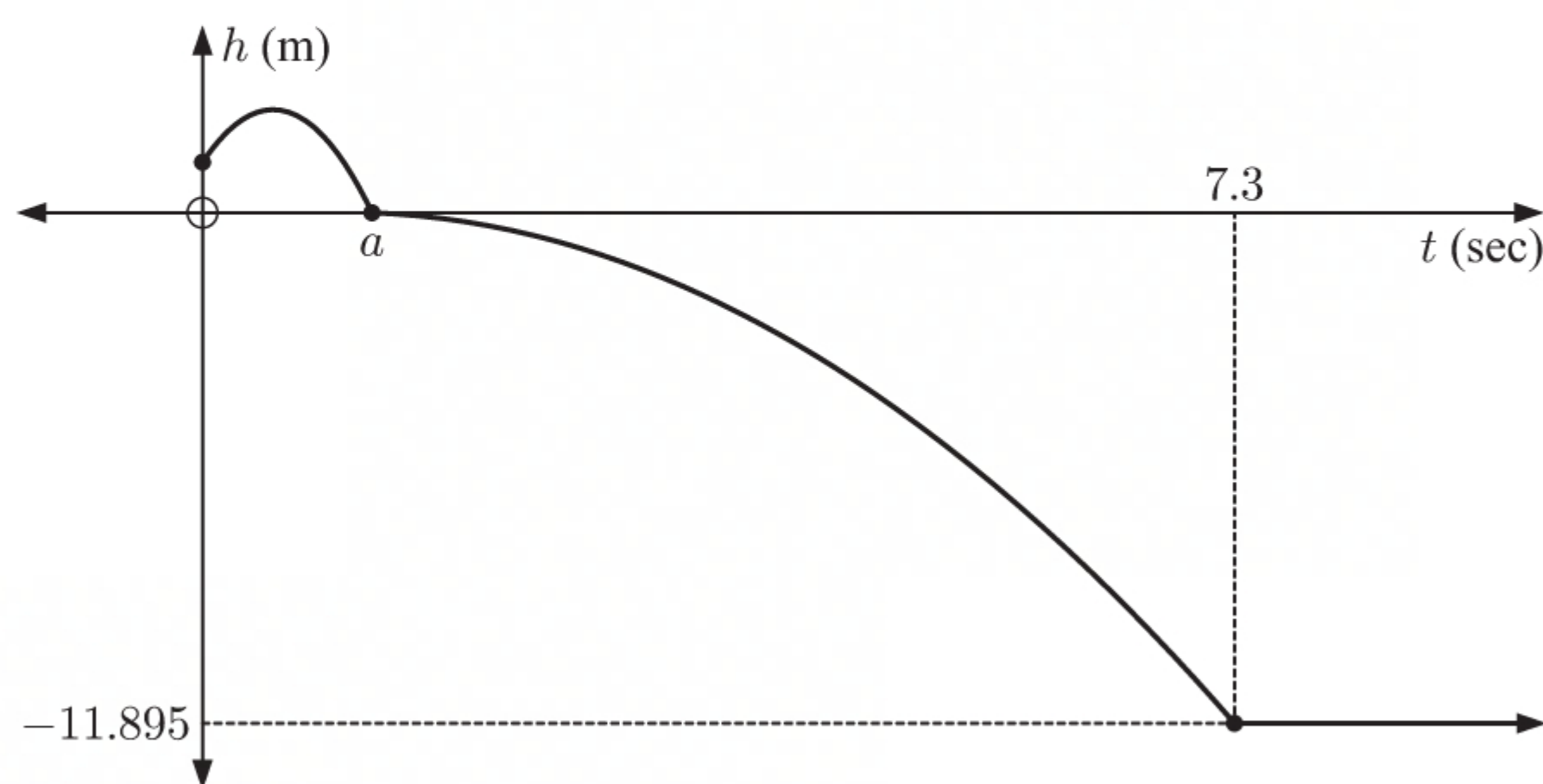
$\therefore$  the value after 7 years is

$$\begin{aligned} u_{84} &= u_0 \times (1 + i)^{84} \\ &\approx 2000 \times (1.006667)^{84} \quad \{0.6667\% = 0.006667\} \\ &\approx 3495 \end{aligned}$$

The total value of Kapil's investment on January 1st 2019 would be 3495 rupees.

$\therefore$  investing in the account paying 8% per annum interest compounded monthly is the better option.

**6**



- a i** From the graph, it appears that the rock reaches the bottom of the lake after 7.3 seconds.

- ii** When  $t \geq 7.3$ ,  $h(t) = -11.895$

$\therefore$  the lake is 11.895 m deep.

- b** From the graph, when  $t = a$ ,  $-4.9t^2 + 4.9t + 1.176 = -0.3t^2 + 0.6t - 0.288$

$$\therefore -4.9a^2 + 4.9a + 1.176 = -0.3a^2 + 0.6a - 0.288$$

$$\therefore -4.6a^2 + 4.3a + 1.464 = 0$$

$$\text{Using technology, } a = 1.2 \quad \{a > 0\}$$

The rock hits the surface of the lake after 1.2 seconds.

- c** 
$$h(t) = \begin{cases} -4.9t^2 + 4.9t + 1.176, & 0 \leq t < 1.2 \\ -0.3t^2 + 0.6t - 0.288, & 1.2 \leq t < 7.3 \\ -11.895, & t \geq 7.3 \end{cases}$$

- i** 
$$\begin{aligned} h(1) &= -4.9(1)^2 + 4.9(1) + 1.176 \quad \{0 \leq t < 1.2\} \\ &= 1.176 \end{aligned}$$

After 1 second, the rock is 1.176 m above the surface of the lake.



$$\text{ii } h(5) = -0.3(5)^2 + 0.6(5) - 0.288 \quad \{1.2 \leq t < 7.3\}$$

$$= -4.788$$

After 5 seconds, the rock is 4.788 m below the surface of the lake.

$$\text{iii } h(9) = -11.895 \quad \{t \geq 7.3\}$$

After 9 seconds, the rock is 11.895 m below the surface of the lake.

$$\text{d For } 0 \leq t < 1.2, \quad h(t) = -4.9t^2 + 4.9t + 1.176$$

$$\therefore h'(t) = -9.8t + 4.9$$

$$\text{Now } h'(t) = -1 \text{ when } -9.8t + 4.9 = -1$$

$$\therefore 9.8t = 5.9$$

$$\therefore t = \frac{5.9}{9.8} \approx 0.602$$

$$\text{For } 1.2 \leq t < 7.3, \quad h(t) = -0.3t^2 + 0.6t - 0.288$$

$$\therefore h'(t) = -0.6t + 0.6$$

$$\text{Now } h'(t) = -1 \text{ when } -0.6t + 0.6 = -1$$

$$\therefore 0.6t = 1.6$$

$$\therefore t = \frac{1.6}{0.6} \approx 2.67$$

$$\text{For } t \geq 7.3, \quad h(t) = -11.895$$

$$\therefore h'(t) = 0 \text{ which is never } -1.$$

So, the rock is falling at  $1 \text{ m s}^{-1}$  after about 0.602 seconds and 2.67 seconds.

$$\text{7 Volume of revolution about } x\text{-axis } V_X = \pi \int_0^{\sqrt{k}} y^2 dx$$

$$= \pi \int_0^{\sqrt{k}} (k - x^2)^2 dx$$

$$= \pi \int_0^{\sqrt{k}} (k^2 - 2kx^2 + x^4) dx$$

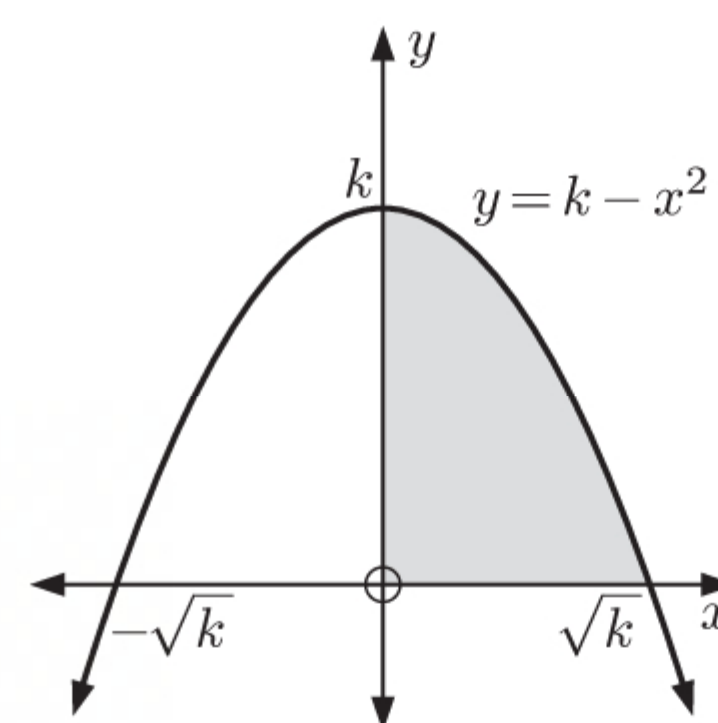
$$= \pi \left[ k^2x - \frac{2}{3}kx^3 + \frac{1}{5}x^5 \right]_0^{\sqrt{k}}$$

$$= \pi \left( k^2\sqrt{k} - \frac{2}{3}k(\sqrt{k})^3 + \frac{1}{5}(\sqrt{k})^5 \right)$$

$$= \pi \left( k^2\sqrt{k} - \frac{2}{3}k^2\sqrt{k} + \frac{1}{5}k^2\sqrt{k} \right)$$

$$= \pi k^2\sqrt{k} \left( 1 - \frac{2}{3} + \frac{1}{5} \right)$$

$$= \frac{8}{15}\pi k^2\sqrt{k} \text{ units}^3$$



$$\text{Volume of revolution about } y\text{-axis } V_Y = \pi \int_0^k x^2 dy$$

$$= \pi \int_0^k (k - y) dy$$

$$= \pi \left[ ky - \frac{1}{2}y^2 \right]_0^k$$

$$= \pi \left( k^2 - \frac{1}{2}k^2 \right)$$

$$= \frac{1}{2}\pi k^2 \text{ units}^3$$

$$\text{Now } V_X = V_Y$$

$$\therefore \frac{8}{15}\pi k^2\sqrt{k} = \frac{1}{2}\pi k^2$$

$$\therefore \frac{8}{15}\sqrt{k} = \frac{1}{2} \quad \{k > 0\}$$

$$\therefore \sqrt{k} = \frac{15}{16}$$

$$\therefore k = \frac{225}{256}$$

$$\text{8 a } \mathbf{T} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$$



$$\begin{aligned}
\mathbf{b} \quad & \text{If } \det(\lambda \mathbf{I} - \mathbf{T}) = 0 \text{ then } \begin{vmatrix} \lambda - 0.9 & -0.3 \\ -0.1 & \lambda - 0.7 \end{vmatrix} = 0 \\
& \therefore (\lambda - 0.9)(\lambda - 0.7) - 0.3 = 0 \\
& \therefore \lambda^2 - 1.6\lambda + 0.63 - 0.03 = 0 \\
& \therefore \lambda^2 - 1.6\lambda + 0.6 = 0 \\
& \therefore (\lambda - 1)(\lambda - 0.6) = 0 \\
& \therefore \lambda = 1 \text{ or } 0.6
\end{aligned}$$

The eigenvalues are 1 and 0.6.

For  $\lambda_1 = 1$ , consider  $(\lambda \mathbf{I} - \mathbf{T})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{aligned}
\therefore \begin{pmatrix} 0.1 & -0.3 \\ -0.1 & 0.3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\therefore 0.1a - 0.3b &= 0 \\
\therefore a - 3b &= 0
\end{aligned}$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = 3t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$  choosing  $t = 1$ ,  $\mathbf{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue  $\lambda_1 = 1$ .

For  $\lambda_2 = 0.6$ , consider  $(\lambda \mathbf{I} - \mathbf{T})\mathbf{x} = \mathbf{0}$  with  $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{aligned}
\therefore \begin{pmatrix} -0.3 & -0.3 \\ -0.1 & -0.1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\therefore -0.1a - 0.1b &= 0 \\
\therefore a + b &= 0
\end{aligned}$$

Letting  $b = t$ ,  $t \neq 0$ , then  $a = -t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$  choosing  $t = 1$ ,  $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue  $\lambda_2 = 0.6$ .

$\mathbf{c}$  The matrix  $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$  diagonalises  $\mathbf{T}$  with  $\mathbf{P}^{-1}\mathbf{TP} = \begin{pmatrix} 1 & 0 \\ 0 & 0.6 \end{pmatrix}$ .

$$\mathbf{d} \quad \mathbf{s}_0 = \begin{pmatrix} 4000 \\ 1000 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{i} \quad \mathbf{T}^n &= \mathbf{P} \begin{pmatrix} 1^n & 0 \\ 0 & (0.6)^n \end{pmatrix} \mathbf{P}^{-1} \\
&= \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.6)^n \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -(0.6)^n & 3 \times (0.6)^n \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 3 + (0.6)^n & 3 - 3 \times (0.6)^n \\ 1 - (0.6)^n & 1 + 3 \times (0.6)^n \end{pmatrix}
\end{aligned}$$

Now  $\mathbf{s}_n = \mathbf{T}^n \mathbf{s}_0$

$$\begin{aligned}
&= \frac{1}{4} \begin{pmatrix} 3 + (0.6)^n & 3 - 3 \times (0.6)^n \\ 1 - (0.6)^n & 1 + 3 \times (0.6)^n \end{pmatrix} \begin{pmatrix} 4000 \\ 1000 \end{pmatrix} \\
&= \begin{pmatrix} 3 + (0.6)^n & 3 - 3 \times (0.6)^n \\ 1 - (0.6)^n & 1 + 3 \times (0.6)^n \end{pmatrix} \begin{pmatrix} 1000 \\ 250 \end{pmatrix} \\
&= \begin{pmatrix} 3000 + 1000(0.6)^n + 750 - 750(0.6)^n \\ 1000 - 1000(0.6)^n + 250 + 750(0.6)^n \end{pmatrix} \\
&= \begin{pmatrix} 3750 + 250(0.6)^n \\ 1250 - 250(0.6)^n \end{pmatrix}
\end{aligned}$$

$\therefore$  the number of employed adults after  $n$  years is  $3750 + 250(0.6)^n$ .



- ii As  $n \rightarrow \infty$ ,  $(0.6)^n \rightarrow 0$   
 $\therefore 3750 + 250(0.6)^n \rightarrow 3750$

In the long term, the town will have 3750 employed adults.

- 9 a The net at U is closest to P.

b

	① P	⑤ Q	⑥ R	③ S	④ T	② U
P	—	620	900	750	600	500
Q	620	—	500	800	280	730
R	900	500	—	350	300	600
S	750	800	350	—	310	300
T	600	280	300	310	—	340
U	500	730	600	300	340	—

From P, the nearest net is at U.

From U, the nearest unvisited net is at S.

From S, the nearest unvisited net is at T.

From T, the nearest unvisited net is at Q.

From Q, we must visit the net at R, then return to P.

The Hamiltonian cycle is  $P \rightarrow U \rightarrow S \rightarrow T \rightarrow Q \rightarrow R \rightarrow P$ .

So, an upper bound for the distance travelled is  $500 + 300 + 310 + 280 + 500 + 900 = 2790$  m.

- c The route  $P \rightarrow Q \rightarrow T \rightarrow R \rightarrow U \rightarrow S \rightarrow P$  has distance  $620 + 280 + 300 + 600 + 300 + 750 = 2850$  m.

This is larger than the upper bound in b, so this route cannot be the shortest route.

- d Deleting Q and the edges connected to it, the minimum spanning tree has edges RT, SU, ST, and PU.

The minimum spanning tree has weight  $300 + 300 + 310 + 500 = 1410$ .

The two shortest deleted edges have weight 280 and 500.

So, a lower bound for the distance travelled is  $1410 + 280 + 500 = 2190$  m.

- 10 a i  $X \sim B(4, 0.5)$

ii (1)  $P(X = 0) = 0.0625$

(2)  $P(\text{at least two})$

$= P(X \geq 2)$

$= 1 - P(X \leq 1)$

$= 1 - 0.3125$

$= 0.6875$

(3)  $P(X = 4) = 0.0625$

iii  $P(\text{all girls or all boys}) = P(\text{all girls}) + P(\text{no girls})$

$= 0.0625 + 0.0625$

$= 0.125$

- b i Under  $H_0$ ,  $X \sim B(4, 0.5)$  {from a i}

From a iii,  $p = P(\text{all girls or all boys}) = 0.125$

$\therefore$  the null hypothesis can be written as  $H_0: p = 0.125$ .

- ii A value of  $p > 0.125$  indicates that the probability of a newborn child being a girl is *not* 0.5.

- iii Step 1: The hypotheses to be considered are:

$H_0: p = 0.125$

$H_1: p > 0.125$

Step 2: The significance level is  $\alpha = 0.1$ .

Step 3: The observed value of the test statistic is  $y = 21$ .

Step 4: The null distribution is  $Y \sim B(120, 0.125)$ . The alternative hypothesis is  $H_1: p > 0.125$ , so we use the upper tail of the null distribution.

$p\text{-value} = P(Y \geq 21)$

$\approx 0.0692$  {technology}

Step 5: Since  $p\text{-value} < 0.1 = \alpha$ , we have enough evidence to reject  $H_0$  in favour of  $H_1$  on the  $\alpha = 0.1$  significance level. We therefore accept  $H_1$ .

Step 6: Since we have accepted  $H_1$ , we conclude that the probability of a newborn child being a girl is not 0.5.



## MIXED QUESTIONS SET 20

**1 a i**  $P = 1000 + ae^{kn}$

The initial population was 2000.

So, when  $n = 0$ ,  $P = 2000$

$$\therefore 2000 = 1000 + ae^0$$

$$\therefore a = 1000$$

**b** Using **a**,  $P = 1000 + 1000 \times 3^{\frac{n}{12}}$

Now  $P = 15\,000$  when  $15\,000 = 1000 + 1000 \times 3^{\frac{n}{12}}$

Using technology,  $n \approx 28.8$ .

$\therefore$  it will take about 28.8 months for the population to reach 15 000.

**ii**  $P = 1000 + 1000e^{kn}$

After 1 year, the population was 4000.

So, when  $n = 12$ ,  $P = 4000$

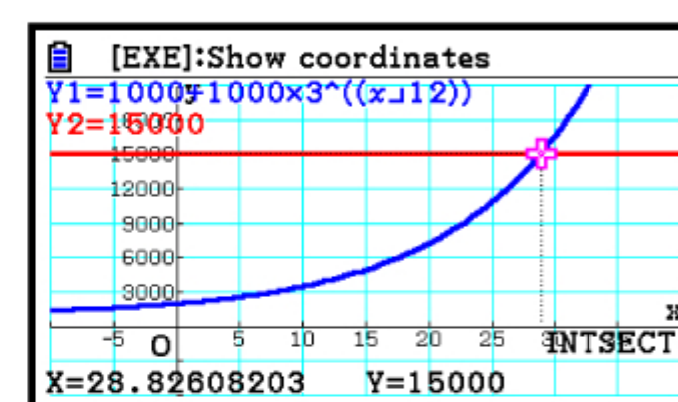
$$\therefore 4000 = 1000 + 1000e^{k \times 12}$$

$$\therefore 3000 = 1000e^{12k}$$

$$\therefore e^{12k} = 3$$

$$\therefore e^k = 3^{\frac{1}{12}}$$

$$\therefore k = \frac{1}{12} \ln 3 \approx 0.0916$$



**2 a**

		Die 2			
		1	2	3	4
Die 1	1	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

**b i**  $P(S = 5) = \frac{4}{16} = \frac{1}{4}$

**ii**  $P(S > 5) = \frac{6}{16} = \frac{3}{8}$

**iii**  $P(S = 5 \mid S > 3) = \frac{P(S = 5 \cap S > 3)}{P(S > 3)}$   
 $= \frac{P(S = 5)}{P(S > 3)}$   
 $= \frac{4}{13}$

**c**

Value of $S$	$S = 2$	$3 \leq S \leq 5$	$S > 5$
Number of points won or lost	32	16	-8
Probability	$\frac{1}{16}$	$\frac{9}{16}$	$\frac{3}{8}$

**i** The expected number of points  $= 32 \times \frac{1}{16} + 16 \times \frac{9}{16} + (-8) \times \frac{3}{8}$   
 $= 2 + 9 + (-3)$   
 $= 8$

**ii** Let  $k$  be the number of points for  $S = 2$ .

For the expectation to be zero,  $k \times \frac{1}{16} + 16 \times \frac{9}{16} + (-8) \times \frac{3}{8} = 0$

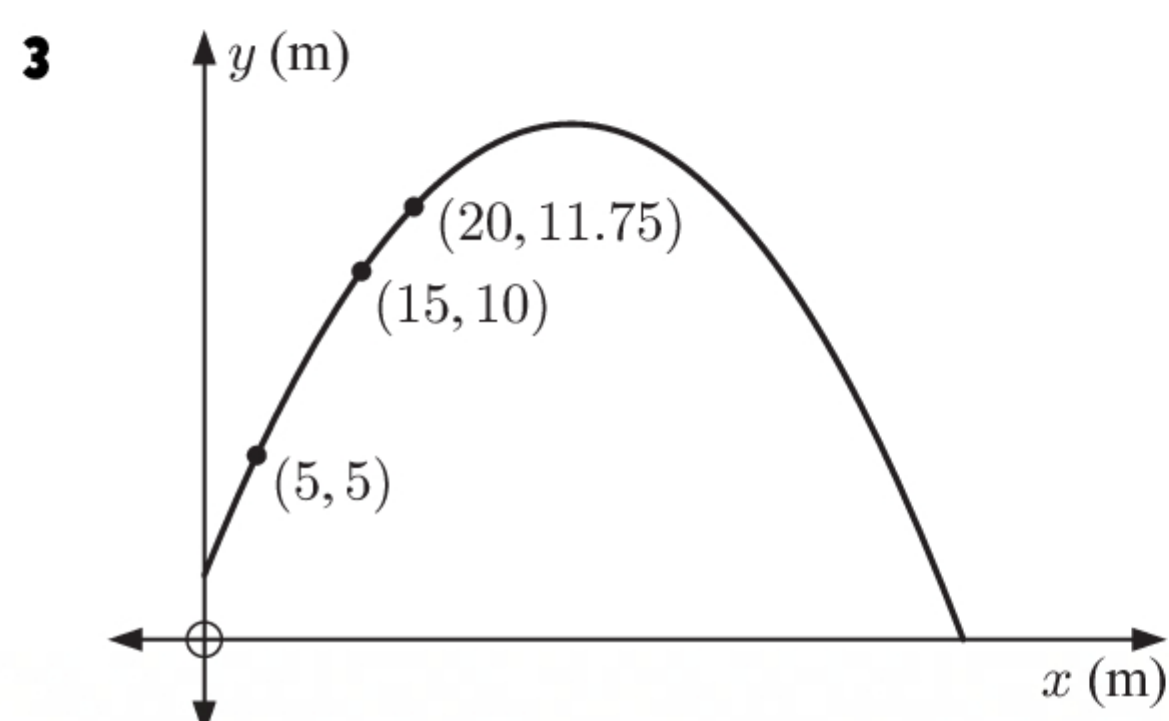
$$\therefore \frac{1}{16}k + 9 - 3 = 0$$

$$\therefore \frac{1}{16}k = -6$$

$$\therefore k = -96$$

So,  $S = 2$  should correspond to a 96 point loss.





**a**  $y = ax^2 + bx + c$

When  $x = 5$ ,  $y = 5$   $\therefore 5 = a(5)^2 + b(5) + c$  or  $25a + 5b + c = 5$

When  $x = 15$ ,  $y = 10$   $\therefore 10 = a(15)^2 + b(15) + c$  or  $225a + 15b + c = 10$

When  $x = 20$ ,  $y = 11.75$   $\therefore 11.75 = a(20)^2 + b(20) + c$  or  $400a + 20b + c = 11.75$

**b** We solve the system of equations

$$\begin{cases} 25a + 5b + c = 5 \\ 225a + 15b + c = 10 \\ 400a + 20b + c = 11.75 \end{cases}$$

simultaneously using technology.

We find that  $a = -0.01$ ,  $b = 0.7$ , and  $c = 1.75$ .

**c**  $y = -0.01x^2 + 0.7x + 1.75$  {using **b**}

Since  $a = -0.01 < 0$ , the shape is .

The maximum height occurs when  $x = -\frac{b}{2a} = -\frac{0.7}{2 \times (-0.01)} = 35$

When  $x = 35$ ,  $y = -0.01(35)^2 + 0.7(35) + 1.75$   
 $= 14$

So, the discus reached a maximum height of 14 m.

**d** When  $y = 0$ ,  $-0.01x^2 + 0.7x + 1.75 = 0$

Using technology,  $x \approx -2.42$  or  $\approx 72.4$

$\therefore x \approx 72.4$   $\{x \geq 0\}$

$\therefore$  the discus travelled about 72.4 metres before it hit the ground.

	a	b	c	d
1	25	5	1	5
2	225	15	1	10
3	400	20	1	11.75

	a	b	c	d
X	-0.01			
Y		0.7		
Z			1.75	

	a	b	c
X	-0.01	0.7	1.75

	a	b	c
X1	72.416		
X2	-2.416		

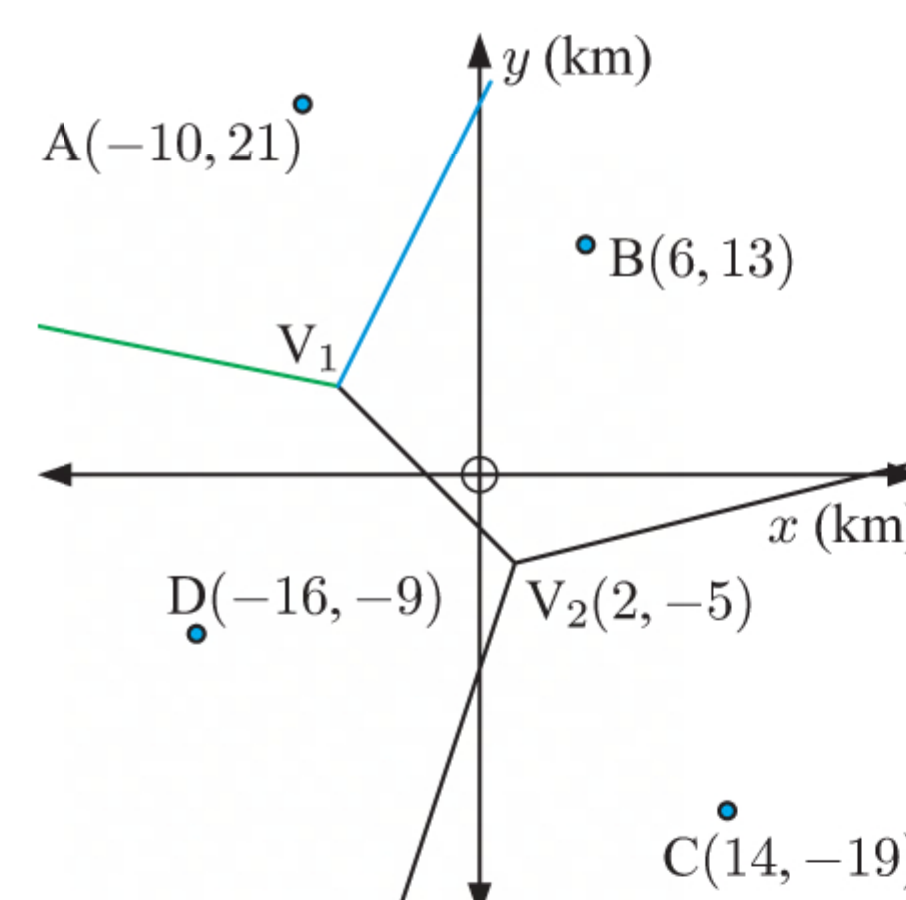
**4 a i** The blue edge is the perpendicular bisector of [AB].

The midpoint of [AB] is  $\left(\frac{-10+6}{2}, \frac{21+13}{2}\right)$  or  $(-2, 17)$ .

The gradient of [AB] is  $\frac{13-21}{6-(-10)} = \frac{-8}{16} = -\frac{1}{2}$ .

So, the blue edge has gradient 2.

$\therefore$  its equation is  $y = 2(x - (-2)) + 17$   
 $= 2(x + 2) + 17$   
 $= 2x + 4 + 17$   
 $= 2x + 21$





- ii** The green edge is the perpendicular bisector of [AD].

The midpoint of [AD] is  $\left(\frac{-10 + (-16)}{2}, \frac{21 + (-9)}{2}\right)$  or  $(-13, 6)$ .

The gradient of [AD] is  $\frac{-9 - 21}{-16 - (-10)} = \frac{-30}{-6} = 5$ .

So, the green edge has gradient  $-\frac{1}{5}$ .

$$\begin{aligned}\therefore \text{ its equation is } y &= -\frac{1}{5}(x - (-13)) + 6 \\ &= -\frac{1}{5}(x + 13) + 6 \\ &= -\frac{1}{5}x - \frac{13}{5} + 6 \\ &= -\frac{1}{5}x + \frac{17}{5}\end{aligned}$$

- b** The blue edge and green edge intersect where  $2x + 21 = -\frac{1}{5}x + \frac{17}{5}$   
 $\therefore \frac{11}{5}x = -\frac{88}{5}$   
 $\therefore 11x = -88$   
 $\therefore x = -8$

$$\begin{aligned}\text{When } x = -8, \quad y &= 2(-8) + 21 \quad \{\text{using the blue line}\} \\ &= -16 + 21 \\ &= 5\end{aligned}$$

$\therefore V_1$  has coordinates  $(-8, 5)$ .

- c i**  $V_1$  is equidistant from A, B, and D.

$$\begin{aligned}V_1A &= \sqrt{(-10 - (-8))^2 + (21 - 5)^2} \\ &= \sqrt{(-2)^2 + 16^2} \\ &= \sqrt{260} \text{ km}\end{aligned}$$

- $V_2$  is equidistant from B, C, and D.

$$\begin{aligned}V_2B &= \sqrt{(6 - 2)^2 + (13 - (-5))^2} \\ &= \sqrt{4^2 + 18^2} \\ &= \sqrt{340} \text{ km}\end{aligned}$$

So, the largest empty circle has centre  $V_2(2, -5)$  and radius  $\sqrt{340}$  km.

$\therefore$  Ivan should open the weekend retreat at  $V_2(2, -5)$ .

- ii** Towns B, C, and D are closest to the retreat.

- 5 a** From the graph,  $f(-2) = -1$

$$\begin{aligned}\therefore -2 + \frac{k}{(-2)^2} &= -1 \\ \therefore \frac{k}{4} &= 1 \\ \therefore k &= 4\end{aligned}$$

- b**  $f(x) = x + \frac{4}{x^2} = x + 4x^{-2}$

$$\therefore f'(x) = 1 - 8x^{-3} = 1 - \frac{8}{x^3}$$

- c** P is a stationary point of  $f(x)$ .

$$\begin{aligned}\text{Now } f'(x) = 0 \text{ where } 1 - \frac{8}{x^3} &= 0 \\ \therefore x^3 &= 8 \\ \therefore x &= 2\end{aligned}$$

$$\begin{aligned}f(2) &= 2 + \frac{4}{2^2} \\ &= 2 + 1 \\ &= 3\end{aligned}$$

So, P has coordinates  $(2, 3)$ .

$$\begin{aligned}\mathbf{d} \quad \int f(x) dx &= \int (x + 4x^{-2}) dx \\ &= \frac{1}{2}x^2 + \frac{4}{-1}x^{-1} + c \\ &= \frac{1}{2}x^2 - \frac{4}{x} + c\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad \text{Shaded area} &= \int_2^6 f(x) dx \\ &= \left[\frac{1}{2}x^2 - \frac{4}{x}\right]_2^6 \quad \{\text{using } \mathbf{d}\} \\ &= \left(\frac{1}{2}(6)^2 - \frac{4}{6}\right) - \left(\frac{1}{2}(2)^2 - \frac{4}{2}\right) \\ &= \frac{52}{3} - 0 \\ &= \frac{52}{3} \text{ units}^2\end{aligned}$$



- 6 a i** Let  $H_1(t)$  be given by the cosine model  
 $H_1(t) = a \cos(b(t - c)) + d, t \geq 0$ .

$$\text{The mean height of Alysa} = \frac{0+6}{2} = 3 \text{ m} \\ \therefore d = 3$$

$$\text{The amplitude of Alysa's height} = \frac{6}{2} = 3 \text{ m} \\ \therefore |a| = 3$$

$$\text{The period is 40 seconds, so } b = \frac{2\pi}{40} = \frac{\pi}{20}.$$

$$\text{The minimum occurs at } t = 0, \text{ so } c = 0 \text{ and } a = -3.$$

$$\therefore H_1(t) = 3 - 3 \cos\left(\frac{\pi}{20}t\right), t \geq 0.$$

**b**  $H_1(10) = 3 - 3 \cos \frac{\pi}{2} = 3$

$$H_2(10) = 5 - 5 \cos\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = 0$$

When Martha gets on the Ferris wheel, Alysa is 3 m higher than Martha.

$$\begin{aligned} \text{c } H_1(t) - H_2(t) &= 3 - 3 \cos\left(\frac{\pi}{20}t\right) - \left[5 - 5 \cos\left(\frac{\pi}{20}t - \frac{\pi}{2}\right)\right] \\ &= 5 \cos\left(\frac{\pi}{20}t - \frac{\pi}{2}\right) - 3 \cos\left(\frac{\pi}{20}t\right) - 2 \\ &= \operatorname{Re}\left(5 \operatorname{cis}\left(\frac{\pi}{20}t - \frac{\pi}{2}\right) - 3 \operatorname{cis}\left(\frac{\pi}{20}t\right)\right) - 2 \\ &= \operatorname{Re}\left(5e^{(\frac{\pi}{20}t - \frac{\pi}{2})i} - 3e^{\frac{\pi}{20}ti}\right) - 2 \end{aligned}$$

$$\begin{aligned} \text{Now } 5e^{(\frac{\pi}{20}t - \frac{\pi}{2})i} - 3e^{\frac{\pi}{20}ti} &= e^{\frac{\pi}{20}ti}(5e^{-\frac{\pi}{2}i} - 3) \\ &\approx e^{\frac{\pi}{20}ti}(5.83e^{-2.11i}) \\ &\approx 5.83e^{(\frac{\pi}{20}t - 2.11)i} \end{aligned}$$

$$\begin{aligned} \text{So, } H_1(t) - H_2(t) &\approx \operatorname{Re}\left(5.83e^{(\frac{\pi}{20}t - 2.11)i}\right) - 2 \\ &\approx 5.83 \cos\left(\frac{\pi}{20}t - 2.11\right) - 2 \end{aligned}$$

- d i**  $H_1(t) - H_2(t)$  has maximum  $\approx 5.83 - 2 \approx 3.83$ .

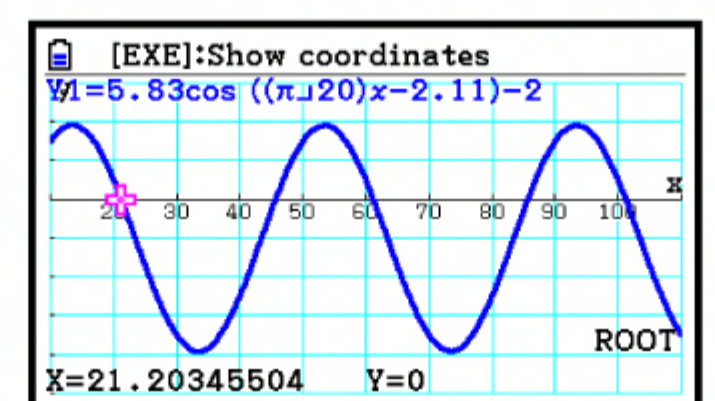
$\therefore$  the maximum amount by which Alysa is higher than Martha is about 3.83 m.

- ii**  $H_1(t) - H_2(t)$  has minimum  $\approx -5.83 - 2 \approx -7.83$ .

$\therefore$  the maximum amount by which Martha is higher than Alysa is about 7.83 m.

- e** Using technology, the first time where  $H_1(t) - H_2(t) = 0$  is  $t \approx 21.2 \quad \{t \geq 10\}$ .

$\therefore$  the first time at which Alysa and Martha are at the same height occurs about 21.2 seconds after Alysa gets on the Ferris wheel.



- 7 a** Emma's direction vector  $4\mathbf{i} + 4\mathbf{j} + \mathbf{k}$  has length  $\sqrt{4^2 + 4^2 + 1^2} = \sqrt{33}$ .

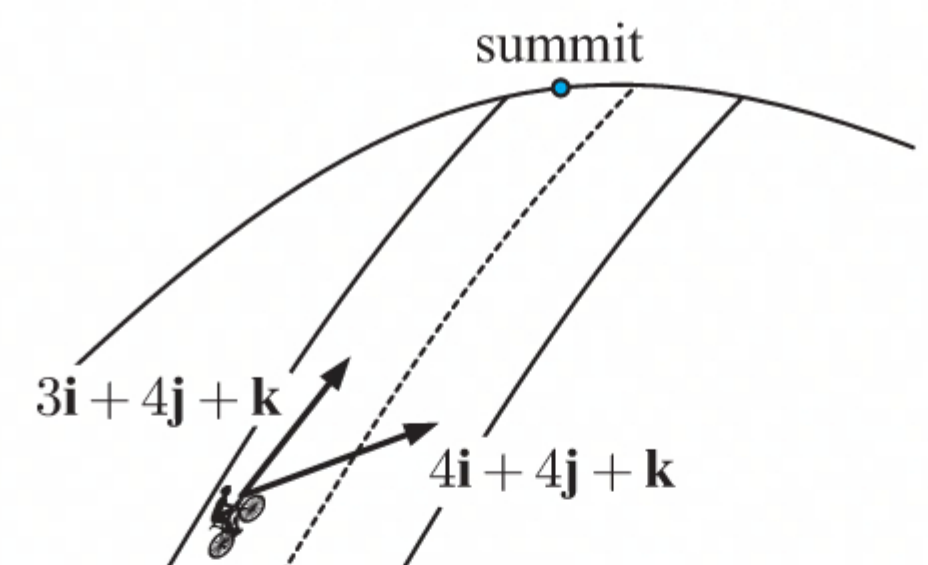
She cycles with speed  $3 \text{ m s}^{-1}$ .

$$\therefore \text{Emma's velocity vector is } \mathbf{a} = \frac{3}{\sqrt{33}} \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} = \frac{\sqrt{33}}{11} \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}.$$

- b** Let  $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$  be the direction of the summit.

$$\begin{aligned} \text{Component of } \mathbf{a} \text{ in the direction } \mathbf{b} &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \\ &= \frac{\frac{\sqrt{33}}{11}(4(3) + 4(4) + 1(1))}{\sqrt{3^2 + 4^2 + 1^2}} \\ &= \frac{29\sqrt{33}}{11\sqrt{26}} \approx 2.97 \end{aligned}$$

So, Emma is travelling about  $2.97 \text{ m s}^{-1}$  in the direction of the summit.





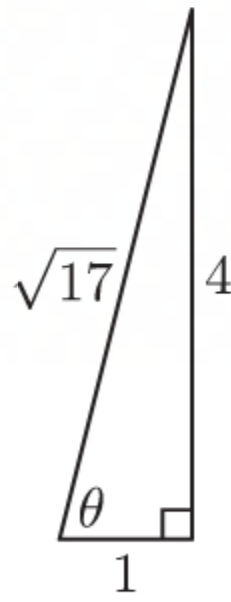
$$\begin{aligned}
 \mathbf{c} \quad \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{4\sqrt{33}}{11} & \frac{4\sqrt{33}}{11} & \frac{\sqrt{33}}{11} \\ 3 & 4 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} \frac{4\sqrt{33}}{11} & \frac{\sqrt{33}}{11} \\ 4 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{4\sqrt{33}}{11} & \frac{\sqrt{33}}{11} \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{4\sqrt{33}}{11} & \frac{4\sqrt{33}}{11} \\ 3 & 4 \end{vmatrix} \mathbf{k} \\
 &= -\frac{\sqrt{33}}{11} \mathbf{j} + \frac{4\sqrt{33}}{11} \mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{component of } \mathbf{a} \text{ perpendicular to } \mathbf{b} &= \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|} \\
 &= \frac{\sqrt{\left(-\frac{\sqrt{33}}{11}\right)^2 + \left(\frac{4\sqrt{33}}{11}\right)^2}}{\sqrt{26}} \\
 &= \frac{\sqrt{\frac{51}{11}}}{\sqrt{26}} \approx 0.422
 \end{aligned}$$

So, Emma is travelling about  $0.422 \text{ m s}^{-1}$  across the road.

$\therefore$  it will take her about  $\frac{4.5}{0.422} \approx 10.7$  seconds to cycle from one side of the road to the other.

**8 a** If  $\tan \theta = 4$ ,  
 then  $\cos \theta = \frac{1}{\sqrt{17}}$   
 $\therefore \cos^2 \theta = \frac{1}{17}$



**b i**

$$\begin{aligned}
 f(x) &= a\sqrt{\tan x} + b \\
 &= a(\tan x)^{\frac{1}{2}} + b \\
 \therefore f'(x) &= \frac{1}{2}a(\tan x)^{-\frac{1}{2}} \times \frac{1}{\cos^2 x} \\
 &= \frac{a}{2\cos^2 x \sqrt{\tan x}} \\
 \therefore f'\left(\frac{\pi}{4}\right) &= \frac{a}{2\cos^2 \frac{\pi}{4} \sqrt{\tan \frac{\pi}{4}}} \\
 &= \frac{a}{2\left(\frac{1}{\sqrt{2}}\right)^2 \sqrt{1}} \\
 &= a
 \end{aligned}$$

The gradient of the normal to the curve at  $x = \frac{\pi}{4}$  will be  $-\frac{1}{a}$ .

However, the equation of the normal is  $x + \sqrt{\pi}y = \pi$  or  $y = -\frac{1}{\sqrt{\pi}}x + \sqrt{\pi}$  which has gradient  $-\frac{1}{\sqrt{\pi}}$ .

$$\begin{aligned}
 \therefore -\frac{1}{a} &= -\frac{1}{\sqrt{\pi}} \\
 \therefore a &= \sqrt{\pi}
 \end{aligned}$$

Also, at  $x = \frac{\pi}{4}$  the normal line intersects the curve.

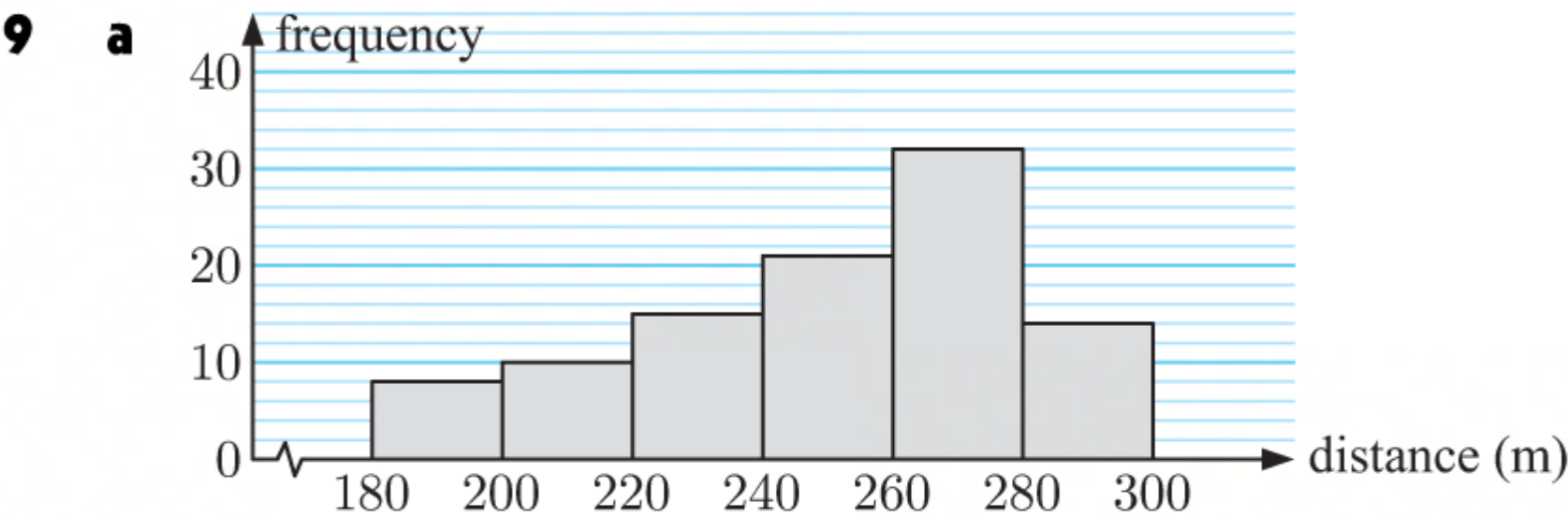
$$\begin{aligned}
 \therefore a\sqrt{\tan \frac{\pi}{4}} + b &= -\frac{1}{\sqrt{\pi}}\left(\frac{\pi}{4}\right) + \sqrt{\pi} \\
 \therefore \sqrt{\pi} \times \sqrt{1} + b &= -\frac{\sqrt{\pi}}{4} + \sqrt{\pi} \\
 \therefore b &= -\frac{\sqrt{\pi}}{4}
 \end{aligned}$$



ii From i,  $f'(x) = \frac{\sqrt{\pi}}{2 \cos^2 x \sqrt{\tan x}}$

$$\begin{aligned}\therefore f'(\tan^{-1}(4)) &= \frac{\sqrt{\pi}}{2 \cos^2(\tan^{-1}(4)) \sqrt{\tan(\tan^{-1}(4))}} \\ &= \frac{\sqrt{\pi}}{2 \times \frac{1}{17} \times \sqrt{4}} \quad \{\text{using a}\} \\ &= \frac{17\sqrt{\pi}}{4}\end{aligned}$$

$\therefore$  the gradient of the normal to  $f(x)$  at the point where  $x = \tan^{-1}(4)$  is  $-\frac{4}{17\sqrt{\pi}}$ .



b From the histogram in a, the data does not appear to be normally distributed as it is negatively skewed.

c Using technology,  $\mu \approx \bar{x} \approx 250.2$ .

Distance (d m)	Midpoint	Frequency
$180 \leq d < 200$	190	8
$200 \leq d < 220$	210	10
$220 \leq d < 240$	230	15
$240 \leq d < 260$	250	21
$260 \leq d < 280$	270	32
$280 \leq d < 300$	290	14
Total		100

d Using c,  $D \sim N(250.2, 30^2)$ .

Distance (d m)	Probability	$f_{\text{obs}}$	$f_{\text{exp}}$
$180 \leq d < 200$	$\approx 0.0471$	8	$\approx 4.71$
$200 \leq d < 220$	$\approx 0.1099$	10	$\approx 10.99$
$220 \leq d < 240$	$\approx 0.2099$	15	$\approx 20.99$
$240 \leq d < 260$	$\approx 0.2611$	21	$\approx 26.11$
$260 \leq d < 280$	$\approx 0.2117$	32	$\approx 21.17$
$280 \leq d < 300$	$\approx 0.1603$	14	$\approx 16.03$

$\leftarrow < 5$

There are expected frequencies less than 5, so we combine “categories” appropriately:

Distance (d m)	$f_{\text{obs}}$	$f_{\text{exp}}$	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
$180 \leq d < 220$	18	$\approx 15.70$	$\approx 0.3355$
$220 \leq d < 240$	15	$\approx 20.99$	$\approx 1.7084$
$240 \leq d < 260$	21	$\approx 26.11$	$\approx 1.0005$
$260 \leq d < 280$	32	$\approx 21.17$	$\approx 5.5421$
$280 \leq d < 300$	14	$\approx 16.03$	$\approx 0.2565$
Total			$\approx 8.8430$

e Step 1: The hypotheses are:

$$\begin{aligned}H_0: & \text{the data is from } N(250.2, 30^2) \\ H_1: & \text{the data is not from } N(250.2, 30^2)\end{aligned}$$

Step 2: The significance level is  $\alpha = 0.05$ .

Step 3: From d,  $\chi^2_{\text{calc}} \approx 8.84$ .



Step 4:  $df = 5 - 1 - 1 = 3$

Rad	Norm1	d/c	Real
List 1	List 2	List 3	List 4
SUB			
1	18	15.704	
2	15	20.988	
3	21	26.111	
4	32	21.168	
			18
GRAPH	CALC	TEST	INTR
			DIST

Rad	Norm1	d/c	Real
$\chi^2$ GOF Test			
Observed: List1			
Expected: List2			
df: 3			
CNTRB: List3			
Save Res: None			
GphColor: Blue			

Rad	Norm1	d/c	Real
$\chi^2$ GOF Test			
$\chi^2 = 8.84295121$			
$p = 0.03145347$			
df = 3			
CNTRB: List3			

Using technology,  $p$ -value  $\approx 0.0315$ .

Step 5: Since  $p$ -value  $< 0.05 = \alpha$ , we have sufficient evidence to reject  $H_0$  in favour of  $H_1$ . We therefore accept  $H_1$ .

Step 6: Since we have accepted  $H_1$ , we conclude that the data is not normally distributed with standard deviation 30 m.

**10 a** Let  $y = \frac{dp}{dt}$

$$\therefore \frac{dy}{dt} = \frac{d^2p}{dt^2}$$

$$\therefore \text{the system is } \begin{cases} \frac{dp}{dt} = y \\ \frac{dy}{dt} = -\frac{1}{2}y. \end{cases}$$

**b**

$$\frac{dy}{dt} = -\frac{1}{2}y$$

$$\therefore \frac{1}{y} \frac{dy}{dt} = -\frac{1}{2}$$

$$\therefore \int \frac{1}{y} \frac{dy}{dt} dt = \int -\frac{1}{2} dt$$

$$\therefore \int \frac{1}{y} dy = \int -\frac{1}{2} dt$$

$$\therefore \ln|y| = -\frac{1}{2}t + c$$

$$\therefore y = \pm e^{-\frac{1}{2}t+c}$$

$$\therefore y(t) = Ae^{-\frac{1}{2}t} \quad \{A = \pm e^c\}$$

**c**  $\frac{dp}{dt} = y = Ae^{-\frac{1}{2}t} \quad \{\text{from b}\}$

$$\therefore p = \int Ae^{-\frac{1}{2}t} dt$$

$$\therefore p(t) = -2Ae^{-\frac{1}{2}t} + d$$

$$\therefore p(t) = Be^{-\frac{1}{2}t} + d \quad \{B = -2A\}$$

**d i**  $p(0) = 0$

$$\therefore B + d = 0$$

$$\therefore d = -B \quad \dots (*)$$

$$p(2) = 0.45$$

$$\therefore Be^{-1} + d = 0.45$$

$$\therefore -de^{-1} + d = 0.45 \quad \{\text{using } (*)\}$$

$$\therefore d(1 - e^{-1}) = 0.45$$

$$\therefore d = \frac{0.45}{1 - e^{-1}} \quad \text{and} \quad B = -\frac{0.45}{1 - e^{-1}}$$

$$\therefore p(t) = -\frac{0.45}{1 - e^{-1}} e^{-\frac{1}{2}t} + \frac{0.45}{1 - e^{-1}}$$

$$= \frac{0.45}{1 - e^{-1}} \left(1 - e^{-\frac{1}{2}t}\right)$$

**ii** As  $t \rightarrow \infty$ ,  $e^{-\frac{1}{2}t} \rightarrow 0$

$$\therefore p(t) \rightarrow \frac{0.45}{1 - e^{-1}} \approx 0.712$$

$\therefore$  about  $(1 - 0.712) \times 100\% \approx 28.8\%$  of the wildlife will never be observed by the scientists.



# TRIAL EXAMINATION 1

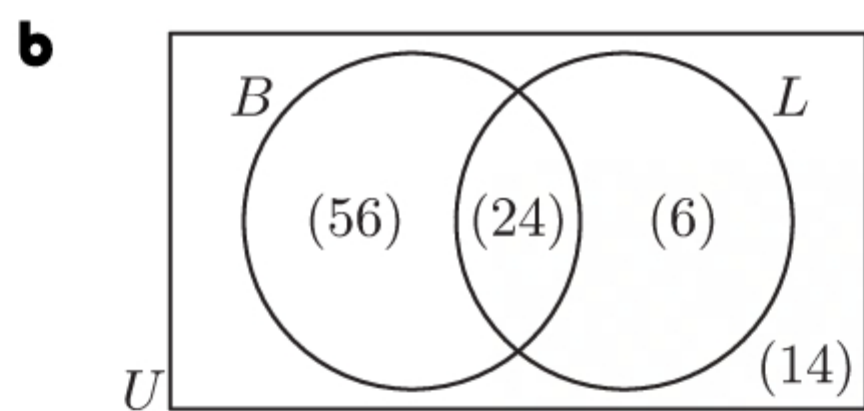
## PAPER 1

**1 a**  $n(B \cup L) = n(B) + n(L) - n(B \cap L)$

$\therefore 100 - 14 = 80 + 30 - n(B \cap L)$

$\therefore n(B \cap L) = 24$

**A1**



$$\begin{aligned} P(B \mid L') &= \frac{P(B \cap L')}{P(L')} \\ &= \frac{0.56}{0.7} \\ &= 0.8 \end{aligned}$$

**M1**

**A1**

**c**  $P(B \cap L) = 0.24$  and  $P(B) \times P(L) = 0.8 \times 0.3 = 0.24$

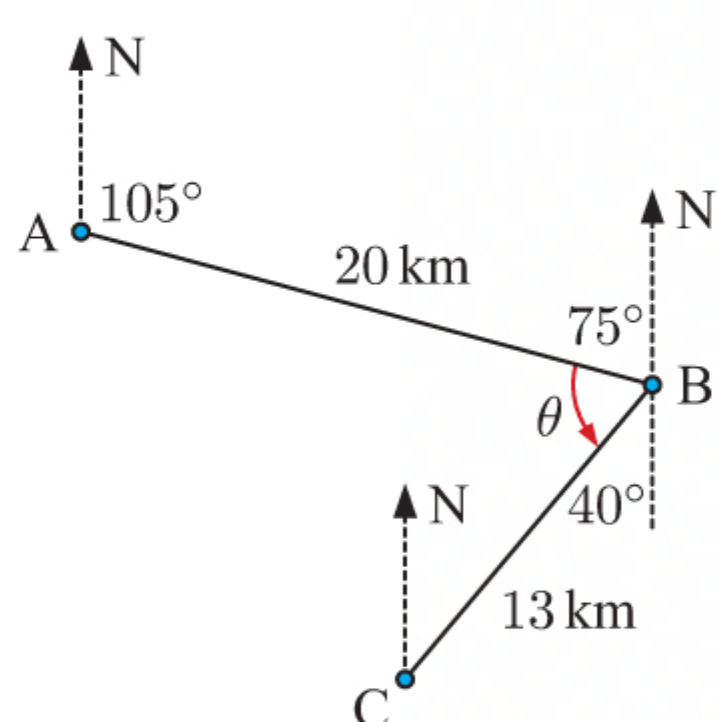
**M1**

$P(B \cap L) = P(B) \times P(L)$ , so  $B$  and  $L$  are independent.

**R1**

**Total [5 marks]**

**2 a**

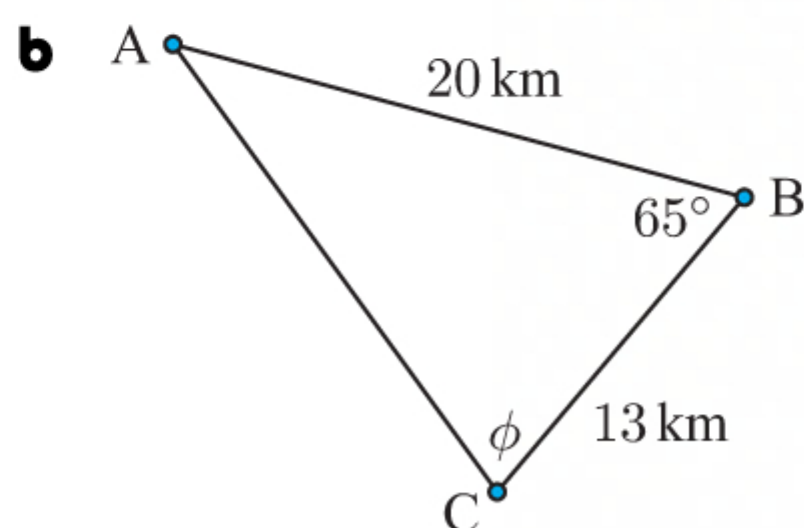


$$\begin{aligned} \widehat{ABN} &= 180^\circ - 105^\circ && \{\text{cointerior angles}\} \\ &= 75^\circ \end{aligned}$$

$$\begin{aligned} \therefore \theta &= 180^\circ - 75^\circ - 40^\circ && \{\text{angles on a straight line}\} \\ &= 65^\circ \end{aligned}$$

**R1**

**AG**



$$\begin{aligned} AC^2 &= 20^2 + 13^2 - 2 \times 20 \times 13 \times \cos 65^\circ \\ \therefore AC &\approx 18.7 \text{ km} \end{aligned}$$

**(M1)A1**

**A1**

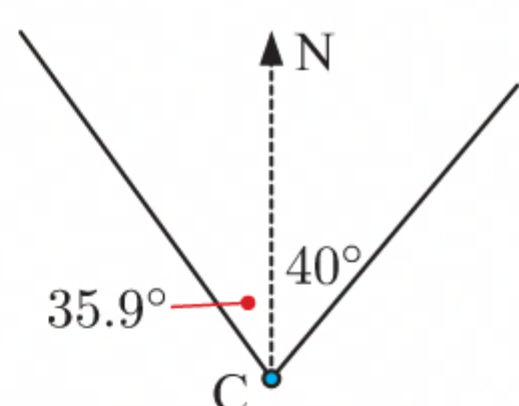
So, the ship must sail about 18.7 km.

**c**  $\cos \phi \approx \frac{13^2 + 18.7^2 - 20^2}{2 \times 13 \times 18.7}$

**(M1)**

$\therefore \phi \approx 75.9^\circ$

**A1**



$$\begin{aligned} \text{Bearing from C to A} & \\ &\approx 360^\circ - (75.9^\circ - 40^\circ) \\ &\approx 324^\circ \end{aligned}$$

**A1**

So, the ship must sail on a bearing of about  $324^\circ$ .

**Total [7 marks]**

**3 a**  $22^\circ\text{C}$

**A1**

**b** When  $t = 5$ ,  $T = 22 + 78(0.85)^5$

**(M1)**

$\approx 56.6^\circ\text{C}$

**A1**

So, after 5 minutes the soup is about  $56.6^\circ\text{C}$ .



**c** We need to find when  $T = 22 + 78(0.85)^t = 30$

$$\therefore (0.85)^t = \frac{8}{78} \quad (\text{M1})$$

$$\therefore t = \frac{\log(\frac{8}{78})}{\log(0.85)}$$

$$\therefore t \approx 14.0 \quad \text{A1}$$

$\therefore$  it takes  $\approx 14.0$  minutes for the temperature of the soup to drop below  $30^\circ\text{C}$ .

**Total [5 marks]**

**4 a** Greg's salary forms a geometric sequence with  $u_1 = 600$  and  $r = 1.1$ .

His salary in February 2021 is given by  $u_{14} = 600 \times (1.10)^{13}$  (M1)

$$\approx \$2071.36 \quad \text{A1}$$

**b** Ann's salary forms an arithmetic sequence with  $u_1 = 1000$  and  $d = 50$ .

The amount she receives in the first 12 months is given by  $S_{12} = \frac{12}{2}(2 \times 1000 + 11 \times 50)$  (M1)

$$= \$15\,300 \quad \text{A1}$$

**c** Total from Ann's contract after  $n$  months  $= \frac{n}{2}[2000 + 50(n - 1)]$  M1

$$= 975n + 25n^2$$

Total from Greg's contract after  $n$  months  $= \frac{600(1.1^n - 1)}{1.1 - 1}$  M1

$$= 6000(1.1^n - 1)$$

Using technology,  $6000(1.1^n - 1) > 975n + 25n^2$  when  $n > 16.48$  A1

$\therefore$  total amount from Greg's contract first exceeds the total amount from Ann's contract when  $n = 17$ , A1  
which is May 2021.

**Total [8 marks]**

**5 a** Let  $X$  kg be the mass of a randomly selected pumpkin.

$$X \sim N(4.7, 0.9^2) \quad \text{A1}$$

$\therefore P(X < 3) \approx 0.0295$  A1

**b** If  $P(X > a) = 0.1$  then  $a \approx 5.85$ . M1A1

The lowest mass of a "Super Pumpkin" is about 5.85 kg.

**c**  $P(\text{one too light and one "Super"}) \approx 2 \times 0.0295 \times 0.1$  M1

$$\approx 0.00589 \quad \text{A1}$$

**Total [6 marks]**

**6 a**  $N = 36, I = 4.8, PV = 5000, FV = 0, P/Y = 12, C/Y = 12$  (M1)

$$\therefore PMT \approx -\$149.41 \quad \text{A1}$$

Anya must pay \$149.41 each month.

**b**  $N = 12, I = 4.8, PV = 5000, PMT = 149.41, P/Y = 12, C/Y = 12$  (M1)

$$\therefore FV \approx -\$3412.46 \quad \text{A1}$$

After 1 year, Anya has \$3412.46 still to be repaid.

**c**  $N = 24, I = 5.3, PV = 3412.46, FV = 0, P/Y = 12, C/Y = 12$  (M1)

$$\therefore PMT \approx -\$150.17 \quad \text{A1}$$

Anya must pay \$150.17 each month for the last two years.

**Total [6 marks]**

**7 a**  $\text{Area} = \int_0^{10} \frac{1}{50}x(x - 10)(x - 16) dx$  M1

$$= 36\frac{2}{3} \text{ m}^2 \quad \{\text{using technology}\} \quad (\text{A1})$$

Volume  $= 100 \times 36\frac{2}{3}$

$$= 3666\frac{2}{3} \text{ m}^3 \quad \text{A1}$$



**b** Area  $\approx \frac{2}{2}[0 + 2(4.48 + 5.76 + 4.8 + 2.56) + 0]$  (M1)  
 $\approx 35.2 \text{ m}^2$  (A1)  
 Volume  $\approx 100 \times 35.2$   
 $\approx 3520 \text{ m}^3$  A1

**c** Percentage error  $= \frac{\left| 3520 - 3666\frac{2}{3} \right|}{3666\frac{2}{3}} \times 100\%$   
 $= 4\%$  A1

**Total [7 marks]**

**8**  $s(t) = t^2 \sin 2t, \quad 0 \leq t \leq 4$

**a**  $v(t) = s'(t)$   
 $= (t^2)(2 \cos 2t) + (2t)(\sin 2t)$  (M1)(M1)  
 $= 2t(t \cos 2t + \sin 2t), \quad 0 \leq t \leq 4$  A1

**b** Stationary when  $v(t) = 0$  (M1)  
 $\therefore t = 0, t \approx 1.14, t \approx 2.54$  A1

**c** Distance travelled  $= \int_0^4 |v(t)| dt$  (M1)  
 $= \int_0^4 |2t(t \cos 2t + \sin 2t)| dt$   
 $\approx 29.8 \text{ mm}$  A1

**Total [7 marks]**

**9 a**

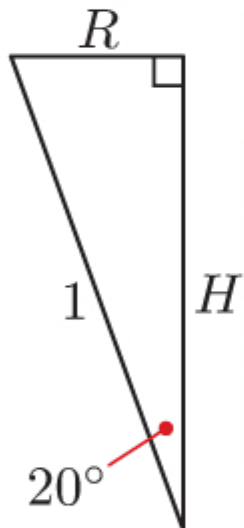
$x$	1	2	5	10
$P(X = x)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

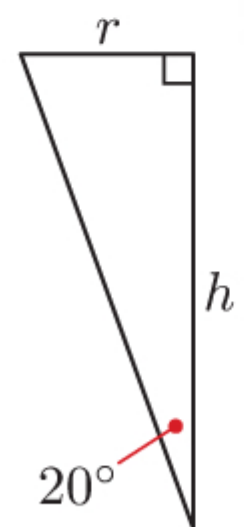
 $E(X) = \frac{3}{8} + \frac{6}{8} + \frac{5}{8} + \frac{10}{8}$  (M1)  
 $= 3$  A1

**b**  $E(P) = E(2X + 5)$   
 $= 2E(X) + 5$   
 $= 11$  A1

**c**  $\text{Var}(P) = \text{Var}(2X + 5)$   
 $= 2^2 \text{Var}(X)$  (M1)  
 $= 34$  A1

**Total [5 marks]**

**10 a**   $R = \sin 20^\circ, H = \cos 20^\circ$  (M1)  
 $\therefore \text{maximum volume} = \frac{1}{3}\pi(\sin 20^\circ)^2(\cos 20^\circ)$  (M1)  
 $\approx 0.115111 \text{ m}^3$   
 $\approx 115111 \text{ mL}$  A1

**b**   $\tan 20^\circ = \frac{r}{h} \Rightarrow r = h \tan 20^\circ$   
 $\therefore V = \frac{1}{3}\pi(h \tan 20^\circ)^2 \times h$   
 $= \frac{\pi}{3}(\tan 20^\circ)^2 \times h^3$  (M1)  
 $\therefore \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$   
 $= \pi(\tan 20^\circ)^2 h^2 \times \frac{dh}{dt}$  (M1)

Now when  $V = \frac{1}{2} \times \text{maximum volume} \approx \frac{1}{2} \times 115111 \approx 57555.5 \text{ mL},$

$57555.5 \approx \frac{\pi}{3}(\tan 20^\circ)^2 \times h^3$

$\therefore h \approx 74.58 \text{ cm}$

**A1**



Also  $\frac{dV}{dt} = 500$  when the cone is half full.

$$\therefore 500 \approx \pi(\tan 20^\circ)^2 \times 74.58^2 \times \frac{dh}{dt} \quad \text{M1}$$

$$\therefore \frac{dh}{dt} \approx 0.216 \text{ cm/s} \quad \text{A1}$$

So, the water level is rising at about 0.216 cm/s when the cone is filled to half its maximum volume.

**Total [8 marks]**

**11 a** When  $t = 0$ ,  $\mathbf{v} = \begin{pmatrix} 14 \\ 35 \end{pmatrix}$  (M1)

$$\therefore \text{initial speed} = \sqrt{14^2 + 35^2} \approx 37.7 \text{ m/s} \quad \text{A1}$$

**b** Maximum height occurs when the vertical velocity  $v_y = 0$

$$\therefore 35 - 20t = 0$$

$$\therefore t = 1.75 \text{ s} \quad \text{A1}$$

$$\begin{aligned} \text{The height at time } t, \quad h(t) &= \int 35 - 20t \, dt \\ &= 35t - 10t^2 + c \end{aligned}$$

But  $h(0) = 1$ , so  $c = 1$

$$\therefore h(t) = 35t - 10t^2 + 1 \quad \text{A1}$$

$$\therefore h(1.75) = 31.625 \text{ m} \quad \text{A1}$$

**c**  $h(t) = 2.5$  when  $t \approx 0.043$  or  $3.457$  A1

The fielder would not be positioned directly in front of the batter.

$$\therefore \text{horizontal distance to the fielder} \approx 14 \times 3.457$$

$$\approx 48.4 \text{ m} \quad \text{A1}$$

**Total [7 marks]**

**12 a**  $V = \pi \int [f(x)]^2 \, dx$  (M1)

$$= \pi \int_0^{10} \frac{x^2}{400} x(10 - x) \, dx$$

$$\approx 39.3 \text{ mm}^3 \quad \text{A2}$$

**b** Using technology, the maximum value of  $f(x)$  is  $\approx 1.624$ , which occurs when  $x = 7.5$ . (M1)

$$\therefore \text{maximum circumference} \approx 2\pi \times 1.624 \quad \text{M1}$$

$$\approx 10.2 \text{ mm} \quad \text{A1}$$

**Total [6 marks]**

**13**  $\frac{dA}{dt} \propto \sqrt{A}$

$$\therefore \frac{dA}{dt} = k\sqrt{A}$$

$$\therefore \int \frac{1}{\sqrt{A}} \, dA = \int k \, dt \quad \text{M1}$$

$$\therefore 2\sqrt{A} = kt + c \quad \text{M1}$$

$$\text{When } t = 0, A = 900 \quad \therefore 60 = c \quad \text{A1}$$

$$\text{When } t = 3, A = 1296 \quad \therefore 72 = 3k + 60$$

$$\therefore k = 4 \quad \text{A1}$$

$$\therefore 2\sqrt{A} = 4t + 60$$

$$10\% \text{ of area} = 0.1 \times 50 \times 400 = 2000 \text{ m}^2$$

$$\therefore 2\sqrt{2000} = 4t + 60 \quad \text{M1}$$

$$\therefore t \approx 7.36 \text{ weeks}$$

$$\approx 52 \text{ days} \quad \text{A1}$$

**Total [6 marks]**



**14 a** (M1)

$x$	$I$	<i>Residuals</i>
10	118	$118 - 120 = -2$
20	46.1	$46.1 - 45 = 1.1$
40	25.9	$25.9 - 26.25 = -0.35$
100	21.3	$21.3 - 21 = 0.3$

$$SS_{\text{res}} = (-2)^2 + (1.1)^2 + (-0.35)^2 + (0.3)^2 = 5.4225 \quad \text{A1}$$

Relatively low value of  $SS_{\text{res}} \Rightarrow$  Good fit for the data R1

**b** As  $x \rightarrow \infty$ ,  $I \rightarrow 20$  A1

So, the limit of sound intensity as the engineer moves a large distance from the speaker is about 20 W/m<sup>2</sup>.

**c** 
$$I = \frac{10\,000}{x^2} + 20$$

$$\therefore I - 20 = \frac{10\,000}{x^2}$$

$$\therefore x^2 = \frac{10\,000}{I - 20} \quad \{I \neq 20\} \quad \text{(M1)}$$

$$\therefore x = \frac{100}{\sqrt{I - 20}} \quad \{x > 0\} \quad \text{A1}$$

**d** 
$$x \geq 10$$

$$\therefore \frac{100}{\sqrt{I - 20}} \geq 10$$

$$\therefore \sqrt{I - 20} \leq 10$$

$$\therefore I \leq 120$$

So, the range of sound intensities predicted by the model is  $20 < I \leq 120$ . A1A1

**Total [8 marks]**

**15 a**  $|\mathbf{M}| = 1$ , and  $\mathbf{M}$  has the form  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ .

$$\therefore \text{we have a rotation with } \cos \theta = \frac{\sqrt{3}}{2}, \sin \theta = -\frac{1}{2}.$$

$$\therefore \theta = -30^\circ$$

The transformation is a rotation of  $30^\circ$  clockwise about the origin. A1A1

**b** 
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{3} + 3 \\ -2 + 3\sqrt{3} \end{pmatrix} \quad \text{M1}$$

$$\therefore \text{the image of } (4, 6) \text{ is } (2\sqrt{3} + 3, -2 + 3\sqrt{3}). \quad \text{A1}$$

**c**  $\mathbf{M}^{-1}$  is a rotation of  $30^\circ$  anticlockwise about the origin, which is equivalent to a rotation of  $330^\circ$  clockwise about the origin, or 11 rotations of  $30^\circ$  clockwise about the origin. R1

$$\therefore n = 11 \quad \text{A1}$$

**Total [6 marks]**

**16 a** Area of sector  $= \frac{1}{2}r^2\theta = \frac{1}{2} \times 20^2 \times \theta = 200\theta$  M1

$$\therefore \text{volume} = 200\theta h = 1400$$

$$\Rightarrow h = \frac{1400}{200\theta} = \frac{7}{\theta} \quad \text{AG}$$

**b** Arc length  $= r\theta = 20\theta$  M1

$$\begin{aligned} \text{Total surface area } A &= 2(200\theta) + 2(20h) + 20\theta h \\ &= 400\theta + 40 \times \frac{7}{\theta} + 20\theta \times \frac{7}{\theta} \\ &= 400\theta + \frac{280}{\theta} + 140 \end{aligned} \quad \begin{array}{l} \text{M1} \\ \text{AG} \end{array}$$



$$\begin{aligned} \text{c } \frac{dA}{d\theta} &= 400 - 280\theta^{-2} & \text{A1A1} \\ &= 400 - \frac{280}{\theta^2} \end{aligned}$$

$$\begin{aligned} \text{d } \frac{dA}{d\theta} &= 0 \text{ when } 400 = \frac{280}{\theta^2} \\ &\therefore \theta^2 = 0.7 \\ &\therefore \theta \approx 0.837^c \quad \{\theta > 0\} & \text{A1} \end{aligned}$$

When  $\theta \approx 0.837^c$ ,  $A \approx 809 \text{ cm}^2$  A1

So, the minimum possible surface area is about  $809 \text{ cm}^2$ , which occurs when  $\theta \approx 0.837^c$ .

Total [7 marks]

$$\begin{aligned} \text{17 a Using technology, } \hat{\mu} &\approx \bar{x} \approx 10.3 & \text{A1} \\ \widehat{\sigma^2} &\approx s^2 \approx 0.5807^2 \approx 0.337 & \text{A1} \end{aligned}$$

$$\begin{aligned} \text{b } H_0: \mu &= 10 & \text{M1} \\ H_1: \mu &> 10 \end{aligned}$$

We conduct a one-tailed  $t$ -test at the 5% level.

$t \approx 2.04$ , so using technology the  $p$ -value  $\approx 0.0332$ . A1

$0.0332 < 5\%$ , so we reject  $H_0$  in favour of  $H_1$  at the 5% significance level. A1

We therefore accept  $H_1$ , and conclude that the company's claim is not valid. A1

Total [6 marks]

## PAPER 2

$$\begin{aligned} \text{1 a } d &= \frac{4.18 + 1.73}{2} & \text{M1} \\ &= 2.955 & \text{AG} \end{aligned}$$

$$\begin{aligned} \text{b } a &= \frac{4.18 - 1.73}{2} \\ &= 1.225 & \text{A1} \end{aligned}$$

$$\begin{aligned} \text{c Period} &= 12.5 \text{ hours} & \text{M1} \\ \therefore b &= \frac{2\pi}{12.5} = \frac{4\pi}{25} & \text{A1} \end{aligned}$$

$$\begin{aligned} \text{d The graph cuts the principal axis halfway between } t = 7 \text{ and } t = 13.25, \text{ which is } t = 10.125 \\ \therefore c &= 10.125 & \text{M1} \\ & & \text{A1} \end{aligned}$$

$$\text{e Stretch horizontally by a scale factor of } \frac{25}{4\pi}. \quad \text{A1}$$

Translate in the positive horizontal direction by 10.125 hours. A1

Stretch vertically by a scale factor of 1.225. A1

Translate in the positive vertical direction by 2.955 metres. A1

A1 : Correct order

$$\text{f High tide occurs every 12.5 hours.} \quad \text{(M1)}$$

$$\therefore \text{ high tide occurs at } t = 13.25 + 12.5N, \quad N \in \mathbb{N}. \quad \text{(M1)}$$

For the afternoon of 5th June,  $108 < t < 120$ .

High tide occurs when  $t = 113.25$ , which is 17:15 on 5th June. A1

Total [14 marks]

$$\text{2 a Midpoint} = \left( \frac{2+8}{2}, \frac{10+8}{2} \right) = (5, 9) \quad \text{A1}$$

$$\begin{aligned} \text{b Gradient} &= \frac{8-10}{8-2} \\ &= -\frac{1}{3} & \text{A1} \end{aligned}$$



**c** The perpendicular bisector has gradient  $-\frac{1}{-\frac{1}{3}} = 3$

**M1**

$\therefore$  its equation is  $y = 3x + c$

**M1**

$(5, 9)$  lies on the line, so  $9 = 3(5) + c$

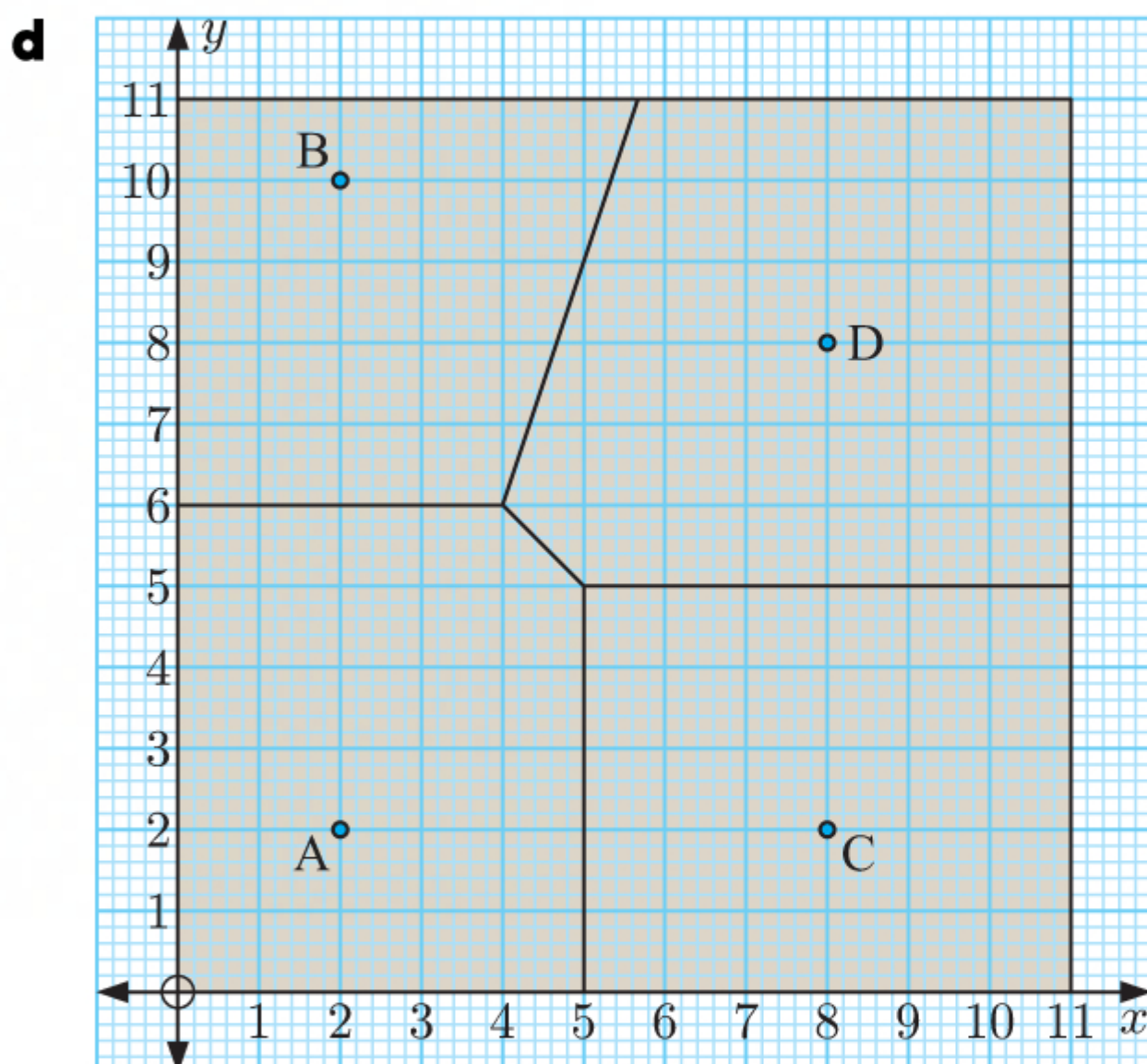
$\therefore c = -6$

**M1**

$\therefore$  the perpendicular bisector of BD has equation  $y = 3x - 6$

$\therefore y - 2x + 6 = 0$

**AG**



**(M1) : 2 correct PBs**

**(M1) : 4 correct PBs**

**(M1) : 5 or 6 correct PBs**

**A1 : 2 correct Voronoi regions**

**A1 : 4 correct Voronoi regions**

**e** Optimum position for new station is at the vertex with the largest radius of circle to its adjacent sites.

For vertex  $(4, 6)$ , radius  $= \sqrt{2^2 + 4^2} = \sqrt{20}$

**M1**

For vertex  $(5, 5)$ , radius  $= \sqrt{3^2 + 3^2} = \sqrt{18}$

**M1**

$\therefore$  optimal position is  $(4, 6)$ .

**A1**

**Total [13 marks]**

**3 a** Velocity  $= \frac{1}{3} \left( \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

**M1**

Initial position  $= \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$

**M1**

$\therefore \mathbf{r}_b = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

**AG**

**b** Speed  $= |\text{velocity}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$= \sqrt{6} \times 100 \text{ metres per minute}$

$\approx 245 \text{ metres per minute}$

**A1**

**c** At time  $t$ , the helicopter has been travelling for  $t - 5$  minutes,  $t \geq 5$ .

**R1**

$\therefore \mathbf{r}_h = \begin{pmatrix} 0 \\ 20 \\ 1 \end{pmatrix} + (t - 5) \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$

**AG**

**d**  $\mathbf{r}_h = \begin{pmatrix} 0 \\ 20 \\ 1 \end{pmatrix} + (t - 5) \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}, \quad \mathbf{r}_b = \begin{pmatrix} -4 + 2t \\ t \\ t \end{pmatrix}$

Helicopter is due east of balloon when  $y$ -values are equal.

**M1**

$\therefore 20 + (t - 5)(-3) = t$

$\therefore t = 8.75$

**A1**

When  $t = 8.75$ ,  $\mathbf{r}_h = \begin{pmatrix} 15 \\ 8.75 \\ 8.5 \end{pmatrix}$

$\therefore$  height of helicopter  $= 8.5 \times 100 = 850 \text{ m}$

**A1**



**e** At 08:10,  $t = 10$

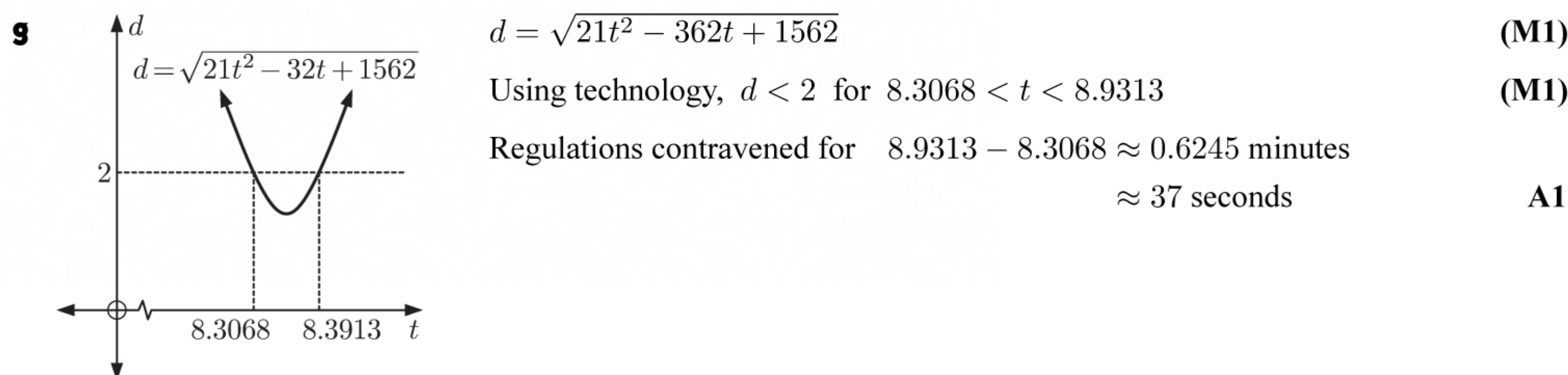
$$\therefore \mathbf{r}_b = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + 10 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 10 \\ 10 \end{pmatrix} \quad \text{A1}$$

$$\text{and } \mathbf{r}_h = \begin{pmatrix} 0 \\ 20 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \\ 11 \end{pmatrix} \quad \text{A1}$$

$$\begin{aligned} \therefore \text{distance between helicopter and balloon} &= \left| \begin{pmatrix} 20 \\ 5 \\ 11 \end{pmatrix} - \begin{pmatrix} 16 \\ 10 \\ 10 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix} \right| \\ &= \sqrt{4^2 + (-5)^2 + 1^2} \\ &= \sqrt{42} \text{ units} \\ &= \sqrt{42} \times 100 \text{ m} \approx 648 \text{ m} \quad \text{A1} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \text{ At time } t, \quad \mathbf{r}_b - \mathbf{r}_h &= \begin{pmatrix} -4 + 2t \\ t \\ t \end{pmatrix} - \begin{pmatrix} 0 + 4(t-5) \\ 20 - 3(t-5) \\ 1 + 2(t-5) \end{pmatrix} \quad \text{(M1)} \\ &= \begin{pmatrix} 16 - 2t \\ -35 + 4t \\ 9 - t \end{pmatrix} \quad \text{A1} \end{aligned}$$

$$\begin{aligned} \therefore d^2 &= (16 - 2t)^2 + (-35 + 4t)^2 + (9 - t)^2 \quad \text{(M1)} \\ &= (256 - 64t + 4t^2) + (1225 - 280t + 16t^2) + (81 - 18t + t^2) \quad \text{M1} \\ &= 21t^2 - 362t + 1562 \quad \text{AG} \end{aligned}$$



**Total [17 marks]**

**4 a**  $84.5 + (10 - 2) \times 6.9 = 139.7 \text{ cm}$  A1

**b**  $h(t) = 84.5 + 6.9(t - 2)$  (M1)  
 $= 6.9t + 70.7, \quad 2 \leq t \leq 13$  A1

**c**  $h'(t) = \frac{32}{5(t-12)}$

$$\therefore h(t) = \frac{32}{5} \int \frac{1}{t-12} dt \quad \text{(M1)}$$

$$= 6.4 \ln(t-12) + c \quad \{t-12 > 0\} \quad \text{A1}$$

Under the linear model,  $h(13) = 6.9 \times 13 + 70.7$   
 $= 160.4$  (M1)

$$\therefore 6.4 \ln(13 - 12) + c = 160.4$$

$$\therefore c = 160.4 \quad \text{A1}$$

$$\therefore h(t) = 6.4 \ln(t - 12) + 160.4 \text{ for } 13 < t \leq 20 \quad \text{A1}$$

**d**  $g(0) = \frac{165}{1 + 4e^0} = \frac{165}{5} = 33 \text{ cm}$  A1



**e** As  $t \rightarrow \infty$ ,  $g(t) \rightarrow \frac{165}{1+4 \times 0} = 165$  cm A2

**f** When Rose is age  $t$ , the chimpanzee is age  $(t - 5)$ .

$\therefore$  we find  $t$  such that  $g(t - 5) = h(t)$ . (M1)

For  $2 \leq t \leq 13$ ,  $\frac{165}{1+4e^{-0.78(t-5)}} = 6.9t + 70.7$

Using technology,  $t \approx 8.40$   $\{t \leq 13\}$  A1

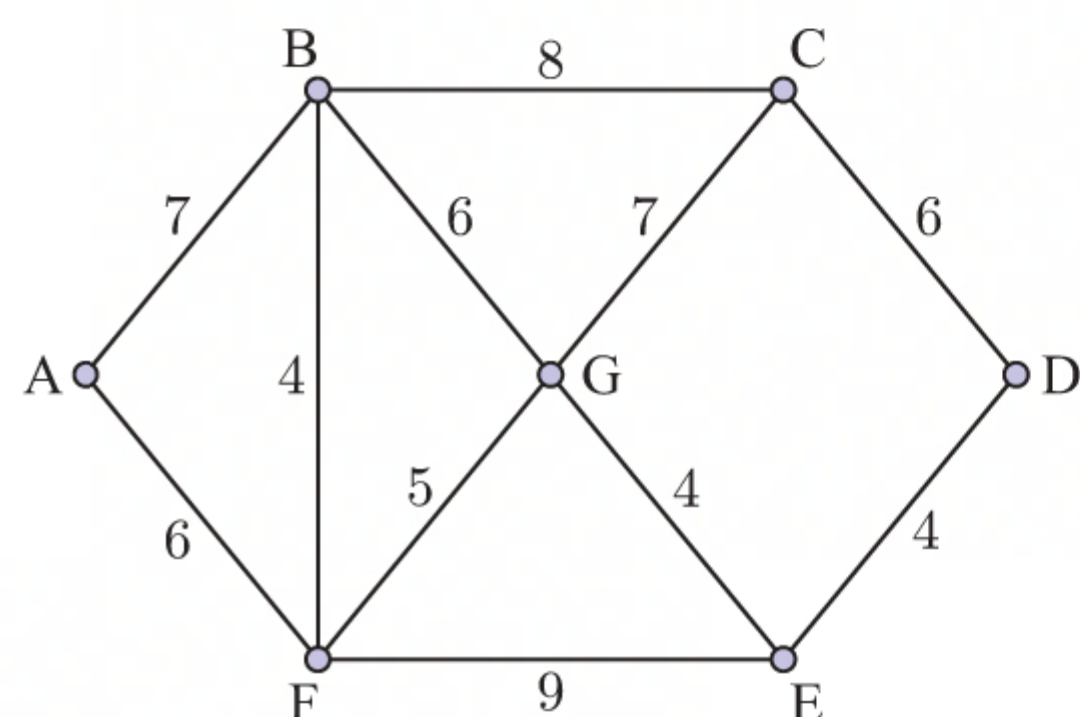
For  $13 < t \leq 20$ ,  $\frac{165}{1+4e^{-0.78(t-5)}} = 6.4 \ln(t - 12) + 160.4$

Using technology,  $t \approx 13.9$  A1

So, Rose and the chimpanzee will be the same height when Rose is 8 and 13 years old.

**Total [14 marks]**

**5 a**



**b** There are vertices of odd degree

$\therefore$  no Eulerian circuit is possible. R1

**c** The vertices of odd degree are C and E.

**R1**

$\therefore$  if she starts at station C then she will finish at station E.

**A1**

**d** The minimum path from C to E is  $6 + 4 = 10$  minutes

**(M1)**

The sum of all paths is  $7 + 6 + 8 + 4 + 6 + 6 + 7 + 4 + 9 + 4 + 5 = 66$  minutes.

$\therefore$  minimum time =  $66 + 10 = 76$  minutes A1

A possible route is  $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow G \rightarrow C \rightarrow D \rightarrow E \rightarrow G \rightarrow B \rightarrow F \rightarrow A$ .

**A2**

**e** Consider the three possible pairings of vertices of odd degree A, C, D, and E.

$AC + ED = 11 + 4 = 15$  M1

$AD + CE = 7 + 10 = 17$  M1

$AE + CD = 8 + 6 = 14$  M1

$\therefore$  new minimum time =  $66 + 4 + 3 + 14$

$= 87$  minutes A1

Percentage increase =  $\frac{|87 - 76|}{76} \times 100\% \approx 14.5\%$  A1

**Total [14 marks]**

**6 a**  $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

**M1**

$\therefore \lambda_1 = -2$  A1

**b** Let  $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix}$

**M1**

$\therefore 2x + y = 3x \Rightarrow x = y$

and  $4x - y = 3y \Rightarrow x = y$

So,  $\lambda_2 = 3$ ,  $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  A1



$$\mathbf{c} \quad \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \frac{1}{100} \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix  $\frac{1}{100} \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$  has eigenvalues and eigenvectors  $\lambda_1 = -\frac{2}{100}$ ,  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$  and **M1**

$$\lambda_2 = \frac{3}{100}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = Ae^{0.03t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-0.02t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} \quad \mathbf{M1}$$

When  $t = 0$ ,  $x = 100$  and  $y = 50$ .

$$\therefore 100 = A + B \text{ and } 50 = A - 4B$$

So,  $A = 90$  and  $B = 10$  **A1**

$$\therefore x = 90e^{0.03t} + 10e^{-0.02t}$$

$$y = 90e^{0.03t} - 40e^{-0.02t} \quad \mathbf{AG}$$

$$\mathbf{d} \quad \text{When } t = 30, \quad x = 90e^{0.9} + 10e^{-0.6} \approx 227 \quad \mathbf{A1}$$

$$y = 90e^{0.9} - 40e^{-0.6} \approx 199 \quad \mathbf{A1}$$

$$\mathbf{e} \quad \text{As } t \rightarrow \infty, \quad x \rightarrow \infty \text{ and } y \rightarrow \infty \quad \mathbf{R1}$$

$$\text{As } t \rightarrow \infty, \quad \frac{x}{y} \rightarrow 1 \quad \mathbf{R1}$$

**f**

$t$	$x_n$	$y_n$	$\frac{dx}{dt}$	$\frac{dy}{dt}$
0	100	50	5	-5
1	105	45	5.78	-4.05
2	110.78	40.95	6.54	-3.21
3	117.32	37.74	7.30	-2.47
4	124.62	35.27		

(M1)

(M1)

So, after 4 months  $x \approx 125$  and  $y \approx 35$ . **A1A1**

$$\mathbf{g} \quad \text{Minimum and maximum values of } y \text{ occur when } \frac{dy}{dt} = 0 \quad \mathbf{(R1)}$$

$$\therefore y(2x - 300) = 0$$

$$\therefore x = 150 \quad \mathbf{A1}$$

$$\mathbf{h} \quad \text{Let } x = 150$$

$$\therefore 300 + y + 100 \ln \left( \frac{50\,000\,000}{150^3 \times y} \right) = 250 \quad \mathbf{(M1)}$$

$$\therefore 50 + y + 100 \ln \left( \frac{400}{27y} \right) = 0$$

Using technology,  $y \approx 34.48$  or  $219.63$  **A1A1**

So, the minimum size of the predator population is about 34, and the maximum size is about 220.

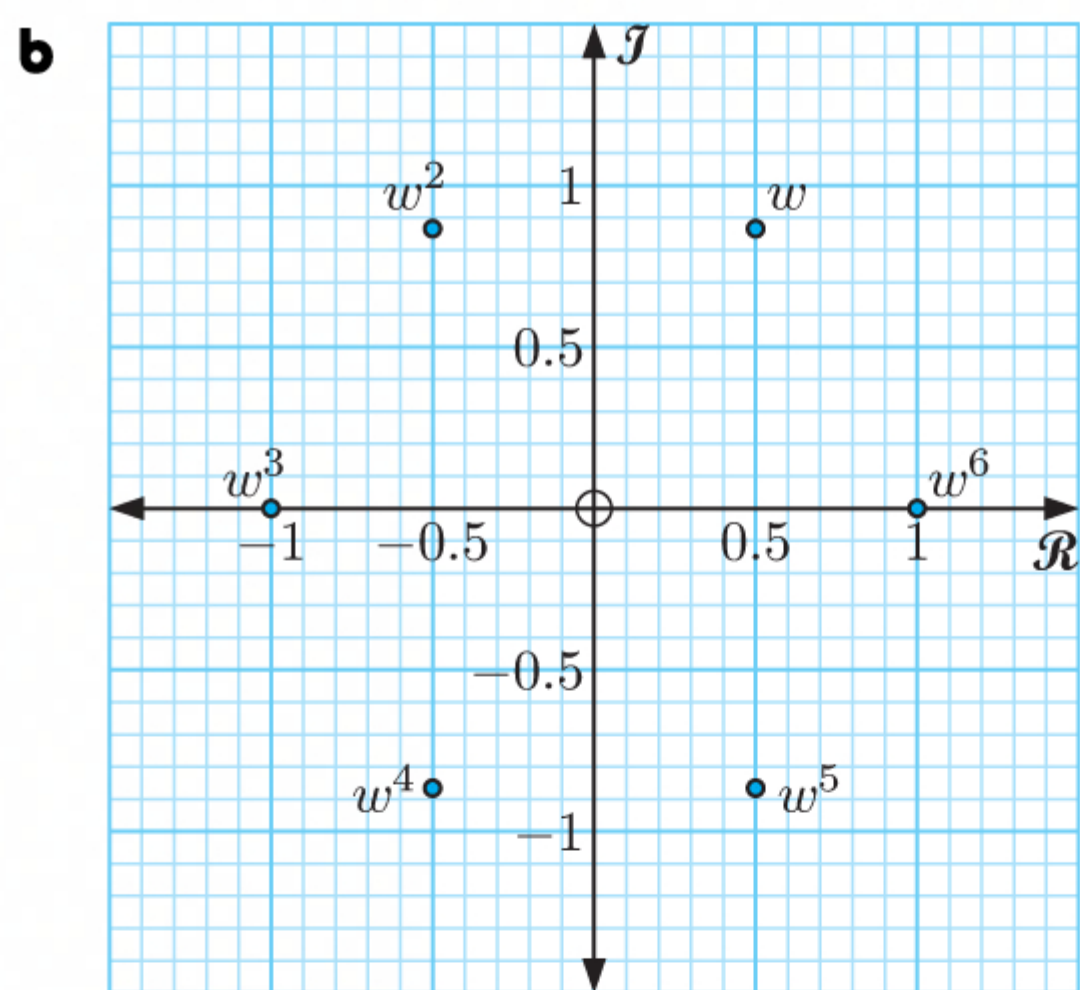
**Total [20 marks]**

**7 a**

	$w$	$w^2$	$w^3$
<i>Cartesian form</i> $a + bi$	$\frac{1}{2} + \frac{\sqrt{3}}{2}i$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}i$	$-1$
<i>Euler's form</i> $e^{i\theta}$ , $0 < \theta < 2\pi$	$e^{\frac{\pi}{3}i}$	$e^{\frac{2\pi}{3}i}$	$e^{\pi i}$

**A5**





A1 :  $w^3, w^6$  correct

A1 :  $w, w^2$  correct

A1 :  $w^4, w^5$  correct

**c** 6 units

A1

**d**  $50f\left(\frac{3}{\pi}\right) = 50e^i$   
 $= 50 \cos 1 + i50 \sin 1$   
 $\approx 27.02 + 42.07i$

M1

M1

AG

**e**  $\mathcal{Re}[20f(t)] = \mathcal{Re}[20e^{\frac{\pi}{3}ti}]$   
 $= \mathcal{Re}[20 \cos(\frac{\pi}{3}t) + i \sin(\frac{\pi}{3}t)]$   
 $= 20 \cos(\frac{\pi}{3}t)$   
 $= p(t)$

M1

M1

AG

**f**  $T(t) = p(t) + q(t)$ , where  $q(t) = 50 \cos(\frac{\pi}{3}t + 1) = \mathcal{Re}[50e^{(\frac{\pi}{3}t+1)i}]$

$\therefore T(t) = \mathcal{Re}[20e^{\frac{\pi}{3}ti} + 50e^{(\frac{\pi}{3}t+1)i}]$  (M1)

$= \mathcal{Re}[e^{\frac{\pi}{3}ti}(20) + e^{\frac{\pi}{3}ti}(50e^i)]$

$= \mathcal{Re}[e^{\frac{\pi}{3}ti}(20 + 50e^i)]$

$\approx \mathcal{Re}[e^{\frac{\pi}{3}ti}(20 + 27.02 + 42.07i)]$

$\approx \mathcal{Re}[e^{\frac{\pi}{3}ti}(47.02 + 42.07i)]$

(M1)

$\approx \mathcal{Re}[e^{\frac{\pi}{3}ti}(63.1e^{0.730i})]$  {using technology}

$\approx \mathcal{Re}[63.1(e^{(\frac{\pi}{3}t+0.730)i})]$

(M1)

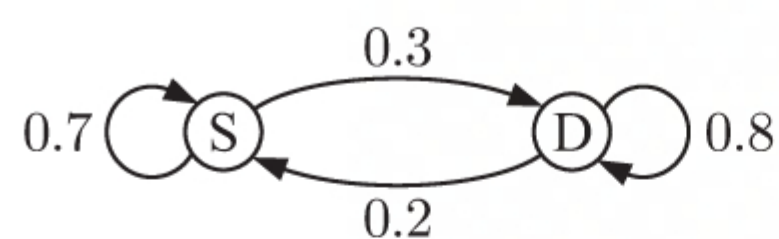
$\approx 63.1 \cos(\frac{\pi}{3}t + 0.730)$

A1A1

Total [18 marks]

### PAPER 3

**1 a**



A1

$T = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$

A1

**b**  $\begin{vmatrix} 0.7 - \lambda & 0.2 \\ 0.3 & 0.8 - \lambda \end{vmatrix} = 0$

(M1)

$(0.7 - \lambda)(0.8 - \lambda) - (0.2)(0.3) = 0$

$\lambda^2 - 1.5\lambda + 0.5 = 0$

$(\lambda - 1)(\lambda - 0.5) = 0$

$\lambda_1 = 1$  and  $\lambda_2 = 0.5$

A1

For  $\lambda_1 = 1$ ,  $\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore 0.7x + 0.2y = x$

$\therefore 0.2y = 0.3x$

$\therefore \mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

A1



$$\begin{aligned}\text{For } \lambda_2 = 0.5, \quad & \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.5 \begin{pmatrix} x \\ y \end{pmatrix} \\ & \therefore 0.7x + 0.2y = 0.5x \\ & \therefore 0.2y = -0.2x \\ & \therefore \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\end{aligned}$$

A1

**c** In the steady state,  $\lambda_1 = 1$

$$\therefore \text{ the corresponding eigenvector is } \mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

R1

$\therefore$  the 24 000 customers are split in ratio 2 : 3.

$$\therefore \text{ number of Standard customers} = \frac{24\,000}{5} \times 2 = 9600$$

A1

$$\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$$

A1

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$$

A1

$$\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$$

M1

$$\mathbf{P}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$$

(M1)

$$\begin{aligned}\therefore \mathbf{T}^n &= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.5^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 2 & 0.5^n \\ 3 & -0.5^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 2 + 3 \times 0.5^n & 2 - 2 \times 0.5^n \\ 3 - 3 \times 0.5^n & 3 + 2 \times 0.5^n \end{pmatrix}\end{aligned}$$

(M1)

(M1)

$$\begin{aligned}\therefore \mathbf{x}_n &= \mathbf{T}^n \mathbf{x}_0 \\ &= \frac{1}{5} \begin{pmatrix} 2 + 3 \times 0.5^n & 2 - 2 \times 0.5^n \\ 3 - 3 \times 0.5^n & 3 + 2 \times 0.5^n \end{pmatrix} \begin{pmatrix} 24\,000 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 + 3 \times 0.5^n & 2 - 2 \times 0.5^n \\ 3 - 3 \times 0.5^n & 3 + 2 \times 0.5^n \end{pmatrix} \begin{pmatrix} 4800 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 9600 + 14\,400 \times 0.5^n \\ 14\,400 - 14\,400 \times 0.5^n \end{pmatrix}\end{aligned}$$

(M1)

$$\therefore S_{\text{two}}(n) = 9600 + 14\,400 \times 0.5^n$$

AG

$$\mathbf{M} = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.2 & 0.4 & 0.3 \\ 0 & 0.4 & 0.7 \end{pmatrix}$$

A1

$$\mathbf{x}_5 = \mathbf{M}^5 \mathbf{x}_0$$

(M1)

$$= \mathbf{M}^5 \begin{pmatrix} 24\,000 \\ 0 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 11\,355 \\ 6580 \\ 6065 \end{pmatrix} \quad \{\text{using technology}\}$$

There are about 11 355 Standard customers after 5 months.

A1



$$\mathbf{h} \quad \mathbf{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (\text{M1})$$

$$\begin{aligned} \therefore 0.8x + 0.2y &= x \\ 0.2x + 0.4y + 0.3z &= y \\ &+ 0.4y + 0.7z = z \end{aligned}$$

$$\text{where } x + y + z = 24\,000 \quad (\text{M1})$$

$$\therefore -0.2x + 0.2y = 0 \quad \dots (1)$$

$$0.2x - 0.6y + 0.3z = 0 \quad \dots (2)$$

$$0.4y - 0.3z = 0 \quad \dots (3)$$

$$x + y + z = 24\,000 \quad \dots (4) \quad (\text{M1})$$

From (1),  $x = y$

$$\text{From (3), } z = \frac{4}{3}y = \frac{4}{3}x$$

$$\therefore \text{ in (4), } x + x + \frac{4}{3}x = 24\,000$$

$$\therefore \frac{10}{3}x = 24\,000$$

$$\therefore x = 7200 \quad \text{AG}$$

$$\mathbf{i} \quad S_3(n) = p + qr^n$$

A steady state can only exist if  $r < 1$ .

In this case the steady state number of customers is  $p$ .

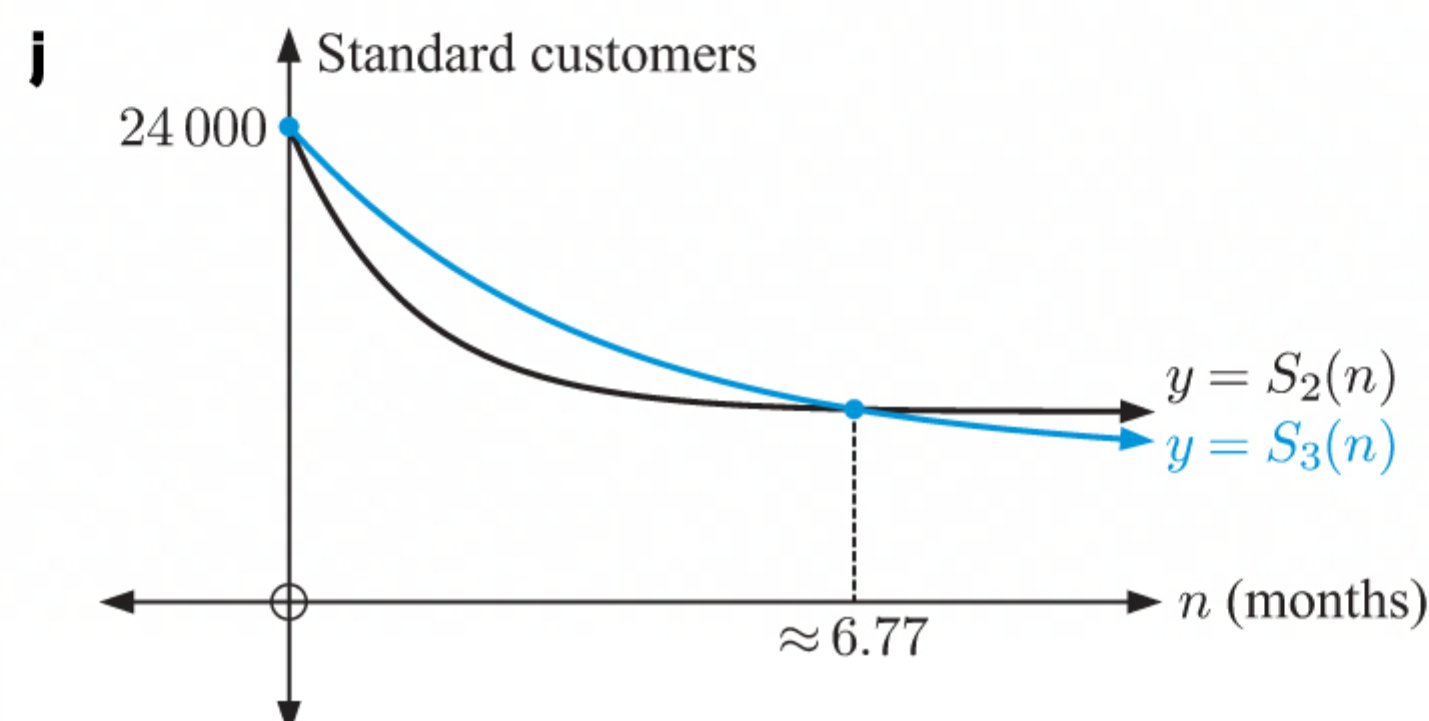
$$\therefore p = 7200 \quad \text{A1}$$

$$\text{In the initial state } S_3(0) = 7200 + q = 24\,000$$

$$\therefore q = 16\,800 \quad \text{A1}$$

$$\text{Now after 5 months } S_3(5) = 7200 + 16\,800r^5 \approx 11\,355$$

$$\therefore r \approx 0.756\,23 \quad \text{A1}$$



**A1 : each graph**

$$\mathbf{k} \quad S_3(n) < S_2(n) \text{ when } 7200 + 16\,800(0.756\,23)^n < 9600 + 14\,400(0.5)^n \quad (\text{M1})$$

$$\therefore n > 6.77$$

So, the number of Standard customers under the three package system will first be less than the number of Standard customers under the two package system in the 7th month. A1

**Total [28 marks]**

**2 a** Find the proportion of each colour in the population, and replicate these proportions for the colours in the sample. R2

**b**  $H_0$ :  $X$  follows a Poisson distribution with mean 1.7. A1

<b>c</b>	<i>Number of flaws</i>	0	1	2	3	$\geq 4$	<span style="float: right;">A2</span>
	<i>Expected frequency</i>	18.27	31.06	26.40	14.96	9.32	

**d** Degrees of freedom =  $5 - 1 = 4$  A1

**e** Using technology and the frequency tables, the  $p$ -value  $\approx 0.216$ . (M1)A1

**f**  $0.216 > 5\%$ , so we have insufficient evidence to reject  $H_0$ . R1

We therefore accept  $H_0$ , and conclude that  $X \sim \text{Po}(1.7)$ . A1



**g** Let  $F$  be the number of flawless scarves.

$$\therefore F \sim B(5, 0.1827) \quad (\text{M1})$$

$$\therefore P(F \geq 3) \approx 0.0455 \quad (\text{A2})$$

**h** The Central Limit Theorem applies since  $n > 30$ , so we can assume that  $\bar{X}$  is normally distributed. R1R1

**i** Let  $\bar{Y}$  be the mean number of flaws per scarf in a single box from the overseas manufacturer.

$H_0$ : No difference in flaws between local and overseas. (M1)

$H_1$ : Less flaws in overseas than local.

We conduct a one-tailed two-sample pooled  $t$ -test at the 5% level.

Using technology, the  $p$ -value  $\approx 0.0420$ . A2

$0.0420 < \alpha = 0.05$ , so we have sufficient evidence to reject  $H_0$  in favour of  $H_1$  at the 5% significance level.

We therefore accept  $H_1$ , and conclude the overseas scarves have fewer flaws than the local scarves. R1

Assumptions:

- $\bar{X}$  and  $\bar{Y}$  are both normally distributed
- $\bar{X}$  and  $\bar{Y}$  have the same variance.

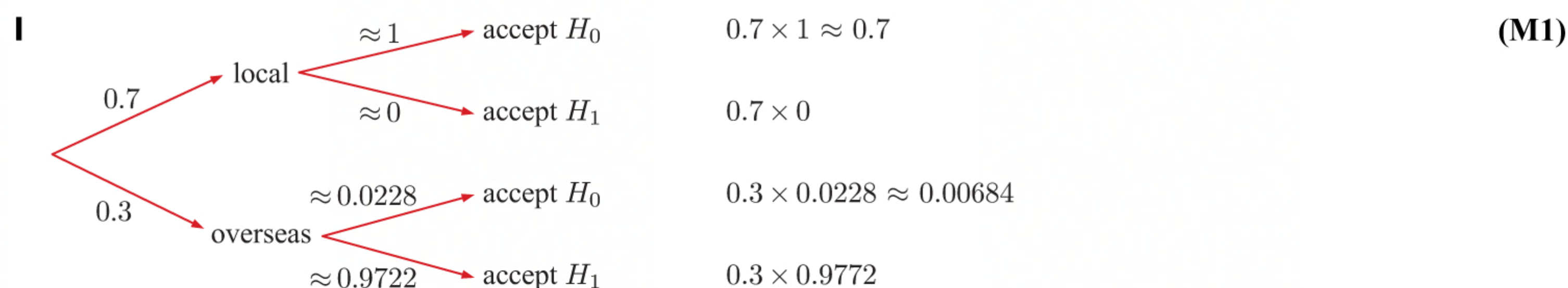
R1

**j**  $\bar{X} \sim N(1.7, (0.025)^2)$

$$\begin{aligned} P(\text{Type I error}) &= P(\text{accept } H_1 \mid H_0 \text{ is true}) \\ &= P(\bar{X} \leq 1.55 \mid \mu = 1.7) \\ &\approx 0.0000 \end{aligned} \quad \begin{array}{l} (\text{M1}) \\ (\text{A1}) \end{array}$$

**k**  $\bar{Y} \sim N(1.5, (0.025)^2)$

$$\begin{aligned} P(\text{Type II error}) &= P(\text{accept } H_0 \mid H_1 \text{ is true}) \\ &= P(\bar{Y} > 1.55 \mid \mu = 1.5) \\ &\approx 0.0228 \end{aligned} \quad \begin{array}{l} (\text{M1}) \\ (\text{A1}) \end{array}$$



$$\therefore P(\text{overseas} \mid \text{accept } H_0) \approx \frac{0.0228}{0.7 + 0.0228} \approx 0.00968 \quad \begin{array}{l} (\text{M1}) \\ (\text{A1}) \end{array}$$

**Total [27 marks]**



# TRIAL EXAMINATION 2

## PAPER 1

**1 a**  $IQR = 14 - 7$  M1  
 $= 7$  A1

**b** Upper boundary  $= 14 + (1.5 \times 7)$  M1  
 $= 24.5$  A1

$\therefore$  the largest value that is not an outlier is 18. A1

**Total [5 marks]**

**2 a i**  $\approx -2.81$  A1

**ii**  $x = 9^2 = 81$  A1

**b**  $(3^2)^x \times (3^{-3})^{1-x} = 3^0$  M1

$\therefore 3^{2x} \times 3^{3x-3} = 3^0$

$\therefore 3^{5x-3} = 3^0$  (M1)

$\therefore 5x - 3 = 0$

$\therefore x = \frac{3}{5}$  A1

**c**  $\ln(1.65^a \times 2.03^b) = \ln 9.8$

$\ln(1.42^a \times 2.59^b) = \ln 13.6$  M1

$\therefore a \ln 1.65 + b \ln 2.03 = \ln 9.8$

$a \ln 1.42 + b \ln 2.59 = \ln 13.6$  M1

Solving simultaneously for  $a$  and  $b$  gives  $a \approx 1.42$ ,  $b \approx 2.22$ . A1A1

**Total [9 marks]**

**3 a** C A1

**b** The vertices are  $V_1(2, 1)$ ,  $V_2(1, 0)$ , and  $V_3(2, 3)$ .

$V_1A = \sqrt{(2 - (-1))^2 + (1 - 3)^2} = \sqrt{9 + 4} = \sqrt{13} \approx 3.6055 \dots$

$V_2A = \sqrt{(1 - (-1))^2 + (0 - 3)^2} = \sqrt{4 + 9} = \sqrt{13} \approx 3.6055 \dots$

$V_3A = 3$  M1M1

The new recycling centre should be placed at  $(2, 1)$  or  $(1, 0)$ . A1

**Total [4 marks]**

**4 a** Using GDC:

$N = 5$ ,  $I\% = 2.5$ ,  $PV = 0$ ,  $PMT = -500$ ,  $P/Y = 1$ ,  $C/Y = 1$  (M1)(A1)

$FV \approx 2628.16$ , so she will have saved £2628.16. A1

**b**  $I\% = 2.5$ ,  $PV = 0$ ,  $PMT = -500$ ,  $FV = 10\,000$ ,  $P/Y = 1$ ,  $C/Y = 1$  (M1)

$N \approx 16.4$ , so it will take 17 years. A1

**Total [5 marks]**

**5 a** The sequence is geometric with  $u_1 = 12$  and  $r = 1.1$ . A1

Solving  $2000 = \frac{12((1.1)^n - 1)}{1.1 - 1}$  (M1)

gives  $n = 30.129 \dots$  {GDC} (A1)

It will take 31 days in total. A1

**b** She will cycle furthest on the 30th day. (M1)

$u_{30} = 12(1.1)^{29}$  (M1)

$\approx 190$  miles A1

**Total [7 marks]**



- 6 a i B and C** A1  
**ii A, B, and D** A1  
**iii B** A1  
**b** Yes. All vertices have even degree. R1A1

**Total [5 marks]**

- 7 a**  $x \neq 3 \quad (x \in \mathbb{R})$  A1  
**b**  $\frac{3x+5}{x-3} = -4 \quad \{f^{-1}(-4) \text{ is the solution to } f(x) = -4\}$  (M1)(A1)  
 $\therefore 3x + 5 = -4x + 12$   
 $\therefore x = 1$  A1  
**c**  $\frac{3x+5}{x-3} = 2^{x+1}$  (M1)  
 Using GDC,  $x \approx -2.36$  or  $3.64$ . A1

**Total [6 marks]**

- 8 a**  $H_0: p_1 = \frac{1}{6}, p_2 = \frac{1}{6}, p_3 = \frac{2}{3}$   
 $H_1: p_1 \neq \frac{1}{6}, \text{ or } p_2 \neq \frac{1}{6}, \text{ or } p_3 \neq \frac{2}{3}$   
 where  $p_1, p_2$ , and  $p_3$  represent the proportions of sweet peas with red, white, and pink flowers respectively. A1  
 OR  
 $H_0$ : the proportions are in line with the theory.  
 $H_1$ : the proportions are not in line with the theory. A1  
**b** 2 A1  
**c** Expected values: red = 20  
 white = 20  
 pink = 80 (M1)  
 $p\text{-value} \approx 0.000\,659$  A1  
**d**  $0.000\,659 < 0.01 \quad \therefore \text{reject } H_0$  R1  
 At a 1% significance level the flower colours do not follow the proportions in the theory. A1

**Total [6 marks]**

- 9 a** After 12 months:  
 $C = 35 + 27(12) = 359$  (M1)  

$$C = \begin{cases} 35 + 27m, & 0 \leq m \leq 12 \\ 359 + 49(m - 12), & m > 12 \end{cases}$$
 A1 : correct equations  
A1 : correct domains  
**b** 3.5 years = 42 months (M1)  
 $\therefore C = 359 + 49(42 - 12) = \text{£}1829$  A1

**Total [5 marks]**

- 10 a**  $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}$  M1  
 $= \begin{pmatrix} -8 - 6 \\ -2 - 12 \\ -18 + 4 \end{pmatrix}$   
 $= \begin{pmatrix} -14 \\ -14 \\ -14 \end{pmatrix}$  A1



**Method 1**

Area of trapezium = area of parallelogram + area of triangle

**M1**

$$= |\mathbf{a} \times \mathbf{b}| + \frac{1}{2} |\mathbf{a} \times \mathbf{b}|$$

**M1**

$$= \frac{3}{2} |\mathbf{a} \times \mathbf{b}|$$

**(A1)**

$$= \frac{3}{2} \sqrt{(-14)^2 + (-14)^2 + (-14)^2}$$

$$= \frac{3}{2} \sqrt{588}$$

$$\approx 36.4 \text{ units}^2$$

**A1**
**Method 2**

 component of  $\mathbf{a}$  perpendicular to  $\mathbf{b} = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$ 
**M1**

$$= \frac{\sqrt{(-14)^2 + (-14)^2 + (-14)^2}}{\sqrt{2^2 + (-6)^2 + 4^2}}$$

$$= \frac{14\sqrt{3}}{2\sqrt{14}}$$

$$= \frac{1}{2} \sqrt{42}$$

**A1**

$$\therefore \text{area of trapezium} = \frac{|\mathbf{b}| + 2|\mathbf{b}|}{2} \times \frac{1}{2} \sqrt{42}$$

**M1**

$$= \frac{3}{2} \times 2\sqrt{14} \times \frac{1}{2} \sqrt{42}$$

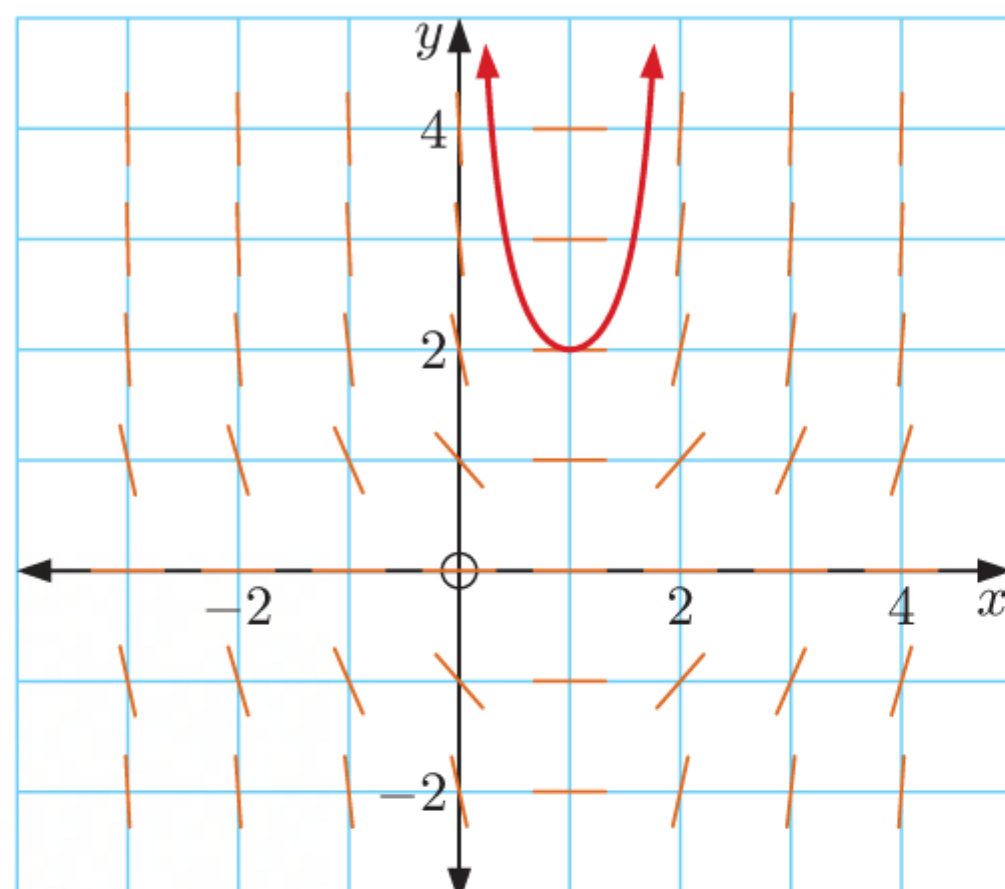
$$= \frac{3}{2} \sqrt{588}$$

$$\approx 36.4 \text{ units}^2$$

**A1**
**Total [6 marks]**
**M1 : tied ranks**
**A1 : correct values in both rows**
**11**

	A	B	C	D	E	F	G	H
True rank	1	2	3	4	5	6	7	8
Butcher's rank	1	2.5	4	2.5	7	7	5	7

 Using GDC,  $r_s \approx 0.835$ .

**M1A1**
**Total [4 marks]**
**12 a**

**A1 : shape**
**A1 : passing through (1, 2)**
**b**

$$\frac{1}{y^2} \frac{dy}{dx} = (x - 1)$$

**M1**

$$\therefore \int y^{-2} dy = \int (x - 1) dx$$

**M1**

$$\therefore -\frac{1}{y} = \frac{x^2}{2} - x + c$$

**A1**

 Substituting in (1, 2) gives  $-\frac{1}{2} = \frac{1}{2} - 1 + c$ 
**(M1)**

$$\therefore c = 0$$

$$\therefore -\frac{1}{y} = \frac{x^2}{2} - x$$

$$\therefore \frac{1}{y} = \frac{2x - x^2}{2}$$

$$\therefore y = \frac{2}{2x - x^2}$$

**A1**
**Total [7 marks]**



- 13 a**  $g(6.33) \approx 4.01$  A1
- b**  $y = \frac{1}{10}x^2, \quad 0 \leq x \leq 6.33$   
 $\therefore$  the inverse function is  $x = \frac{1}{10}y^2$  (A1)  
 $\therefore y^2 = 10x$   
 $\therefore y = \sqrt{10x} \quad \{y \geq 0\}$   
 $\therefore g^{-1}(x) = \sqrt{10x}$  A1
- c** Volume  $\approx \pi \int_0^{4.01} (\sqrt{10x})^2 dx$  M1A1  
 $\approx 252.193 \dots$   
 $\approx 252 \text{ units}^3$  A2
- Total [7 marks]**
- 14 a** Poisson distribution,  $X \sim \text{Po}(2.1)$  A1
- b**  $P(X \geq 2) = 1 - P(X \leq 1)$  (M1)  
 $\approx 1 - 0.37961 \dots$  A1  
 $\approx 0.620$  A1
- c** Let  $Y$  be the number of mistakes in 10 papers.  
 $E(Y) = 2.1 \times 10 = 21$  M1  
 $\therefore Y \sim \text{Po}(21)$  A1  
 $P(Y = 15) \approx 0.0395$  A1
- Total [7 marks]**
- 15 a**  $A' = \begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$   
 $D' = \begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$  M1  
 $A'(4, 2), D'(8, 8)$  A1A1
- b** Area of rectangle ABCD  $= 4 \times 2 = 8 \text{ units}^2$  A1  
Area of image  $= \left| \det \begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix} \right| \times \text{area of original}$  M1  
 $= 8 \times 8$   
 $= 64 \text{ units}^2$  A1
- Total [6 marks]**
- 16 a** The weight of apple boxes  $A \sim N(25.7, 3.1^2)$ .  
The weight of banana boxes  $B \sim N(17.3, 1.5^2)$ .  
The weight of orange boxes  $C \sim N(22.1, 2.2^2)$ .  
 $P(2A > B + C) = P(2A - B - C > 0)$  M1  
Let  $X = 2A - B - C$   
 $E(X) = 2(25.7) - 17.3 - 22.1$   
 $= 12$  A1  
 $\text{Var}(X) = 4(3.1)^2 + (1.5)^2 + (2.2)^2$   
 $= 45.53$  A1  
 $\therefore X \sim N(12, 45.53)$  A1  
 $P(X > 0) \approx 0.962$  A1
- b** For 10 boxes of apples, the mean weight  $\bar{A}_{10} \sim N\left(25.7, \frac{3.1^2}{10}\right)$  M1A1  
 $P(\bar{A}_{10} > 26) \approx 0.380$  A1
- Total [8 marks]**



$$\begin{aligned}
 \mathbf{17} \quad \mathbf{a} \quad \vec{AB} &= \begin{pmatrix} 4 \\ 7.5 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} -1 \\ 0.5 \\ 1 \end{pmatrix}
 \end{aligned}$$

M1

$$\begin{aligned}
 |\vec{AB}| &= \sqrt{(-1)^2 + 0.5^2 + 1^2} \\
 &= 1.5
 \end{aligned}$$

M1

A1

$$\therefore \text{speed} = \frac{1.5}{\frac{1}{3}} = 4.5 \text{ km/min}$$

A1

$$\mathbf{b} \quad \text{We require the angle } \theta \text{ between } \begin{pmatrix} -1 \\ 0.5 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 0.5 \\ 0 \end{pmatrix}.$$

(M1)

$$\cos \theta = \frac{\begin{pmatrix} -1 \\ 0.5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0.5 \\ 0 \end{pmatrix}}{\sqrt{2.25} \sqrt{1.25}}$$

M1A1

$$\therefore \theta \approx 41.8^\circ$$

A1

Total [8 marks]

$$\mathbf{18} \quad t_{n+1} = t_n + 0.1$$

$$x_{n+1} = x_n + 0.1(x_n t_n + 30)$$

M1A1

$n$	$t_n$	$x_n$
0	0	0
1	0.1	3
2	0.2	6.03
3	0.3	9.1506
4	0.4	12.4251
5	0.5	15.9221
6	0.6	19.7182
7	0.7	23.9013
8	0.8	28.5744
9	0.9	33.8604
10	1	39.9078

A2

$$\therefore x(1) \approx 39.9$$

A1

Total [5 marks]

## PAPER 2

$\mathbf{1} \quad \mathbf{a}$	$x$	2	2.5	3	3.5	4	4.5	5	5.5	6	$\mathbf{A2}$
	$y = \sqrt[4]{20x^2 + 1}$	3	3.3504	3.6679	3.9604	4.2328	4.4888	4.7311	4.9616	5.1818	

$$\mathbf{b} \quad h = \frac{1}{2}$$

(A1)

$$\begin{aligned}
 &\int_2^6 \sqrt[4]{20x^2 + 1} \, dx \\
 &\approx \frac{1}{2} \left( \frac{1}{2} \right) [(3 + 5.1818) + 2(3.3504 + 3.6679 + 3.9604 + 4.2328 + 4.4888 + 4.7311 + 4.9616)] \\
 &\approx 16.742
 \end{aligned}$$

(M1)A1

A1

$$\mathbf{c} \quad \int_2^6 \sqrt[4]{20x^2 + 1} \, dx = 16.748331 \dots$$

A2

$$\mathbf{d} \quad \frac{|16.742 - 16.748|}{16.748} \times 100\%$$

(M1)A1

$$\approx 0.0358\%$$

A1



$$\begin{aligned} \text{e Area} &= \int_2^6 \sqrt[4]{20x^2 + 1} \, dx - \text{area of triangle} && \text{(A1)} \\ &\approx 16.748 - \frac{1}{2} \times 4 \times 3 && \text{M1} \\ &\approx 10.748 \, \text{m}^2 && \text{A1} \end{aligned}$$

Total [14 marks]

2 a Systematic A1

$$\text{b } P(> 3 \text{ hours} \mid \text{Year 12}) = \frac{9}{11 + 14 + 9} = \frac{9}{34} \quad \text{M1A1}$$

c i  $H_0$ : Year group and the amount of screen time are independent A1

ii 6 A1

iii  $p\text{-value} \approx 0.770$  (0.769 58 ....) A2iv  $0.770 > 0.1$ , so accept  $H_0$  R1

Year group and the amount of screen time are independent. A1

d i  $H_0: p = 0.2$   
 $H_1: p > 0.2$  A1

ii The null distribution of the test statistic is  $X \sim B(34, 0.2)$ . M1

$$\begin{aligned} p\text{-value} &= P(X \geq 11) && \text{M1} \\ &= 1 - P(X \leq 10) \\ &\approx 1 - 0.9380 \\ &\approx 0.0620 && \text{A1} \end{aligned}$$

iii  $0.0620 < 0.1$ , so reject  $H_0$  R1

The proportion of Year 12 students who have less than 1 hour of screen time each evening has increased. A1

Total [15 marks]

3 a  $35^\circ$  A1

$$\begin{aligned} \text{b } \widehat{FBC} &= 180^\circ - 35^\circ = 145^\circ \\ \widehat{BFC} &= 180^\circ - 145^\circ - 29^\circ = 6^\circ && \text{(A1)(A1)} \end{aligned}$$

$$\begin{aligned} \text{In } \triangle BCF, \quad \frac{FC}{\sin 145^\circ} &= \frac{7.5}{\sin 6^\circ} && \text{M1A1} \\ \therefore FC &= \frac{7.5 \sin 145^\circ}{\sin 6^\circ} \\ &= 41.154 \dots \\ &\approx 41.2 \, \text{m} && \text{A1} \end{aligned}$$

$$\begin{aligned} \text{c In } \triangle ACF, \quad \sin 29^\circ &= \frac{FA}{41.154 \dots} && \text{(M1)A1} \\ \therefore FA &= 19.952 \dots && \text{A1} \end{aligned}$$

The height of the flagpole is about 20.0 m.

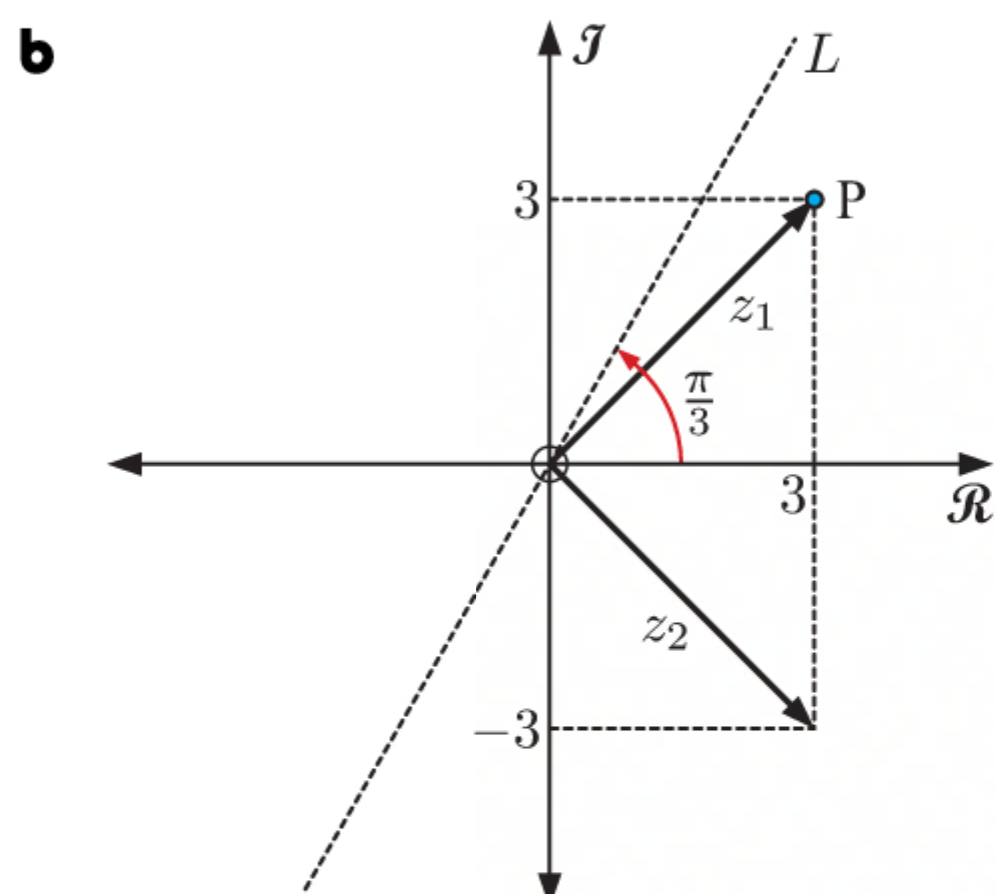
$$\begin{aligned} \text{d In } \triangle ACF, \quad \tan 29^\circ &= \frac{19.952 \dots}{AC} && \text{(M1)} \\ \therefore AC &= \frac{19.952 \dots}{\tan 29^\circ} \\ &= 35.994 \dots && \text{(A1)} \end{aligned}$$

$$\begin{aligned} \therefore \text{total distance} &= 35.994 \dots \times 4 \\ &\approx 144 \, \text{m} && \text{A1} \end{aligned}$$

Total [12 marks]

4 a  $z_2 = 3 - 3i$  A1





A1

**c**

$$\begin{aligned} & (x - (3 + 3i))(x - (3 - 3i)) \\ &= x^2 - 3x + 3xi - 3x - 3xi + 9 + 9 \\ &= x^2 - 6x + 18 \end{aligned}$$

M1A1

A1

A1

**d**

$$\begin{aligned} \arg z &= \tan^{-1}\left(\frac{3}{3}\right) \\ &= \frac{\pi}{4} \\ |z| &= \sqrt{3^2 + 3^2} = 3\sqrt{2} \\ z &= 3\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right) \end{aligned}$$

A1

A1

**e i**

$$\alpha = \frac{\pi}{3}, \text{ so } 2\alpha = \frac{2\pi}{3}$$

$$\therefore \cos 2\alpha = -\frac{1}{2} \text{ and } \sin 2\alpha = \frac{\sqrt{3}}{2}$$

M1

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

M1A1

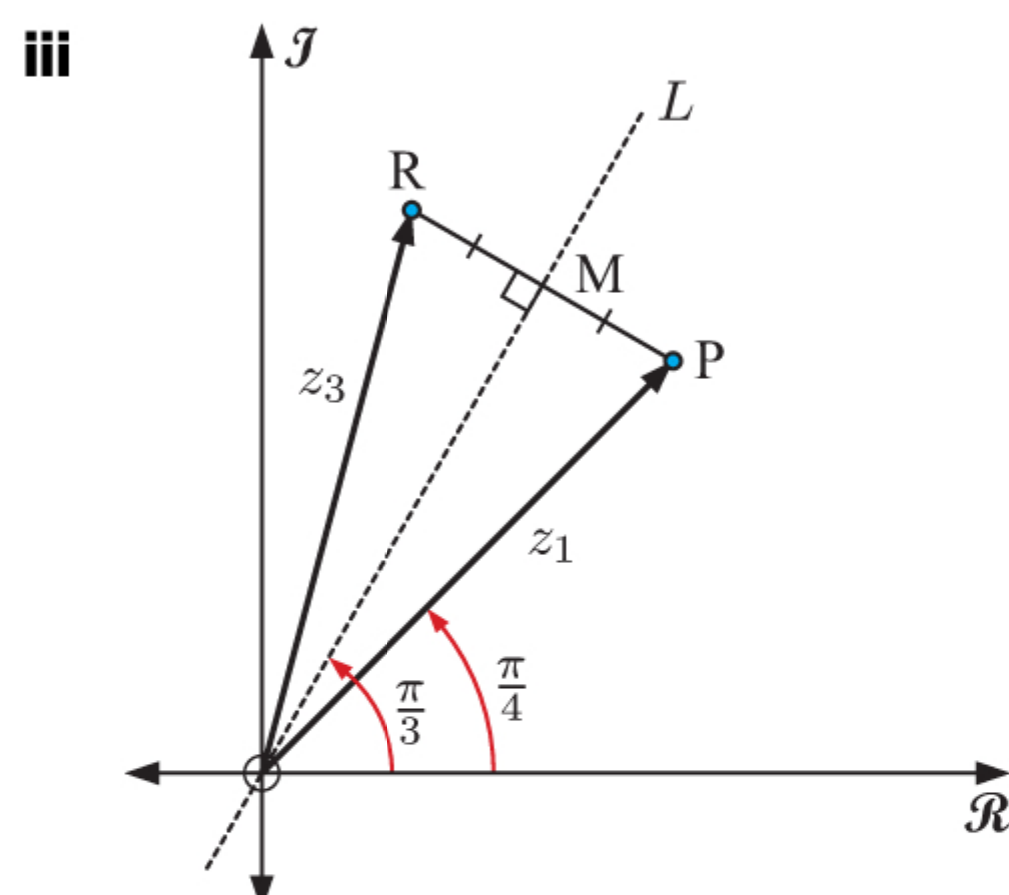
**ii**

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} + \frac{3\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} + \frac{3}{2} \end{pmatrix}$$

$$\therefore R\left(-\frac{3}{2} + \frac{3}{2}\sqrt{3}, \frac{3}{2} + \frac{3}{2}\sqrt{3}\right)$$

M1

A1


 $\triangle OMR$  and  $\triangle OMP$  are congruent.

(M1)

$$\therefore |z_3| = |z_1| = 3\sqrt{2}$$

M1AG

$$\text{and } \widehat{MOR} = \widehat{MOP}$$

(M1)

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

$$\therefore \arg z_3 = \frac{\pi}{3} + \frac{\pi}{12} = \frac{5\pi}{12}$$

M1AG

**iv** From **ii** and **iii**,  $z_3 = 3\sqrt{2} \operatorname{cis} \frac{5\pi}{12} = \left(-\frac{3}{2} + \frac{3}{2}\sqrt{3}\right) + \left(\frac{3}{2} + \frac{3}{2}\sqrt{3}\right)i$

M1

$$\text{Equating real parts, } 3\sqrt{2} \cos \frac{5\pi}{12} = \frac{3\sqrt{3} - 3}{2}$$

M1

$$\begin{aligned} \therefore \cos \frac{5\pi}{12} &= \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

AG

$$\text{Equating imaginary parts, } 3\sqrt{2} \sin \frac{5\pi}{12} = \frac{3\sqrt{3} + 3}{2}$$

M1

$$\begin{aligned} \therefore \sin \frac{5\pi}{12} &= \left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

AG

Total [20 marks]



- 5 a**  $X \sim N(51.6, 7.5^2)$   
 $P(X < 40) = 0.060\,97\dots$  **A1**  
 $\therefore$  about 6.10% of eggs weighed less than 40 g. **A1**
- b**  $P(49 \leq X \leq k) = 0.3569$   
 $\therefore P(X \leq k) - P(X < 49) = 0.3569$  **M1**  
 $\therefore P(X \leq k) - 0.364\,42\dots = 0.3569$  **A1**  
 $\therefore P(X \leq k) = 0.721\,32\dots$  **A1**  
 $\therefore k \approx 56.0$  **A1**
- c** Let  $Y$  be the number of medium eggs selected.  
 $Y \sim B(40, 0.3569)$  **M1**  
 $\frac{2}{5}$  of 40 = 16 **A1**  
 $P(Y = 16) \approx 0.109$  **A1**
- d**  $P(Y \geq 10) = 1 - P(Y \leq 9)$  **M1**  
 $= 1 - 0.053\,95\dots$   
 $\approx 0.946$  **A1**
- e**  $\bar{x} = 53.1$   
 $s \approx 13.0$  **A3**
- f i**  $H_0: \mu_0 = 51.6$   
 $H_1: \mu_0 > 51.6$  **A1**
- ii**  $p\text{-value} \approx 0.267$   
 $t \approx 0.630$  **A2**
- g**  $0.267 > 0.1$ , so we accept  $H_0$  **R1**  
The eggs are not significantly heavier than last year, so the farmer is not correct. **A1**

**Total [19 marks]**

- 6 a**

$\ln D$	0.095 31	1.939	3.191	4.176	4.637
$\ln T$	-0.9163	0.4055	0.7419	1.281	1.482

**A1A1**
- b** Using GDC:  $a \approx 0.511$  **A1**  
 $b \approx -0.835$  **A1**
- c**  $\ln T \approx 0.511 \ln D - 0.835$   
 $\therefore \ln T \approx \ln(D^{0.511}) - \ln(e^{0.835})$  **(M1)**  
 $\therefore \ln T \approx \ln\left(\frac{D^{0.511}}{e^{0.835}}\right)$  **M1**  
 $\therefore T \approx \frac{1}{e^{0.835}} \times D^{0.511}$  **A1**  
 $\therefore n \approx 0.511, k \approx e^{-0.835} \approx 0.434$  **A1A1**
- d**  $T \approx 0.434(150)^{0.511}$  **M1**  
 $\approx 5.62$  seconds **A1**
- e** Since 150 is outside the range of values for  $D$  the prediction in **d** is an extrapolation and is therefore unreliable. **R1A1**

**Total [13 marks]**

- 7 a**  $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$  **A1**
- b**  $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 15 \\ 20 \end{pmatrix} = \begin{pmatrix} 55 \\ 120 \end{pmatrix}$  **M1A1**



$$\begin{aligned} \text{c } \det(\lambda \mathbf{I} - \mathbf{A}) &= \begin{vmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{vmatrix} = 0 & \text{M1} \\ \therefore (\lambda - 1)(\lambda - 3) - 8 &= 0 & \text{A1} \\ \therefore \lambda^2 - 4\lambda - 5 &= 0 \\ \therefore (\lambda + 1)(\lambda - 5) &= 0 \\ \therefore \lambda &= -1 \text{ or } 5 & \text{A1} \end{aligned}$$

$$\begin{aligned} \text{When } \lambda = -1, \quad \begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= 0 \\ \therefore x &= -y \end{aligned}$$

$$\begin{aligned} \text{When } \lambda = 5, \quad \begin{pmatrix} 4 & -2 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= 0 \\ \therefore 2x &= y & \text{M1} \end{aligned}$$

$$\therefore \text{an eigenvector for } \lambda = -1 \text{ is } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{A1}$$

$$\text{an eigenvector for } \lambda = 5 \text{ is } \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{A1}$$

$$\text{d } \mathbf{x} = Ae^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{(M1)}$$

$$\text{When } t = 0, \quad \mathbf{x} = \begin{pmatrix} 15 \\ 20 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 15 \\ 20 \end{pmatrix} = A \begin{pmatrix} 1 \\ -1 \end{pmatrix} + B \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{A1}$$

$$\begin{pmatrix} 15 \\ 20 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{(M1)}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 15 \\ 20 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{M1}$$

$$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ \frac{35}{3} \end{pmatrix}$$

$$\mathbf{x} = \frac{10}{3}e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{35}{3}e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{A1}$$

e After 6 months,  $t = 0.5$ .

$$\mathbf{x} = \frac{10}{3}e^{-0.5} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{35}{3}e^{2.5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 144.1508 \dots \\ 282.2364 \dots \end{pmatrix} \quad \text{M1A1}$$

Height of marigolds  $\approx 1.44$  m

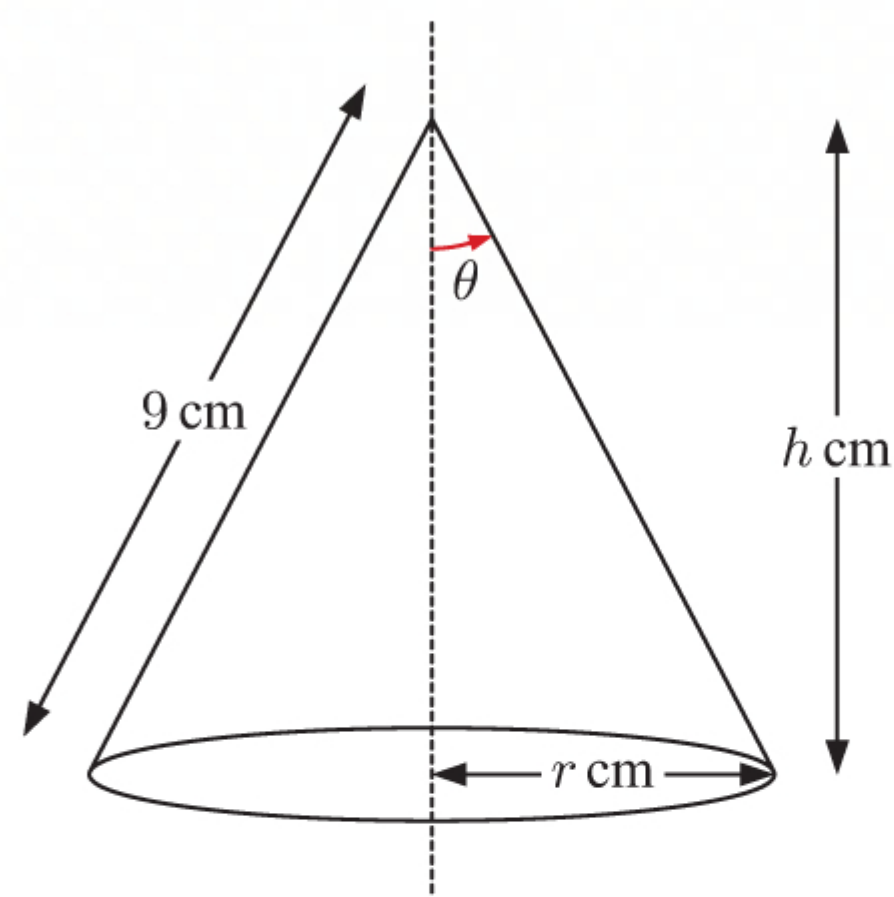
Height of dianthus  $\approx 2.82$  m

Alanna will need to cut back both the marigolds and the dianthus. A1

Total [17 marks]

### PAPER 3

1 a



$$h = 9 \cos \theta \quad \text{A1}$$

$$r = 9 \sin \theta \quad \text{A1}$$

$$\therefore V = \frac{1}{3}\pi(9 \sin \theta)^2(9 \cos \theta) \quad \text{(M1)}$$

$$\therefore V = 243\pi \sin^2 \theta \cos \theta \quad \text{A1}$$



$$\mathbf{b} \quad \frac{dV}{d\theta} = 486\pi \sin \theta \cos \theta (\cos \theta) + 243\pi \sin^2 \theta (-\sin \theta) \quad \mathbf{M1A1}$$

$$\therefore \frac{dV}{d\theta} = 243\pi (2 \sin \theta \cos^2 \theta - \sin^3 \theta) \quad \mathbf{A1}$$

$$\mathbf{c} \quad \frac{dV}{d\theta} = 0 \quad \mathbf{M1}$$

$$\text{Using GDC, } \theta \approx 0.955 \text{ radians. } \{0 < \theta < \frac{\pi}{2}\} \quad \mathbf{A1}$$

$$\mathbf{d} \quad \theta \approx 54.7^\circ \quad \mathbf{A1}$$

$$\mathbf{e} \quad V \approx 243\pi \sin^2 0.955 \cos 0.955 \quad \mathbf{(M1)}$$

$$\approx 294 \text{ cm}^3 \quad \mathbf{A1}$$

$$\mathbf{f} \quad 90\% \text{ of maximum volume} = 264.451 \dots \quad \mathbf{A1}$$

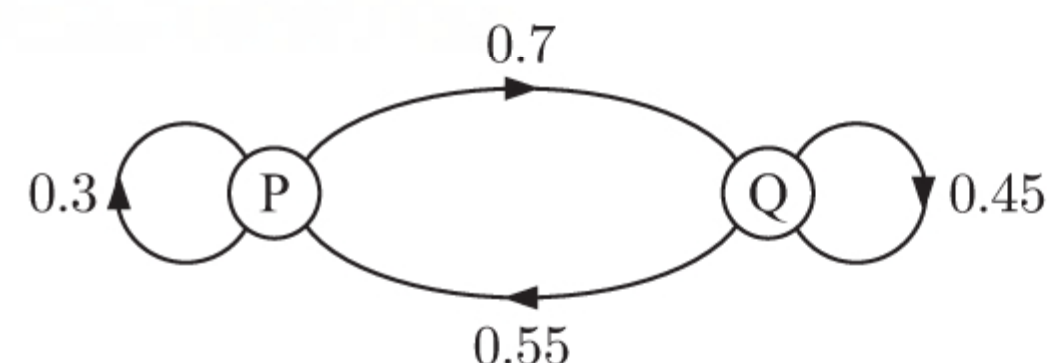
Converting units

$$11.7 \text{ litres} = 11\,700 \text{ cm}^3 \quad \mathbf{M1}$$

$$11\,700 \div 264.451 \dots \approx 44.24 \quad \mathbf{M1}$$

It will fill approximately 44 cups.  $\mathbf{A1}$

$$\mathbf{g} \quad \quad \quad \mathbf{A3}$$



$$\mathbf{h} \quad \mathbf{M} = \begin{pmatrix} 0.3 & 0.55 \\ 0.7 & 0.45 \end{pmatrix} \quad \mathbf{A2}$$

$$\mathbf{i} \quad \mathbf{s}_0 = \begin{pmatrix} 100 \\ 50 \end{pmatrix} \quad \mathbf{A1}$$

$$\mathbf{j} \quad \mathbf{s}_2 = \begin{pmatrix} 0.3 & 0.55 \\ 0.7 & 0.45 \end{pmatrix}^2 \begin{pmatrix} 100 \\ 50 \end{pmatrix} \quad \mathbf{M1A1}$$

$$= \begin{pmatrix} 68.125 \\ 81.875 \end{pmatrix} \quad \mathbf{A1}$$

Approximately 82 people will vote for company Q in two years' time.  $\mathbf{A1}$

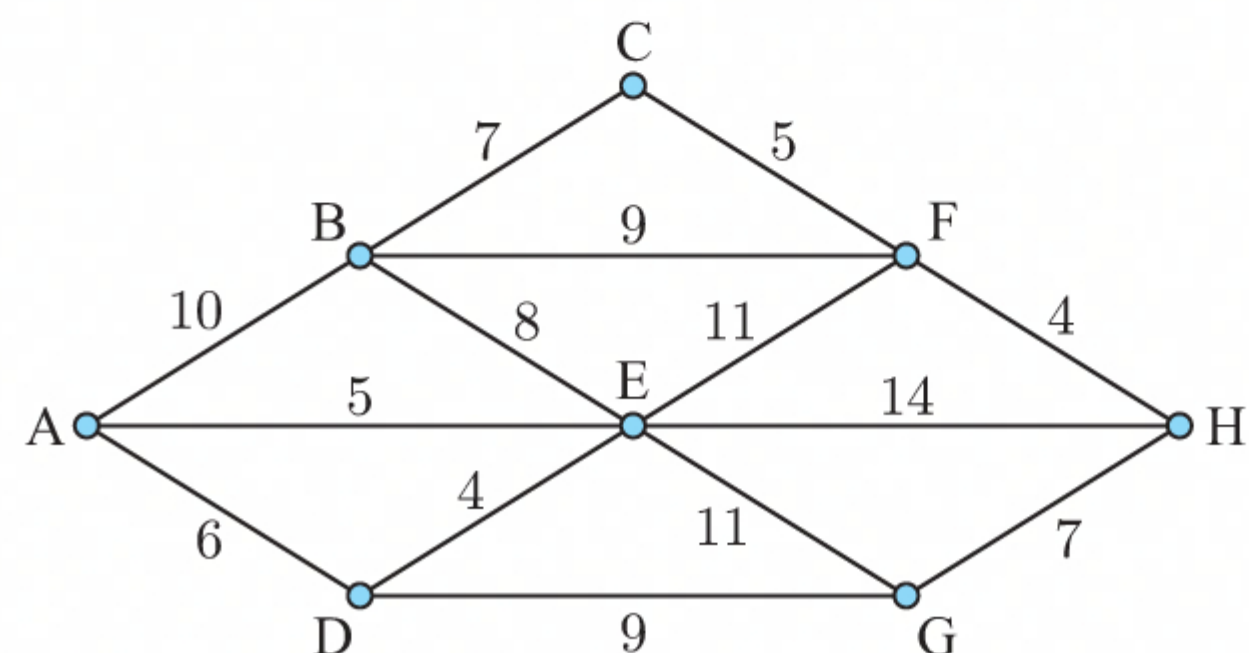
$$\mathbf{k} \quad \mathbf{s}_{100} = \begin{pmatrix} 0.3 & 0.55 \\ 0.7 & 0.45 \end{pmatrix}^{100} \begin{pmatrix} 100 \\ 50 \end{pmatrix} \quad \mathbf{M1}$$

$$\approx \begin{pmatrix} 66 \\ 84 \end{pmatrix} \quad \mathbf{A1}$$

In the long term, about 66 people will vote for company P.  $\mathbf{A1}$

**Total [29 marks]**

$$\mathbf{2} \quad \mathbf{a} \quad \quad \quad \mathbf{A3}$$



$$\mathbf{b} \quad \mathbf{i} \quad \deg(E) = 6 \quad \mathbf{A1}$$

$\mathbf{ii}$  There are 6 sections which can be accessed directly from the chimpanzees.  $\mathbf{A1}$



- c**  $AE = 5$   
 $ED = 4$   
 $EB = 8$   
 $BC = 7$   
 $CF = 5$   
 $FH = 4$   
 $HG = 7$  **M2**
- Weight of minimum spanning tree  $= 5 + 4 + 8 + 7 + 5 + 4 + 7$  **(M1)**  
 $= 40$  **A1**
- d** Chinese postman algorithm. **A1**
- e** Vertices A, D, G, and H have odd degree. **A1**
- Possible combinations:  $AD \text{ and } GH = 6 + 7 = 13$   
 $AG \text{ and } DH = 15 + 16 = 31$   
 $AH \text{ and } DG = 19 + 9 = 28$  **M1A1**
- Total weight of graph  $= 110$  **A1**
- Total weight with AD and GH repeated  $= 110 + 13$  **(M1)A1**
- $\therefore$  shortest time  $= 123$  minutes **A1**
- f**  $123 \text{ minutes} = 2 \text{ hours } 3 \text{ minutes}$
- $\therefore$  they will leave the zoo at 5:33 pm, and reach the restaurant at 5:48 pm. **(M1)**
- $\therefore$  they will make the reservation. **A1**
- g** It is equally close to recycling bins L, M, and P. **A2**
- h** Midpoint of [JN]  $= (4.5, 3.5)$  **A1**
- Gradient of [JN]  $= 1$
- $\therefore$  perpendicular bisector has gradient  $-1$ . **A1**
- Perpendicular bisector has equation  $y - 3.5 = -1(x - 4.5)$  **(M1)**
- $\therefore x + y = 8$  **A1**
- i** Bin J **A1**

**Total [26 marks]**



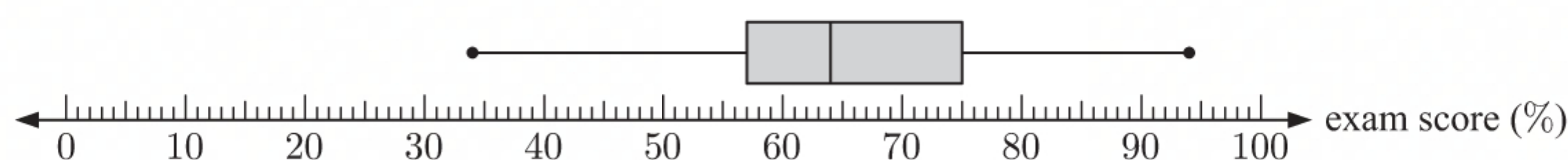
# TRIAL EXAMINATION 3

## PAPER 1

- 1 a**  $129 - 13 = 116$  out of 129, which is about 89.9% of students, obtained at least 50% for their examination. **A1**

- b** The 5-number summary for the results is:
- $$\begin{aligned} \min &= 34 \\ Q_1 &= 57 \\ \text{median} &= 64 \\ Q_3 &= 75 \\ \max &= 94 \end{aligned}$$
- (M1)**

The box and whisker diagram is:



**A1A1**

- c** The interquartile range  $= Q_3 - Q_1$   
 $= 75 - 57$   
 $= 18$  **A1**

The middle half of results are spread across 18%. **A1**

**Total [6 marks]**

- 2 a** Each year, the rent Georgio pays is multiplied by the same constant 1.025.  
 $\therefore$  the rent follows a geometric sequence with common ratio  $r = 1.025$ . **R1**  
**A1**

- b** The rent in 2020 is  $u_1 = €12\,000$   
 $\therefore$  the rent in 2025 is  $u_6 = €12\,000 \times 1.025^5$  **M1**  
 $= €13\,576.90$  **A1**

- c** The total rent Georgio pays from 2020 to 2025 inclusive is  $S_6 = \frac{u_1(r^6 - 1)}{r - 1}$  **M1**  
 $= € \frac{12\,000(1.025^6 - 1)}{0.025}$   
 $= €76\,652.84$  **A1**

**Total [6 marks]**

- 3 a** In a stratified sample, each age group is sampled according to its proportion of the population.  
 Now 28.3% of 250  $= 0.283 \times 250$  **M1**  
 $= 70.75$

$\therefore$  71 customers in the age range 51 - 70 were sampled. **A1**

- b** The total number of customers surveyed  $= 250$   
 $\therefore 55 + x + 56 + 42 + y + 17 + 6 = 250$   
 $\therefore x + y = 74$  .... (1) **A1**

The mean number of visits per week was 3.08.

$$\begin{aligned} \therefore \frac{1 \times 55 + 2x + 3 \times 56 + 4 \times 42 + 5y + 6 \times 17 + 7 \times 6}{250} &= 3.08 \\ \therefore 2x + 5y + 535 &= 770 \\ \therefore 2x + 5y &= 235 \quad \dots (2) \end{aligned}$$

**A1**

- c** Using (1),  $2x + 2y = 148$  .... (3)  
 (2) - (3) gives  $3y = 87$   
 $\therefore y = 29$  **A1**  
 $\therefore x = 74 - 29 = 45$  **A1**

**Total [6 marks]**



- 4 a** The population standard deviation is known, so the normal distribution should be used to construct the confidence interval. **R1A1**
- b** 
$$\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}}$$
 **(M1)**  

$$\therefore 152 - 1.96 \times \frac{30}{\sqrt{50}} \leq \mu \leq 152 + 1.96 \times \frac{30}{\sqrt{50}}$$
  

$$\therefore 143.7 \leq \mu \leq 160.3$$
 **A1**
- Total [4 marks]**
- 5 a** Using technology, volume  $\approx 0.0160 \times (\text{height})^{3.00}$  where  $r \approx 1.00$ . **A1**
- b** Since  $r \approx 1$  for the model in **a**, and the power 3.00 is consistent with volume being proportional to the cube of a length, it is reasonable to believe that the models are mathematically similar. **R1A1**
- c** Using the model from **a**, if height = 5.17 m then volume  $\approx 0.016 \times 5.17^3$   

$$\approx 2.21 \text{ m}^3$$
 **M1**
- This is about 5.3% more than the actual volume of David, so we conclude the replicas are not to scale. **R1**
- Total [6 marks]**
- 6 a** Since  $V \propto r^3$ , if the radius of a sphere is doubled, its volume is multiplied by  $2^3 = 8$ . **A1**
- b** 
$$V = \frac{4}{3}\pi r^3$$
  

$$\therefore r^3 = \frac{3V}{4\pi}$$
  

$$\therefore r = \sqrt[3]{\frac{3V}{4\pi}}$$
 **M1**  

$$\therefore V^{-1}(V) = \sqrt[3]{\frac{3V}{4\pi}}$$
 **A1**  

$$(V^{-1} \circ V)(r) = V^{-1}(V(r))$$
  

$$= \sqrt[3]{\frac{3(\frac{4}{3}\pi r^3)}{4\pi}}$$
 **M1**  

$$= \sqrt[3]{r^3}$$
  

$$= r$$
 **AG**
- c** The function  $V(r)$  maps the radius  $r$  of a sphere to its volume  $V$ . The inverse function  $V^{-1}(V)$  does the reverse. It maps the volume  $V$  of a sphere to its radius  $r$ . **R1**
- Total [5 marks]**
- 7 a** The missing edge is the perpendicular bisector of CD.
- The gradient of CD =  $\frac{3-1}{2-4} = \frac{2}{6} = \frac{1}{3}$  **A1**
- The midpoint of CD is  $\left(\frac{-4+2}{2}, \frac{1+3}{2}\right)$  which is  $(-1, 2)$ . **A1**
- $\therefore$  the missing edge has equation  $y - 2 = -3(x + 1)$
- Rearranging,  $y - 2 = -3x - 3$   
 $\therefore 3x + y + 1 = 0$  **A1**
- b** The missing edge must intersect OP at P.
- Substituting  $y = -\frac{2}{3}x$  into the equation of the missing edge,  $3x + (-\frac{2}{3}x) + 1 = 0$   

$$\therefore \frac{7}{3}x = -1$$
  

$$\therefore x = -\frac{3}{7}$$
 **A1**  

$$\therefore y = \frac{2}{7}$$
  

$$\therefore \text{P is } \left(-\frac{3}{7}, \frac{2}{7}\right)$$
 **A1**



- c** For Hamsika's restaurant to be as far as possible from the other restaurants, it should be located at one of the Voronoi vertices  $(-\frac{3}{7}, \frac{2}{7})$ ,  $(0, 0)$ , or  $(1, 0)$ . (M1)

$$\text{Now } PD = \sqrt{(2 + \frac{3}{7})^2 + (3 - \frac{2}{7})^2} \approx 3.64$$

$$OD = \sqrt{2^2 + 3^2} \approx 3.61$$

$$\text{and } QD = \sqrt{(2 - 1)^2 + 3^2} \approx 3.16$$

P is the most appropriate location, as it is furthest from the existing restaurants.

M1

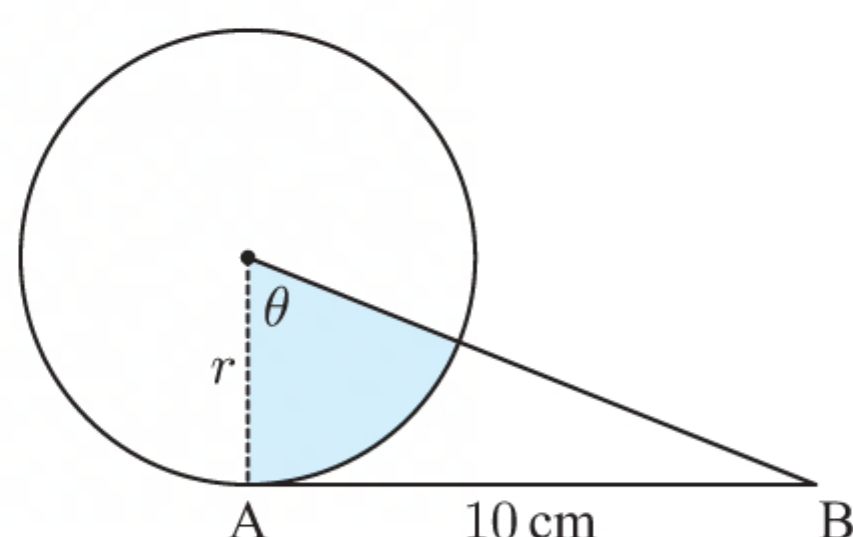
A1

Total [8 marks]

- 8 a**  $H_0: \mu_m = \mu_f$  {males and females have the same mean weight} A1  
 $H_1: \mu_m \neq \mu_f$  {males and females have different mean weights} A1  
**b** Using technology to conduct a two-sample  $t$ -test,  $p$ -value  $\approx 0.954$ . A2  
**c** Since  $p$ -value  $> 0.05$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% significance level. We therefore accept  $H_0$ . R1A1

Total [6 marks]

9



$$\text{Since the shaded area is } 20 \text{ cm}^2, \quad \frac{\theta}{2\pi} \times \pi r^2 = 20$$

$$\therefore \theta r^2 = 40$$

M1A1

$$\text{Since AB is a tangent, } \tan \theta = \frac{10}{r}$$

$$\therefore r = \frac{10}{\tan \theta}$$

M1

A1

$$\therefore \theta \times \frac{100}{\tan^2 \theta} = 40$$

$$\therefore \frac{\theta}{\tan^2 \theta} = \frac{2}{5}$$

M1

$$\text{Using technology, } \theta \approx 1.01 \quad \left\{ \text{since } 0 < \theta < \frac{\pi}{2} \right\}$$

A1

$$\text{and } r = \frac{10}{\tan \theta} \approx 6.30 \text{ cm}$$

A1

Total [7 marks]

**10 a**  $k = 1 - 0.46 - 0.15 - 0.06 - 0.03 - 0.01$   
 $= 0.29$

$\therefore$  29% of people leave without buying any fish.

A1

**b** Expected number of fish types bought

$$= 0 \times 0.29 + 0.46 \times 1 + 0.15 \times 2 + 0.06 \times 3 + 0.03 \times 4 + 0.01 \times 5$$

$$= 1.11$$

M1

A1

**c**  $P(\text{a person buys more than one type of fish}) = 1 - 0.29 - 0.46$   
 $= 0.25$

(M1)

Let  $X$  be the number of people in the sample of 3 who buy more than one type of fish.

$$\therefore X \sim B(3, \frac{1}{4})$$

$$P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - \binom{3}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^3$$

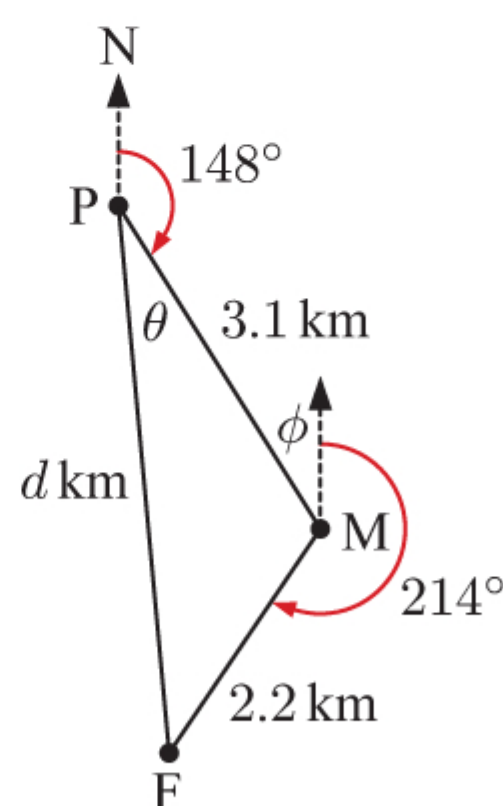
$$\approx 0.578$$

A1

Total [5 marks]



11 a


 Suppose Eleanor runs  $d$  km.

 Let  $\theta$  and  $\phi$  be the angles marked.

$$\begin{aligned}\phi &= 180^\circ - 148^\circ \quad \{\text{co-interior angles}\} \\ &= 32^\circ\end{aligned}$$

(M1)

$$\begin{aligned}\therefore \widehat{FMP} &= 360^\circ - 32^\circ - 214^\circ \\ &= 114^\circ\end{aligned}$$

$$\begin{aligned}\text{Using the cosine rule, } d^2 &= 3.1^2 + 2.2^2 - 2 \times 3.1 \times 2.2 \times \cos 114^\circ \\ \therefore d &\approx 4.47\end{aligned}$$

M1

A1

Eleanor runs about 4.47 km.

$$\begin{aligned}\text{b Using the sine rule } \frac{\sin \theta}{2.2} &= \frac{\sin 114^\circ}{d} \\ \therefore \sin \theta &= \frac{2.2 \sin 114^\circ}{d} \\ \therefore \theta &\approx 26.7^\circ\end{aligned}$$

M1

A1

 $\therefore$  Eleanor runs on the bearing  $\approx 148^\circ + 026.7^\circ$  which is about  $174.7^\circ$ .

A1

$$\begin{aligned}\text{c Morris runs 5.3 km in } \frac{5.3}{8.4} &\approx 0.631 \text{ hours} \\ &\approx 37.86 \text{ minutes}\end{aligned}$$

A1

$$\begin{aligned}\text{Eleanor runs about 4.47 km in } \frac{4.47}{6.9} &\approx 0.648 \text{ hours} \\ &\approx 38.87 \text{ minutes}\end{aligned}$$

A1

Morris arrives first and about 1 minute sooner than Eleanor.

A1

Total [9 marks]

12 a

 If  $P(X)$  is the probability of finding a card featuring Professor X, then  
 $H_0: P(D) = 0.6, P(M) = 0.2, P(T) = 0.1, P(F) = 0.07, \text{ and } P(S) = 0.03.$ 

A1

Professor	D	M	T	F	S
Expected frequency	$150 \times 0.6 = 90$	$150 \times 0.2 = 30$	$150 \times 0.1 = 15$	$150 \times 0.07 = 10.5$	$150 \times 0.03 = 4.5$

A2

$$\text{c } df = 5 - 1 = 4$$

A1

 d Using technology,  $p\text{-value} \approx 0.595$ .

A2

 e Since  $p\text{-value} > 0.05$ , we do not have enough evidence to reject  $H_0$  at a 5% significance level, so we accept that the cards are distributed as claimed.

R1A1

Total [8 marks]

$$\text{13 } \mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ -250 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

A1

$$\therefore \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -250 \end{pmatrix}$$

M1

$$= \begin{pmatrix} 1 \times (-250) - 1 \times 0 \\ 1 \times 0 - 2 \times (-250) \\ 2 \times 0 - 1 \times 0 \end{pmatrix}$$

(M1)

$$= \begin{pmatrix} -250 \\ 500 \\ 0 \end{pmatrix} \text{ Nm}$$

A1

Total [4 marks]



$$\begin{aligned}
 14 \quad a \quad S(t) &= \sqrt{3} \cos 2t + \cos\left(2t + \frac{\pi}{2}\right) \\
 &= \operatorname{Re}\left(\sqrt{3}e^{2ti} + e^{(2t+\frac{\pi}{2})i}\right) && \text{M1} \\
 &= \operatorname{Re}\left(e^{2ti}(\sqrt{3} + e^{\frac{\pi}{2}i})\right) \\
 &= \operatorname{Re}\left(e^{2ti}(\sqrt{3} + i)\right) && \text{A1} \\
 &= \operatorname{Re}\left(e^{2ti}(2e^{\frac{\pi}{6}i})\right) && \text{A1} \\
 &= \operatorname{Re}\left(2e^{(2t+\frac{\pi}{6})i}\right) \\
 &= 2 \cos\left(2t + \frac{\pi}{6}\right) && \text{A1}
 \end{aligned}$$

- b**  $S(t) = 2 \cos(2t + \frac{\pi}{6})$  has maximum value 2.  
 $\therefore$  the maximum sum of the pulses is 2.

R1  
 A1

Total [6 marks]

- 15 a**  $y = a(x - b)^3 + c$  is a translation of  $y = ax^3$  through  $\begin{pmatrix} b \\ c \end{pmatrix}$ .  
 $\therefore$  the stationary inflection is at  $(b, c)$ .

(M1)

This point corresponds to point P on the pot.

$$\therefore b = 20 \text{ and } c = 15.$$

A1

When  $x = 0$ ,  $y = 10$

$$\therefore 10 = a(-20)^3 + 15$$

M1

$$\therefore -8000a = -5$$

$$\therefore a = \frac{1}{1600}$$

A1

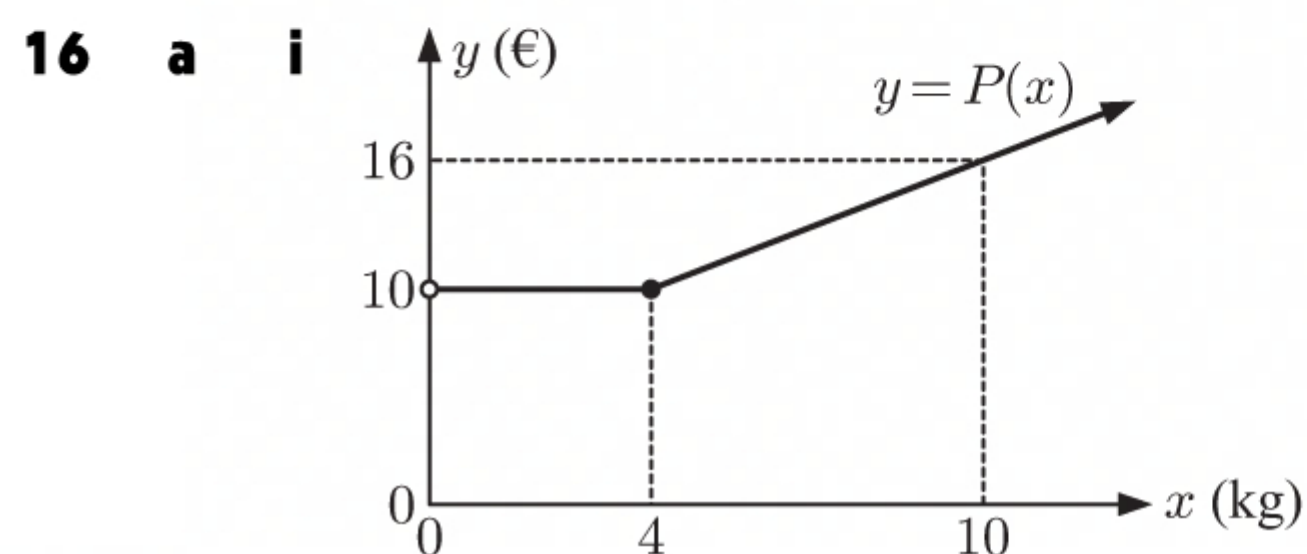
- b** The capacity of the pot  $= \pi \int_0^{40} \left(\frac{1}{1600}(x - 20)^3 + 15\right)^2 dx \text{ cm}^3$

A1

- c** Using technology, the capacity  $\approx 28\,723 \text{ cm}^3$   
 $\approx 28.7 \text{ L}$

A1  
 A1

Total [7 marks]



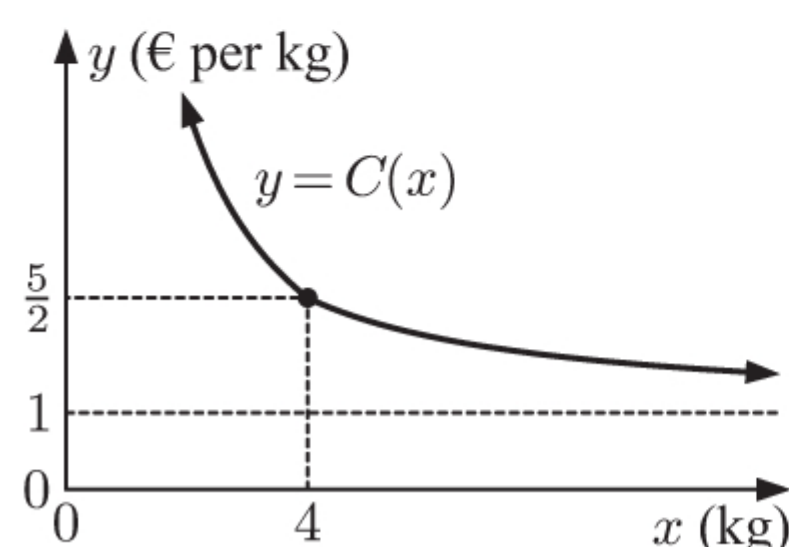
A2

$$\begin{aligned}
 \text{ii} \quad P(x) &= \begin{cases} 10 & 0 < x \leq 4 \\ 10 + (x - 4) & x > 4 \end{cases} \\
 &= \begin{cases} 10 & 0 < x \leq 4 \\ x + 6 & x > 4 \end{cases}
 \end{aligned}$$

A1  
 A1

$$\begin{aligned}
 \text{b} \quad C(x) &= \begin{cases} \frac{10}{x} & 0 < x \leq 4 \\ \frac{x+6}{x} & x > 4 \end{cases} \\
 &= \begin{cases} \frac{10}{x} & 0 < x \leq 4 \\ \frac{6}{x} + 1 & x > 4 \end{cases}
 \end{aligned}$$

(A1)



A2

Total [7 marks]



$$\begin{aligned}
 17 \quad a \quad v_y(t) &= -gt \\
 \therefore y(t) &= \int -gt \, dt \\
 &= -\frac{1}{2}gt^2 + c
 \end{aligned}$$

M1

But  $y(0) = 80$ , so  $c = 80$

$$\therefore y(t) = -\frac{1}{2}gt^2 + 80$$

A1

The supplies hit the water when  $y = 0$

M1

$$\therefore \frac{1}{2}gt^2 = 80$$

$$\therefore t^2 \approx \frac{160}{9.8} \quad \{g \approx 9.8\}$$

$$\therefore t \approx 4.04 \text{ s} \quad \{\text{as } t > 0\}$$

A1

The supplies reach the water after about 4.04 seconds.

$$b \quad \text{After 4.04 seconds, } \mathbf{v} \approx \begin{pmatrix} 40 \\ -g \times 4.04 \end{pmatrix} \approx \begin{pmatrix} 40 \\ -39.6 \end{pmatrix}$$

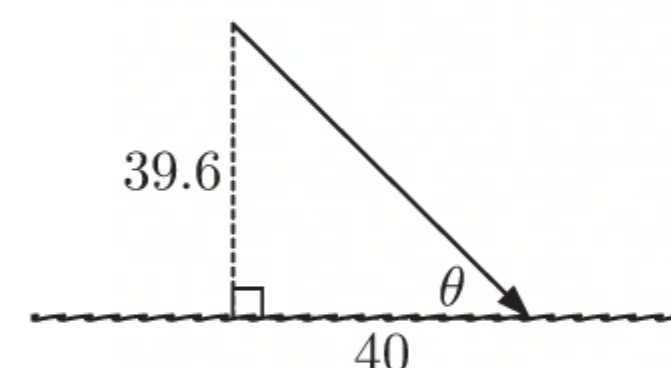
$$\therefore |\mathbf{v}| \approx \sqrt{40^2 + (-39.6)^2} \approx 56.3 \text{ m s}^{-1}$$

M1A1

Let  $\theta$  be the angle from horizontal at which the supplies hit the water.

$$\therefore \tan \theta \approx \frac{39.6}{40}$$

$$\therefore \theta \approx 44.7^\circ$$



M1

A1

So, the supplies strike the water at about  $56.3 \text{ m s}^{-1}$ , at the angle of about  $44.7^\circ$ .

$$c \quad \text{The horizontal displacement } x(t) = 40t \text{ m}$$

M1

$$\therefore x(4.04) \approx 161.6 \text{ m}$$

A1

The pilot should release the supplies when she is (horizontally) about 162 m from the ship.

**Total [10 marks]**

## PAPER 2

$$1 \quad a \quad N = 5 \times 4 = 20$$

$$I\% = 11.8$$

$$PMT = 869.77$$

$$FV = 0$$

$$P/Y = 4$$

$$C/Y = 4$$

(M1)(A1)

$$\therefore PV = -13\,000$$

A1

Corina borrowed \$13 000.

$$b \quad \text{Interest paid} = 20 \times \$869.77 - \$13\,000$$

(M1)

$$= \$4395.40$$

A1

$$c \quad N = 30 \times 12 = 360$$

$$I\% = 5.2$$

$$PV = -865\,400$$

$$FV = 0$$

$$P/Y = 12$$

$$C/Y = 12$$

(A1)

$$\therefore PMT = 4752.00$$

A1

Corina can afford to withdraw \$4752.00 each month.



**d i**  $N = 6 \times 12 = 72$

$$I\% = 5.2$$

$$PV = -865\,400$$

$$PMT = 4752.00$$

$$P/Y = 12$$

$$C/Y = 12$$

(A1)

$$\therefore FV = -780\,952.25$$

A1

There is \$780 952.25 in the account when Corina decides to increase her withdrawals.

**ii**  $I\% = 5.2$

$$PV = -780\,952.25$$

$$PMT = 5400$$

$$FV = 0$$

$$P/Y = 12$$

$$C/Y = 12$$

(A1)

$$\therefore N \approx 228$$

A1

Corina's money will run out in the 228th month after her decision to increase her withdrawals.

At this time, she will be  $66 + \frac{228}{12} = 85$  years old.

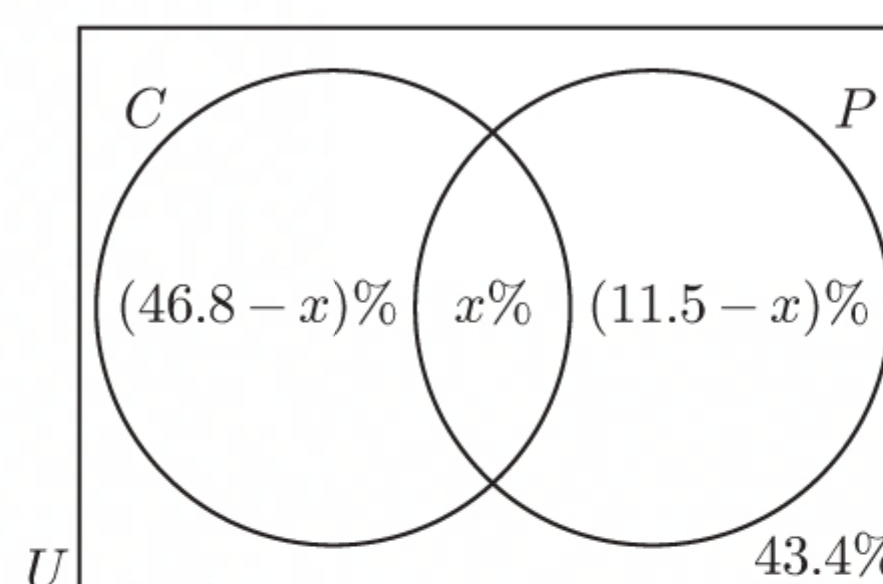
Total [11 marks]

**2 a** Let the percentage in  $C \cap P$  be  $x$ .

$\therefore$  the percentage in  $C \cap P'$  is  $46.8 - x$  and the percentage in  $C' \cap P$  is  $11.5 - x$ .

$$(46.8 - x) + x + (11.5 - x) + 43.4 = 100$$

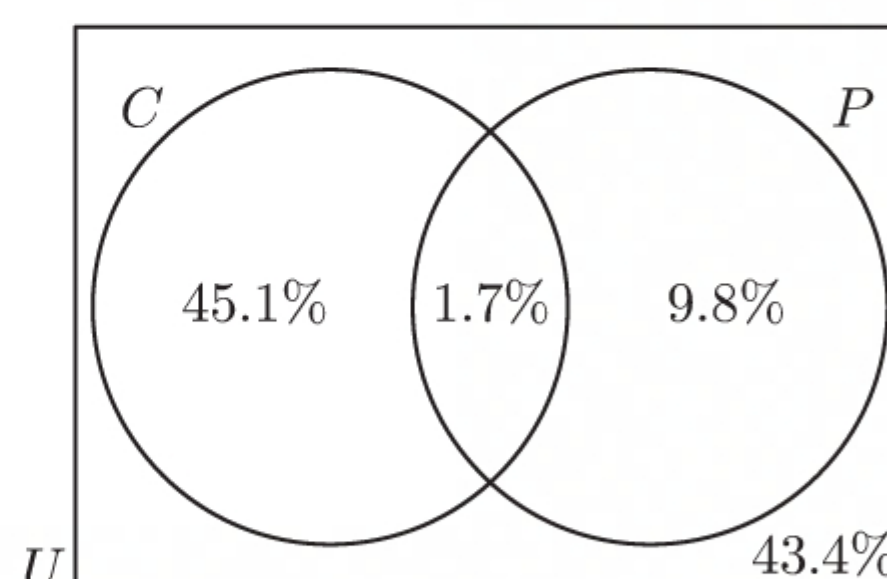
$$\therefore x = 1.7$$



(M1)

(A1)

The completed Venn diagram is:



A2

**b**  $P(C \cap P) = 1.7\% = 0.017$

A1

**c**  $P(P' | C') = \frac{P(P' \cap C')}{P(C')}$

M1

$$= \frac{0.434}{0.098 + 0.434}$$

$$\approx 0.816$$

A1

**d i**  $n(C' \cap P) = 500 - 9 - 236 - 205$   
 $= 50$

A1

**ii** The alternative hypothesis is:

$$H_1: \text{at least one of } p_1 \neq 0.017, p_2 \neq 0.451, p_3 \neq 0.098, p_4 \neq 0.434$$

A1

**iii**

Group	Expected frequency
$C \cap P$	$0.017 \times 500 = 8.5$
$C \cap P'$	$0.451 \times 500 = 225.5$
$C' \cap P$	$0.098 \times 500 = 49$
$C' \cap P'$	$0.434 \times 500 = 217$

A2



$$\begin{aligned}
 \text{iv } \chi_{\text{calc}}^2 &= \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}} \\
 &= \frac{(9 - 8.5)^2}{8.5} + \frac{(236 - 225.5)^2}{225.5} + \frac{(50 - 49)^2}{49} + \frac{(205 - 217)^2}{217} \\
 &\approx 1.20
 \end{aligned}$$

A2

v Since  $\chi_{\text{calc}}^2 < \chi_{\text{crit}}^2$ , we do not have enough evidence to reject  $H_0$  in favour of  $H_1$  on a 5% level of significance. We therefore accept  $H_0$ .

R1A1

$\therefore$  this sample is suitable to answer the extensive questionnaire.

Total [15 marks]

$$\text{3 a i } \mathbf{T} = \begin{matrix} & \begin{matrix} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} \end{matrix} \\ \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \\ \text{F} \end{matrix} & \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix} \end{matrix}$$

A1A1A1

$$\text{ii } (1 \ 0 \ 0 \ 0 \ 0 \ 0) \mathbf{T}^3 \approx (0.0556 \ 0.204 \ 0.296 \ 0.0926 \ 0.139 \ 0.213)$$

M1

The visitor is most likely to be travelling to C after three roads.

A1

iii In reality, visitors are more likely to choose roads which take them to attractions they have not visited, rather than choosing roads at random, which may lead them back to an attraction they have already visited.

R1

**b**

Route	Distance	Probability
A $\rightarrow$ F $\rightarrow$ E	240 + 160 = 400	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
A $\rightarrow$ F $\rightarrow$ C	240 + 200 = 440	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
A $\rightarrow$ C $\rightarrow$ F	180 + 200 = 380	$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$
A $\rightarrow$ C $\rightarrow$ E	180 + 120 = 300	$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$
A $\rightarrow$ C $\rightarrow$ D	180 + 160 = 340	$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$
A $\rightarrow$ B $\rightarrow$ D	190 + 170 = 360	$\frac{1}{3} \times 1 = \frac{1}{3}$

A1A1A1

$$\begin{aligned}
 \text{Expected distance} &= \frac{1}{6} \times 400 + \frac{1}{6} \times 440 + \frac{1}{9} \times 380 + \frac{1}{9} \times 300 + \frac{1}{9} \times 340 + \frac{1}{3} \times 360 \\
 &= 373\frac{1}{3} \text{ m}
 \end{aligned}$$

A1

c A  $\rightarrow$  B  $\rightarrow$  D  $\rightarrow$  C  $\rightarrow$  E  $\rightarrow$  F  $\rightarrow$  A

A1

$$\text{Weight} = 190 + 170 + 160 + 120 + 160 + 240 = 1040$$

A1

d i The vertices A, D, F, and M are of odd degree, so the graph is not Eulerian.

R1

ii The shortest way to connect the odd vertices is

$$A \rightarrow F \text{ (distance 240) and } D \rightarrow E \rightarrow M \text{ (distance 240)}$$

M1

$\therefore$  the shortest distance the street cleaner needs to travel is

$$240 + 160 + 200 + 120 + 40 + 200 + 160 + 180 + 190 + 170 + 240 + 240 = 2140$$

A1

iii M  $\rightarrow$  E  $\rightarrow$  D  $\rightarrow$  B  $\rightarrow$  A  $\rightarrow$  F  $\rightarrow$  A  $\rightarrow$  C  $\rightarrow$  F  $\rightarrow$  E  $\rightarrow$  C  $\rightarrow$  D  $\rightarrow$  E  $\rightarrow$  M

A1A1

Total [17 marks]

$$\text{4 a } X \sim N(5.38, 0.62^2)$$

M1

$$P(X < k) = 0.4$$

Using technology,  $k \approx 5.22$

A1

This means that 40% of the cherries have mass less than about 5.22 g.

A1

b i Using technology,  $P(X > 6) \approx 0.1587 \approx 0.159$

A1

ii Christina would expect  $\approx 50 \times 0.1587 \approx 8$  cherries to have mass greater than 6 g.

M1A1



- iii Let  $Y$  be the number of cherries in Christina's sample with mass greater than 6 g.

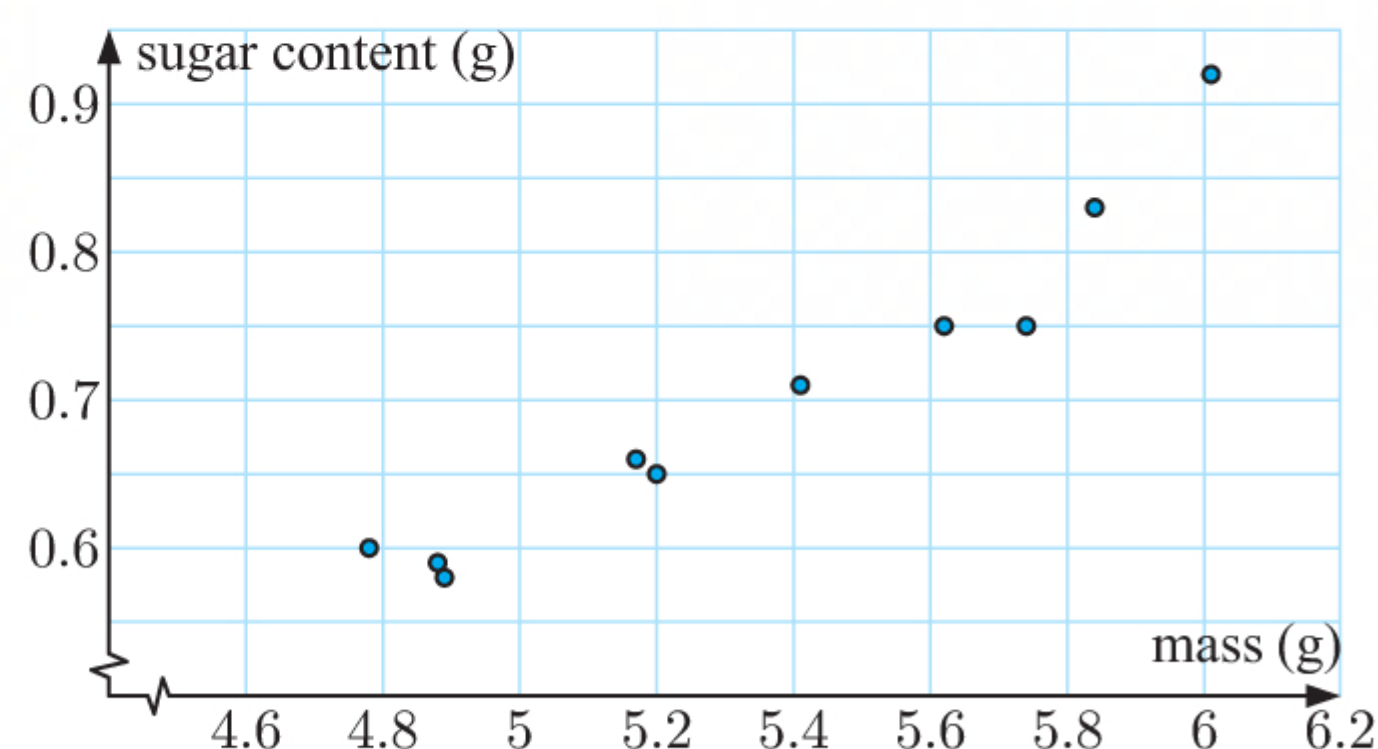
$$\therefore Y \sim B(50, 0.1587)$$

Using technology,  $P(Y \geq 4) \approx 0.967$

M1

A1

c i



A2

- ii From the scatter diagram, the data appears non-linear. We therefore favour Spearman's rank correlation coefficient rather than applying Pearson's correlation coefficient directly.

R1

iii	Mass (g)	5.17	5.84	6.01	5.74	4.88	5.41	5.62	4.78	4.89	5.20
	rank of mass	4	9	10	8	2	6	7	1	3	5
	Sugar content (g)	0.66	0.83	0.92	0.75	0.59	0.71	0.75	0.60	0.58	0.65
	rank of sugar content	5	9	10	7.5	2	6	7.5	3	1	4

A1

A1

- iv Using technology,  $r_s \approx 0.936$

A2

- v There is a strong positive correlation between the variables.

A1

Total [16 marks]

- 5 a After  $n$  iterations, there will be  $4^n$  Bettys.

A1

- b i To obtain Betty i, we rotate the shaded Betty through  $\pi$  about the origin, then translate  $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ .

A1

The rotation matrix is  $\begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ .

So, the transformation equation is  $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ .

A1

- ii To obtain Betty ii, we reflect in the line  $y = x$ , then translate  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

A1

The line  $y = x$  is  $y = (\tan \frac{\pi}{4})x$ , so the reflection matrix is  $\begin{pmatrix} \cos \frac{\pi}{2} & \sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & -\cos \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

So, the transformation equation is  $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

A1

- iii To obtain Betty iii, we reflect in the line  $y = -x$ , then translate  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ .

A1

The line  $y = -x$  is  $y = \tan(-\frac{\pi}{4})x$ , so the reflection matrix is

$$\begin{pmatrix} \cos(-\frac{\pi}{2}) & \sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & -\cos(-\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

So, the transformation equation is  $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ .

A1

- c i The rotation is the same as for iteration 1, but the translation vector is  $\begin{pmatrix} 2^n \\ 2^n \end{pmatrix}$ .

A1

So, the transformation equation is  $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2^n \\ 2^n \end{pmatrix}$ .

A1

- ii The reflection is the same as for iteration 1, but the translation vector is  $\begin{pmatrix} 0 \\ 2^{n-1} \end{pmatrix}$ .

A1

So, the transformation equation is  $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 2^{n-1} \end{pmatrix}$ .

A1



- iii The reflection is the same as for iteration 1, but the translation vector is  $\begin{pmatrix} 2^n \\ 2^{n-1} \end{pmatrix}$ . A1

So, the transformation equation is  $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2^n \\ 2^{n-1} \end{pmatrix}$ . A1

**Total [13 marks]**

- 6 a Substituting the known values,  $\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 41q = E(t)$   
 $\therefore \frac{dI}{dt} + 10I + 41q = E(t)$

The system is:  $\begin{cases} \frac{dq}{dt} = I \\ \frac{dI}{dt} = -41q - 10I + E(t) \end{cases}$  A1

- b i Suppose  $E(t) = 0$ .

Letting  $\mathbf{x} = \begin{pmatrix} q \\ I \end{pmatrix}$ ,

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -41 & -10 \end{pmatrix} \mathbf{x}$$
A1

- ii When  $t = 0$ ,  $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  A1

$$\therefore \dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -41 & -10 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -10 \end{pmatrix}$$
A1

- iii Let  $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -41 & -10 \end{pmatrix}$ .

If  $|\lambda \mathbf{I} - \mathbf{A}| = 0$  then  $\begin{vmatrix} \lambda & -1 \\ 41 & \lambda + 10 \end{vmatrix} = 0$  M1

$$\therefore \lambda(\lambda + 10) + 41 = 0$$
A1

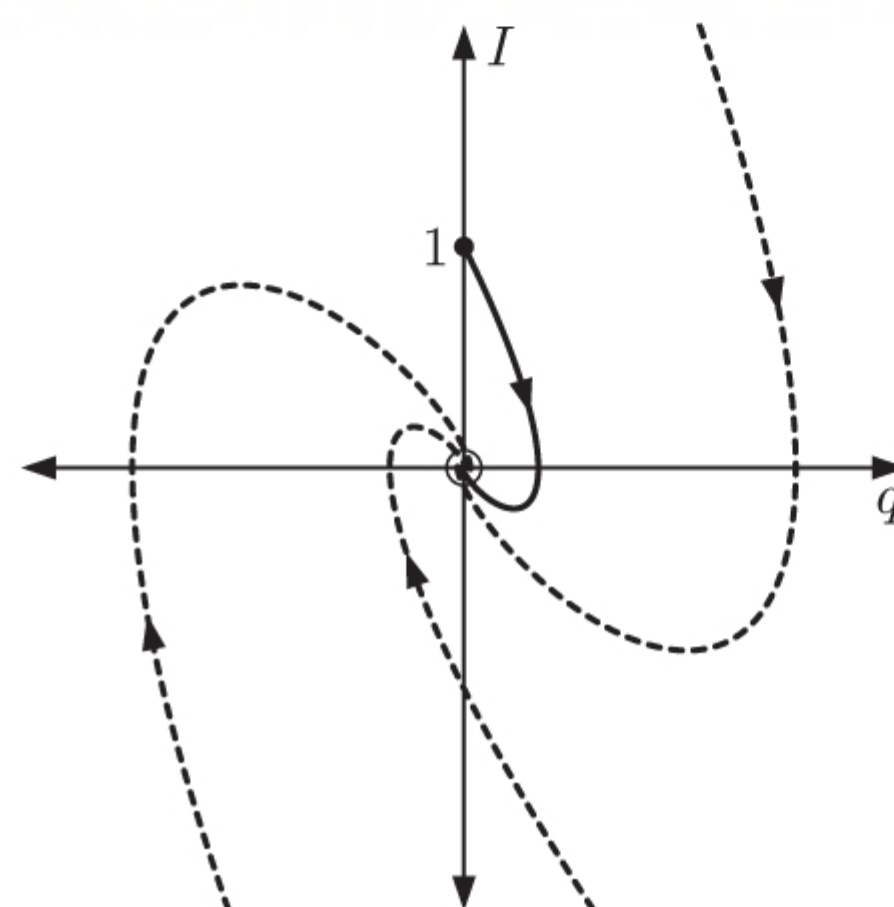
$$\therefore \lambda^2 + 10\lambda + 41 = 0$$

$$\therefore \lambda = \frac{-10 \pm \sqrt{100 - 164}}{2}$$

$$\therefore \lambda = -5 \pm 4i$$
A1

- iv The eigenvalues are complex with negative real part, so the system is a stable spiral.

Since  $\dot{\mathbf{x}} = \begin{pmatrix} 1 \\ -10 \end{pmatrix}$  at  $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , the system rotates clockwise.



- v The system follows a stable spiral to the origin. Without a power source, the circuit will eventually lose both its current and all charge stored within it. A1A1



**c i** Letting  $E(t) = 300$  V, the system is now 
$$\begin{cases} \frac{dq}{dt} = I \\ \frac{dI}{dt} = -41q - 10I + 300. \end{cases}$$

In the steady state,  $\frac{dq}{dt}$  and  $\frac{dI}{dt}$  are both zero.

$$\therefore I = 0 \text{ A}$$

and  $-41q + 300 = 0$

$$\therefore q = \frac{300}{41} \approx 7.32 \text{ coulombs}$$

A1

A1

**ii (1)** Applying Euler's method with  $h = 0.05$  and initial conditions  $t_0 = 0$ ,  $q_0 = 0$ ,  $I_0 = 0$ , we have:

$$t_k = t_{k-1} + 0.05$$

$$q_k = q_{k-1} + 0.05I_{k-1}$$

$$I_k = I_{k-1} + 0.05(-41q_{k-1} - 10I_{k-1} + 300)$$

$$= -2.05q_{k-1} + 0.5I_{k-1} + 15$$

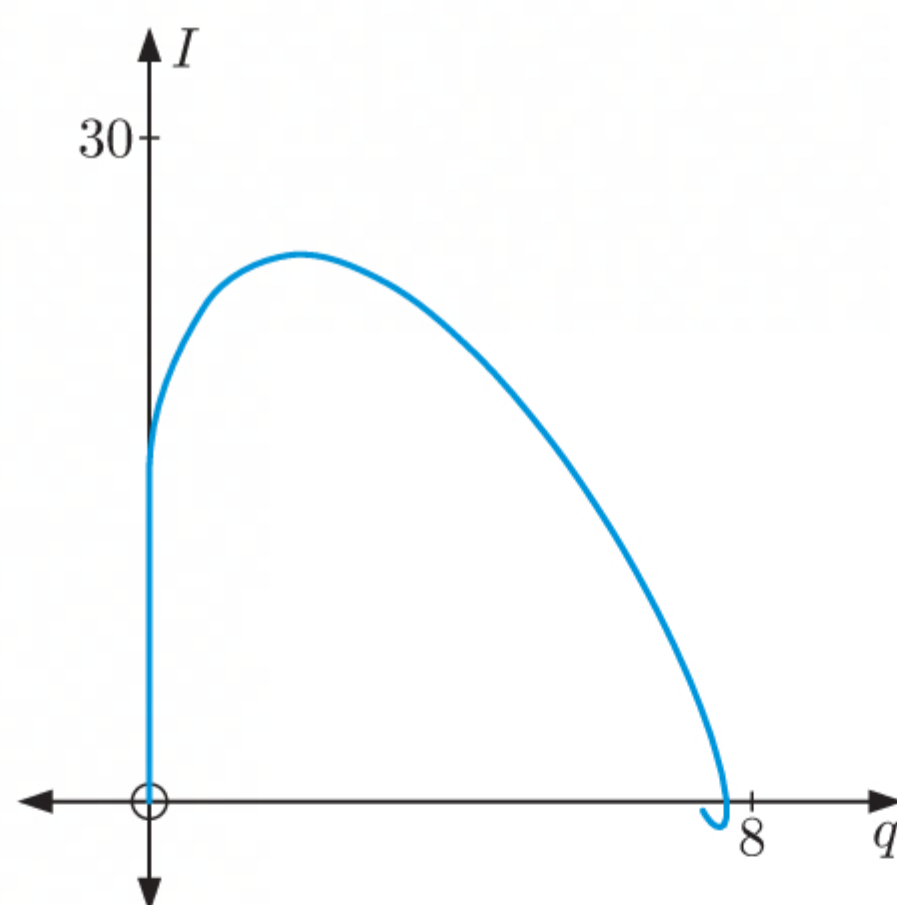
A1

A1

**(2)**  $q(1) \approx 7.35$ ,  $I(1) \approx -0.415$

A1A1

**(3)**



A1A1

Total [20 marks]

**7 a**  $P = A \times b^t$

Now  $P(0) = 38$ , so  $A = 38$

A1

and  $P(5) = 55$ , so  $38 \times b^5 = 55$

M1

$$\therefore b^5 = \frac{55}{38}$$

$$\therefore b = \sqrt[5]{\frac{55}{38}} \approx 1.08$$

A1

**b** Using this model,  $P(10) = 38 \times b^{10}$

$$= 38 \times \left(\frac{55}{38}\right)^2$$

M1

$$\approx 79.6$$

A1

The model predicts there will be about 80 female condor after 10 years.

**c** The percentage error in the estimate  $= \frac{80 - 68}{68} \times 100\%$

M1

$$\approx 17.6\%$$

A1

**d** The exponential model used by the biologist predicts that the population of condor will continue to grow forever. However, in reality the population will be limited by available resources, so a logistic model is more appropriate.

R1



- e i**  $P = \frac{L}{1 + ae^{-kt}}$
- $$\therefore P + ae^{-kt}P = L$$
- $$\therefore ae^{-kt}P = L - P$$
- $$\therefore ae^{-kt} = \frac{L - P}{P}$$
- $$\therefore \ln(ae^{-kt}) = \ln\left(\frac{L - P}{P}\right) \quad \text{A1}$$
- $$\therefore \ln a + \ln(e^{-kt}) = \ln\left(\frac{L - P}{P}\right)$$
- $$\therefore \ln a - kt \ln e = \ln\left(\frac{L - P}{P}\right) \quad \text{A1}$$
- $$\therefore \ln a = \ln\left(\frac{L - P}{P}\right) + kt \quad \text{AG}$$
- ii** When  $t = 0$ ,  $P = 38$
- $$\therefore \ln a = \ln\left(\frac{L - 38}{38}\right) \quad \dots (1)$$
- When  $t = 5$ ,  $P = 55$
- $$\therefore \ln a = \ln\left(\frac{L - 55}{55}\right) + 5k \quad \dots (2)$$
- When  $t = 10$ ,  $P = 68$
- $$\therefore \ln a = \ln\left(\frac{L - 68}{68}\right) + 10k \quad \dots (3) \quad \text{A1}$$
- (1) + (3) gives  $2 \ln a = \ln\left(\frac{L - 38}{38}\right) + \ln\left(\frac{L - 68}{68}\right) + 10k \quad \text{A1}$
- $2 \times (2)$  gives  $2 \ln a = 2 \ln\left(\frac{L - 55}{55}\right) + 10k \quad \text{A1}$
- $$\therefore \ln\left(\frac{L - 38}{38}\right) + \ln\left(\frac{L - 68}{68}\right) + 10k = 2 \ln\left(\frac{L - 55}{55}\right) + 10k \quad \text{AG}$$
- iii** Using **ii**,  $\ln\left(\frac{L - 38}{38}\right) + \ln\left(\frac{L - 68}{68}\right) = 2 \ln\left(\frac{L - 55}{55}\right)$
- $$\therefore \ln\left(\frac{(L - 38)(L - 68)}{38 \times 68}\right) = \ln\left(\frac{(L - 55)^2}{55^2}\right) \quad \text{(M1)}$$
- $$\therefore \frac{(L - 38)(L - 68)}{38 \times 68} = \frac{(L - 55)^2}{55^2}$$
- $$\therefore 55^2(L - 38)(L - 68) = 38 \times 68(L - 55)^2 \quad \text{A1}$$
- $$\therefore 3025L^2 - 320650L + 7816000 = 2584L^2 - 284240L + 7816000$$
- $$\therefore 441L^2 - 36410L = 0 \quad \text{A1}$$
- iv**  $L(441L - 36410) = 0$
- Since  $L > 0$ ,  $L = \frac{36410}{441} \approx 82.56 \quad \text{A1}$
- In the long-term, we predict there will be about 83 female condor in the region. A1

**Total [18 marks]**

### PAPER 3

- 1 a**  $\begin{pmatrix} 0 & 1000\alpha \\ \beta & \beta \end{pmatrix}$  E(female fry produced by each female adult each year) =  $\frac{1}{2} \times 2000 = 1000$ . R1
- Each fry has probability  $\alpha$  of surviving to become a juvenile.
- $\therefore$  E(female juveniles per female adult) =  $1000\alpha$ . R1
- Each juvenile and adult has probability  $\beta$  of surviving the year.
- b**  $\mathbf{s}_n = \mathbf{L}\mathbf{s}_{n-1}$
- $$= \mathbf{L}(\mathbf{L}\mathbf{s}_{n-2})$$
- $$= \mathbf{L}^2\mathbf{s}_{n-2}$$
- $$= \mathbf{L}^2(\mathbf{L}\mathbf{s}_{n-3})$$
- $$= \mathbf{L}^3\mathbf{s}_{n-3}$$
- $$\vdots$$
- $$= \mathbf{L}^n\mathbf{s}_0 \quad \text{A1}$$
- AG**

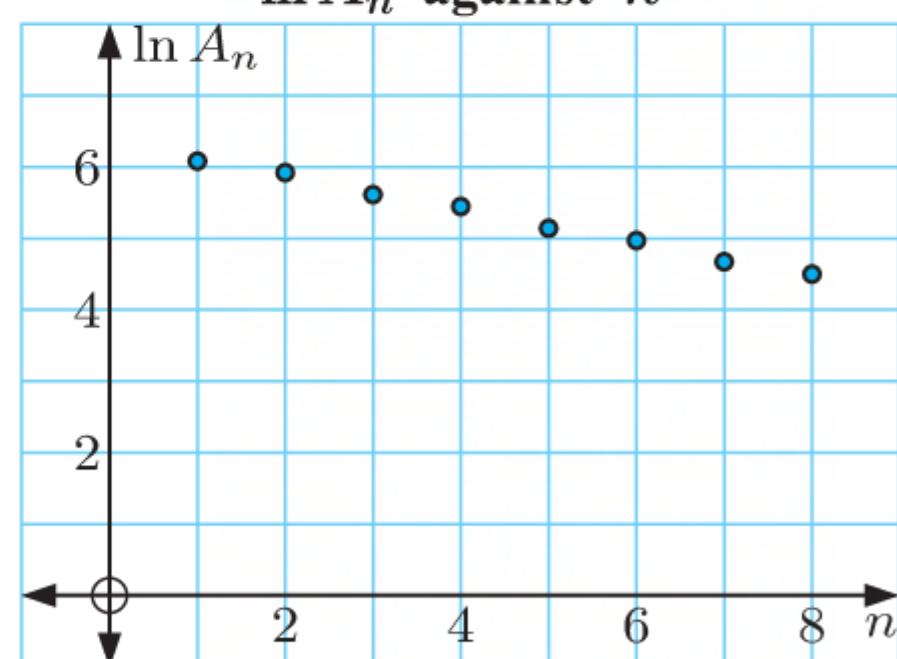


$$\mathbf{c} \quad \mathbf{i} \quad \mathbf{s}_0 = \begin{pmatrix} 14\,000 \\ 600 \end{pmatrix} \quad \text{A1}$$

$$\mathbf{L} = \begin{pmatrix} 0 & 1000\alpha \\ \beta & \beta \end{pmatrix} = \begin{pmatrix} 0 & 1000 \times 0.02 \\ 0.03 & 0.03 \end{pmatrix} \\ = \begin{pmatrix} 0 & 20 \\ 0.03 & 0.03 \end{pmatrix} \quad \text{A1}$$

$$\mathbf{ii} \quad \begin{array}{c|ccccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline A_n & 438 & 373 & 274 & 232 & 171 & 144 & 107 & 90 \end{array} \quad \text{A1A1}$$

$\mathbf{iii} \quad \ln A_n \text{ against } n \quad \text{A1A1}$



$\mathbf{iv} \quad \text{The graph of } \ln A_n \text{ against } n \text{ appears linear, with } r^2 \approx 0.996. \quad \text{R1}$

$\therefore \text{ there is an exponential relationship between } A_n \text{ and } n. \quad \text{AG}$

$\mathbf{v} \quad \text{Using linear regression, } \ln A_n \approx -0.2328n + 6.3411 \quad \text{M1}$

$$\therefore A_n \approx e^{-0.2328n+6.3411}$$

$$\therefore A_n \approx 567 \times 0.792^n \quad \text{A1}$$

$\mathbf{d} \quad \mathbf{i} \quad \text{Let } a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{s}_0$

$$\therefore \begin{pmatrix} 25.32 & -26.32 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 14\,000 \\ 600 \end{pmatrix} \quad \text{M1}$$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 25.32 & -26.32 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 14\,000 \\ 600 \end{pmatrix} \\ \approx \begin{pmatrix} 576.9 \\ 23.1 \end{pmatrix}$$

$$\therefore \mathbf{s}_0 \approx 577\mathbf{v}_1 + 23\mathbf{v}_2 \quad \text{A1}$$

$$\mathbf{ii} \quad \mathbf{s}_n = \mathbf{L}^n \mathbf{s}_0 \\ \approx \mathbf{L}^n (577\mathbf{v}_1 + 23\mathbf{v}_2) \quad \text{M1}$$

$$\approx 577\mathbf{L}^n \mathbf{v}_1 + 23\mathbf{L}^n \mathbf{v}_2$$

$$\approx 577\lambda_1^n \mathbf{v}_1 + 23\lambda_2^n \mathbf{v}_2 \quad \{\text{property of eigenvalues and eigenvectors}\} \quad \text{M1}$$

$$\approx 577\lambda_1^n \left( \mathbf{v}_1 + \frac{23}{577} \left( \frac{\lambda_2}{\lambda_1} \right)^n \mathbf{v}_2 \right) \quad \text{A1}$$

$\mathbf{iii} \quad |\lambda_1| > |\lambda_2|$

$$\therefore \text{ as } n \rightarrow \infty, \quad \left( \frac{\lambda_2}{\lambda_1} \right)^n \rightarrow 0 \quad \text{M1}$$

$$\therefore \mathbf{s}_n \rightarrow 577\lambda_1^n \mathbf{v}_1 \quad \text{A1}$$

$$\therefore A_n \rightarrow 577\lambda_1^n \approx 577 \times 0.7897^n \quad \text{A1}$$

which is very close to the exponential model in  $\mathbf{c} \mathbf{v}$ .

$\mathbf{e} \quad \mathbf{i} \quad \text{The population will be sustainable in the long-term if the positive eigenvalue is 1.} \quad \text{A1}$

$\mathbf{ii} \quad \text{The eigenvalues of } \mathbf{L} \text{ are the solutions to } |\lambda \mathbf{I} - \mathbf{L}| = 0 \quad \text{M1}$

$$\therefore \begin{vmatrix} \lambda & -20 \\ -\beta & \lambda - \beta \end{vmatrix} = 0$$

$$\therefore \lambda(\lambda - \beta) - 20\beta = 0$$

If  $\lambda = 1$  is a solution then  $1 - \beta - 20\beta = 0 \quad \text{M1}$

$$\therefore -21\beta = -1$$

$$\therefore \beta = \frac{1}{21} \quad \text{A1}$$



- f i** Using **e ii**,  $\lambda^2 - \lambda\beta - 20\beta = 0$   
 $\therefore \lambda^2 - 0.1\lambda - 2 = 0$  M1  
 Using technology,  $\lambda \approx 1.465$  or  $-1.365$ . A1
- ii** For  $\lambda \approx 1.465$ , consider  $(\lambda \mathbf{I} - \mathbf{L}) \begin{pmatrix} c \\ d \end{pmatrix} = \mathbf{0}$  M1  
 $\therefore \begin{pmatrix} 1.465 & -20 \\ -0.1 & 1.365 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$   
 $\therefore 1.465c - 20d = 0$  M1  
 $\therefore c \approx \frac{20}{1.465} d$   
 $\approx 13.65d$   
 $\therefore$  the corresponding eigenvector is  $\begin{pmatrix} 13.7 \\ 1 \end{pmatrix}$ . A1
- iii** This eigenvector tells us that in the long-term, the ratio of juvenile to adult female salmon in the population is about 13.7 : 1. R1

**Total [31 marks]**

- 2 a**  $a(t) = v'(t)$   
 $= \frac{4}{3}e^{-\frac{t}{24}} \sin 2t - 64e^{-\frac{t}{24}} \cos 2t$  M1A1  
 $\therefore a(0) = -64 \text{ cm s}^{-2}$  A1  
 The pendulum is accelerating at  $64 \text{ cm s}^{-2}$  towards its mean position.
- b** If  $s(t) = e^{-\frac{t}{24}}(a \sin 2t + b \cos 2t)$   
 then  $v(t) = s'(t)$   
 $= -\frac{1}{24}e^{-\frac{t}{24}}(a \sin 2t + b \cos 2t) + e^{-\frac{t}{24}}(2a \cos 2t - 2b \sin 2t)$  M1A1  
 $= e^{-\frac{t}{24}}\left(\left(-\frac{1}{24}a - 2b\right) \sin 2t + \left(-\frac{1}{24}b + 2a\right) \cos 2t\right)$   
 Equating coefficients of  $\sin 2t$  and  $\cos 2t$ , M1  
 $-\frac{1}{24}a - 2b = -32$  and  $-\frac{1}{24}b + 2a = 0$  M1  
 $\therefore \frac{1}{24}b = 2a$   
 $\therefore b = 48a$  A1  
 $\therefore -\frac{1}{24}a - 96a = -32$  M1  
 $\therefore 96a \approx 32$   
 $\therefore a \approx \frac{1}{3}$  AG  
 $\therefore b \approx 48 \times \frac{1}{3}$  M1  
 $\approx 16$  AG
- c** The pendulum changes direction the first two times when  $2t = \pi$  and  $2t = 2\pi$   
 $\therefore t = \frac{\pi}{2}$  and  $t = \pi$  A1A1
- d i** Using the approximation in **b**,  
 $s(t) \approx e^{-\frac{t}{24}}\left(\frac{1}{3} \sin 2t + 16 \cos 2t\right)$   
 $\therefore s(0) \approx 16 \text{ cm}$  A1  
 and  $s\left(\frac{\pi}{2}\right) \approx e^{-\frac{\pi}{48}}\left(\frac{1}{3} \sin \pi + 16 \cos \pi\right) \approx -14.99 \text{ cm}$  A1  
 $\therefore$  the first swing has length  $\approx 16 + 14.99 \approx 30.99 \text{ cm}$ . A1
- ii**  $s(\pi) \approx e^{-\frac{\pi}{24}}\left(\frac{1}{3} \sin 2\pi + 16 \cos 2\pi\right) \approx 14.04 \text{ cm}$  A1  
 The swing back towards the initial position has length  $\approx 14.99 + 14.04 \approx 29.03 \text{ cm}$ . A1
- e i** The ratio of lengths for the first two swings  $\approx \frac{29.03}{30.99} \approx 0.9368$  M1  
 $\therefore$  we estimate we will lose  $\approx 6.32\%$  distance with each swing. A1



- ii Assuming this loss continues, which is not unreasonable considering the exponential decay term in  $s(t)$  and  $v(t)$ , we model the distance travelled after  $n$  swings with the geometric series

$$S_n = \sum_{i=1}^n 30.99(0.9368)^i \quad \text{M1}$$

As  $n \rightarrow \infty$ , this series converges to the sum

$$S = 30.99 \times \frac{1}{1 - 0.9368} \approx 490 \text{ cm} \quad \text{M1A1}$$

We expect the pendulum to travel a total distance of about 4.90 m.

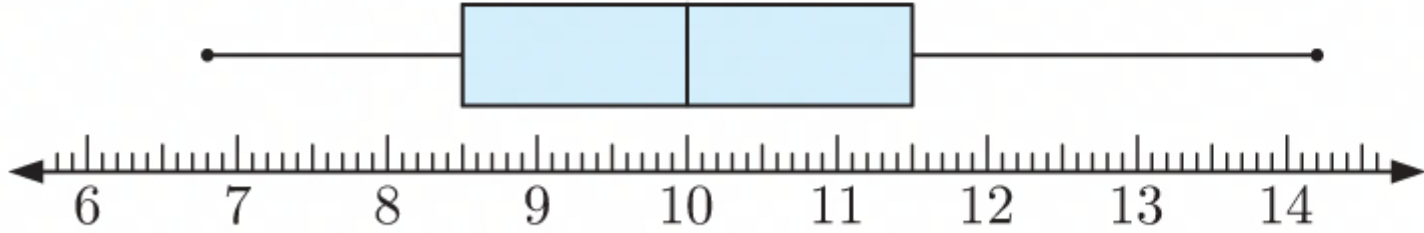
$$\begin{aligned} \text{f Total distance travelled} &= \int_0^\infty |v(t)| dt & \text{M1} \\ &= \int_0^\infty 32e^{-\frac{t}{24}} |\sin 2t| dt \text{ cm} & \text{A1} \end{aligned}$$

**Total [24 marks]**



# TRIAL EXAMINATION 4

## PAPER 1

- 1 a** Volume of Saturn  $\approx \frac{4}{3}\pi r^3$   
 $\approx \frac{4}{3}\pi(5.8232 \times 10^4)^3$   
 $\approx 8.2713 \times 10^{14} \text{ km}^3$  which agrees with the NASA data. **M1A1**
- b** Radius of outer ring  $R \approx 5.8232 \times 10^4 + 2.82 \times 10^5 \approx 3.40232 \times 10^5 \text{ km}$  **A1**  
 Circumference of orbit  $\approx 2\pi R$   
 $\approx 2\pi \times (3.40232 \times 10^5)$  **M1**  
 $\approx 2.14 \times 10^6 \text{ km}$  **A1**  
**Total [5 marks]**
- 2 a** There are  $6 \times 6 = 36$  possible outcomes when rolling the dice.  
 Of these, the outcomes with sum 5 are: (1, 4), (2, 3), (3, 2), (4, 1). **(A1)**  
 So, the probability that the sum is 5 is  $\frac{4}{36} = \frac{1}{9}$ . **A1**
- b** When the pair of dice is rolled 10 times, let  $X$  be the number of times that the sum of the dice is 5.  
 $\therefore X \sim B(10, \frac{1}{9})$  **M1A1**  
 Using technology,  $P(X \geq 2) \approx 0.307$  **A1**  
**Total [5 marks]**
- 3 a i** minimum = 6.8  
 $Q_1 = 8.5$  **A1**  
 median = 10.0  
 $Q_3 = 11.5$  **A1**  
 maximum = 14.2 **A1**
- ii**  **A1 : scale, A1 : box, A1 : whiskers**
- b i**  $IQR = 11.3 - 8 = 3.3$  **A1**
- ii** Each statistic of the 5-number summary for sample B is lower than the corresponding statistic for sample A, and the IQR for sample A is less than the IQR for sample B.  
 $\therefore$  sample A had better and more reliable growing conditions. **R1**  
**Total [9 marks]**
- 4 a**  $H_0$ : An adult's *age group* is independent of their *opinion* about their government's handling of the COVID-19 pandemic. **A1**
- b** Using technology,  $p\text{-value} \approx 0.495$ . **A2**
- c**  $p\text{-value} > 0.1$ , so we do not have enough evidence to reject the null hypothesis, and therefore should accept it. **R1A1**  
 We conclude that at a 10% level of significance, *age group* and *opinion* are independent.  
**Total [5 marks]**
- 5 a** As  $t$  becomes very large  $(\frac{1}{2})^{\frac{t}{400}}$  approaches zero.  
 $\therefore M$  approaches  $A$ . **R1**  
 $A = 97.8\%$  of  $18.61 \text{ g}$   
 $\therefore A \approx 18.2 \text{ g}$  **A1**
- b**  $M(0) = 18.61$  **M1**  
 $\therefore 18.61 \approx 18.2 + B(\frac{1}{2})^0$   
 $\therefore B \approx 0.41$  **A1**



- c** We assume  $2 \text{ years} = 2 \times 365.25 \text{ days}$   
 $= 730.5 \text{ days}$

$$M(730.5) \approx 18.2 + 0.41\left(\frac{1}{2}\right)^{\frac{730.5}{400}}$$

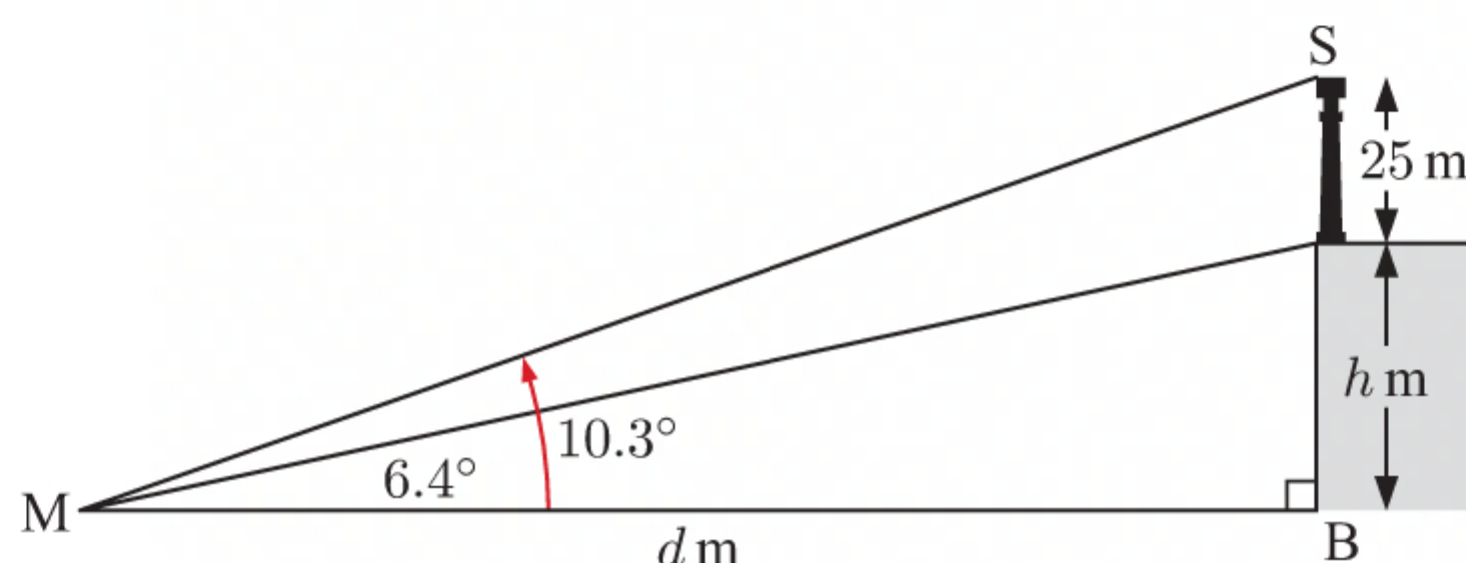
$$\approx 18.3$$

$\therefore$  after 2 years, the mass is about 18.3 g.

M1

A1

Total [6 marks]

**6 a**

Suppose the cliff is  $h$  m high and that Markus is  $d$  m from the base of the cliff.

$$\therefore \tan 6.4^\circ = \frac{h}{d} \text{ and } \tan 10.3^\circ = \frac{h + 25}{d}$$

(M1)

$$\therefore \tan 10.3^\circ = \tan 6.4^\circ + \frac{25}{d}$$

(M1)

$$\therefore \frac{25}{d} = \tan 10.3^\circ - \tan 6.4^\circ$$

$$\therefore d = \frac{25}{\tan 10.3^\circ - \tan 6.4^\circ}$$

$$\therefore d \approx 359.39$$

A1

$\therefore$  Markus is about 359 m from the base of the cliff.

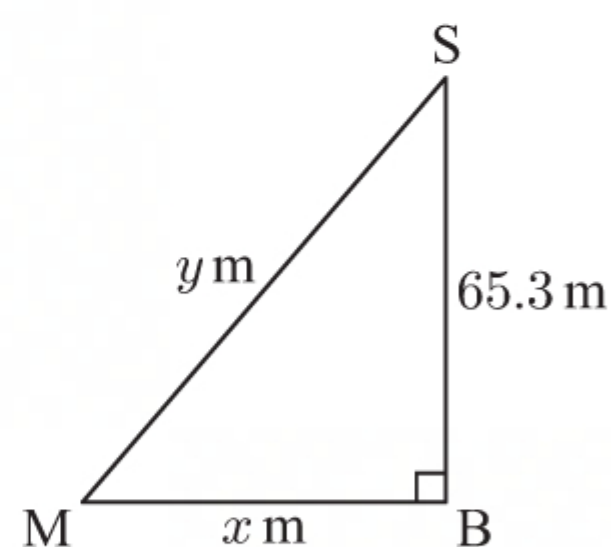
- b**  $h = d \tan 6.4^\circ$

$$\approx 40.3$$

A1

$\therefore$  the cliff is about 40.3 m high.

- c** Let the distance from Markus to the base of the cliff be  $x$  m,  
 and the distance from Markus to Sam be  $y$  m.



$$\frac{dx}{dt} = -2 \text{ m s}^{-1}$$

By Pythagoras' theorem,  $y^2 = x^2 + 65.3^2$

$$\therefore y = \sqrt{x^2 + 4264.09} \quad \{y > 0\}$$

M1

$$\therefore \frac{dy}{dx} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + 4264.09}}$$

M1

$$= \frac{x}{\sqrt{x^2 + 4264.09}}$$

Now  $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$  {chain rule}

$$\therefore \text{when } x = 100 \text{ m, } \frac{dy}{dt} = \frac{100}{\sqrt{10\,000 + 4264.09}} \times -2$$

M1

$$\approx -1.67 \text{ m s}^{-1}$$

A1

So, the distance between Markus and Sam is decreasing at about  $1.67 \text{ m s}^{-1}$ .

Total [8 marks]

**7 a**  $M = \frac{2}{3} \log_{10}(7.8 \times 10^{13}) - 3.6$   
 $\approx 5.66$

(M1)

A1

The magnitude of the earthquake is about 5.66.

**b** When  $M = 2.6$ ,  $2.6 = \frac{2}{3} \log_{10} E - 3.6$

(M1)

Using technology,  $E \approx 2.00 \times 10^9$

A1

The earthquake releases about  $2.00 \times 10^9 \text{ J}$  of energy.

Total [4 marks]



8 a  $r \approx 0.610$  A1

There is a weak positive correlation between the variables. R1

b  $r^2 \approx (0.610)^2 \approx 0.372$  A1

This means that approximately 37.2% of the variation in the *number of accidents* can be explained by the variation in the *number of tickets*. R1

c Using technology, the  $p$ -value  $\approx 0.0351$ . A1

Since the  $p$ -value  $< 0.05$ , we reject  $H_0$  on the 5% level of significance.

$\therefore$  we conclude that there is a non-zero correlation between the variables. A1

Total [6 marks]

9 a The lines intersect where 
$$\begin{cases} 1 - s = 1 + 2t & \dots (1) \\ 5 + 3s = -1 & \dots (2) \\ -s = a + t & \dots (3) \end{cases}$$
 M1

Using (2),  $3s = -6$

$$\therefore s = -2$$

Using (1),  $1 + 2t = 3$

$$\therefore t = 1$$

Using (3),  $a = -s - t = 1$  A1

b The lines have direction vectors  $\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ . M1

If  $\theta$  is the acute angle between the lines, then

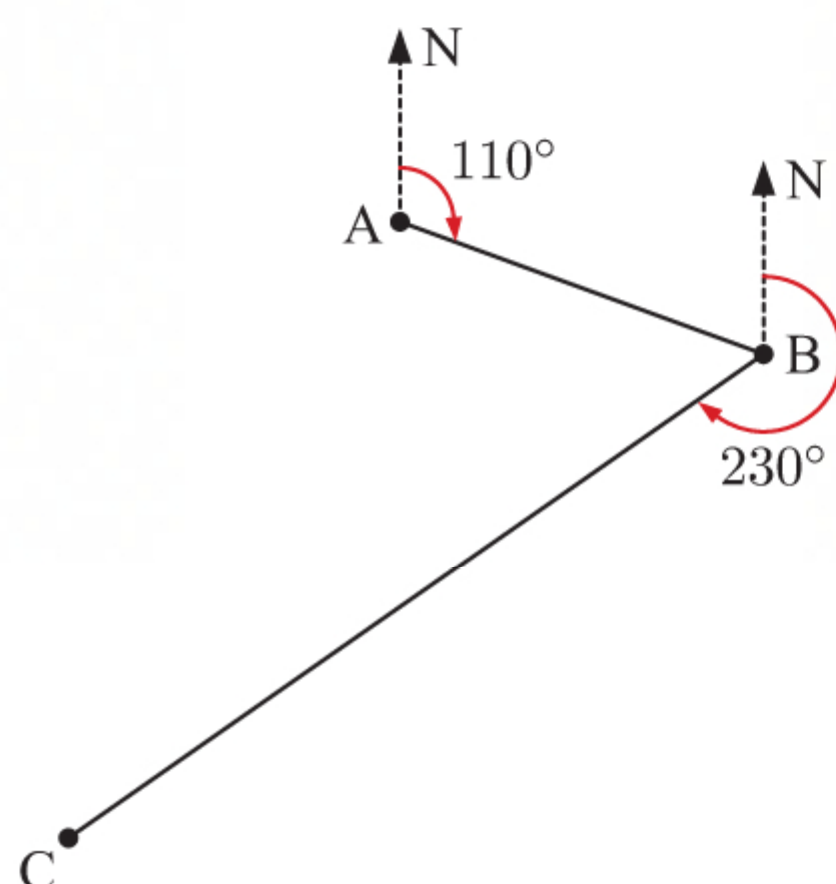
$$\cos \theta = \frac{\left| \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right|} = \frac{|-2 + 0 - 1|}{\sqrt{1+9+1}\sqrt{4+0+1}} = \frac{3}{\sqrt{11}\sqrt{5}}$$
 M1

$$\therefore \theta \approx 66.1^\circ$$
 A1

Total [6 marks]

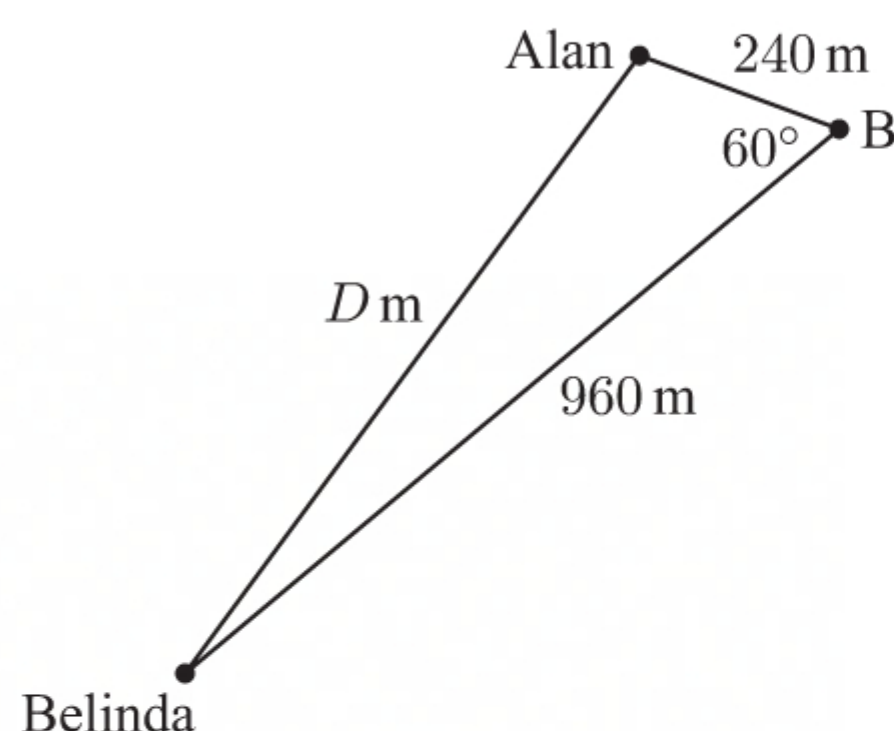
10 a  $\widehat{ABC} = 360^\circ - 230^\circ - (180 - 110)^\circ$  M1

$$= 60^\circ$$
 A1



b After 2 minutes, Alan is  $(600 - 3 \times 120) = 240$  m from B. M1

In the same time, Belinda cycles  $8 \times 120 = 960$  m. M1



Let the distance between Alan and Belinda be  $D$  m.

Using the cosine rule,

$$D^2 = 240^2 + 960^2 - 2 \times 240 \times 960 \times \cos 60^\circ$$
 M1

$$= 748\,800$$

$$\therefore D \approx 865$$
 A1

After 2 minutes, the distance between Alan and Belinda is about 865 m.

Total [6 marks]



- 11 a** The rate at which vehicles cross the intersection = 31 vehicles per hour  
 $= (31 \times 2.5) \text{ vehicles per } 2\frac{1}{2} \text{ hours}$   
 $= 77.5 \text{ vehicles per } 2\frac{1}{2} \text{ hours}$

$$\therefore X \sim \text{Po}(77.5)$$

A1

**b**  $E(X) = 77.5 \text{ vehicles}$

A1

$$\text{Var}(X) = E(X) = 77.5$$

A1

**c i**  $P(X \leq 70) \approx 0.215 \quad \{\text{using technology}\}$

A1

**ii**  $P(X > 70) = 1 - P(X \leq 70)$

$$\approx 1 - 0.215$$

$$\approx 0.785$$

A1

Total [5 marks]

- 12 a** The midpoint of [AB] is  $\left(\frac{-5+1}{2}, \frac{3+5}{2}\right)$  which is  $(-2, 4)$ .

M1

$$\text{The gradient of [AB] is } \frac{5-3}{1-(-5)} = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \text{the gradient of the perpendicular bisector is } -3.$$

M1

$$\therefore \text{the equation of the perpendicular bisector is } y - 4 = -3(x + 2) \text{ which is } y = -3x - 2.$$

A1

- b i** The gradient of the perpendicular bisector of [AC] is  $\frac{2}{3}$ .

$$\therefore \text{the gradient of [AC] is } -\frac{3}{2}.$$

M1

$$\therefore \frac{k-3}{-1-(-5)} = -\frac{3}{2}$$

$$\therefore k - 3 = -6$$

$$\therefore k = -3$$

A1

- ii** P lies at the intersection of  $y = -3x - 2$  and  $y = \frac{2}{3}x + 2$ .

M1

$$\text{Equating values of } y, \quad -3x - 2 = \frac{2}{3}x + 2$$

$$\therefore -\frac{11}{3}x = 4$$

$$\therefore x = -\frac{12}{11}$$

A1

$$\therefore y = -3\left(-\frac{12}{11}\right) - 2 = \frac{36-22}{11} = \frac{14}{11}$$

A1

$$\therefore \text{P is at } \left(-\frac{12}{11}, \frac{14}{11}\right).$$

- c** Q lies at the intersection between B, C, and D, so  $\frac{z + 2307 + 2683}{3} = 2602$

M1

$$\therefore z + 4990 = 7806$$

$$\therefore z = 2816$$

A1

Total [10 marks]

**13**  $T = \int -\frac{28}{r} dr$

M1

$$= -28 \ln|r| + c$$

A1

$$\text{When } r = 1, \quad T = 68$$

$$\therefore c = 68$$

$$\therefore T = -28 \ln|r| + 68$$

A1

Let  $d$  be the minimum necessary thickness of foam.

$$\text{When } r = d + 1, \quad T = 40$$

$$\therefore -28 \ln(d + 1) + 68 = 40 \quad \{d + 1 > 0\}$$

M1

$$\therefore -28 \ln(d + 1) = -28$$

$$\therefore \ln(d + 1) = 1$$

$$\therefore d + 1 = e$$

A1

$$\therefore d = e - 1 \approx 1.72 \text{ cm}$$

So, the minimum necessary thickness of foam is about 1.72 cm.

A1

Total [6 marks]



$$\begin{array}{c}
 \text{Finishing vertex} \\
 \begin{array}{cccc}
 \text{A} & \text{B} & \text{C} & \text{F} \\
 \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{F} \end{array} \\
 \text{Starting vertex}
 \end{array}
 \end{array}
 \quad \text{14 a} \quad \mathbf{A} = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} \quad \text{A1A1}$$

$$\text{14 b} \quad \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 9 & 19 & 9 & 17 \\ 19 & 14 & 17 & 16 \\ 9 & 17 & 9 & 19 \\ 17 & 16 & 19 & 14 \end{pmatrix} \quad \text{A1}$$

This matrix gives the number of possible paths for travel between vertices in one, two, or three steps. A1

14 c If the runner starts at A, then the initial state matrix is  $\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$ .

$$\text{Now } \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \mathbf{A}^3 = \begin{pmatrix} 4 & 16 & 5 & 14 \end{pmatrix} \quad \text{M1A1}$$

$$\therefore \text{P(runner is at the fountain)} = \frac{14}{4 + 16 + 5 + 14} = \frac{14}{39} \quad \text{A1}$$

Total [7 marks]

$$\begin{array}{l}
 \text{15 a When } s = 0, \quad v = \frac{(5-0)^2}{5} \\
 \quad \quad \quad = 5 \text{ cm s}^{-1}
 \end{array} \quad \begin{array}{l} \text{M1} \\ \text{A1} \end{array}$$

$$\begin{array}{l}
 \text{15 b } a = v \frac{dv}{ds} \\
 \quad = \frac{(5-s)^2}{5} \times \left( \frac{2(5-s)(-1)}{5} \right) \\
 \quad = -\frac{2(5-s)^3}{25} \text{ cm s}^{-2}
 \end{array} \quad \begin{array}{l} \text{M1} \\ \text{M1} \\ \text{A1} \end{array}$$

$$\begin{array}{l}
 \text{15 c } a = -0.64 \text{ cm s}^{-2} \text{ when } -\frac{2(5-s)^3}{25} = -\frac{16}{25} \\
 \quad \quad \quad \therefore (5-s)^3 = 8 \\
 \quad \quad \quad \therefore 5-s = 2 \\
 \quad \quad \quad \therefore s = 3 \text{ cm}
 \end{array} \quad \begin{array}{l} \text{M1} \\ \\ \text{A1} \end{array}$$

Total [7 marks]

$$\text{16 a } H_1: p > \frac{1}{4} \quad \text{A1}$$

$$\begin{array}{l}
 \text{16 b significance level } \alpha = \text{P(Type I error)} \\
 \quad = \text{P(Reject } H_0 \mid H_0 \text{ true)} \\
 \quad = \text{P}(X \geq 10 \mid X \sim \text{B}(15, \frac{1}{4})) \\
 \quad \approx 0.000795
 \end{array} \quad \begin{array}{l} \text{M1} \\ \\ \text{A1} \end{array}$$

$$\begin{array}{l}
 \text{16 c } p\text{-value} = \text{P}(X \geq 11 \mid X \sim \text{B}(15, \frac{1}{4})) \\
 \quad \approx 0.000115
 \end{array} \quad \begin{array}{l} \text{M1} \\ \text{A1} \end{array}$$

$$\begin{array}{l}
 \text{16 d } \text{P(Type II error)} = \text{P(Retain } H_0 \mid H_0 \text{ false)} \\
 \quad = \text{P}(X + 7 < 10 \mid X \sim \text{B}(8, \frac{1}{2})) \\
 \quad = \text{P}(X < 3 \mid X \sim \text{B}(8, \frac{1}{2})) \\
 \quad \approx 0.145
 \end{array} \quad \begin{array}{l} \text{M1} \\ \text{M1} \\ \text{A1} \end{array}$$

Total [9 marks]

$$\begin{array}{l}
 \text{17 a } x^2 + y^2 = \left(\frac{1}{2} \cos \frac{t}{10}\right)^2 + \left(-\frac{1}{2} \sin \frac{t}{10}\right)^2 \\
 \quad = \left(\frac{1}{2}\right)^2 (\cos^2 \frac{t}{10} + \sin^2 \frac{t}{10}) \\
 \quad = \frac{1}{4}
 \end{array} \quad \begin{array}{l} \text{M1} \\ \\ \text{A1} \end{array}$$

$\therefore$  the train travels in a circle with centre O and radius  $\frac{1}{2}$  m. AG



$$\begin{aligned}
 \text{b } x'(t) &= -\frac{1}{20} \sin \frac{t}{10}, \quad y'(t) = -\frac{1}{20} \cos \frac{t}{10} & \text{A1} \\
 \therefore v^2 &= (x'(t))^2 + (y'(t))^2 & \text{M1} \\
 &= \left(-\frac{1}{20}\right)^2 \left(\sin^2 \frac{t}{10} + \cos^2 \frac{t}{10}\right) \\
 &= \frac{1}{400} & \text{A1} \\
 \therefore F_C &= \frac{mv^2}{r} = \frac{0.38\left(\frac{1}{400}\right)}{\frac{1}{2}} = 0.0019 \text{ N} & \text{A1}
 \end{aligned}$$

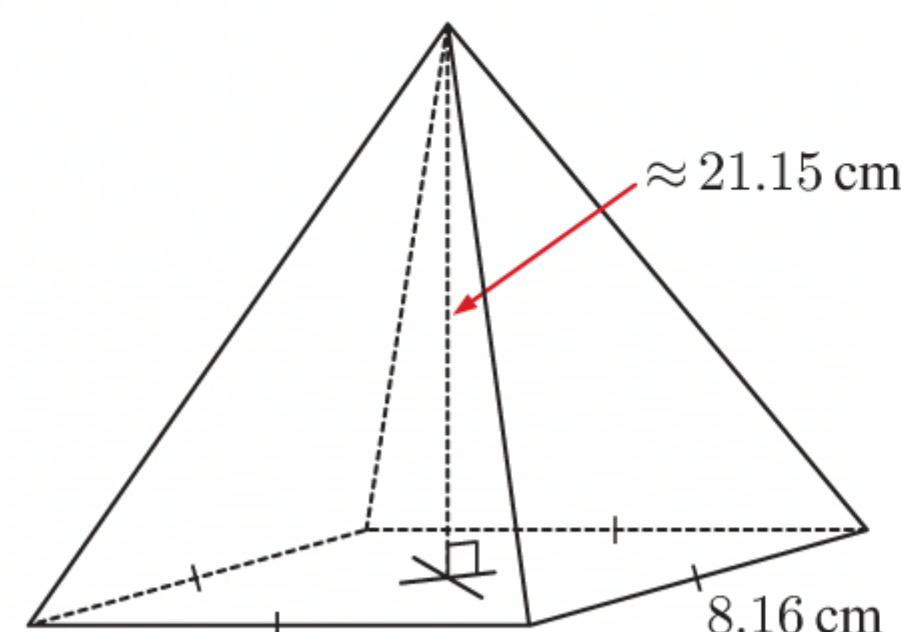
Total [6 marks]

## PAPER 2

$$\begin{aligned}
 \text{1 a i Height of the model} &= 32\,400 \times \frac{8}{12\,500} & \text{M1} \\
 &\approx 20.736 & \\
 &\approx 20.7 \text{ cm} & \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii The case has base with sides } 8 \times 1.02 &= 8.16 \text{ cm and} \\
 \text{height } &\approx 20.736 \times 1.02 \\
 &\approx 21.15 \text{ cm}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{the volume of the glass pyramid} &\approx \frac{1}{3} \times 8.16^2 \times 21.15 \\
 &\approx 469 \text{ cm}^3
 \end{aligned}$$



$$\begin{aligned}
 \text{b i If the shop owner does not buy any models, he cannot sell any models, so he makes neither profit nor loss.} & & \text{R1} \\
 \therefore P(0) &= 0 \\
 \therefore d &= 0
 \end{aligned}$$

ii Using the table of values,

$$\begin{aligned}
 P(4) &= 4^3a + 4^2b + 4c = 32\,000 \\
 \therefore 64a + 16b + 4c &= 32\,000 \quad \dots (1) & \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 P(9) &= 9^3a + 9^2b + 9c = 85\,500 \\
 \therefore 729a + 81b + 9c &= 85\,500 \quad \dots (2) & \text{A1}
 \end{aligned}$$

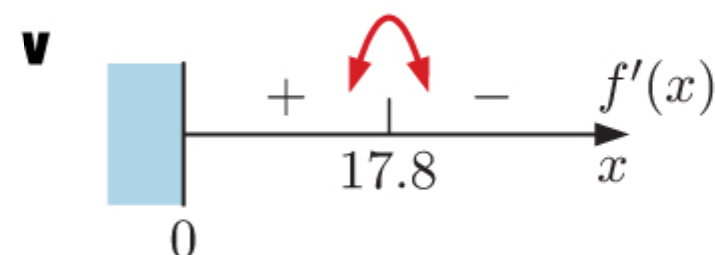
$$\begin{aligned}
 P(12) &= 12^3a + 12^2b + 12c = 115\,200 \\
 \therefore 1728a + 144b + 12c &= 115\,200 \quad \dots (3) & \text{A1}
 \end{aligned}$$

Solving these simultaneously, using technology,  $a = -\frac{100}{3}$ ,  $b = \frac{2200}{3}$ ,  $c = 5600$ 

$$\therefore P(x) = -\frac{100}{3}x^3 + \frac{2200}{3}x^2 + 5600x \text{ euros} \quad \text{A1}$$

$$\text{iii } P'(x) = -100x^2 + \frac{4400}{3}x + 5600 \quad \text{A1}$$

$$\text{iv Using technology, } P'(x) = 0 \text{ when } x \approx -3.14 \text{ or } 17.8 \quad \text{A2}$$



The maximum profit will occur when the shop owner buys about 17 800 models. A1

The profit in this case is  $P(17.8) \approx \text{€}144\,000$  A1

Total [15 marks]

$$\text{2 a Investment A is simple interest, so it results in an arithmetic sequence.} \quad \text{A1}$$

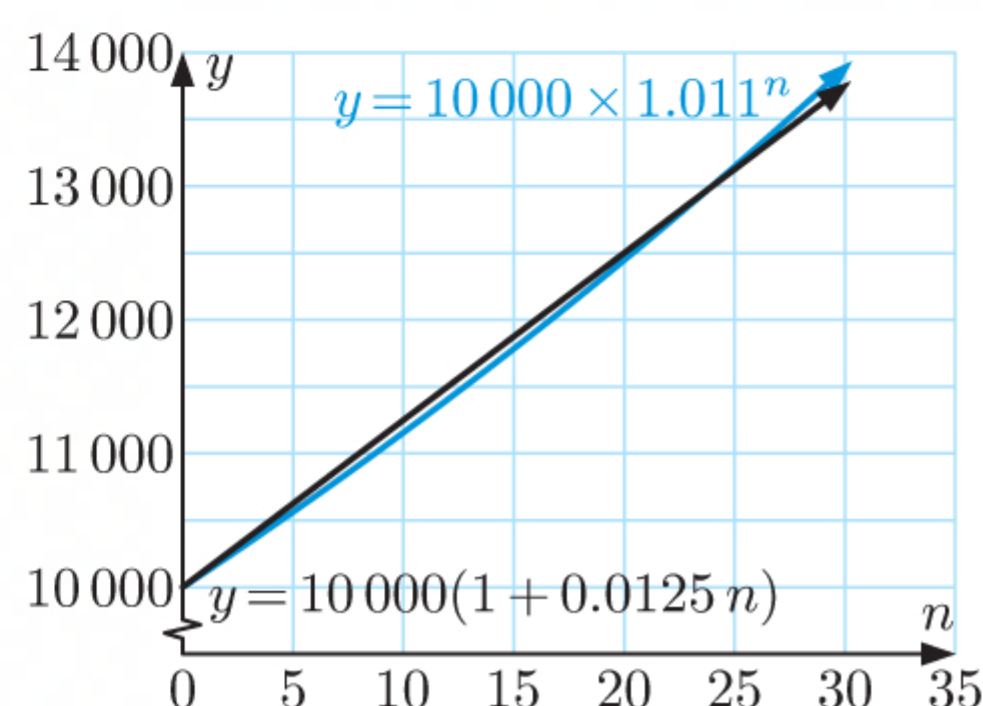
Investment B is compound interest, so it results in a geometric sequence. A1

$$\text{b Each quarter, the interest paid is } \text{€}10\,000 \times \frac{0.05}{4} \quad \text{A1}$$

$$\begin{aligned}
 \therefore \text{after } n \text{ quarters, the investment is worth } &\text{€}10\,000 + n \times \text{€}10\,000 \times \frac{0.05}{4} & \text{A1} \\
 &= \text{€}10\,000(1 + 0.0125n)
 \end{aligned}$$



- c** Each quarter, the value of the investment is multiplied by  $1 + \frac{0.044}{4} = 1.011$  A1  
 $\therefore$  after  $n$  quarters, the investment is worth  $\text{€}10\,000 \times 1.011^n$  A1
- d** We use technology to compare the graphs of  
 $y = 10\,000(1 + 0.0125n)$   
 and  $y = 10\,000 \times 1.011^n$  M1  
 The graphs intersect when  $n \approx 23.85$  A1  
 $\therefore$  it will take 24 quarters, which is 6 years, for the compound interest investment to be the better option. A1
- e i** 15 years =  $15 \times 4$  quarters = 60 quarters  
 The starting balance is  $\text{€}10\,000 \times 1.011^{60}$  M1  
 $= \text{€}19\,278.33$  A1
- ii**  $N = 5 \times 12 = 60$   
 $I\% = 2.8$   
 $PV = -19\,278.33$   
 $FV = 0$   
 $P/Y = 12$   
 $C/Y = 12$  M1A1  
 $\therefore PMT \approx 334.69$  A1  
 Portia can afford to withdraw  $\text{€}334.69$  each month.



Total [14 marks]

- 3 a** Systematic sampling A1
- b i**  $\bar{x} \approx 162.5$  (162.4643) A1  
 $s \approx 5.01$  (5.008 85) A1
- ii** A one-tailed  $t$ -test for a population mean, as James needs to be confident that the mean weight is *more than* 160 kg, and he only knows the *sample* standard deviation. R1A1
- iii**  $H_1: \mu_J > 160$  A1
- iv** Using technology,  $p$ -value  $\approx 0.0443$  A1  
 Since  $p$ -value  $< 0.05$ , there is sufficient evidence to reject  $H_0$  in favour of  $H_1$  at the 5% significance level. R1A1  
 James can therefore confidently sell the bales he is receiving.
- c i** A two-sample  $t$ -test for comparing population means. A1
- ii** James will need to assume that the standard deviation of his sample is the same as Susan's. A1
- iii**  $H_1: \mu_J \neq \mu_S$  A1
- iv** Using technology,  $p$ -value  $\approx 0.312$  A1  
 Since  $p$ -value  $> 0.05$ , there is not sufficient evidence to reject  $H_0$  in favour of  $H_1$  at the 5% significance level. R1A1  
 James therefore concludes that the mean weight of his bales is not significantly different from the mean weight of Susan's bales.

Total [15 marks]

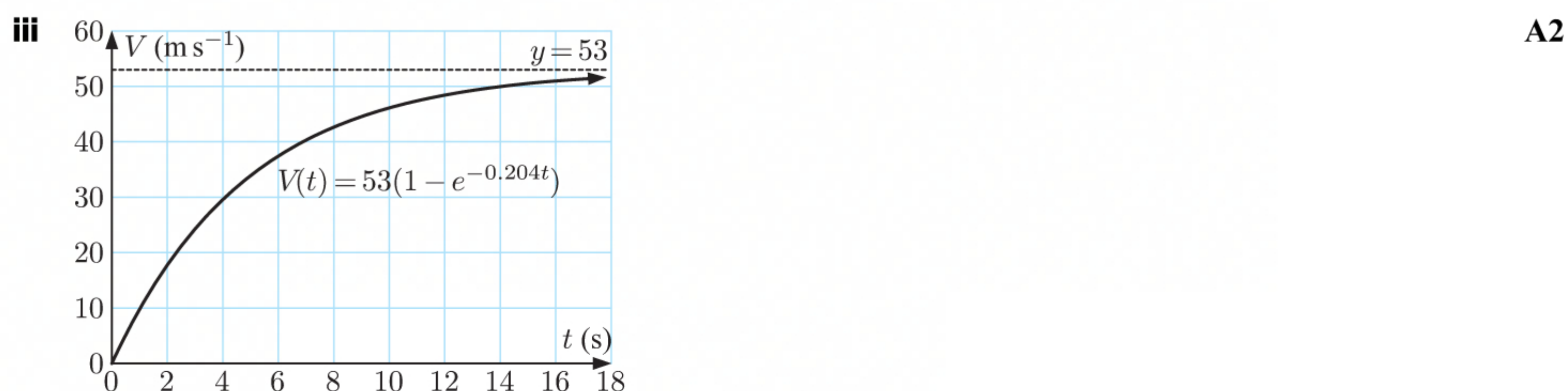
- 4 a** Distance fallen  $= \int_0^{\tau} v(t) dt$  metres. A1
- b i**  $v \propto t$   
 $\therefore v = \beta t$  for some constant  $\beta$   
 Now  $v(1) = 9.8$  (M1)  
 $\therefore \beta = 9.8$   
 So, the proportionality constant is 9.8, and  $v = 9.8t \text{ m s}^{-1}$ . A1A1



$$\begin{aligned}
 \text{ii Distance fallen in the first 2 seconds} &= \int_0^2 9.8t \, dt && \text{M1} \\
 &= [4.9t^2]_0^2 && \text{A1} \\
 &= 4.9 \times 4 - 0 \\
 &= 19.6 \text{ m} && \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{c i } V(0) &= 53(1 - e^0) = 53(1 - 1) = 0 && \text{A1} \\
 \text{This matches } v(0) &= 0 \text{ and the fact that at time 0 seconds, the skydiver jumps and has therefore} && \text{R1} \\
 &\text{not yet fallen.}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } V(1) &= 9.8 \\
 \therefore 53(1 - e^{-k}) &= 9.8 && \text{(M1)} \\
 \text{Solving using technology, } k &\approx 0.204 && \text{A1}
 \end{aligned}$$



Over time, the speed of the skydiver approaches  $53 \text{ m s}^{-1}$ . A1

$$\begin{aligned}
 \text{iv Distance fallen in the first 5 seconds} &= \int_0^5 53(1 - e^{-0.204t}) \, dt && \text{M1A1} \\
 &\approx 99 \text{ m} \quad \{\text{using technology}\} && \text{A1}
 \end{aligned}$$

**Total [17 marks]**

5 a The states are “left” and “right”. A1

$$\text{b i } \mathbf{T} = \begin{pmatrix} \begin{matrix} \text{left} & \text{right} \\ 0.4 & 0.7 \\ 0.6 & 0.3 \end{matrix} \end{pmatrix} \begin{matrix} \text{left} \\ \text{right} \end{matrix} && \text{A1}$$

$$\text{ii } \mathbf{s}_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \quad \{\text{both directions equally likely}\} && \text{A1}$$

$$\begin{aligned}
 \text{c } \mathbf{s}_5 &= \mathbf{T}^5 \mathbf{s}_0 && \text{M1} \\
 &= \begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix}^5 \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \\
 &\approx \begin{pmatrix} 0.539 \\ 0.461 \end{pmatrix} && \text{A1}
 \end{aligned}$$

The probability that the 5th step will be to the left is about 0.539, and the probability that it will be to the right is about 0.461. R1

d The eigenvalues of  $\mathbf{T}$  are the solutions to  $\det(\lambda \mathbf{I} - \mathbf{T}) = 0$  M1

$$\begin{aligned}
 \therefore \begin{vmatrix} \lambda - 0.4 & -0.7 \\ -0.6 & \lambda - 0.3 \end{vmatrix} &= 0 \\
 \therefore (\lambda - 0.4)(\lambda - 0.3) - 0.42 &= 0 && \text{M1} \\
 \therefore \lambda^2 - 0.7\lambda + 0.12 - 0.42 &= 0 \\
 \therefore \lambda^2 - 0.7\lambda - 0.3 &= 0 \\
 \therefore (\lambda - 1)(\lambda + 0.3) &= 0 \\
 \therefore \lambda &= 1 \text{ or } -0.3 && \text{A1}
 \end{aligned}$$

$\therefore$  the eigenvalues of  $\mathbf{T}$  are  $\lambda_1 = 1$ ,  $\lambda_2 = -0.3$ . AG

e The steady state is the eigenvector corresponding to the eigenvalue  $\lambda_1 = 1$ . R1



**f** If  $\mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix}$  is the steady state matrix, then  $\mathbf{T}\mathbf{s} = \mathbf{s}$  M1

$$\therefore \begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0.4a + 0.7b \\ 0.6a + 0.3b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

A1

Using either row,  $0.6a + 0.3b = b$

$$\therefore 0.6a = 0.7b$$

$$\therefore a = \frac{7}{6}b$$

A1

Now  $a + b = 1$  { $\mathbf{s}$  is a matrix of probabilities}

M1

$$\therefore \frac{7}{6}b + b = 1$$

$$\therefore \frac{13}{6}b = 1$$

$$\therefore b = \frac{6}{13} \quad \text{and} \quad a = \frac{7}{13}$$

$$\therefore \mathbf{s} = \begin{pmatrix} \frac{7}{13} \\ \frac{6}{13} \end{pmatrix}$$

A1

**g** From **f**, the long-term probability that Ryu5 will move to the left is greater than the probability that it will move to the right.

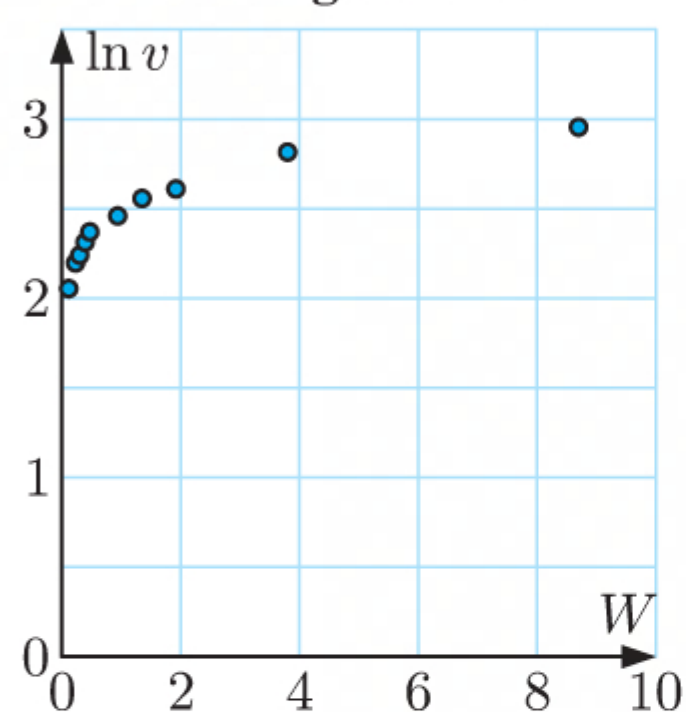
$\therefore$  in the long term, Ryu5 will be to the left of the origin.

R1

Total [16 marks]

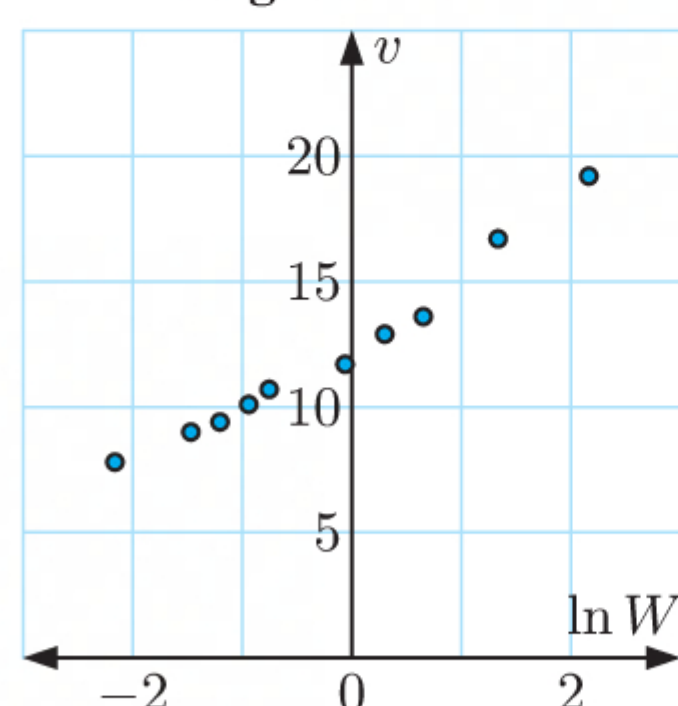
**6 a i**  $\ln v$  against  $W$

A1A1



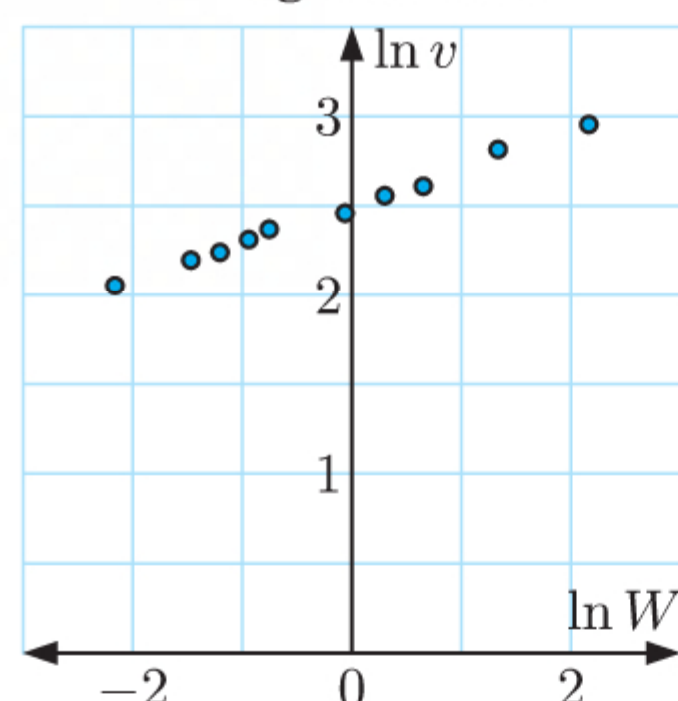
**ii**  $v$  against  $\ln W$

A1A1



**iii**  $\ln v$  against  $\ln W$

A1A1



**b** The graph of  $\ln v$  against  $W$  is clearly non-linear, so we reject an exponential model.

The graph of  $v$  against  $\ln W$  has  $r^2 \approx 0.972$  which is very high, but there appears to be a curve.

R1

The graph of  $\ln v$  against  $\ln W$  has  $r^2 \approx 0.995$  and appears to form a straight line.

R1

We conclude there is a linear relationship between  $\ln v$  and  $\ln W$ , and therefore a power relationship between  $v$  and  $W$ .

AG



**c** Using linear regression,  $\ln v \approx 0.209 \ln W + 2.502$ . **A1**

**d** From **c**,  $v \approx e^{0.209 \ln W + 2.502}$   
 $\approx e^{2.502} \times W^{0.209}$  **M1**  
 $\approx 12.2W^{0.209}$  **A1**

**e** Letting  $W = 2.8$  kg  
 $v \approx 12.2 \times 2.8^{0.209}$  **M1**  
 $\therefore v \approx 15.1 \text{ m s}^{-1}$

The mean cruising velocity of the sooty albatross is about  $15.1 \text{ m s}^{-1}$ . **A1**

Since this is an interpolation and the value of  $r^2$  is very close to 1, we expect this estimate to be reliable. **R1**

**f**  $\sqrt{S} \propto v$   
 $\therefore S \propto v^2$   
 $\therefore S \propto (W^{0.209})^2$  {using **d**} **M1**  
 $\therefore S \propto W^{0.418}$   
 $\therefore n \approx 0.418$  **A1**

**Total [16 marks]**

**7 a**  $f_2(0) = 0$   
 $\therefore 2 \ln(\cos 0 + 1) + a = 0$  **M1**  
 $\therefore a = -2 \ln 2$  **A1**

**b**  $f_2(k) = -4$   
 $\therefore 2 \ln(\cos \frac{k\pi}{12} + 1) - 2 \ln 2 = -4$  **M1A1**  
 Using technology,  $k \approx 9.12$ . **A1**

**c**  $f_1(x) = \ln(\cos \frac{\pi x}{12} + 2)$   
 $\therefore f_1'(x) = \frac{-\frac{\pi}{12} \sin \frac{\pi x}{12}}{\cos \frac{\pi x}{12} + 2}$  **M1A1**  
 $\therefore f_1''(x) = \frac{-\frac{\pi^2}{144} \cos \frac{\pi x}{12} (\cos \frac{\pi x}{12} + 2) + \frac{\pi}{12} \sin \frac{\pi x}{12} (-\frac{\pi}{12} \sin \frac{\pi x}{12})}{(\cos \frac{\pi x}{12} + 2)^2}$  **M1A1**  
 $\therefore f_1''(x) = 0$  when  $-\frac{\pi^2}{144} (\cos^2(\frac{\pi x}{12}) + \sin^2(\frac{\pi x}{12}) + 2 \cos \frac{\pi x}{12}) = 0$  **M1**  
 $\therefore -\frac{\pi^2}{144} (1 + 2 \cos \frac{\pi x}{12}) = 0$   
 $\therefore \cos \frac{\pi x}{12} = -\frac{1}{2}$  **A1**  
 $\therefore \frac{\pi x}{12} = \pm \frac{2\pi}{3}$  {for  $-12 \leq x \leq 12$ }  
 $\therefore x = \pm 8$  **A1**

Now  $f'(-8) = \frac{\frac{\pi}{12} \sin(-\frac{2\pi}{3})}{\cos(-\frac{2\pi}{3}) + 2} = \frac{-\frac{\pi}{12}(-\frac{\sqrt{3}}{2})}{\frac{3}{2}} = \frac{\pi}{12\sqrt{3}}$  **A1**

$\therefore$  the maximum gradient of the road is  $\frac{\pi}{12\sqrt{3}}$  at distance 8 m from the centre of the bridge, whichever side you are approaching from.

**d** Since the defining functions are both symmetric about the  $y$ -axis,

shaded area

$= 2 \left( \int_0^k (f_1(x) - f_2(x)) dx + \int_k^{12} (f_1(x) - (-4)) dx \right)$  **M1A1A1**  
 $\approx 2 \left( \int_0^{9.12} (\ln(\cos \frac{\pi x}{12} + 2) - 2 \ln(\cos \frac{\pi x}{12} + 1) + 2 \ln 2) dx + \int_{9.12}^{12} (\ln(\cos \frac{\pi x}{12} + 2) + 4) dx \right)$   
 $\approx 2(17.66 + 11.77)$   
 $\approx 58.9 \text{ m}^2$  **A1**

**Total [17 marks]**



## PAPER 3

1 a i

$$x = \frac{a}{1 + be^{-kt}}$$

$$\therefore \frac{dx}{dt} = \frac{-a(-bke^{-kt})}{(1 + be^{-kt})^2}$$

M1

$$= \frac{a}{1 + be^{-kt}} \left( \frac{bke^{-kt}}{1 + be^{-kt}} \right)$$

$$= x \left( \frac{k(1 + be^{-kt}) - k}{1 + be^{-kt}} \right)$$

M1

$$= kx \left( 1 - \frac{1}{1 + be^{-kt}} \right)$$

$$= kx \left( 1 - \frac{x}{a} \right)$$

$$\therefore k = 0.15 \text{ and } a = 80\,000$$

A1A1

$$\text{Now } x(0) = 500$$

$$\therefore \frac{80\,000}{1 + b} = 500$$

M1

$$\therefore b + 1 = \frac{80\,000}{500}$$

$$\therefore b = 159$$

A1

$$\text{ii } x(t) = \frac{80\,000}{1 + 159 \times e^{-0.15t}}$$

$$\therefore x(20) = \frac{80\,000}{1 + 159 \times e^{-3}} \approx 8972$$

M1A1

The population is now about 8972 deer.

$$\text{iii As } t \rightarrow \infty, e^{-0.15t} \rightarrow 0$$

$$\therefore x(t) \rightarrow 80\,000$$

A1

The limiting population is 80 000 deer.

$$\text{b i } X(0) = 8972, Y(0) = 20$$

$$\text{When } T = 0, \frac{dX}{dT} \approx 0.15 \times 8972 \left( 1 - \frac{8972}{80\,000} - \frac{20}{150} \right)$$

$$\approx 1015$$

A1

$$\text{and } \frac{dY}{dT} \approx 20 \left( \frac{8972}{10\,000} - 1 - \frac{20}{1000} \right)$$

$$\approx -2.46$$

A1

$\therefore$  we expect that in the first year after the wolves are introduced, the deer population will grow by about 1000, and the wolf population might decrease by 2 or 3.

R1

$$\text{ii The equilibrium points occur when } \frac{dX}{dT} \text{ and } \frac{dY}{dT} \text{ are both zero.}$$

M1

One equilibrium point is the origin

$$\therefore \text{ at the equilibrium point where } X > 0 \text{ and } Y > 0, \begin{cases} 1 - \frac{X}{80\,000} - \frac{Y}{150} = 0 \\ \frac{X}{10\,000} - 1 - \frac{Y}{1000} = 0 \end{cases}$$

M1

$$\therefore \begin{pmatrix} -\frac{1}{80\,000} & -\frac{1}{150} \\ \frac{1}{10\,000} & -\frac{1}{1000} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

M1

$$\therefore \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\frac{1}{80\,000} & -\frac{1}{150} \\ \frac{1}{10\,000} & -\frac{1}{1000} \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 11\,288 \\ 129 \end{pmatrix}$$

The equilibrium point is (11 288, 129).

A1

From the phase portrait, the equilibrium point is a stable anticlockwise spiral.

A1



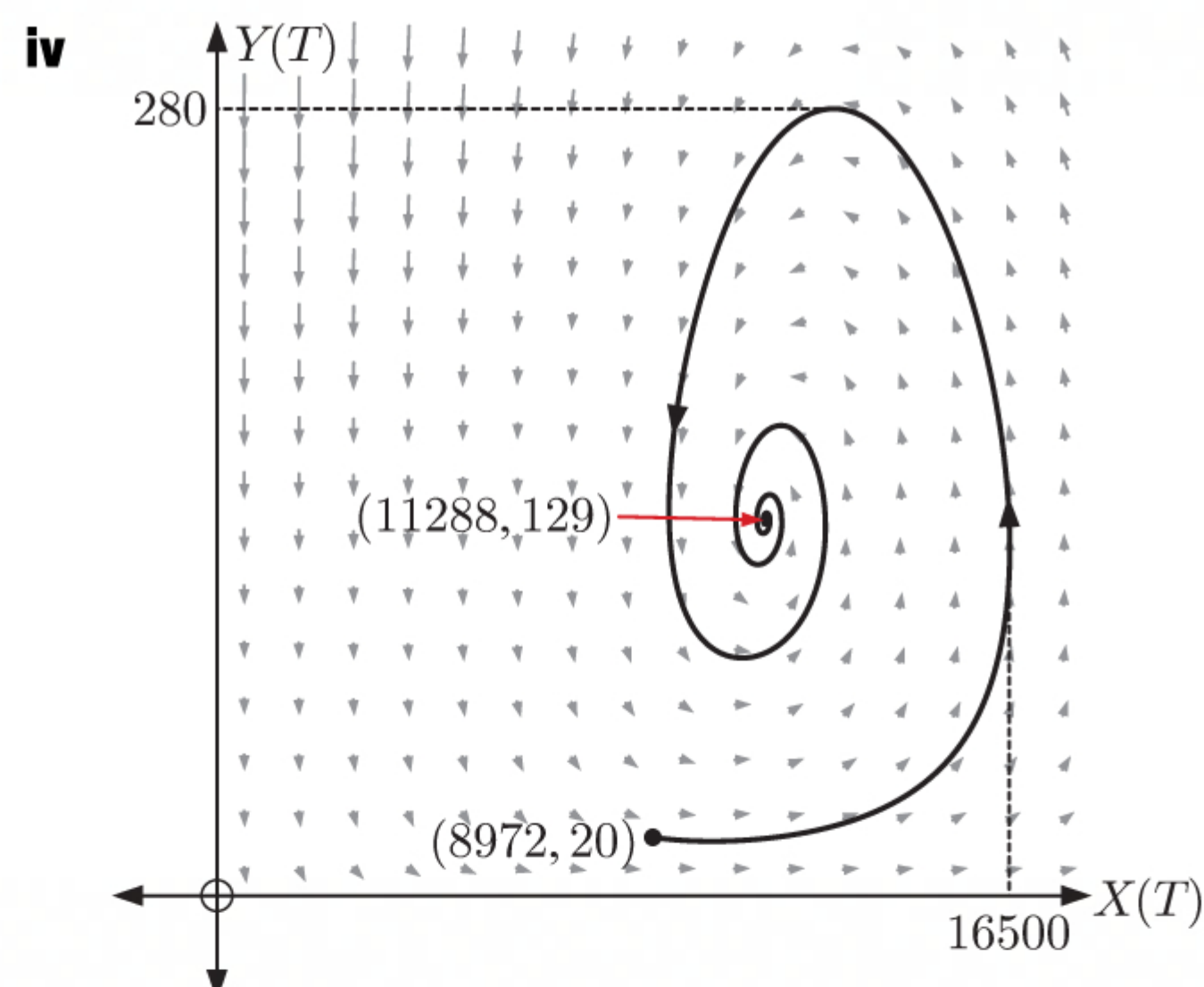
iii Using technology:

(1) The maximum population of deer  $\approx 16\,500$   
which occurs in the 7th year.

A1  
A1

(2) The maximum population of wolves  $\approx 280$   
which occurs in the 10th year.

A1  
A1



A1A1

v Ranger Roy has used a step size which is too large.

R1

Since Euler's method follows the tangent to the actual solution curve at each iteration, a spiral will never turn quickly enough under Euler's method.

In this case, the step size is so large that it appears the equilibrium point is unstable and the trajectory spirals outwards.

R1

Total [25 marks]

2 a  $p_{0,1} = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

A1

$p_{0,2} = e^{i\frac{4\pi}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

A1

b i  $p_{0,n} e^{i\frac{\pi}{6}} = e^{i\frac{2n\pi}{3}} e^{i\frac{\pi}{6}}$   
 $= e^{i(\frac{2n\pi}{3} + \frac{\pi}{6})}$

M1

The modulus of  $p_{0,n}$  is unchanged, but its argument has increased by  $\frac{\pi}{6}$ .

$\therefore p_{0,n}$  has been rotated anticlockwise through  $\frac{\pi}{6}$  about O.

R1

ii  $p_{1,1}$  lies on the side of  $T_0$  between  $p_{0,1}$  and  $p_{0,2}$ .

This side is vertical with real part  $-\frac{1}{2}$ .

$\therefore p_{1,1}$  has real part  $-\frac{1}{2}$ .

R1

iii  $p_{1,1} = p_{0,1} \times ke^{i\frac{\pi}{6}}$   
 $= e^{i\frac{2\pi}{3}} \times ke^{i\frac{\pi}{6}}$   
 $= ke^{i(\frac{2\pi}{3} + \frac{\pi}{6})}$   
 $= ke^{i\frac{5\pi}{6}}$   
 $= k\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

M1

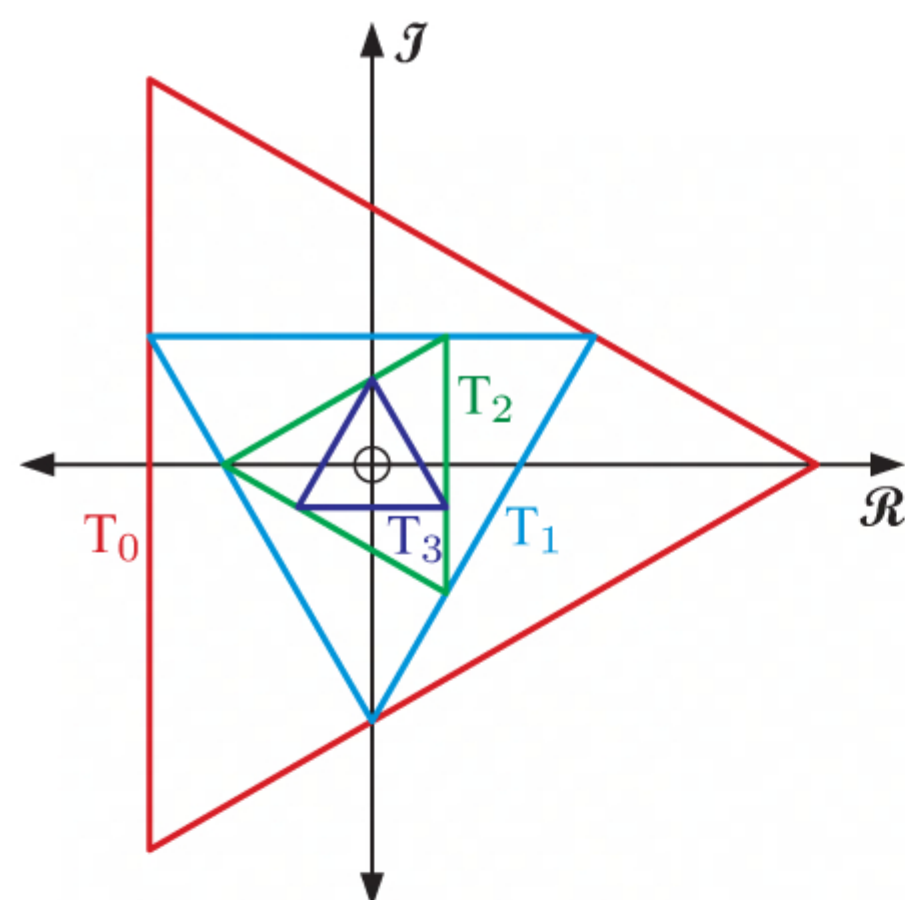
$\therefore \operatorname{Re}(p_{1,1}) = -\frac{\sqrt{3}}{2}k$

Using ii,  $-\frac{\sqrt{3}}{2}k = -\frac{1}{2}$

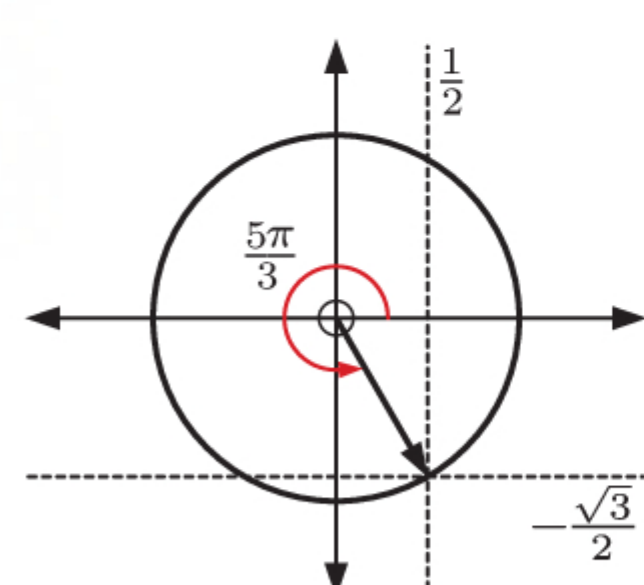
$\therefore k = \frac{1}{\sqrt{3}}$

A1



**c i**

**A1A1**

$$\begin{aligned}
 \text{ii } p_{m,n} &= p_{m-1,n} \times 3^{-\frac{1}{2}} e^{i\frac{\pi}{6}} \\
 &= p_{m-2,n} \times \left(3^{-\frac{1}{2}} e^{i\frac{\pi}{6}}\right)^2 \\
 &\vdots \\
 &= p_{0,n} \times \left(3^{-\frac{1}{2}} e^{i\frac{\pi}{6}}\right)^m \\
 &= e^{i\frac{2n\pi}{3}} \times 3^{-\frac{m}{2}} \times e^{i\frac{m\pi}{6}} \\
 &= 3^{-\frac{m}{2}} e^{i\pi(\frac{2n}{3} + \frac{m}{6})}
 \end{aligned}$$

**M1**
**A1**
**AG**
**iii**


$$\begin{aligned}
 p_{10,0} &= 3^{-5} e^{i\pi(0 + \frac{10}{6})} \\
 &= 3^{-5} e^{i\frac{5\pi}{3}} \\
 &= 3^{-5} \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\
 &= \frac{1}{486} - \frac{\sqrt{3}}{486}i
 \end{aligned}$$

**M1**
**A1**

$$\begin{aligned}
 \text{d i } \mathbf{A} &= \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{6} & -\sin \frac{\pi}{6} \\ \sin \frac{\pi}{6} & \cos \frac{\pi}{6} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2} \end{pmatrix}
 \end{aligned}$$

**M1M1**
**A1**

$$\begin{aligned}
 \text{ii } \mathbf{p}_{4,2} &= \mathbf{A}^4 \mathbf{p}_{0,2} \\
 &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2} \end{pmatrix}^4 \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{9} \\ 0 \end{pmatrix}
 \end{aligned}$$

**M1**

$$\therefore p_{4,2} = \frac{1}{9}$$

**A1**

$$\begin{aligned}
 \text{iii } \det \mathbf{A} &= \frac{1}{2} \left(\frac{1}{2}\right) - \left(-\frac{1}{2\sqrt{3}}\right) \left(\frac{1}{2\sqrt{3}}\right) \\
 &= \frac{1}{4} + \frac{1}{12} \\
 &= \frac{1}{3}
 \end{aligned}$$

**M1**
**A1**

$$\begin{aligned}
 \text{iv } \frac{\text{Area of } T_m}{\text{Area of } T_0} &= (\det \mathbf{A})^m \\
 &= \frac{1}{3^m}
 \end{aligned}$$

**A1**

$$\text{e i } X \sim B(m, q)$$

**A1**

**ii (1)** After  $m$  iterations,  $p_{0,0}$  has been rotated  $X$  times through  $\frac{\pi}{6}$  and  $m - X$  times through  $-\frac{\pi}{6}$ .

**M1**

Since  $p_{0,0} = 1$  has argument 0, the argument of  $p_{m,0}$  is  $Y = \frac{\pi}{6}X - \frac{\pi}{6}(m - X)$

$$\therefore Y = \frac{\pi}{3}X - \frac{\pi}{6}m$$

**A1**



$$\begin{aligned}
 \textbf{(2)} \quad E(Y) &= \frac{\pi}{3} E(X) - \frac{\pi}{6} m && \textbf{M1} \\
 &= \frac{\pi}{3} mq - \frac{\pi}{6} m \\
 &= \frac{\pi}{6} m(2q - 1) && \textbf{A1}
 \end{aligned}$$

$$\begin{aligned}
 \textbf{iii} \quad \text{If } m = 10 \text{ and } q = \frac{2}{5} \text{ then } X \sim B(10, \frac{2}{5}) \text{ and } Y &= \frac{\pi}{3} X - \frac{5\pi}{3} \\
 &= \frac{\pi}{3} (X - 5). && \textbf{M1}
 \end{aligned}$$

Since  $X \in \{0, 1, 2, 3, \dots, 9, 10\}$ ,

$$Y \in \left\{ -\frac{5\pi}{3}, -\frac{4\pi}{3}, -\frac{3\pi}{3}, \dots, \frac{4\pi}{3}, \frac{5\pi}{3} \right\}$$

From **ii**, the argument of  $p_{10,0}$  is  $Y$ .

$$\therefore p_{10,0} \text{ lies above the real axis if } -2\pi < Y < -\pi \quad \text{or} \quad 0 < Y < \pi \quad \textbf{M1}$$

$$\therefore -2\pi < \frac{\pi}{3}(X - 5) < -\pi \quad \text{or} \quad 0 < \frac{\pi}{3}(X - 5) < \pi$$

$$\therefore -6 < X - 5 < -3 \quad \text{or} \quad 0 < X - 5 < 3$$

$$\therefore -1 < X < 2 \quad \text{or} \quad 5 < X < 8 \quad \textbf{A1}$$

$$P(p_{10,0} \text{ lies above the real axis}) = P(X \leq 1) + P(X = 6) + P(X = 7)$$

$$\approx 0.0464 + 0.111 + 0.0425$$

$$\approx 0.200$$

**A1**

**Total [30 marks]**



## PAPER 3 PRACTICE

**1 a i**  $h(t) = a \cos(bt)^\circ + d$  metres

The amplitude is  $\frac{3.3 - 0.9}{2} = 1.2$  m, so  $|a| = 1.2$ .

When  $t = 0$  there is a high tide, so  $a > 0$   
 $\therefore a = 1.2$

The period is 12 h 25 min  $\approx 12.417$  h

$$\therefore b \approx \frac{360}{12.417} \approx 29.0$$

The average tide height is  $\frac{3.3 + 0.9}{2} = 2.1$  m  
 $\therefore d = 2.1$

$$\therefore h(t) \approx 1.2 \cos(29.0t)^\circ + 2.1 \text{ m}$$

**ii**  $h(t) \geq 3.0$  m when  $1.2 \cos(29.0t)^\circ + 2.1 \geq 3.0$   
 $\therefore 1.2 \cos(29.0t)^\circ \geq 0.9$   
 $\therefore \cos(29.0t)^\circ \geq \frac{3}{4}$

For  $0^\circ \leq \theta \leq 360^\circ$ ,  $\cos \theta = \frac{3}{4}$  when  $\theta \approx 41.4^\circ$   
and  $\theta \approx 360^\circ - 41.4^\circ \approx 318.6^\circ$

$\therefore \cos \theta \geq \frac{3}{4}$  when  $0^\circ \leq \theta \leq 41.4^\circ$  and  $318.6^\circ \leq \theta \leq 360^\circ$ , which is about  
 $\frac{2 \times 41.4}{360} \approx 23.0\%$  of the time.

$\therefore h(t) \geq 3.0$  m about 23.0% of the time.

Now  $0.230 \times 24$  hours  $\approx 5.52$  hours  
 $\approx 5$  h 31 min

$\therefore$  on average, the fishing boats can pass over the reef for about 5 h 31 min per day.

**b i**  $a(t) = 0.4 \cos\left(\frac{15}{14}t\right)^\circ + 1.2$  m is an appropriate model because:

- its maximum is 0.4 m above its mean value, which is consistent with the 3.7 m “spring” high tide being 0.4 m above average
- its period is  $\frac{360}{\frac{15}{14}} \text{ h} = 336 \text{ h} = 14$  days, which is consistent with the 14 days between “new” and “full” moons
- its mean value is 1.2 m, which is the amplitude used in Herlina’s first model.

**ii (1)**  $f_1(t) = 2.1 + a(t)$   
 $= 0.4 \cos\left(\frac{15}{14}t\right)^\circ + 3.3$  shows the fluctuation of high tides with the phases of the moon.  
 $f_2(t) = 2.1 - a(t)$   
 $= -0.4 \cos\left(\frac{15}{14}t\right)^\circ + 0.9$  shows the fluctuation of low tides with the phases of the moon.

If time  $t$  corresponds to a high tide then  $f_1(t)$  tells us its height.

If time  $t$  corresponds to a low tide then  $f_2(t)$  tells us its height.

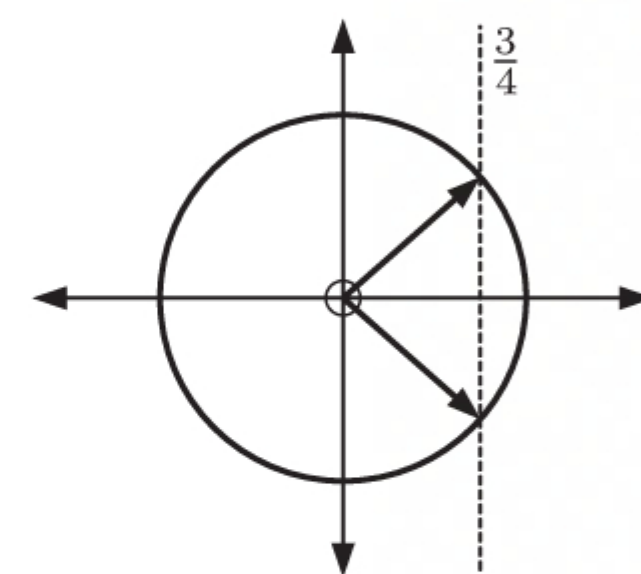
**(2)** 28 days = 672 h

From the graph, there appear to be 12 high tides under 3 m in a lunar month, at which the fishing boats will be unable to cross the reef.

**c** The first “neap” high tide is at 173.8 hours.

$$H(172.3) \approx 2.683 \text{ and } H(175.3) \approx 2.681$$

$\therefore$  if the reef is cut back 42 cm to 2.68 m, the fishermen will have the necessary 3 hours to safely cross the reef, even at “neap” high tide.





**2 a i**  $\frac{dI}{dt} = \frac{\beta SI}{N} = \frac{\beta(N-I)I}{N}$

**ii** Suppose  $I = \frac{N}{1 + ae^{-\beta t}} = N(1 + ae^{-\beta t})^{-1}$

$$\begin{aligned}\therefore \frac{dI}{dt} &= \frac{-N(-a\beta e^{-\beta t})}{(1 + ae^{-\beta t})^2} \\ &= \beta \frac{N}{1 + ae^{-\beta t}} \frac{ae^{-\beta t}}{1 + ae^{-\beta t}} \\ &= \beta I \frac{1 + ae^{-\beta t} - 1}{1 + ae^{-\beta t}} \\ &= \beta I \left(1 - \frac{1}{1 + ae^{-\beta t}}\right) \\ &= \frac{\beta I}{N} \left(N - \frac{N}{1 + ae^{-\beta t}}\right) \\ &= \frac{\beta(N-I)I}{N}\end{aligned}$$

$\therefore I = \frac{N}{1 + ae^{-\beta t}}$  satisfies the differential equation.

**iii**  $S = N - I$

$\therefore S = N \left(1 - \frac{1}{1 + ae^{-\beta t}}\right)$

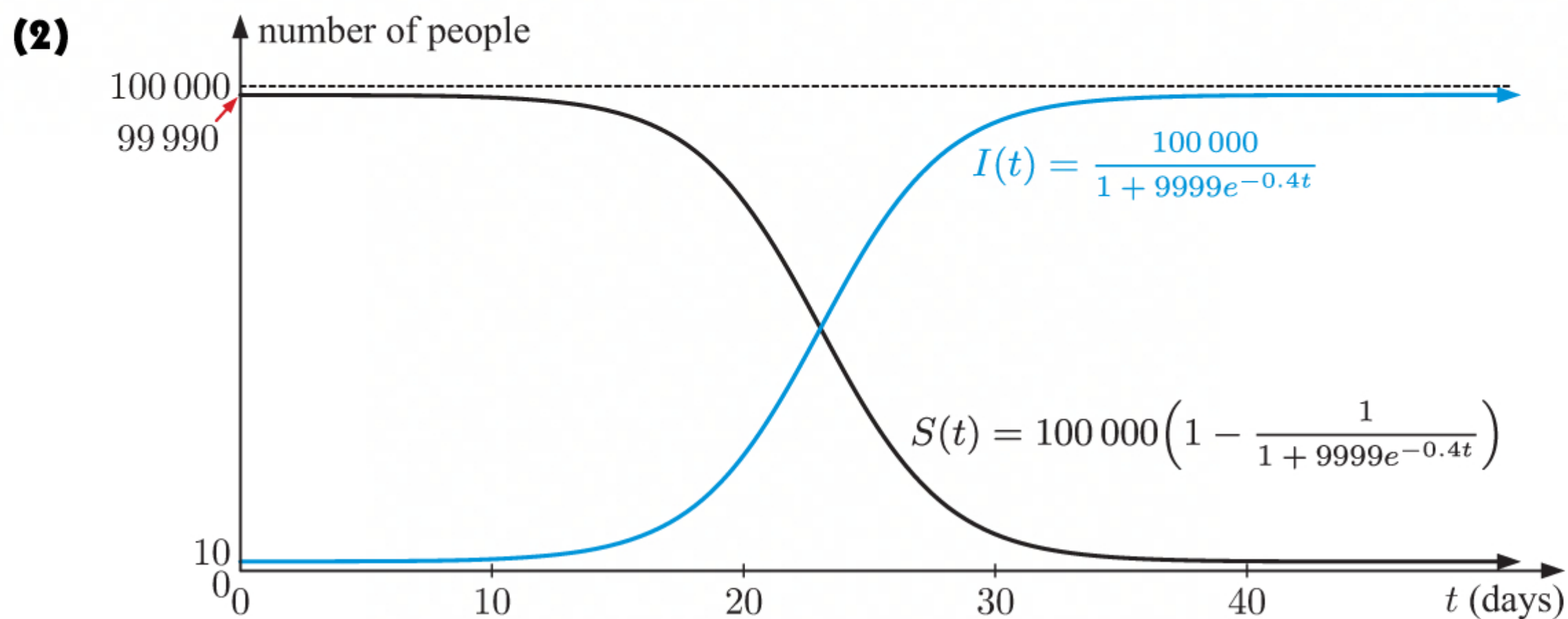
**iv (1)**  $N = S_0 + I_0 = 100\,000$

$\therefore I_0 = \frac{100\,000}{1 + ae^0} = 10$

$\therefore 10 + 10a = 100\,000$

$\therefore 10a = 99\,990$

$\therefore a = 9999$



**(3)** If  $I(t) = 0.5N$  then  $\frac{100\,000}{1 + 9999e^{-0.4t}} = 50\,000$

$\therefore 1 + 9999e^{-0.4t} = 2$

$\therefore 9999e^{-0.4t} = 1$

Using technology,  $t \approx 23.0$  days

**b i** Equilibrium points occur when both  $\frac{dS}{dt} = 0$  and  $\frac{dI}{dt} = 0$ .

However,  $\frac{dI}{dt} = -\frac{dS}{dt}$  so satisfying one will satisfy both.

$\therefore \frac{\beta SI}{N} - \gamma I = 0$

$\therefore I \left( \frac{\beta S}{N} - \gamma \right) = 0$

$\therefore I = 0$  or  $\frac{\beta S}{N} = \gamma$

If  $I = 0$  then  $S = N - I = N$ .

If  $\frac{\beta S}{N} = \gamma$  then  $S = \frac{\gamma}{\beta} N$  and  $I = N - S = N \left(1 - \frac{\gamma}{\beta}\right)$ .

So, the equilibrium points are  $(S = N, I = 0)$  and  $\left(S = \frac{\gamma}{\beta} N, I = \left(1 - \frac{\gamma}{\beta}\right) N\right)$ .



- ii** If  $\gamma > \beta$  then for the second equilibrium point,  $S > N$  and  $I < 0$ . This is impossible.

In this case, the rate of recovery  $\gamma$  is faster than the rate at which the virus is spread  $\beta$ . The infection will therefore die out.

**iii (1)**  $\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I = I \left( \frac{\beta S}{N} - \gamma \right)$

Assuming  $I > 0$ , or in other words that the virus is indeed present in the population,  $\frac{dI}{dt} > 0$  if  $\frac{S}{N} > \frac{\gamma}{\beta}$ .

- (2)** The number of infected individuals  $I$  will increase if  $\frac{S}{N}$  is greater than some critical proportion  $\frac{\gamma}{\beta}$  of the population, and decrease if  $\frac{S}{N}$  is less than this critical proportion.

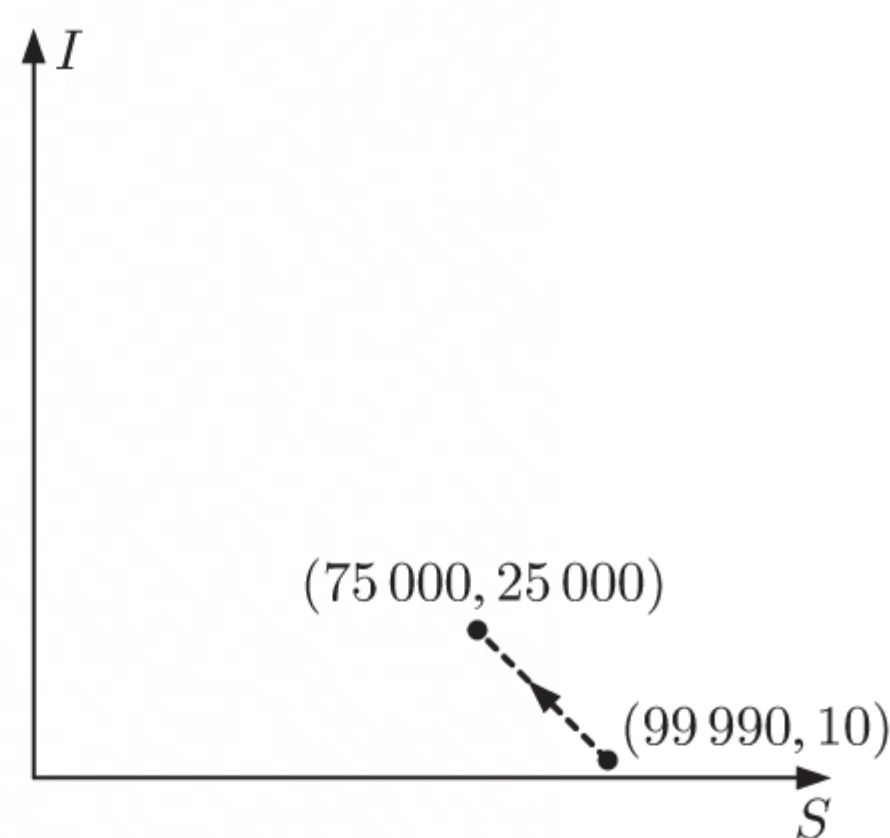
So, the equilibrium point  $\left( S = \frac{\gamma}{\beta} N, I = \left( 1 - \frac{\gamma}{\beta} \right) N \right)$  is stable, and the equilibrium point  $(S = N, I = 0)$  is unstable.

- iv (1)** If  $\frac{\gamma}{\beta} = 0.75$  and  $N = 100\,000$  then the stable equilibrium point is  $(S = 75\,000, I = 25\,000)$ .

$\frac{S_0}{N} > \frac{\gamma}{\beta}$ , so  $\frac{dI}{dt} > 0$ .

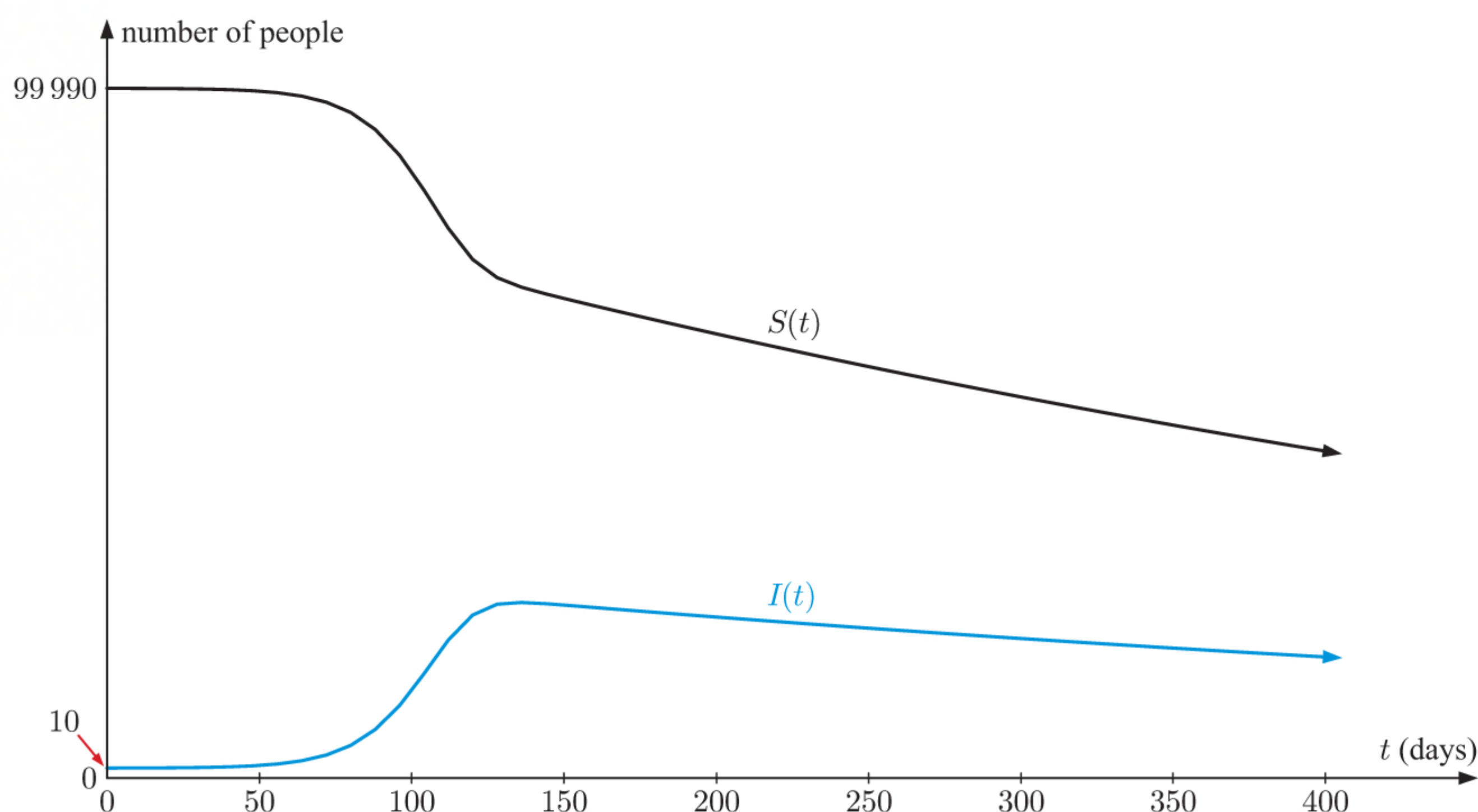
The number of infected individuals increases to a quarter of the population.

- (2)** Since  $S + I = N$  is constant, the trajectory is a straight line.



- c i (1)**  $S_0 = 99\,990$ ,  $I_0 = 10$ , time step = 8 days

$$\begin{cases} \frac{dS}{dt} = -0.4 \frac{SI}{S+I} + 0.294I \\ \frac{dI}{dt} = 0.4 \frac{SI}{S+I} - 0.3I \end{cases}$$



- (2)**  $S(240) \approx 60\,564$  and  $I(240) \approx 20\,600$

$\therefore$  about  $100\,000 - 60\,564 - 20\,600 \approx 18\,836$  people have died so far.

- (3)** Since  $\gamma < \beta$ , the virus will remain in the population, and eventually wipe the population out.

The population is destined for extinction.

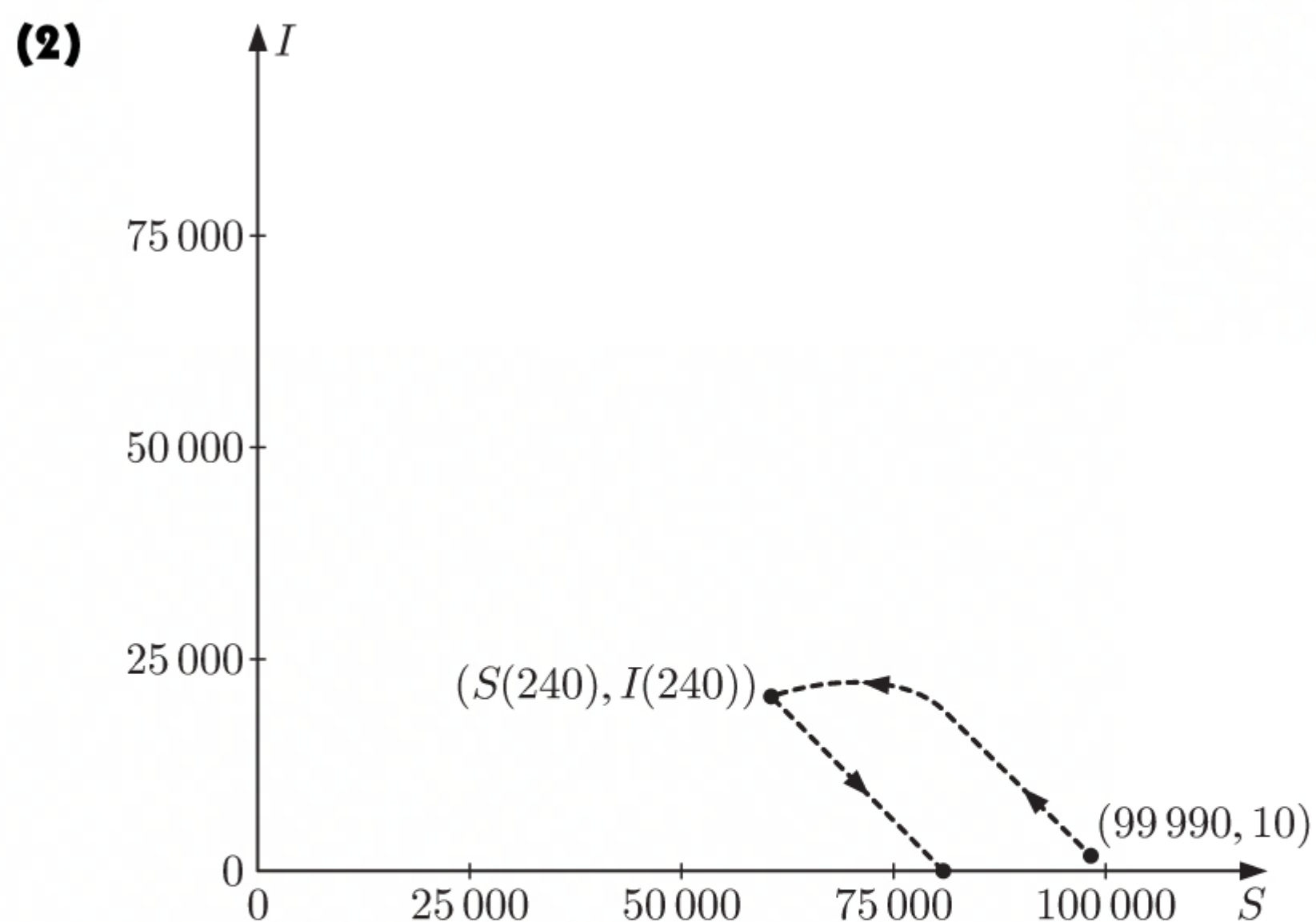


- ii Reducing  $\mu$  alone will not be sufficient to remove the virus from the population. This can only be achieved if  $\gamma > \beta$ .

So, the population is still destined for extinction, but more slowly.

- iii (1) The phase portrait **B** must correspond to  $\beta = 0.4$ , because we can see the trajectory taking the population to extinction.

The phase portrait **A** corresponds to  $\beta = 0.28$ . In this case  $\gamma > \beta$ , so the trajectory takes us to an equilibrium point where the infection is eliminated and the population survives.



- (3) The only way to ensure the rate of spread  $\beta$  is reduced to a value lower than the rate of recovery, is by the population cooperating with social distancing, wearing protective equipment, and staying in quarantine.